物理統計学

5 [I] または [II] のどちらか一方を選択して解答せよ．

[I]

$X(t)$ は次のランジュバン方程式に従うとする．

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\frac{1}{\gamma}((X(t))^3 - X(t)) + F(t),$$

ここで，γ は正の定数．$F(t)$ は白色雑音で，$\langle F(t)\rangle = 0$，$\langle F(t)F(s)\rangle = 2D\delta(t-s)$．（$\langle A\rangle$ は A の平均を表し，D は正の定数，$\delta(t)$ はディラックのデルタ関数.）$X(t)$ の確率密度関数を $f(x,t)$ とし，$\lim_{x\to\pm\infty} f(x,t) = \lim_{x\to\pm\infty}\frac{\partial f(x,t)}{\partial x} = 0$ とする．$p(z,t+\Delta t|x,t)$ を，$X(t)=x$ という条件付きの $X(t+\Delta t)$ の確率密度関数とする．すなわち，$X(t)=x$ という条件のもとで無限小の $\Delta z\ (>0)$ に対して，$z < X(t+\Delta t) \leqq z+\Delta z$ である確率が $p(z,t+\Delta t|x,t)\Delta z$ である．$a_n := \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}z(z-x)^n p(z,t+\Delta t|x,t)\ (n=1,2,\cdots)$ とすると，$F(t)$ が白色雑音で あるので，$a_n = 0\ (n \geqq 3)$ であることが証明できる．以下の問いに答えよ．

(i) $\Delta X(t) := X(t+\Delta t) - X(t)$ を $(X(t))^3 - X(t)$，$F(t)$ の積分を用いて表せ．

(ii) a_1 と a_2 を求めよ．

(iii) $\{X(t)\}$ はマルコフ過程であるので，確率密度関数 $f(x,t)$ は $\int \mathrm{d}x g(x)\frac{\partial f(x,t)}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}x g(x)(\int \mathrm{d}z p(x,t+\Delta t|z,t)f(z,t) - f(x,t))$ を満たす．ここで，$g(x)$ は適当に滑らかで，上の積分が存在するような任意の関数とする．$g(x)$ を z のまわりでテーラー展開して，$f(x,t)$ に対するフォッカー・プランク方程式を導け．

(iv) (iii) で導いたフォッカー・プランク方程式の定常解 $f_{\mathrm{st}}(x)$ を求めよ．

物理統計学

5 [I] または [II] のどちらか一方を選択して解答せよ．

[II]

m 次 Chebyshev 多項式 $T_m(x), (\cos m\theta = T_m(\cos\theta)), T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, \cdots$ によって与えられる $\Omega = [-1, 1]$ 上の写像力学系 $X_{n+1} = T_m(X_n), m \geqq 2,$ によって生成される離散時間上の確率過程 $\{X_n\}_{n\geqq 0}$ を考える．

以下の問いに答えよ．

(i) 写像 $T_m, m \geqq 2,$ は, $\Omega = [-1, 1]$ 上の任意の可測集合 B に対して定義される確率測度 $\mu(B) = \int_B d\mu(x) = \int_B \frac{dx}{\pi\sqrt{1-x^2}}$ を保存する保測変換であることを示せ．

(ii) Chebyshev 多項式 $T_m(x), m \geqq 0,$ が直交性を持つことを証明せよ．ここで, 直交性とは, 以下の等式が成立することをいう．

$$\langle T_l, T_m \rangle \equiv \int_{-1}^{1} T_l(x) T_m(x) d\mu(x) = \frac{1+\delta_{l,0}}{2}\delta_{l,m} \tag{1}$$

ここに, $\delta_{l,m}$ は, Kronecker のデルタ関数である．

(iii) $\Omega = [-1, 1]$ 上の実数値をとる可測関数で絶対値の２乗が可積分なものの全体は, 内積

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) d\mu(x) \tag{2}$$

により Hilbert 空間をなす．それを $L^2(\Omega, \mu)$ とする．いま, 確率過程 $\{X_n\}$ の時間発展演算子 U_m を, $f \in L^2(\Omega, \mu)$ に対して, $U_m f(x) = f(T_m(x)) = f \circ T_m(x)$ と定義する．f が高々 L 次であるとき, f の Chebyshev 多項式展開

$$f(x) = a_0 + \sum_{i=1}^{L} a_i T_i(x) \tag{3}$$

に対して, 展開係数 $\{a_0, a_1, \cdots, a_L\}$ を用いて $\|f\|^2 = \langle f, f \rangle < \infty$ を表わし, 任意の $l\,(\geqq 0)$ に対して, $\|U_m^l f\|^2 = \|f\|^2$ となることを示せ．

(iv) ２次以上の Chebyshev 多項式で与えられる写像力学系 $X_{n+1} = T_m(X_n)$ が混合的であることを証明せよ．ただし, 混合的とは, 任意の $f, g \in L^2(\Omega, \mu)$ に対して,

$$\int_{-1}^{1} f(X_{n+l}) g(X_n) d\mu(X_n) = \langle U_m^l f, g \rangle \to \langle f, 1 \rangle \langle g, 1 \rangle, \quad l \to \infty \tag{4}$$

が成立することを意味する．ここで, 任意の $f \in L^2(\Omega, \mu)$ は, 適当な L 個の有限個の Chebyshev 多項式の線型和により十分近次できることを用いてよい．

