基礎数学 I

1

変数 x の関数 $f(x)=\tan^{-1}x$, $(f(0)=0)$ の n 次導関数を $f^{(n)}(x)$ とかく．以下の問いに答えよ．

(i) 任意の自然数 n について

$$f^{(n)}(x)=(n-1)!\cos^n y\cdot\sin\left(n(y+\frac{\pi}{2})\right),\quad (y=\tan^{-1}x)$$

を示せ．

(ii)

$$f^{(n)}(0)=\begin{cases}(-1)^m(2m)! & (n=2m+1)\\ 0 & (n=2m)\end{cases}$$

が成り立つことを示せ．

(iii) 剰余項 $R_{2n}(x)$ を用いて

$$\tan^{-1}x=\sum_{k=0}^{n-1}(-1)^k\frac{1}{2k+1}x^{2k+1}+R_{2n}(x),$$
$$R_{2n}(x)=\frac{1}{2n}\cos^{2n}z\cdot\sin\left(2n(z+\frac{\pi}{2})\right)\cdot x^{2n},\quad (z=\tan^{-1}\theta x,\quad 0<{}^{\exists}\theta<1)$$

とおくとき，$|x|\leqq 1$ ならば

$$\lim_{n\to\infty}|R_{2n}(x)|=0$$

となることを示せ．

(iv)

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

を示せ．

Basic Mathematics I

1

Let $f^{(n)}(x)$ be the n-th derivative of the function $f(x) = \tan^{-1} x$ with $f(0) = 0$. Answer the following questions.

(i) Show that

$$f^{(n)}(x) = (n-1)! \cos^n y \cdot \sin\left(n(y + \frac{\pi}{2})\right) \quad \text{with} \quad y = \tan^{-1} x$$

for any natural number n.

(ii) Show that

$$f^{(n)}(0) = \begin{cases} (-1)^m (2m)! & \text{for} \quad n = 2m+1 \\ 0 & \text{for} \quad n = 2m \end{cases}.$$

(iii) Let $R_{2n}(x)$ be the remainder defined by

$$\tan^{-1} x = \sum_{k=0}^{n-1} (-1)^k \frac{1}{2k+1} x^{2k+1} + R_{2n}(x),$$

$$R_{2n}(x) = \frac{1}{2n} \cos^{2n} z \cdot \sin\left(2n(z + \frac{\pi}{2})\right) \cdot x^{2n} \quad \text{with} \quad z = \tan^{-1} \theta x \quad \text{and} \quad 0 < {}^{\exists}\theta < 1.$$

Show that

$$\lim_{n \to \infty} |R_{2n}(x)| = 0 \quad \text{for} \quad |x| \leqq 1.$$

(iv) Show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$