

線形計画

3

以下の (i), (ii) に答えよ.

(i) 次の線形計画問題 (P1) とその双対問題 (D1) を考える.

$$\begin{array}{ll} \text{(P1): Minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \begin{array}{ll} \text{(D1): Maximize} & \mathbf{b}^\top \mathbf{w} \\ \text{subject to} & \mathbf{A}^\top \mathbf{w} \leq \mathbf{c} \end{array}$$

ここで, \mathbf{A} は $m \times n$ 定数行列, \mathbf{b} は m 次元定数ベクトル, \mathbf{c} は n 次元定数ベクトル, \mathbf{x} は n 次元変数ベクトル, \mathbf{w} は m 次元変数ベクトルであり, $^\top$ は転置記号を表す. 問題 (P1) と (D1) は最適解 \mathbf{x}^* と \mathbf{w}^* をもつとする. さらに $\mathbf{y}^* = \mathbf{c} - \mathbf{A}^\top \mathbf{w}^*$ とする. このとき, $x_i^* > 0$ であれば, $y_i^* = 0$ が成り立つことを示せ.

(ii) 次の線形計画問題を考える.

$$\begin{array}{ll} \text{(P2): Maximize} & x_5 \\ \text{subject to} & \sum_{i=1}^4 x_i \leq 1 \\ & \sum_{i=k+1}^4 x_i \leq kx_k \quad (k = 1, 2, 3) \\ & x_5 \leq 4x_4 \end{array}$$

問題 (P2) の最適解を \mathbf{x}^* とする. 問題 (P2) の双対問題の最適解を求めよ. さらに,

$$\sum_{i=1}^4 x_i^* = 1$$

が成り立つことを示せ.

An English Translation:

Linear Programming

3

Answer the following questions (i) and (ii).

(i) Consider the following linear programming problem (P1) and its dual problem (D1):

$$\begin{array}{ll} \text{(P1) : Minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \quad \begin{array}{ll} \text{(D1) : Maximize} & \mathbf{b}^\top \mathbf{w} \\ \text{subject to} & \mathbf{A}^\top \mathbf{w} \leq \mathbf{c}, \end{array}$$

where \mathbf{A} is an $m \times n$ constant matrix, \mathbf{b} is an m -dimensional constant vector, \mathbf{c} is an n -dimensional constant vector, \mathbf{x} is an n -dimensional vector of variables, \mathbf{w} is an m -dimensional vector of variables, and $^\top$ denotes transposition. Suppose that problems (P1) and (D1) have optimal solutions \mathbf{x}^* and \mathbf{w}^* , respectively. Let $\mathbf{y}^* = \mathbf{c} - \mathbf{A}^\top \mathbf{w}^*$. Then show that $y_i^* = 0$ if $x_i^* > 0$.

(ii) Consider the following linear programming problem:

$$\begin{array}{ll} \text{(P2) : Maximize} & x_5 \\ \text{subject to} & \sum_{i=1}^4 x_i \leq 1 \\ & \sum_{i=k+1}^4 x_i \leq kx_k \quad (k = 1, 2, 3) \\ & x_5 \leq 4x_4. \end{array}$$

Let \mathbf{x}^* be an optimal solution to problem (P2). Obtain an optimal solution to the dual problem of problem (P2). Moreover, show that

$$\sum_{i=1}^4 x_i^* = 1.$$