力学系数学

6

微分方程式系 $\dfrac{dx}{dt} = f(x,y)$, $\dfrac{dy}{dt} = g(x,y)$, ただし, $x = x(t)$, $y = y(t)$, に対して $x(t) \equiv x_0$, $y(t) \equiv y_0$ がその定数解となるとき，xy-平面上の点 (x_0, y_0) をこの微分方程式系の不動点という．以下の問いに答えよ．

(i) 微分方程式

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x^3 - x \tag{1}$$

の不動点を全て求めよ．

(ii) (i) で求めた不動点の近傍における (1) の解の挙動を調べて，不動点が沈点，源点，渦心点，鞍点，その他のいずれであるか，理由を添えて答えよ．

(iii) $x = x(t)$, $y = y(t)$ が微分方程式 (1) の解であるとき，関数

$$H(x,y) = 2y^2 - (x^2-1)^2$$

は t によらない定数であることを示せ．さらに，(ii) および $H(x,y)$ を用いて，xy-平面における (1) の解曲線 $(x(t), y(t))$ のグラフをえがけ．

(iv) 微分方程式

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = x^3 - x \tag{2}$$

の不動点を全て求めよ．

(v) (iv) で求めた不動点の近傍における (2) の解の挙動を調べて，不動点が沈点，源点，渦心点，鞍点，その他のいずれであるか，理由を添えて答えよ．

An English Translation:

Mathematics for Dynamical Systems

6

For the system of differential equations $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$ with $x = x(t)$ and $y = y(t)$, if $x(t) \equiv x_0$ and $y(t) \equiv y_0$ give a constant solution, then the point (x_0, y_0) on the xy-plane is called a fixed point of the system of differential equations. Answer the following questions.

(i) Find all of the fixed points of the system of differential equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x^3 - x. \tag{1}$$

(ii) Investigate the local behavior of solutions of (1) in the neighborhood of each fixed point found in the question (i) to classify these fixed points into sink, source, center, saddle point or others, giving reason for the answer.

(iii) Show that the function

$$H(x, y) = 2y^2 - (x^2 - 1)^2$$

is independent from t for any solution $x = x(t)$ and $y = y(t)$ of (1). Next, graph trajectories $(x(t), y(t))$ of solutions of (1) on the xy-plane using (ii) and $H(x, y)$.

(iv) Find all of the fixed points of the system of differential equations

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = x^3 - x. \tag{2}$$

(v) Investigate the local behavior of solutions of (2) in the neighborhood of each fixed point found in (iv) to classify these fixed points into sink, source, center, saddle point or others, giving reason for the answer.