# A Polynomial-space Exact Algorithm for TSP in Degree-6 Graphs

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**Abstract:** This paper presents the first polynomial-space exact algorithm for TSP in graphs with degree at most 6. We develop a set of branching rules to aid the analysis of the branching algorithm, and we use the measure-and-conquer method to effectively analyze our branching algorithm, and we obtain a running time of $O^*(2.7467^n)$, still advantageous over other known polynomial-space algorithms for the TSP in general graphs.

**Keywords:** Traveling Salesman Problem, Exact Exponential Algorithm, Branch-and-Reduce, Measure-and-Conquer.

## 1 Introduction

The Traveling Salesman Problem is one of the most extensively studied problems in combinatorial optimization. Beside being a well-known NP-hard combinatorial optimization problem, it also has great practical importance. Present-day computers have only limited memory and algorithms which use exponential execution space run out of memory well before time. For this reason, we limit this exposition to algorithms which require merely polynomially bounded execution space.

Gurevich and Shelah [3] gave the first polynomial-space exact algorithm for the TSP, whose running time in a general $n$-vertex graph is bounded by $O^*(4^n n^{\log n})$ (the $O^*$ notation suppresses polynomial factors). This time bound has only recently been improved, but only for graphs of limited degree. From this viewpoint, let degree-$i$ graph stand for a graph in which vertices have maximum number of incident edges at most $i$. For any graph with maximum degree at most $d$, the TSP can be solved in $O(n(d-1)^n)$-time and $O(dn)$-space by generating paths from a vertex.

There are a number of studies have been done for the TSP in degree bounded graphs. The currently fastest polynomial-space algorithms for the TSP in degree-3 and degree-4 graphs were given by Xiao and Nagamochi [8, 9], running in time $O^*(1.2312^n)$ and $O^*(1.692^n)$, respectively. These are the previous studies of the TSP in degree-3 and degree-4 graphs, [1, 4, 5].

To the best of our knowledge, presently the only investigation on the TSP in graphs of degree up to 5 has been done by Md Yunos et al. [6], giving an $O^*(2.4531^n)$-time algorithm.

[1]Technical Report 2015-003, June 30, 2015.

Furthermore, there exist no reports in the literature of exact algorithms specialized to the TSP in degree-6 graphs. Therefore, this paper presents the first algorithm for the TSP in degree-6 graphs, and shows that the algorithm runs in $O^*(2.7467^n)$-time.

# 2 Preliminaries

We consider a generalization of the TSP, named the *forced* Traveling Salesman Problem. We defined an instance $I = (G, F)$ that consists of a simple, edge weighted, undirected graph $G$, and a subset $F$ of edges in $G$, called *forced*. A vertex is called *forced* if exactly one of its incident edges is forced. Similarly, it is called *unforced* if no forced edge is incident to it. A Hamiltonian cycle in $G$ is called a *tour* if it passes through all the forced edges in $F$. Under these circumstances, the forced TSP requests to find a minimum cost tour of an instance $(G, F)$.

For a graph $G$, let $V(G)$ (resp., $E(G)$) denote the set of vertices (resp., the set of edges) in $G$. A pair of vertices $v$ and $t$ are called neighbors if $v$ and $t$ are adjacent by an edge $vt$ in $E(G)$. We denote the set of neighbors of a vertex $v$ by $N(v)$, and denote by $d(v)$ the cardinality $|N(v)|$ of $N(v)$, also called the *degree* of $v$. For a subset of vertices $W \subseteq V(G)$, let $N(v; W) = N(v) \cap W$. For a subset of edges $E' \subseteq E(G)$, let $N_{E'}(v) = N(v) \cap \{u \mid uv \in E'\}$, and let $d_{E'}(v) = |N_{E'}(v)|$. Analogously, let $N_{E'}(v; W) = N_{E'}(v) \cap W$, and $d_{E'}(v, W) = |N_{E'}(v, W)|$. Also, for a subset $E'$ of $E(G)$, we denote by $G - E'$ the graph $(V, E \setminus E')$ obtained from $G$ by removing the edges in $E'$.

In this paper, we assume that the maximum degree of a vertex in $G$ is at most 6. We refer to a forced (resp., unforced) vertex of degree $i$ by f$i$ (resp., u$i$). In this paper, we assume that the maximum degree of a vertex in $G$ is at most 6. Vertices of degree 1 and 2 are treated as a special cases, and this implies eight types of vertices in an instance of $(G, F)$, namely u6, f6, u5, f5, u4, f4, u3 and f3-vertices. For $i = 3, 4, 5, 6$, let $V_{\mathrm{f}i}$ (resp., $V_{\mathrm{u}i}$) denote the set of f$i$-vertices (resp., u$i$-vertices) in $(G, F)$.

## 2.1 Branching Algorithms and Measure-and-Conquer Method

In this paper, we use the same branching algorithm and analysis method as in TSP in degree-5 graphs [6]. We use a branching algorithm for generating a solution space from an initial instance and we derive an upper bound on the number of instances generated by branching operations.

To illustrate the execution of the branching rules, we can represent the solution space in our branching algorithm as a search tree. The search tree is obtained by assigning the input instance of a problem as a root node, and recursively assigning a child to a node for each smaller instance obtained by applying the branching rules. For a single node of the search tree, the algorithm takes time polynomial in the size of the node instance, which in turn, is smaller than or equal to the original instance size. Thus, we can conclude that the running time of the branching algorithm is equal to the number of nodes of the search tree times a polynomial of the original input instance size.

To aid the time analysis of the branching algorithm, let $I$ be a given instance with size $\mu$, and let $I'$ and $I''$ be instances obtained from $I$ by a branching operation. We use $T(\mu)$ to denote the maximum number of nodes in the search tree of an input of size $\mu$ when we execute our branching algorithm. Let $a$ and $b$ be the amounts of decrease in size of instances $I'$ and $I''$, respectively; these values directly determine the performance of the algorithm. Then, we call $(a, b)$ the branching vector of the branching rules, and this implies the linear recurrence:

$$T(\mu) \leq T(\mu - a) + T(\mu - b). \tag{1}$$

To evaluate the performance of this branching vector, we can use any standard method for linear recurrence relations. In fact, it is known that $T(\mu)$ is of the form $O(\tau^{\mu})$, where $\tau$ is the unique positive real root of the function $f(x) = 1 - \left(x^{-a} + x^{-b}\right)$ [2]. The value $\tau$ is called the *branching factor* (of a given branching vector), and the running time of the algorithm decreases with the value of this branching factor.

To effectively analyze our branching algorithm, we use the measure-and-conquer method. The basic idea behind this method is to assign a measure to an instance, as opposed to using simply its size when analyzing the branching vectors of the branching operations. The reader might refer to the book of Fomin and Kratsch [2] for a solid description of branching algorithms and the measure-and-conquer method.

For a given problem instance $I$ of size $\mu$, let $W(I)$ be the measure of $I$. When considering a branch-and-reduce algorithm for the concerned problem, intuitively we seek for a measure which satisfies the following properties

(i) $W(I) = 0$ if and only if $I$ can be solved in polynomial time;

(ii) If $I'$ is a sub-instance of $I$ obtained through a reduction or a branching operation, then $W(I') \leq W(I)$.

We call a measure $W$ satisfying conditions (i) and (ii) above a *proper measure.*

# 3 A Polynomial-Space Branching Algorithm

Our algorithm consists of two major steps which are repeated iteratively. In the first step, the algorithm applies reduction rules until no further reduction is possible. In the second step, the algorithm applies branching rules in a reduced instance to search for a solution.

## 3.1 Reduction Rules

Reduction is a process of transforming an instance to a smaller instance. It takes polynomial-time to construct a solution of an original instance from a solution to a smaller instance obtained through reduction.

If an instance has no tour, we called it *infeasible.* Lemma 1 gives two sufficient conditions for an instance to be infeasible. These two sufficient conditions will be checked when executing the reduction rules.

**Lemma 1 ([6])** *If one of the following conditions holds, then the instance $(G, F)$ is infeasible.*

(i) $d(v) \leq 1$ *for some vertex* $v \in V(G)$.

(ii) $d_F(v) \geq 3$ *for some vertex* $v \in V(G)$.

In this paper, we apply two reduction rules as stated in Md Yunos et al. [6, Lemma 2]. The reduction rules as stated in Lemma 2 preserve the minimum cost tour of an instance and it is applied in each of the branching operation.