An English Translation:

Physical Statistics

5 Select either [I] or [II], and answer it.

[I]

Let $X(t)$ obey the following Langevin equation,

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\frac{1}{\gamma}((X(t))^3 - X(t)) + F(t),$$

where γ is a positive constant and $F(t)$ the white noise, whose mean and correlation function are given by $\langle F(t)\rangle = 0$ and $\langle F(t)F(s)\rangle = 2D\delta(t-s)$, respectively. Here, $\langle A\rangle$ denotes the average of A, D a positive constant and $\delta(t)$ Dirac's delta function. Let $f(x,t)$ denote the probability density function for $X(t)$ and $\lim_{x\to\pm\infty} f(x,t) = \lim_{x\to\pm\infty} \frac{\partial f(x,t)}{\partial x} = 0$ is assumed. Let $p(z, t+\Delta t|x,t)$ be the conditional probability density function for $X(t+\Delta t)$ with the condition that $X(t) = x$. That is, $p(z,t+\Delta t|x,t)\Delta z$ for a positive infinitesimal amount Δz denotes the probability such that $z < X(t+\Delta t) \leqq z + \Delta z$ conditioning on $X(t) = x$. Define $a_n := \lim_{\Delta t\to 0} \frac{1}{\Delta t}\int \mathrm{d}z(z-x)^n p(z,t+\Delta t|x,t)$, where $n = 1, 2, \cdots$. Since $F(t)$ is the white noise, it can be proved that $a_n = 0$ for $n \geqq 3$. Answer the following questions.

(i) Express $\Delta X(t) := X(t+\Delta t) - X(t)$ with the use of the integrals of $(X(t))^3 - X(t)$ and $F(t)$.

(ii) Calculate a_1 and a_2.

(iii) Since $\{X(t)\}$ is a Markov process, the probability density function $f(x,t)$ satisfies $\int \mathrm{d}x g(x)\frac{\partial f(x,t)}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}x g(x)(\int \mathrm{d}z p(x,t+\Delta t|z,t)f(z,t) - f(x,t))$, where $g(x)$ is an arbitrary function for which the existence of the above integrals and its appropriate differentiability are assumed. With the Taylor series of $g(x)$ around z, derive the Fokker-Planck equation for $f(x,t)$.

(iv) Find the stationary solution $f_{\mathrm{st}}(x)$ of the Fokker-Planck equation derived in (iii).

An English Translation:

Physical Statistics

5 Select either [I] or [II], and answer it.

[II]

Let us consider discrete-time stochastic processes $\{X_n\}_{n \geqq 0}$ generated by the dynamical systems $X_{n+1} = T_m(X_n)$ on the finite interval $\Omega = [-1, 1]$ given by the m-th Chebyshev polynomials $T_m(x)$ defined by the relations $\cos m\theta = T_m(\cos\theta), m \geqq 2$. Example of the Chebyshev polynomials are given by $T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, \ldots$.

Answer the following questions.

(i) Prove that the probabilistic measure $\mu(B)$ for any measurable set B on Ω defined by $\mu(B) = \int_B d\mu(x) = \int_B \frac{dx}{\pi\sqrt{1-x^2}}$ is preserved by the transformations $T_m, m \geqq 2$.

(ii) Prove that the Chebyshev polynomials $T_m, m \geqq 0$, satisfy the orthogonal relation

$$\langle T_l, T_m \rangle \equiv \int_{-1}^{1} T_l(x) T_m(x) d\mu(x) = \frac{1 + \delta_{l,0}}{2} \delta_{l,m}, \tag{1}$$

where $\delta_{l,m}$ is Kronecker's delta function.

(iii) A set of measurable functions which are square integrable over the interval $\Omega = [-1, 1]$ can be considered as a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) d\mu(x). \tag{2}$$

Here, we call the Hilbert space $L^2(\Omega, \mu)$. Let us consider a time evolution operator U_m for a stochastic process $\{X_n\}$ by the operation $U_m f(x) = f(T_m(x)) = f \circ T_m(x)$ for $f \in L^2(\Omega, \mu)$. Let us consider an expansion of a polynomial f of order L in terms of the Chebyshev polynomials

$$f(x) = a_0 + \sum_{i=1}^{L} a_i T_i(x). \tag{3}$$

Compute $\|f\|^2 = \langle f, f \rangle < \infty$ by using $\{a_0, a_1, \ldots, a_L\}$ and prove that $\|U_m^l f\|^2 = \|f\|^2$ for any $l \, (\geqq 0)$.

(iv) Prove that the dynamical systems $X_{n+1} = T_m(X_n)$ defined by the Chebyshev polynomials $T_m, m \geqq 2$, is mixing. Here, by mixing it means that the following relation

$$\int_{-1}^{1} f(X_{n+l}) g(X_n) d\mu(X_n) = \langle U_m^l f, g \rangle \to \langle f, 1 \rangle \langle g, 1 \rangle, \quad l \to \infty \tag{4}$$

holds for any $f, g \in L^2(\Omega, \mu)$.

(Hint: It is noted that an arbitrary function f with the condition $f \in L^2(\Omega, \mu)$ can be well approximated by a finite sum of the Chebyshev polynomials.)