応用数学

1

パラメータ $0 \leqq r < 1$ をもつ関数

$$K_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}, \quad 0 \leqq \theta \leqq 2\pi$$

を考える. また, 周期 2π の連続な周期関数 $u(\theta)$ と $K_r(\theta)$ との合成積を

$$(u * K_r)(\theta) = \int_0^{2\pi} u(\phi)K_r(\theta-\phi)d\phi$$

で定義する. 以下の問いに答えよ.

(i) $\displaystyle\int_0^{2\pi} K_r(\phi)d\phi = 1, \int_0^{2\pi} K_r(\theta-\phi)d\phi = 1$ を示せ.

(ii) $r \to 1$ のとき, $K_r(\theta)$ は θ の関数として連続とはならないことを示せ.

(iii) $r \to 1$ のとき, $(u * K_r)(\theta) \to u(\theta)$ を証明せよ. ヒント：適当な $\delta > 0$ をとって, 単位円周上の積分を, 円弧 $|\theta - \phi| \leqq \delta$ 上の積分と円弧の補集合上の積分とに分けて評価する.

Applied Mathematics

1

Let

$$K_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}, \quad 0 \leqq \theta \leqq 2\pi,$$

where r is a real parameter with $0 \leqq r < 1$. The convolution of continuous periodic functions $u(\theta)$ and $K_r(\theta)$ of period 2π is defined to be

$$(u * K_r)(\theta) = \int_0^{2\pi} u(\phi) K_r(\theta - \phi) d\phi.$$

Answer the following questions.

(i) Show that $\displaystyle\int_0^{2\pi} K_r(\phi) d\phi = 1$ and $\displaystyle\int_0^{2\pi} K_r(\theta - \phi) d\phi = 1$.

(ii) Show that $K_r(\theta)$ will not be continuous in θ as $r \to 1$.

(iii) Show that $(u * K_r)(\theta) \to u(\theta)$ as $r \to 1$. Hint: Take a positive constant $\delta > 0$, and divide the unit circle into the union of the arc determined by $|\theta - \phi| \leqq \delta$ and the complement to the arc. Then, evaluate an appropriate integral according to the division of the unit circle.