**Lemma 2 ([6])** *Each of the following reductions preserves the feasibility and a minimum cost tour of an instance* $(G, F)$*.*

(i) *If* $d(v) = 2$ *for a vertex* $v$*, then add to* $F$ *any unforced edge incident to vertex* $v$*; and*

(ii) *If* $d(v) > 2$ *and* $d_F(v) = 2$ *for a vertex* $v$*, then remove from* $G$ *any unforced edge incident to vertex* $v$*.*

Our reduction algorithm is described in Figure 1. An instance $(G, F)$ is called *reduced* if it does not satisfy any of the conditions in Lemma 1 and Lemma 2.

**Input**: An instance $(G, F)$ such that the maximum degree of $G$ is at most 6.
**Output**: A message for the infeasibility of $(G, F)$; or a reduced instance $(G', F')$ of $(G, F)$.

**Initialize** $(G', F') := (G, F)$;

**while** $(G', F')$ is not a reduced instance **do**

  **If** there is a vertex $v$ in $(G', F')$ such that $d(v) \leq 1$ or $d_{F'}(v) \geq 3$ **then**
    **Return** message "Infeasible"

  **Elseif** there is a vertex $v$ in $(G', F')$ such that $2 = d(v) > d_{F'}(v)$ **then**
    Let $E^\dagger$ be the set of unforced edges incident to all such vertices;
    **Set** $F' := F' \cup E^\dagger$

  **Elseif** there is a vertex $v$ in $(G', F')$ such that $d(v) > d_{F'}(v) = 2$ **then**
    Let $E^\dagger$ be the set of unforced edges incident to all such vertices;
    **Set** $G' := G' - E^\dagger$

**End while**;

Return $(G', F')$.

Figure 1: Algorithm Red$(G, F)$

### 3.2 Branching Rules

Our branching algorithm is based on a set of branching rules. The choice of an edge to branch on plays key role in the analysis of our branching algorithm. To this effect, in an instance $(G, F)$, we assign the following priority in choosing an edge $e = vt$ to branch on. Without loss of generality, let $v$ be a vertex of degree 6 and $t$ its neighbor. For the choice of a vertex of degree 6, an f6 vertex takes precedence over a u6 vertex. Otherwise, vertices of smaller degree take precedence over vertices of higher degree, forced vertices over unforced, and a pair of neighbors $vt$ with a common neighborhood of lower cardinality (or zero) precede those with more neighbors in common. If such an edge $e = vt$ of highest priority exists, it is called *optimal.* Otherwise, there exist no more vertices of degree 6 in the given instance, and we can make use of an algorithm specialized to TSP instances of maximum degree up to 5, e.g., the algorithm of Md Yunos et al. [6]. We refer to this priority in choosing an edge $e = vt$ to branch on as the *branching rules*. A list giving the above priorities is given in Figure 2, where the condition (c-$i$) with minimum index $i$ is optimal, over all unforced edge $vt$ in $(G, F)$.

The collective set of branching rules for conditions c-1 to c-12 is illustrated in Figure 3 and the collective set of branching rules for conditions c-13 to c-19 is illustrated in Figure 4.

When the reduced instance has no more vertex $v$ of degree 6, this means that the maximum degree of the reduced instance at this point is at most 5. Then, we can call a polynomial space exact algorithm for the TSP that is specialized for degree-5 graphs. Here, we call the algorithm specialized for degree-5 graphs by Md Yunos et al. [6]. Details of our branching algorithm is described in Figure 5.

(c-1) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
(c-2) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
(c-3) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{u3}})$;
(c-4) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f4}})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
(c-5) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f4}})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
(I) $|N_U(v) \cap N_U(t)| = 1$; and
(II) $|N_U(v) \cap N_U(t)| = 2$;
(c-6) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{u4}})$;
(c-7) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f5}})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
(c-8) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f5}})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
(I) $|N_U(v) \cap N_U(t)| = 1$;
(II) $|N_U(v) \cap N_U(t)| = 2$; and
(III) $|N_U(v) \cap N_U(t)| = 3$;
(c-9) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{u5}})$;
(c-10) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f6}})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
(c-11) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{f6}})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
(I) $|N_U(v) \cap N_U(t)| = 1$;
(II) $|N_U(v) \cap N_U(t)| = 2$;
(III) $|N_U(v) \cap N_U(t)| = 3$; and
(IV) $|N_U(v) \cap N_U(t)| = 4$;
(c-12) $v \in V_{\text{f6}}$ and $t \in N_U(v; V_{\text{u6}})$;
(c-13) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{f3}})$;
(c-14) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{u3}})$;
(c-15) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{f4}})$;
(c-16) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{u4}})$;
(c-17) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{f5}})$;
(c-18) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{u5}})$; and
(c-19) $v \in V_{\text{u6}}$ and $t \in N_U(v; V_{\text{u6}})$.

