基礎数学 II

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複素数を成分とする $n \times n$ 行列 $A = (a_{i,j})_{1 \leqq i,j \leqq n}$ と $B = (b_{i,j})_{1 \leqq i,j \leqq n}$ は，それぞれ，非対角成分が全て非零の三重対角行列と対角行列である．すなわち，行列成分 $a_{i,j}, b_{i,j} \in \mathbb{C}$ は

$$a_{i,j} = 0 \ (|i-j| > 1), \quad a_{i,j} \neq 0 \ (|i-j| = 1), \quad b_{i,j} = 0 \ (|i-j| \geqq 1)$$

を満たす．ここで，同一の正則行列 P を用いた相似変換 $A \mapsto P^{-1}AP, B \mapsto P^{-1}BP$ によって行列 A と B は，それぞれ，対角行列と非対角成分が全て非零の三重対角行列に変換されるものとする．行列 A の固有値を λ_i，単位行列を I，零行列を O で表す．このとき，以下の問いに答えよ．

(i) $c_0, c_1, \ldots, c_{n-1}$ を定数とする．

$$\sum_{k=0}^{n-1} c_k A^k = O$$

が成立するのは $c_0 = c_1 = \cdots = c_{n-1} = 0$ のときのみであることを示せ．

(ii) 行列 A の固有値は全て相異なることを示せ．

(iii) $i, j \in \{1, 2, \ldots, n\}$ とし，行列 E_i を

$$E_i = \prod_{1 \leqq k \leqq n, k \neq i} \frac{1}{\lambda_i - \lambda_k}(A - \lambda_k I)$$

で定める．このとき，

$$\sum_{k=1}^{n} E_k = I, \qquad E_i E_j = \delta_{i,j} E_i$$
$$E_i B E_j = O \ (|i-j| > 1), \quad E_i B E_j \neq O \ (|i-j| = 1)$$

が成り立つことを示せ．ただし，$\delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$ とする．

An English Translation:

Basic Mathematics II

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Let $n \times n$ complex matrices $A = (a_{i,j})_{1 \leqq i,j \leqq n}$ and $B = (b_{i,j})_{1 \leqq i,j \leqq n}$ be a tri-diagonal matrix whose off-diagonal entries are non-zero and a diagonal matrix, respectively. Equivalently, the entries $a_{i,j}, b_{i,j} \in \mathbb{C}$ of A and B satisfy

$$a_{i,j} = 0 \ (|i-j| > 1), \quad a_{i,j} \neq 0 \ (|i-j| = 1), \quad b_{i,j} = 0 \ (|i-j| \geqq 1).$$

Suppose that A and B can be converted into a diagonal matrix and a tri-diagonal matrix whose off-diagonal entries are non-zero, respectively, by a similarity transformation with a common regular matrix P: $A \mapsto P^{-1}AP, B \mapsto P^{-1}BP$.

Let λ_i be the eigenvalues of A. Hereafter, I denotes the identity matrix, and O denotes the zero matrix. Answer the following questions.

(i) Let $c_0, c_1, \ldots, c_{n-1}$ be constants. Show that

$$\sum_{k=0}^{n-1} c_k A^k = O$$

holds only when $c_0 = c_1 = \cdots = c_{n-1} = 0$.

(ii) Show that all the eigenvalues of A are mutually distinct.

(iii) Let $i, j \in \{1, 2, \ldots, n\}$ and let E_i be the matrix defined by

$$E_i = \prod_{1 \leqq k \leqq n, k \neq i} \frac{1}{\lambda_i - \lambda_k}(A - \lambda_k I).$$

Show that

$$\sum_{k=1}^{n} E_k = I, \qquad E_i E_j = \delta_{i,j} E_i,$$

$$E_i B E_j = O \ (|i-j| > 1), \quad E_i B E_j \neq O \ (|i-j| = 1),$$

where $\delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$.