

基礎数学 I

1

指数関数 $E(t, x) = e^{2xt-t^2}$ の展開

$$E(t, x) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!}$$

を用いて関数 $f_n(x)$ ($n = 0, 1, 2, \dots$) を定める. 以下の問いに答えよ.

(i) 関数 $f_n(x)$ ($n = 0, 1, 2, \dots$) は

$$\frac{df_{n+1}(x)}{dx} = 2(n+1)f_n(x)$$

を満たすことを示せ.

(ii) 関数 $f_n(x)$ ($n = 0, 1, 2, \dots$) は

$$f_{n+2}(x) = 2xf_{n+1}(x) - 2(n+1)f_n(x)$$

を満たすことを示せ.

(iii) 関数 $f_n(x)$ ($n = 0, 1, 2, \dots$) は

$$\frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} f_n(x) \right) = -2ne^{-x^2} f_n(x)$$

を満たすことを示せ.

(iv) 関数 $f_n(x)$ ($n = 0, 1, 2, \dots$) について等式

$$\int_{-\infty}^{\infty} e^{-x^2} f_n(x)^2 dx = 2^n n! \sqrt{\pi}$$

を示せ. ただし, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ を用いてよい.

An English Translation:

Basic Mathematics I

1

Define the functions $f_n(x)$ ($n = 0, 1, 2, \dots$) by using the following expansion of the exponential function $E(t, x) = e^{2xt-t^2}$

$$E(t, x) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!}.$$

Answer the following questions.

(i) Show that

$$\frac{df_{n+1}(x)}{dx} = 2(n+1)f_n(x)$$

for any $n = 0, 1, 2, \dots$

(ii) Show that

$$f_{n+2}(x) = 2xf_{n+1}(x) - 2(n+1)f_n(x)$$

for any $n = 0, 1, 2, \dots$

(iii) Show that

$$\frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} f_n(x) \right) = -2ne^{-x^2} f_n(x)$$

for any $n = 0, 1, 2, \dots$

(iv) Show the equality

$$\int_{-\infty}^{\infty} e^{-x^2} f_n(x)^2 dx = 2^n n! \sqrt{\pi}$$

for any $n = 0, 1, 2, \dots$. You may use the equality $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ without proof.