Figure 2: Preferences Conditions

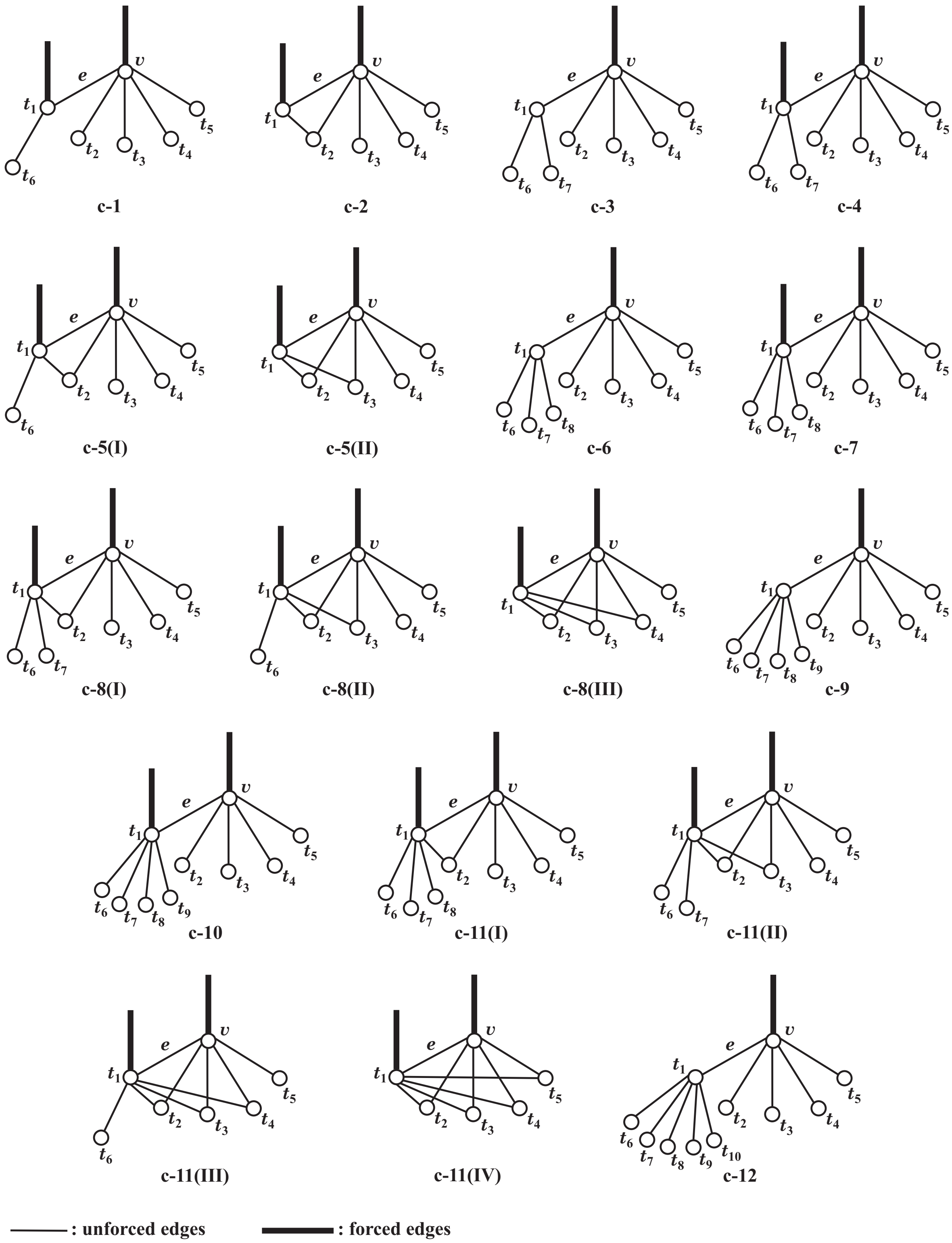



Figure 3: Illustration of the Branching Rules Around f6-vertices

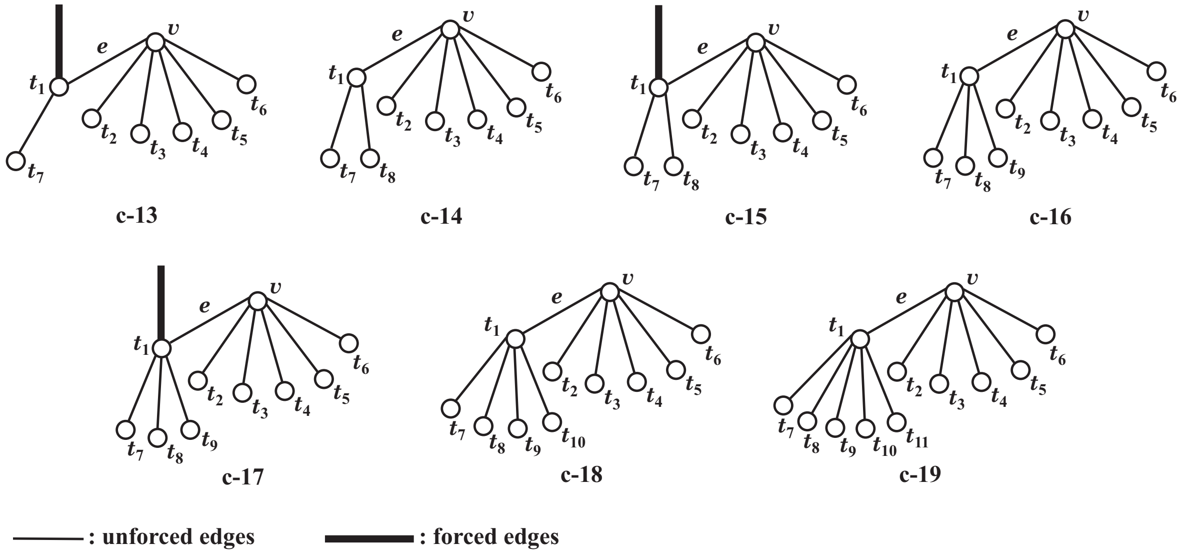



Figure 4: Illustration of the Branching Rules around u6-vertices

**Input**: An instance $(G, F)$ such that the maximum degree of $G$ is at most 6.
**Output**: The minimum cost of a tour of $(G, F)$; or a message for the infeasibility of $(G, F)$.

**Run** Red$(G, F)$;

**If** Red$(G', F')$ returns message "Infeasible" **then**

  **Return** message "Infeasible"

**Else**

  Let $(G', F') := \text{Red}(G, F)$;

  **If** $V_{\text{u6}} \cup V_{\text{f6}} \neq \emptyset$ **then**

    Choose an optimal unforced edge $e$

    **Return** $\min\{\text{tsp6}(G', F' \cup \{e\}), \text{tsp6}(G' - \{e\},\ F')\}$

  **Else** /* there is no vertex of degree 6 in $(G', F')$ */

    **Return** tsp5$(G', F')$.

**Note:** The input and output of algorithm tsp5$(G, F)$ are as follows
Input: An instance $(G, F)$ such that the maximum degree of $G$ is at most 5.
Output: The minimum cost of a tour of $(G, F)$; or a message for the infeasibility of $(G, F)$.

Figure 5: Algorithm tsp6$(G, F)$

## 4 Analysis

### 4.1 Main Result

Let the vertex weight function $\omega(v)$ be chosen as follows:

$$\omega(v) = \begin{cases} w_6 = 1 & \text{for a u6-vertex } v \\ w_{6'} = 0.532091 & \text{for an f6-vertex } v \\ w_5 = 0.458479 & \text{for a u5-vertex } v \\ w_{5'} = 0.220838 & \text{for an f5-vertex } v \\ w_4 = 0.333150 & \text{for a u4-vertex } v \\ w_{4'} = 0.147225 & \text{for an f4-vertex } v \\ w_3 = 0.155400 & \text{for a u3-vertex } v \\ w_{3'} = 0.073612 & \text{for an f3-vertex } v \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

**Lemma 3** *If the vertex weight function $\omega(v)$ is set as in Eq. (2), then each branching operation in Figure 5 has a branching factor not greater than 2.746706.*

A proof of Lemma 3 will be derived analytically in the several subsections which follow. Form the Lemma, we get our main result as stated in Theorem 1

**Theorem 1** *The TSP in a degree-6 graph can be solved in $O^*(2.7467^n)$-time and polynomial space.*

### 4.2 Weight Constraints

For $i = \{3, 4, 5\}$, we denote $w_i$ to be the weight of a u$i$-vertex, and $w_{i'}$ to be the weight of an f$i$-vertex. The conditions for a proper measure require that the measure of an instance obtained through a branching or a reduction operation will not be greater than the measure of the original instance. Thus, the vertex weight for vertices of degree less than 3 is set to be 0, and other vertex weights should satisfy the following relations:

$$w_6 \le 1, \tag{3}$$
$$w_{6'} \le w_6, \tag{4}$$
$$w_{5'} \le w_5, \tag{5}$$
$$w_{4'} \le w_4, \tag{6}$$
$$w_{3'} \le w_3, \tag{7}$$
$$w_3 \le w_4 \le w_5 \le w_6, \text{ and} \tag{8}$$
$$w_{3'} \le w_{4'} \le w_{5'} \le w_{6'}. \tag{9}$$

As a result of the reduction and branching operations, the degree of some vertices will decrease, while the degree of other vertices will remain unchanged. A forced edge will never disappear, neither by the reduction nor branching operations. The other way round goes to an unforced edge where it may be erased or become forced by either of the reduction or branching operation. Thus, the measure of an instance obtained through a reduction or branching operation will not be greater than the measure of the original instance. By setting vertex weights which satisfy the conditions of Eqs. (4) to (9) is sufficient to obtain a proper measure, and Lemma 4 has been proved in [6, Lemma 4].

**Lemma 4** *If the weights of vertices are chosen as in Eqs.* (4) *to* (9)*, then the measure* $W(I)$ *never increases as a result of the reduction or the branching operations of Figure 1 and Figure 5.*

To simplify some arguments, we introduce the following notation:

$$\Delta_3 = w_3 - w_{3'}, \tag{10}$$
$$\Delta_4 = w_4 - w_{4'}, \tag{11}$$
$$\Delta_5 = w_5 - w_{5'}, \tag{12}$$
$$\Delta_6 = w_6 - w_{6'}, \tag{13}$$
$$\Delta_{4-3} = w_4 - w_3, \tag{14}$$
$$\Delta_{5-4} = w_5 - w_4, \tag{15}$$
$$\Delta_{5-3} = w_5 - w_3, \tag{16}$$
$$\Delta_{6-5} = w_6 - w_5, \tag{17}$$
$$\Delta_{6-4} = w_6 - w_4, \tag{18}$$
$$\Delta'_{4-3} = w_{4'} - w_{3'}, \tag{19}$$
$$\Delta'_{5-4} = w_{5'} - w_{4'}, \tag{20}$$
$$\Delta'_{5-3} = w_{5'} - w_{3'}, \tag{21}$$
$$\Delta'_{6-5} = w_{6'} - w_{5'}, \text{ and} \tag{22}$$
$$\Delta'_{6-4} = w_{6'} - w_{4'}. \tag{23}$$

The differences between each of Eq. (10) to Eq. (23) cannot be less than 0.

