応用数学

1

z を $\mathbb{C}$ 上の複素変数とする．z の多項式 $A(z), B(z), C(z)$ と定数 λ を用いて，次の 2 階の常微分方程式を定義する．

$$A(z)\frac{d^2}{dz^2}\phi(z) + B(z)\frac{d}{dz}\phi(z) + C(z)\phi(z) = \lambda\phi(z) \tag{1}$$

このとき，恒等的に 0 でない関数 $\xi(z)$ による変換 $\phi(z) \mapsto \xi(z)\phi(z)$ について考える．この変換にともない，式 (1) は，

$$\frac{1}{\xi(z)}\left(A(z)\frac{d^2}{dz^2}(\xi(z)\phi(z)) + B(z)\frac{d}{dz}(\xi(z)\phi(z)) + C(z)\xi(z)\phi(z)\right) = \lambda\phi(z)$$

と変換され，さらに次の形式に書き表される．

$$\tilde{A}(z)\frac{d^2}{dz^2}\phi(z) + \tilde{B}(z)\frac{d}{dz}\phi(z) + \tilde{C}(z)\phi(z) = \lambda\phi(z)$$

次に，向き付けられた単純閉曲線 γ に依存する関数 ξ_γ を，定数倍の不定性を除いて

$$\frac{\xi'_\gamma(z)}{\xi_\gamma(z)} = \frac{1}{2\pi i}\oint_\gamma \frac{B(\zeta) - A'(\zeta)}{A(\zeta)}\frac{d\zeta}{\zeta - z}$$

によって定義する．ここで，ξ'_γ と A' はそれぞれ ξ_γ と A の導関数を表す．また，向き付けられた単純閉曲線 γ は，γ に沿って動くとき，任意定数に固定された点 z が γ の右側領域に位置するよう $\mathbb{C} \setminus \{\text{zeros of } A(\zeta), z\}$ 上に選ぶこととする．定数 $\alpha \neq 0$ を用いて，$A(z) = z, B(z) = \alpha + 1 - z, C(z) = 0$ とする．このとき，以下の問いに答えよ．

(i) γ を反時計回りの単位円 $|\zeta| = 1$ とするとき，$\tilde{A}(z), \tilde{B}(z), \tilde{C}(z)$ を求めよ．

(ii) 十分大きい正の定数 R に対し，関数 $f(z)$ は領域 $R < |z| < \infty$ で正則であり，$f(z)$ の $z = \infty$ での留数を $\operatorname*{Res}_{z=\infty} f(z)$ で表わすとする．このとき，$\operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i}\oint_{|z|=r>R} f(z)dz$ が成り立つ．ここで，$|z| = r$ は反時計回りの周回積分路とする．$\lim_{z\to\infty} zf(z)$ が収束する場合，$\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} zf(z)$ が成り立つことを示せ．(Hint: 変数変換 $z = 1/y$ を用いてよい)

(iii) 点 z を右側領域にみる $\mathbb{C} \setminus \{0, z\}$ 上の任意の向き付けられた単純閉曲線 γ に対して，$\tilde{A}(z), \tilde{B}(z), \tilde{C}(z)$ が z の多項式であることを示せ．

An English Translation:

Applied Mathematics

1

Let z be a complex variable on $\mathbb{C}$. For polynomials $A(z), B(z), C(z)$ in z and a constant λ, define the second order differential equation

$$A(z)\frac{d^2}{dz^2}\phi(z) + B(z)\frac{d}{dz}\phi(z) + C(z)\phi(z) = \lambda\phi(z). \tag{1}$$

Consider the transformation of ϕ: $\phi(z) \mapsto \xi(z)\phi(z)$ with a given function $\xi(z)$ which is not identically zero. Then, eq.(1) is transformed, accordingly, into

$$\frac{1}{\xi(z)}\left(A(z)\frac{d^2}{dz^2}\left(\xi(z)\phi(z)\right) + B(z)\frac{d}{dz}\left(\xi(z)\phi(z)\right) + C(z)\xi(z)\phi(z)\right) = \lambda\phi(z),$$

which can be put in the form

$$\tilde{A}(z)\frac{d^2}{dz^2}\phi(z) + \tilde{B}(z)\frac{d}{dz}\phi(z) + \tilde{C}(z)\phi(z) = \lambda\phi(z).$$

Draw an oriented, simple closed curve γ in $\mathbb{C}\backslash\{\text{zeros of } A(\zeta),\ z\}$ in such a manner that one looks at the point z, fixed to an arbitrary constant, in the right-hand side region when one moves along γ. For a given γ, we define a function ξ_γ depending on γ within constant multipliers through

$$\frac{\xi'_\gamma(z)}{\xi_\gamma(z)} = \frac{1}{2\pi i}\oint_\gamma \frac{B(\zeta) - A'(\zeta)}{A(\zeta)}\frac{d\zeta}{\zeta - z},$$

where ξ'_γ and A' denote the derivatives of ξ_γ and A, respectively. Let $A(z) = z, B(z) = \alpha + 1 - z, C(z) = 0$, with $\alpha \neq 0$ a constant. Answer the following questions.

(i) Let γ be the unit circle $|\zeta| = 1$ with counter-clockwise orientation. Compute $\tilde{A}(z), \tilde{B}(z)$ and $\tilde{C}(z)$.

(ii) Suppose that a function $f(z)$ is regular in the region $R < |z| < \infty$ for a sufficiently large constant $R > 0$. Let us denote the residue of $f(z)$ at $z = \infty$ by $\operatorname*{Res}_{z=\infty} f(z)$. Then one has $\operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i}\oint_{|z|=r>R} f(z)dz$, where the closed curve $|z| = r$ is traversed counter-clockwise. Show that, if $\lim_{z\to\infty} zf(z)$ converges, then $\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} zf(z)$. (Hint: You may use the variable transformation $z = 1/y$.)

(iii) Show that $\tilde{A}(z), \tilde{B}(z), \tilde{C}(z)$ are polynomials in z for any oriented, simple closed curve γ in $\mathbb{C}\backslash\{0,\ z\}$, as long as γ is subject to the condition: when one moves along γ, one looks at z in the right-hand side region.