線形計画

3

以下の (i), (ii) に答えよ.

(i) 次の線形計画問題 (P1) とその双対問題 (D1) を考える.

$$
\text{(P1)}: \begin{array}{ll} \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geqq \boldsymbol{0} \end{array} \qquad \text{(D1)}: \begin{array}{ll} \text{Maximize} & \boldsymbol{b}^\top \boldsymbol{w} \\ \text{subject to} & \boldsymbol{A}^\top \boldsymbol{w} \leqq \boldsymbol{c} \end{array}
$$

ここで, $\boldsymbol{A}$ は $m \times n$ 定数行列, $\boldsymbol{b}$ は m 次元定数ベクトル, $\boldsymbol{c}$ は n 次元定数ベクトル, $\boldsymbol{x}$ は n 次元変数ベクトル, $\boldsymbol{w}$ は m 次元変数ベクトルであり, $^\top$ は転置記号を表す. 問題 (P1) と (D1) は最適解 $\boldsymbol{x}^*$ と $\boldsymbol{w}^*$ をもつとする. さらに $\boldsymbol{y}^* = \boldsymbol{c} - \boldsymbol{A}^\top \boldsymbol{w}^*$ とする. このとき, $x_i^* > 0$ であれば, $y_i^* = 0$ が成り立つことを示せ.

(ii) 次の線形計画問題を考える.

$$
\text{(P2)}: \begin{array}{ll} \text{Maximize} & x_5 \\ \text{subject to} & \displaystyle\sum_{i=1}^{4} x_i \leqq 1 \\ & \displaystyle\sum_{i=k+1}^{4} x_i \leqq k x_k \quad (k = 1, 2, 3) \\ & x_5 \leqq 4 x_4 \end{array}
$$

問題 (P2) の最適解を $\boldsymbol{x}^*$ とする. 問題 (P2) の双対問題の最適解を求めよ. さらに,

$$
\sum_{i=1}^{4} x_i^* = 1
$$

が成り立つことを示せ.

An English Translation:

Linear Programming

3

Answer the following questions (i) and (ii).

(i) Consider the following linear programming problem (P1) and its dual problem (D1):

$$
\begin{array}{llll}
\text{(P1)}: & \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\
& \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& & \boldsymbol{x} \geqq 0,
\end{array}
\qquad
\begin{array}{lll}
\text{(D1)}: & \text{Maximize} & \boldsymbol{b}^\top \boldsymbol{w} \\
& \text{subject to} & \boldsymbol{A}^\top \boldsymbol{w} \leqq \boldsymbol{c},
\end{array}
$$

where $\boldsymbol{A}$ is an $m \times n$ constant matrix, $\boldsymbol{b}$ is an m-dimensional constant vector, $\boldsymbol{c}$ is an n-dimensional constant vector, $\boldsymbol{x}$ is an n-dimensional vector of variables, $\boldsymbol{w}$ is an m-dimensional vector of variables, and $^\top$ denotes transposition. Suppose that problems (P1) and (D1) have optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{w}^*$, respectively. Let $\boldsymbol{y}^* = \boldsymbol{c} - \boldsymbol{A}^\top \boldsymbol{w}^*$. Then show that $y_i^* = 0$ if $x_i^* > 0$.

(ii) Consider the following linear programming problem:

$$
\begin{array}{lll}
\text{(P2)}: & \text{Maximize} & x_5 \\
& \text{subject to} & \displaystyle\sum_{i=1}^{4} x_i \leqq 1 \\
& & \displaystyle\sum_{i=k+1}^{4} x_i \leqq k x_k \quad (k = 1, 2, 3) \\
& & x_5 \leqq 4x_4.
\end{array}
$$

Let $\boldsymbol{x}^*$ be an optimal solution to problem (P2). Obtain an optimal solution to the dual problem of problem (P2). Moreover, show that

$$\sum_{i=1}^{4} x_i^* = 1.$$