To simplify the list of our branching vectors, we use the following notation:

$$m_1 = \min\left\{w_{3'},\ w_3,\ \Delta'_{4-3},\ \Delta_{4-3},\ \Delta'_{5-4},\ \Delta_{5-4},\ \Delta'_{6-5},\ \Delta_{6-5}\right\}, \tag{24}$$
$$m_2 = \min\{w_3,\ \Delta'_{4-3},\ \Delta_{4-3},\ \Delta'_{5-4},\ \Delta_{5-4},\ \Delta'_{6-5},\ \Delta_{6-5}\}, \tag{25}$$
$$m_3 = \min\{w_{3'},\ \Delta_3,\ w_{4'},\ \Delta_4,\ w_{5'},\ \Delta_5,\ w_{6'},\ \Delta_6\}, \tag{26}$$
$$m_4 = \min\{\Delta'_{4-3},\ \Delta_{4-3},\ \Delta'_{5-4},\ \Delta_{5-4},\ \Delta'_{6-5},\ \Delta_{6-5}\}, \tag{27}$$
$$m_5 = \min\{w_{4'},\ w_4,\ \Delta'_{5-3},\ \Delta_{5-3},\ \Delta'_{6-4},\ \Delta_{6-4}\}, \tag{28}$$
$$m_6 = \min\{\Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}, \tag{29}$$
$$m_7 = \min\{\Delta'_{5-4},\ \Delta_{5-4},\ \Delta'_{6-5},\ \Delta_{6-5}\}, \tag{30}$$
$$m_8 = \min\{\Delta'_{5-3},\ \Delta_{5-3},\ \Delta'_{6-4},\ \Delta_{6-4}\}, \tag{31}$$
$$m_9 = \min\{\Delta_{5-4},\ \Delta'_{6-5},\ \Delta_{6-5}\}, \tag{32}$$
$$m_{10} = \min\{\Delta'_{6-5},\ \Delta_{6-5}\}, \tag{33}$$
$$m_{11} = \min\{\Delta'_{6-4},\ \Delta_{6-4}\}, \tag{34}$$
$$m_{12} = \min\{w_{3'},\ \Delta_3,\ w_{4'},\ \Delta_4,\ w_{5'},\ \Delta_5,\ \Delta_6\}, \text{ and} \tag{35}$$
$$m_{13} = \min\{w_{3'},\ w_3,\ \Delta'_{4-3},\ \Delta_{4-3},\ \Delta'_{5-4},\ \Delta_{5-4},\ \Delta_{6-5}\}. \tag{36}$$

In the remainder of the analysis, for an optimal edge $e = vt_1$, we refer to $N_U(v)$ by $\{t_1, t_2, \ldots, t_a\}$, $a = d_U(v)$, and to $N_U(t_1) \setminus \{v\}$ by $\{t_{a+1}, t_{a+1}, \ldots, t_{a+b}\}$, $b = d_U(t_1) - 1$. We assume without loss of generality that $t_{1+i} = t_{a+i}$ for $i = 1, 2, \ldots, c$, where $c = |N_U(v) \cap N_U(t_1)|$, the number of good neighbors that $v$ and $t_1$ have in common.

If there exists an f3-vertex $t_{a+i} \setminus \{t_1\}$ in $N_U(t_1)$, let $x = N_U(t_{a+i})$. We see that the choice of vertex $x$ is unique, because $t_{a+i}$ is of type f3. This vertex $x$ will be included in the analysis since it is also involved in the branching operation and reduction rules as shown in Figure 6.

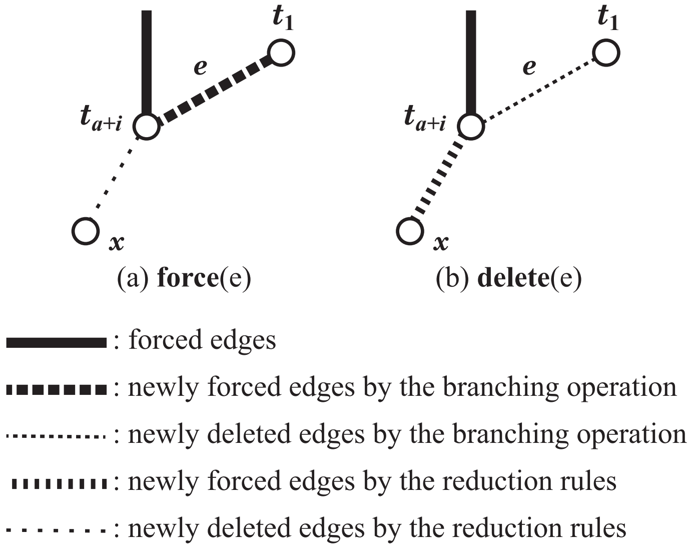



Figure 6: Illustration of forced and deleted edge by branching operation and reduction rules for f3 vertex.

### 4.3 Branching on Edges Around f6-vertices

We will show how we derive the branching vectors for branchings on an optimal edge $e = vt$, incident to a vertex of degree 6. The way we analyze the branching vectors is the same as the method used in TSP in degree-5 graphs by Md Yunos et al. [6]. For this section, there are 12 cases for branching on edges around f6-vertices (c-1 to c-12) in deriving branching vectors for branchings on an optimal edge $e = vt$, incident to an f6-vertex $v$.

**Case c-1.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f3}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$ (see Figure 7): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$ and $t_1t_6$ will be deleted from $G'$ by the reduction rules. Both $v$ and $t_1$ will become vertices of degree 2. From Eq. (2), the weight of vertices of degree 2 is 0. Hence, the weight of vertex $v$ decreases by $w_{5'}$ and the weight of vertex $t_1$ decreases by $w_{3'}$. Each of the vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be either a type f3, u3, f4, u4, f5, or u5-vertex, and each of their weights would decrease by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$.

There are two sub-cases for the vertex type of vertex $t_6$. First, if vertex $t_6$ is an f3-vertex

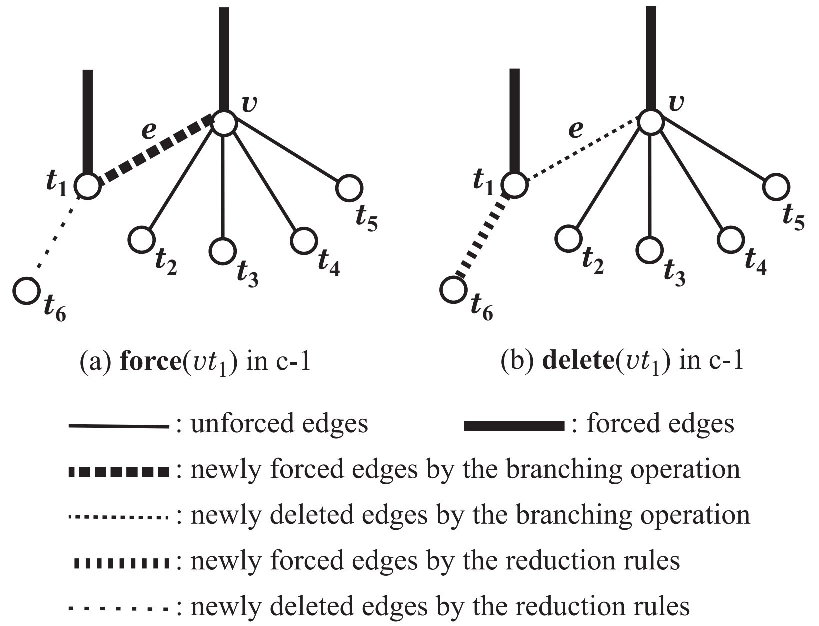



Figure 7: Illustration of branching rule c-1, where vertex $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$.

(see Figure 6), then the weight of vertex $t_6$ decreases by $w_{3'}$. If vertex $x$ is an f3-vertex (resp., u3, f4, u4, f5, u5, f6 or a u6-vertex), then the weight decrease $\beta_2$ of vertex $x$ would be $w_{3'}$ (resp., $\Delta_3$, $w_{4'}$, $\Delta_4$, $w_{5'}$, $\Delta_5$, $w_{6'}$, and $\Delta_6$). Thus the total weight decrease for the first sub-case in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{3'} + 4m_1 + w_{3'} + \beta_2)$.

Second, if vertex $t_6$ is a u3-vertex (resp., f4, u4, f5, u5, f6 or a u6-vertex), then the weight decrease $\alpha_1$ of vertex $t_6$ would be $w_3$ (resp., $\Delta'_{4-3}$, $\Delta_{4-3}$, $\Delta'_{5-4}$, $\Delta_{5-4}$, $\Delta'_{6-5}$, and $\Delta_{6-5}$). Thus, the total weight decrease for the second sub-case in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{3'} + 4m_1 + \alpha_1)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation, and edge $t_1t_6$ will be added to $F'$ by the reduction rules. The weight of vertex $v$ decreases by $\Delta'_{6-5}$ and the weight of vertex $t_1$ decreases by $w_{3'}$.

First, if vertex $t_6$ is an f3-vertex, then the weight of vertex $t_6$ decreases by $w_{3'}$. If vertex $x$ is an f3-vertex (resp., u3, f4, u4, f5, u5, f6 or a u6-vertex), then the weight decrease $\alpha_2$ of vertex $x$ would be $w_{3'}$ (resp., $w_3$, $\Delta'_{4-3}$, $\Delta_{4-3}$, $\Delta'_{5-4}$, $\Delta_{5-4}$, $\Delta'_{6-5}$, and $\Delta_{6-5}$). Thus, the total weight decrease for the first sub-case in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{3'} + w_{3'} + \alpha_2)$.

