基礎数学 I

$\boxed{1}$

n がすべての自然数 $\mathbb{N}$ の上を動くとき，$\dfrac{1}{n}$ の和 $\displaystyle\sum_{n\in\mathbb{N}}\frac{1}{n}$ は $+\infty$ に発散し，$\dfrac{1}{n^2}$ の和 $\displaystyle\sum_{n\in\mathbb{N}}\frac{1}{n^2}$ はある正の値に収束する．p がすべての素数 $\mathbb{P}$ の上を動くとき，$\dfrac{1}{p}$ の和と $1+\dfrac{1}{p}$ の積を，それぞれ，

$$\sum_{p\in\mathbb{P}}\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots,\quad \prod_{p\in\mathbb{P}}\left(1+\frac{1}{p}\right)=\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots,\quad p_1,p_2,\cdots\in\mathbb{P}$$

とかく．自然数 n は適当な非負整数 $k_1,k_2,\cdots\in\{0,1,2,\ldots\}$ を用いて

$$n=p_1^{k_1}\cdot p_2^{k_2}\cdot p_3^{k_3}\cdots$$

と一意に素因数分解できることに注意する．L を 2 以上の自然数とするとき，L 以下の素数のなす有限部分集合を $\mathbb{P}_L$ とかく．以下の問いに答えよ．

(i) 不等式

$$\log\prod_{p\in\mathbb{P}_L}\left(1+\frac{1}{p}\right)<\sum_{p\in\mathbb{P}_L}\frac{1}{p}$$

を示せ．

(ii) $\displaystyle\prod_{p\in\mathbb{P}}\frac{1}{1-\frac{1}{p}}=\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)$ は $+\infty$ に発散することを示せ．

(iii) $\displaystyle\prod_{p\in\mathbb{P}}\frac{1}{1-\frac{1}{p^2}}$ はある正の値に収束することを示せ．

(iv) $\displaystyle\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p}\right)$ の収束，発散について理由をつけて答えよ．

(v) $\displaystyle\sum_{p\in\mathbb{P}}\frac{1}{p}$ の収束，発散について理由をつけて答えよ．

Basic Mathematics I

1

When n runs over the set $\mathbb{N}$ of all natural numbers, the sum $\displaystyle\sum_{n\in\mathbb{N}} \frac{1}{n}$ diverges to $+\infty$ and the sum $\displaystyle\sum_{n\in\mathbb{N}} \frac{1}{n^2}$ converges to a positive constant. Let

$$\sum_{p\in\mathbb{P}} \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots, \quad \prod_{p\in\mathbb{P}} \left(1+\frac{1}{p}\right) = \left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots, \quad p_1, p_2, \cdots \in \mathbb{P}$$

be the sum of $\dfrac{1}{p}$ and the product of $1+\dfrac{1}{p}$, respectively, when p runs over the set $\mathbb{P}$ of all prime numbers. Note that any natural number n admits a unique prime factor decomposition

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots$$

in terms of suitable nonnegative integers $k_1, k_2, \cdots \in \{0, 1, 2, \dots\}$. Let L be a natural number greater than or equal to 2. Let $\mathbb{P}_L$ be the finite set of prime numbers less than or equal to the natural number L. Answer the following questions.

(i) Show the inequality

$$\log \prod_{p\in\mathbb{P}_L} \left(1+\frac{1}{p}\right) < \sum_{p\in\mathbb{P}_L} \frac{1}{p}.$$

(ii) Show that $\displaystyle\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}} = \prod_{p\in\mathbb{P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)$ diverges to $+\infty$.

(iii) Show that $\displaystyle\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^2}}$ converges to a positive constant.

(iv) Determine whether the product $\displaystyle\prod_{p\in\mathbb{P}} \left(1+\frac{1}{p}\right)$ converges or not, giving reason for the answer.

(v) Determine whether the sum $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ converges or not, giving reason for the answer.