Second, if vertex $t_6$ is a u3-vertex (resp., f4, u4, f5, u5, f6 or a u6-vertex), then the weight decrease $\beta_1$ of vertex $t_6$ would be $\Delta_3$ (resp., $w_{4'}$, $\Delta_4$, $w_{5'}$, $\Delta_5$, $w_{6'}$, and $\Delta_6$). Thus, the total weight decrease for the second sub-case in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{3'} + \beta_1)$.

As a result, we get the following eight branching vectors in the first sub-case, and seven branching vectors in the second sub-case, respectively:

$$(w_{6'} + w_{3'} + 4m_1 + w_{3'} + \beta_2,\ w_{6'} - w_{5'} + w_{3'} + w_{3'} + \alpha_2) \tag{37}$$

$$(w_{6'} + w_{3'} + 4m_1 + \alpha_1,\ w_{6'} - w_{5'} + w_{3'} + \beta_1) \tag{38}$$

for $(\alpha_1, \beta_1) \in \{(w_3, \Delta_3),\ (\Delta'_{4-3}, w_{4'}),\ (\Delta_{4-3}, \Delta_4),\ (\Delta'_{5-4}, w_{5'}),\ (\Delta_{5-4}, \Delta_5),\ (\Delta'_{6-5}, w_{6'}),$ $(\Delta_{6-5}, \Delta_6)\}$, and $(\beta_2, \alpha_2) \in \{(w_{3'}, w_{3'}),\ (\Delta_3, w_3),\ (w_{4'}, \Delta'_{4-3}),\ (\Delta_4, \Delta_{4-3}),\ (w_{5'}, \Delta'_{5-4}),$

$(\Delta_5, \Delta_{5-4})$, $(w_{6'}, \Delta'_{6-5})$, $(\Delta_6, \Delta_{6-5})\}$.

**Case c-2.** There exist vertices $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$ (see Figure 8): We branch on edge $vt_1$.

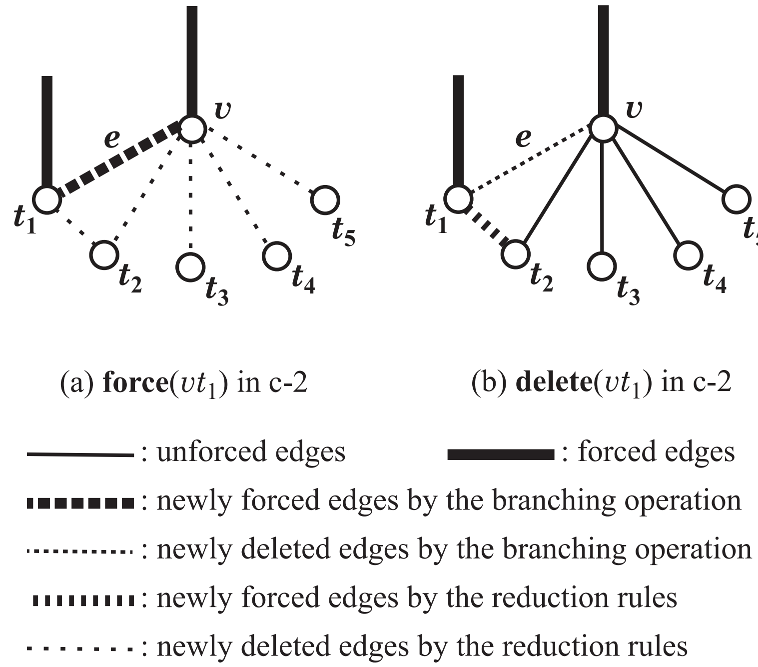



Figure 8: Illustration of branching rule c-2, where vertex $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$.

In the branch of $\mathbf{force}(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$ and $t_1t_2$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{3'}$. Each of the vertices $t_3$, $t_4$ and $t_5$ can be either a type f3, u3, f4, u4, f5, or u5-vertex, and each of their weights would decrease by at least $m_1 = \min\left\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\right\}$.

There are two possible cases for the vertex type of vertex $t_2$. First, let $t_2$ be an f3 or u3-vertex. After performing the branching operation, $t_2$ would become a vertex of degree 1. By Lemma case (i), this is infeasible, and the algorithm will return a message of infeasibility.

Second, let $t_2$ be an f4, u4, f5, or u5-vertex. If $t_2$ is an f4-vertex (resp., u4, f5, u5, f6, or a u6-vertex), then the weight decrease $\alpha$ of vertex $t_2$ would be $w_{4'}$ (resp., $w_4$, $\Delta'_{5-3}$, $\Delta_{5-3}$, $\Delta'_{6-4}$, and $\Delta_{6-4}$). Thus, the total weight decrease in the branch of $\mathbf{force}(vt_1)$ is at least $(w_{6'} + w_{3'} + 3m_1 + \alpha)$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation, and edge $t_1t_2$ will be added to $F'$ by the reduction rules. Hence, the weights of vertices $v$ and $t_1$ decrease by $\Delta'_{6-5}$ and $w_{3'}$, respectively. If vertex $t_2$ is an f4-vertex (resp., u4, f5, u5, f6, or a u6-vertex), then the weight decrease $\beta$ of vertex $t_2$ would be $w_{4'}$ (resp., $\Delta_4$, $w_{5'}$, $\Delta_5$, $w_{6'}$, and $\Delta_6$). Thus, the total weight decrease in the branch of $\mathbf{delete}(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{3'} + \beta)$.

As a result, we get the following six branching vectors:

$$(w_{6'} + w_{3'} + 3m_1 + \alpha,\ w_{6'} - w_{5'} + w_{3'} + \beta) \tag{39}$$

for $(\alpha, \beta) \in \{(w_{4'}, w_{4'}), (w_4, \Delta_4), (\Delta'_{5-3}, w_{5'}), (\Delta_{5-3}, \Delta_5), (\Delta'_{6-4}, w_{6'}), (\Delta_{6-4}, \Delta_6)\}$.

**Case c-3.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{u3}})$ (see Figure 9): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7\}$.

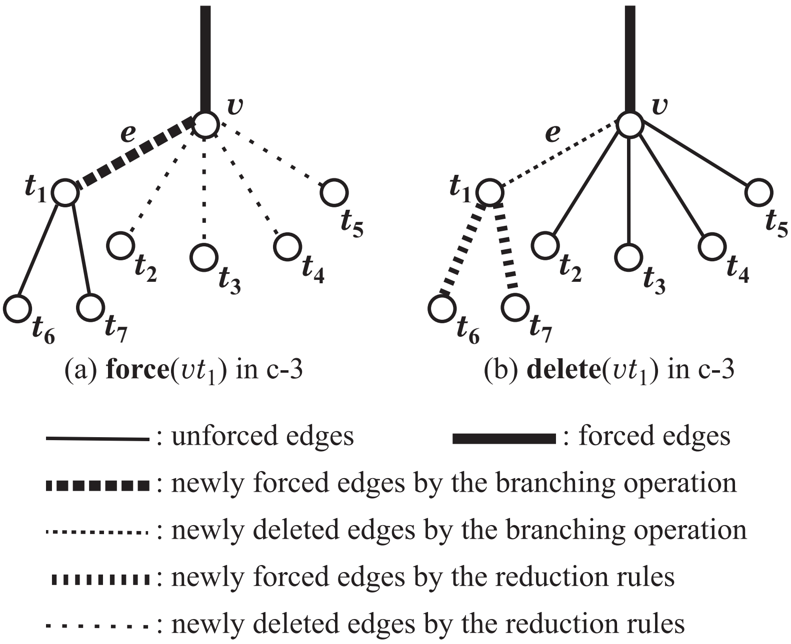



Figure 9: Illustration of branching rule c-3, where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{u3}})$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$ and $vt_5$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $\Delta_3$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be a type u3, f4, u4, f5, u5, f6, or u6-vertex, and each of their weights decrease by at least $m_2 = \min\{w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_3 - w_{3'} + 4m_2)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $w_3$. Each of vertices $t_6$ and $t_7$ can be a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and each of their weights decrease by at least $m_3 = \min\{w_{3'}, \Delta_3, w_{4'}, \Delta_4, w_{5'}, \Delta_5, w_{6'}, \Delta_6\}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} = w_{5'} + w_3 + 2m_3)$.

Then, we get the following branching vector:

$$(w_{6'} + w_3 - w_{3'} + 4m_2,\ w_{6'} = w_{5'} + w_3 + 2m_3)\,. \tag{40}$$

**Case c-4.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f4}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$ (see Figure 10): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$ and $vt_5$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{4'}$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be a type f4, u4, f5, u5, f6, or u6-vertex, and each of their weights decrease by at least $m_4 = \min\{\Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Each of vertices $t_6$

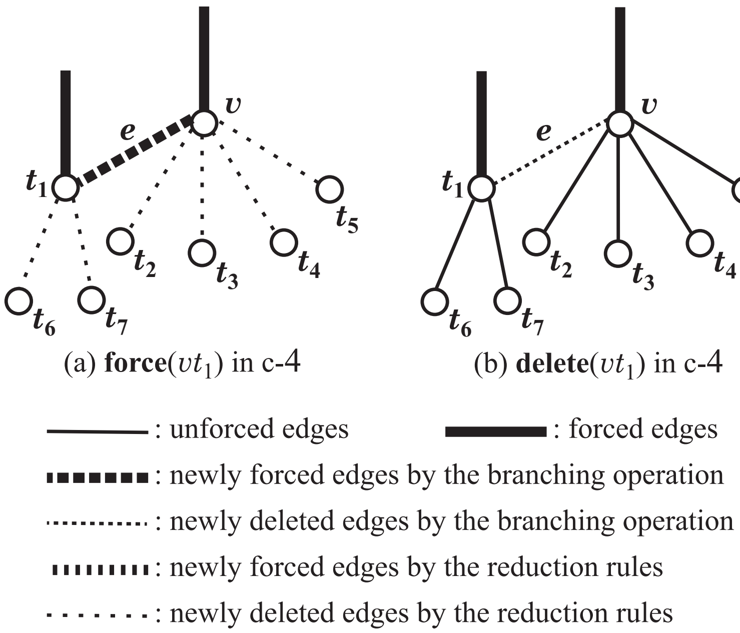



Figure 10: Illustration of branching rule c-4, where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f4}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$.

and $t_7$ can be either a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{4'} + 4m_4 + 2m_1)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{4'} - w_{3'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{4'} + 4m_4 + 2m_1,\ w_{6'} - w_{5'} + w_{4'} - w_{3'})\,. \tag{41}$$

**Case c-5.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f4}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$. We distinguish two sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$, (c-5(I)), $|N_U(v) \cap N_U(t_1)| = 1$, and (c-5(II)), $|N_U(v) \cap N_U(t_1)| = 2$.

**c-5(I).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 11): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6\}$.

In the branch **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_6$ and $t_1t_7$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{4'}$. Vertex $t_2$ and $t_3$ can be either a type f4, u4, f5, u5, f6, or u6-vertex, and its weight would decreases by at least $m_5 = \min\{w_{4'}, w_4, \Delta'5 - 3, \Delta_{5-3}, \Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_3$, $t_4$ and $t_5$ can be either a type f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_4 = \min\{\Delta'_{4-3}, \Delta_{4-3}, \Delta'5 - 4, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Vertex $t_6$ can be either a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and its weight would decreases by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{4'} + m_5 + 3m_4 + m_1)$.

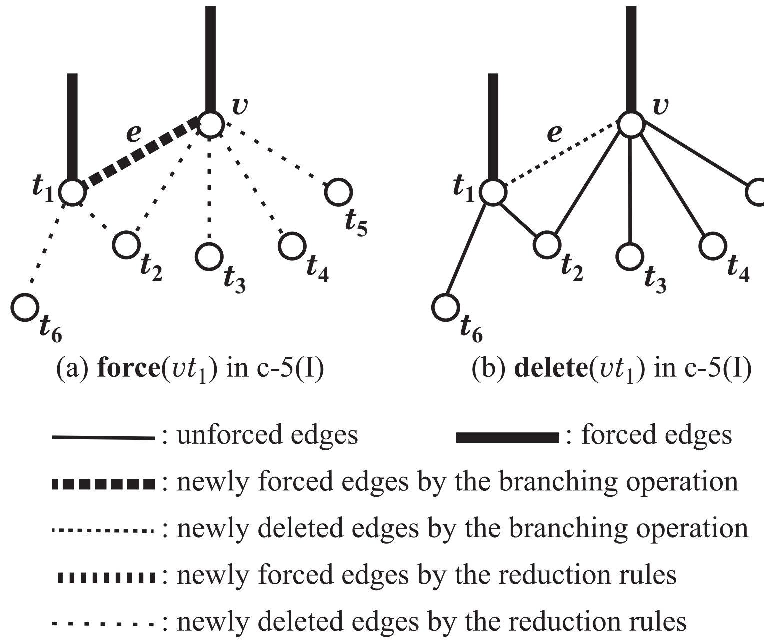



Figure 11: Illustration of branching rule c-5(I), where vertex $v \in V_{f6}$ and $t_1 \in N_U(v; V_{f4})$ such that $N_U(v) \cap N_U(t_1) = \{t_2\}$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{4'} - w_{3'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{4'} + m_5 + 3m_4 + m_1,\ w_{6'} - w_{5'} + w_{4'} - w_{3'}) \tag{42}$$

**c-5(II).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 12): We branch on edge $vt_1$.

In the branch **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$ and $t_1t_3$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{4'}$. Each of vertices $t_2$ and $t_3$ can be either a type f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_5 = \min\{w_{4'}, w_4, \Delta'5 - 3, \Delta_{5-3}, \Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_4$ and $t_5$ can be either a type f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_4 = \min\{\Delta'_{4-3}, \Delta_{4-3}, \Delta'5 - 4, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{4'} + 2m_5 + 2m_4)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_{4'} - w_{3'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{4'} + 2m_5 + 2m_4,\ w_{6'} - w_{5'} + w_{4'} - w_{3'}) \tag{43}$$

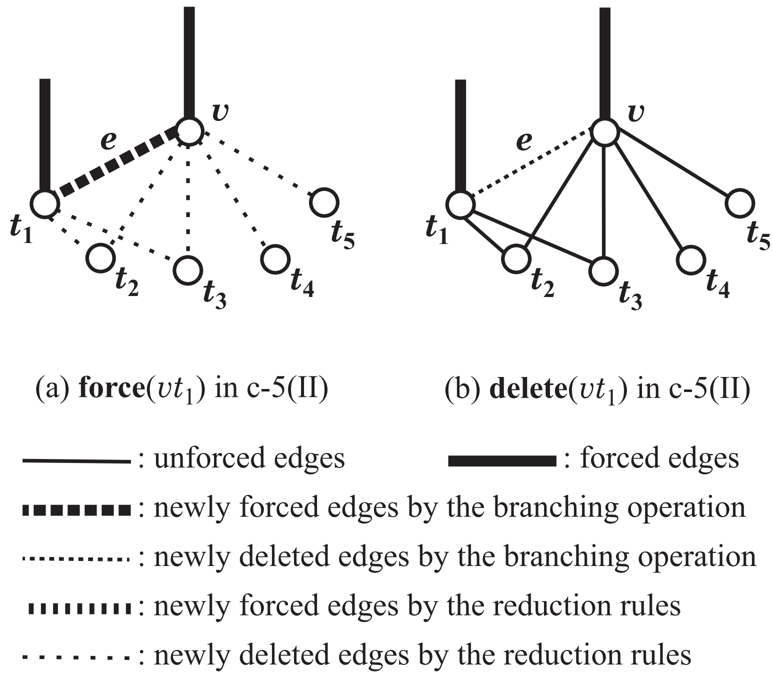



Figure 12: Illustration of branching rule c-5(II), where vertex $v \in V_{\rm f6}$ and $t_1 \in N_U(v; V_{\rm f4})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$.

**Case c-6.** There exist vertices $v \in V_{\rm f6}$ and $t_1 \in N_U(v; V_{\rm u4})$ (see Figure 13: We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8\}$.

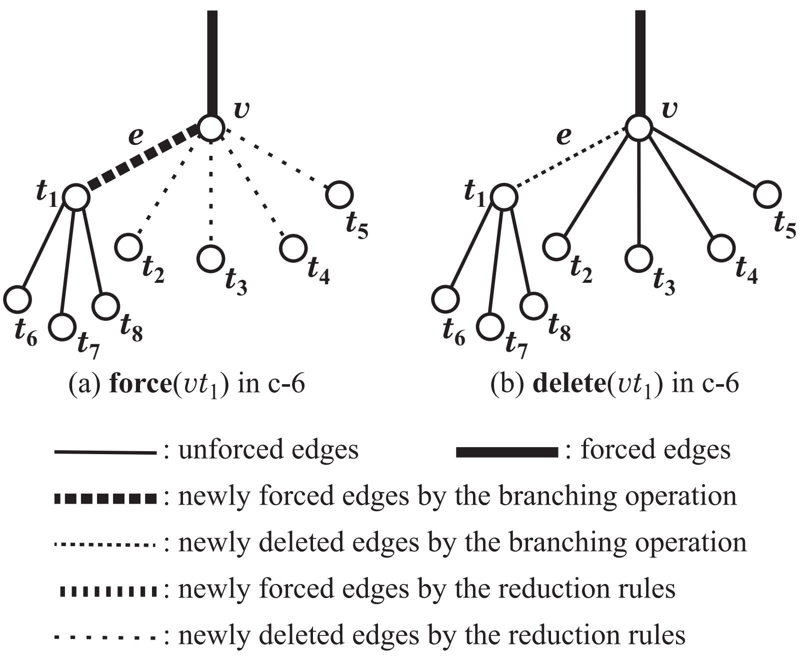



Figure 13: Illustration of branching rule c-6, where vertex $v \in V_{\rm f6}$ and $t_1 \in N_U(v; V_{\rm u4})$.

In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$ and $vt_5$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $\Delta_4$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be a type u4, f5, u5, f6, or u6-vertex, and each of their weights decrease by $m_6 = \min\{\Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_4 - w_{4'} + 4m_6)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_4 - w_3)$.

Then, we get the following branching vector:

$$(w_{6'} + w_4 - w_{4'} + 4m_6,\ w_{6'} - w_{5'} + w_4 - w_3)\,. \tag{44}$$

**Case c-7.** There exist vertices $v \in V_{\mathrm{f}6}$ and $t_1 \in N_U(v; V_{\mathrm{f}5})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$ (see Figure 14): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8\}$.

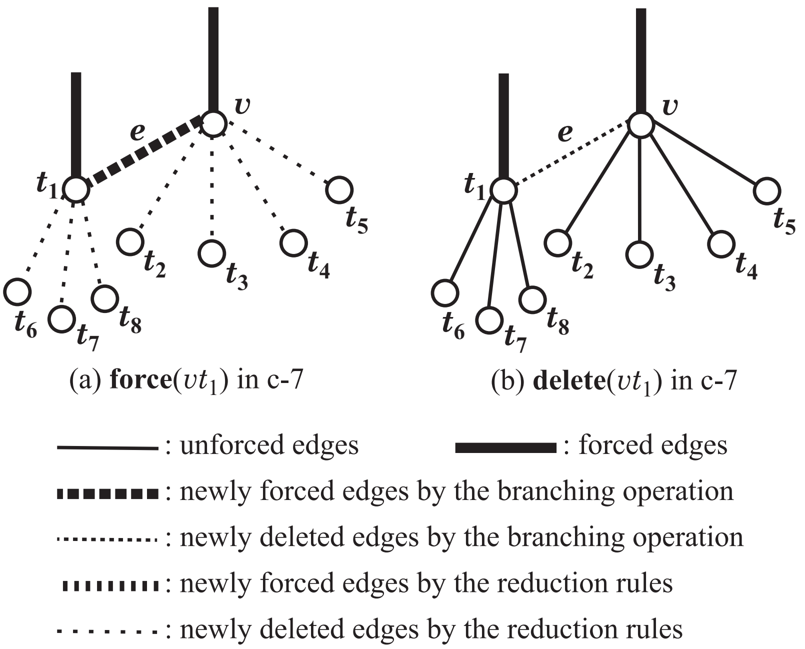



Figure 14: Illustration of branching rule c-7, where vertex $v \in V_{\mathrm{f}6}$ and $t_1 \in N_U(v; V_{\mathrm{f}5})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_6$, $t_1t_7$ and $t_1t_8$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{5'}$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be a type f5, u5, f6, or u6-vertex, and each of their weight decrease by $m_7 = \min\{\Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Each of vertices $t_6$, $t_7$ and $t_8$ can be either a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{5'} + 4m_7 + 3m_1)$.

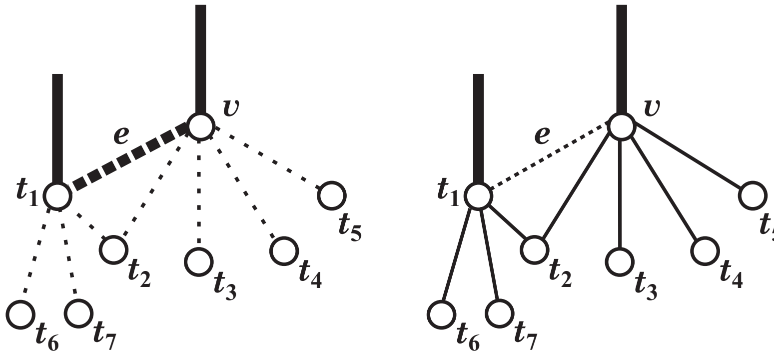
In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{5-4}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{4'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{5'} + 4m_7 + 3m_1,\ w_{6'} - w_{4'})\,. \tag{45}$$

**Case c-8.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f5}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$. We distinguish three sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$, (c-8(I)), $|N_U(v) \cap N_U(t_1)| = 1$, (c-8(II)), $|N_U(v) \cap N_U(t_1)| = 2$, and (c-8(III)), $|N_U(v) \cap N_U(t_1)| = 3$.

**c-8(I).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 15): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7\}$.




(a) **force**($vt_1$) in c-8(I) (b) **delete**($vt_1$) in c-8(I)

: unforced edges : forced edges
: newly forced edges by the branching operation
: newly deleted edges by the branching operation
: newly forced edges by the reduction rules
: newly deleted edges by the reduction rules

Figure 15: Illustration of branching rule c-8(I), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f5}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2\}$.

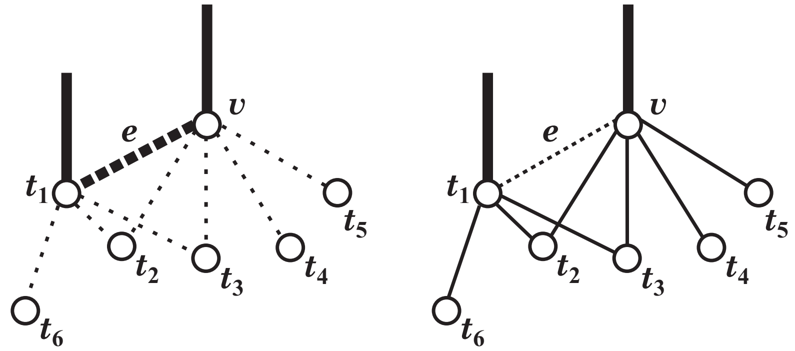
In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_6$ and $t_1t_7$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{5'}$. Vertex $t_2$ can be either a type f5, u5, f6, or u6-vertex, and its weight would decreases by at least $m_8 = \min\{\Delta'_{5-3}, \Delta_{5-3}, \Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_3$, $t_4$ and $t_5$ can be either a type f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_7 = \min\{\Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Each of vertices $t_6$ and $t_7$ can be either a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**($vt_1$) is at least $(w_{6'} + w_{5'} + 2m_8 + 2m_7 + m_1)$.

In the branch of **delete**($vt_1$), edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{5-4}$. Thus, the total weight decrease in the branch of **delete**($vt_1$) is at least $(w_{6'} - w_{4'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{5'} + m_8 + 3m_7 + 2m_1,\ w_{6'} - w_{4'})\,. \tag{46}$$

**c-8(II).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 16): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6\}$.



(a) **force**($vt_1$) in c-8(II) (b) **delete**($vt_1$) in c-8(II)

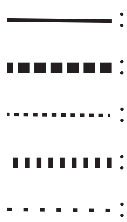



Figure 16: Illustration of branching rule c-8(II), where vertex $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f5}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$.

In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_3$ and $t_1t_6$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{5'}$. Each of vertices $t_2$ and $t_3$ can be either a type f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_8 = \min\{\Delta'_{5-3}, \Delta_{5-3}, \Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_4$ and $t_5$ can be either a type f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_7 = \min\{\Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Vertex $t_6$ can be either a type f3, u3, f4, u4, f5, u5, f6, or u6-vertex, and its weight would decreases by at least $m_1 = \min\{w_{3'}, w_3, \Delta'_{4-3}, \Delta_{4-3}, \Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**($vt_1$) is at least $(w_{6'} + w_{5'} + 2m_8 + 2m_7 + m_1)$.

In the branch of **delete**($vt_1$), edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{5-4}$. Thus, the total weight decrease in the branch of **delete**($vt_1$) is at least $(w_{6'} - w_{4'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{5'} + 2m_8 + 2m_7 + m_1,\ w_{6'} - w_{4'})\,. \tag{47}$$

**c-8(III).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 17): We branch on edge $vt_1$.

In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_3$ and $t_1t_4$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $w_{5'}$. Each of vertices $t_2$, $t_3$ and $t_4$ can be either a type f5, u5, f6, or u6-vertex, and each of their weights would decrease by at least $m_8 = \min\{\Delta'_{5-3}, \Delta_{5-3}, \Delta'_{6-4}, \Delta_{6-4}\}$. Vertex $t_5$

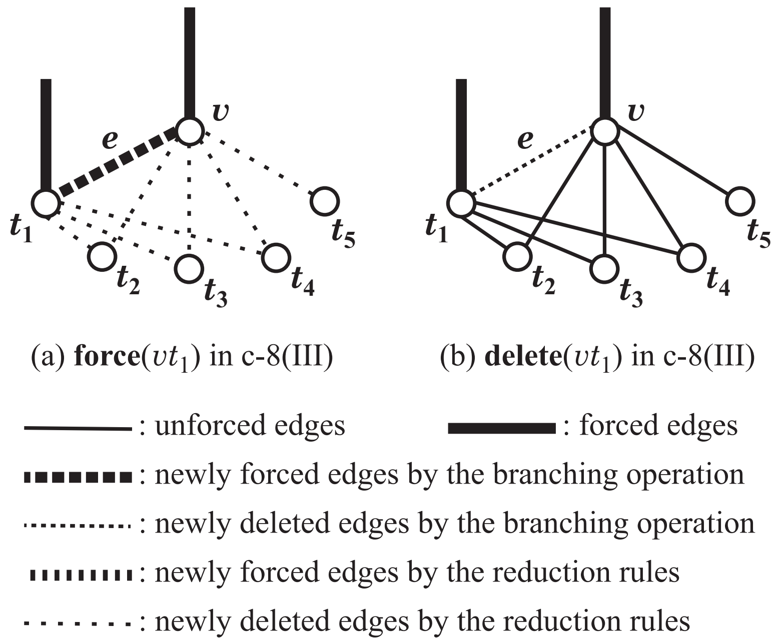



Figure 17: Illustration of branching rule c-8(III), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f5}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$.

can be either a type f5, u5, f6, or u6-vertex, and its weight would decreases by at least $m_7 = \min\{\Delta'_{5-4}, \Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_{5'} + 3m_8 + m_7)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{5-4}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{4'})$.

Then, we get the following branching vector:

$$(w_{6'} + w_{5'} + 3m_8 + m_7,\ w_{6'} - w_{4'})\,. \tag{48}$$

**Case c-9.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{u5}})$ (see Figure 18): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8, t_9\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$ and $vt_5$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $\Delta_5$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be a type u5, f6, or u6-vertex, and each of their weights decrease by $m_9 = \min\{\Delta_{5-4}, \Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_{6'} + w_5 - w_{5'} + 4m_9)$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta_{5-4}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_{6'} - w_{5'} + w_5 - w_4)$.

Then, we get the following branching vector:

$$(w_{6'} + w_3 - w_{5'} + 4m_9,\ w_{6'} - w_{5'} + w_5 - w_4)\,. \tag{49}$$

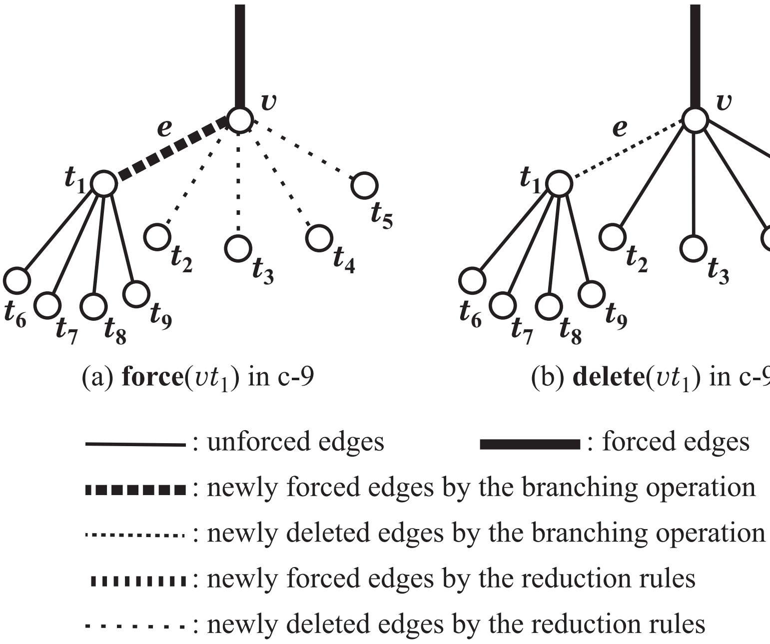



Figure 18: Illustration of branching rule c-9, where vertex $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{u5}})$.

**Case c-10.** There exist vertices $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f6}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$ (see Figure 19): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8, t_9\}$.

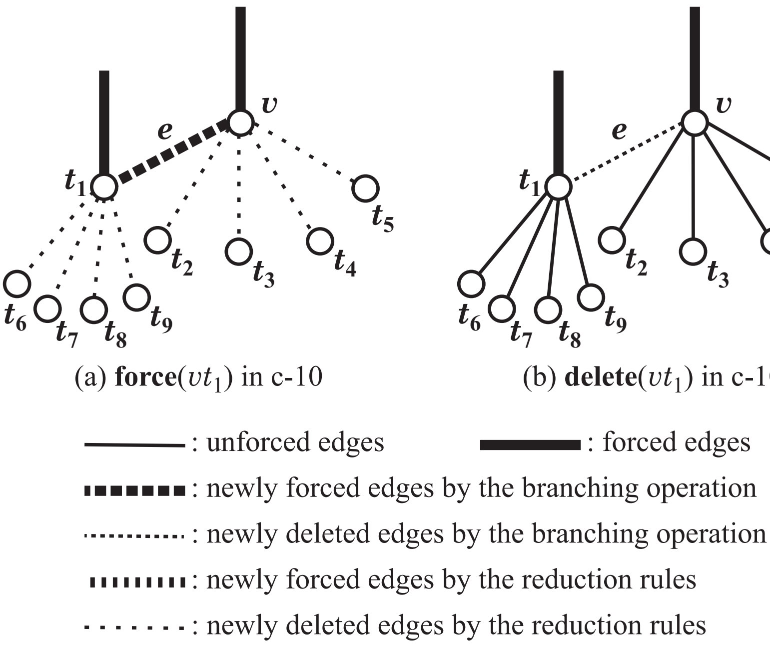



Figure 19: Illustration of branching rule c-10, where vertex $v \in V_{\text{f6}}$ and $t_1 \in N_U(v; V_{\text{f6}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_6$, $t_1t_7$, $t_1t_8$ and $t_1t_9$ will be deleted from $G'$ by the reduction rules. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $w_{6'}$, each. Each of vertices $t_2$, $t_3$, $t_4$, $t_5$, $t_6$, $t_7$, $t_8$ and $t_9$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{10} = \min\{\Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of $\mathbf{force}(vt_1)$ is at least $(w_{6'} + 8m_{10})$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $\Delta'_{6-5}$. Thus, the total weight decrease in the branch of $\mathbf{delete}(vt_1)$ is at least $(2w_{6'} - 2w_{5'})$.

Then, we get the following branching vector:

$$(w_{6'} + 8m_{10},\ 2w_{6'} - 2w_{5'})\,. \tag{50}$$

**Case c-11.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f6}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$. We distinguish four sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$, (c-11(I)), $|N_U(v) \cap N_U(t_1)| = 1$, (c-11(II)), $|N_U(v) \cap N_U(t_1)| = 2$, (c-11(III)), $|N_U(v) \cap N_U(t_1)| = 3$, and (c-11(IV)), $|N_U(v) \cap N_U(t_1)| = 4$.

**c-11(I).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 20): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8\}$.

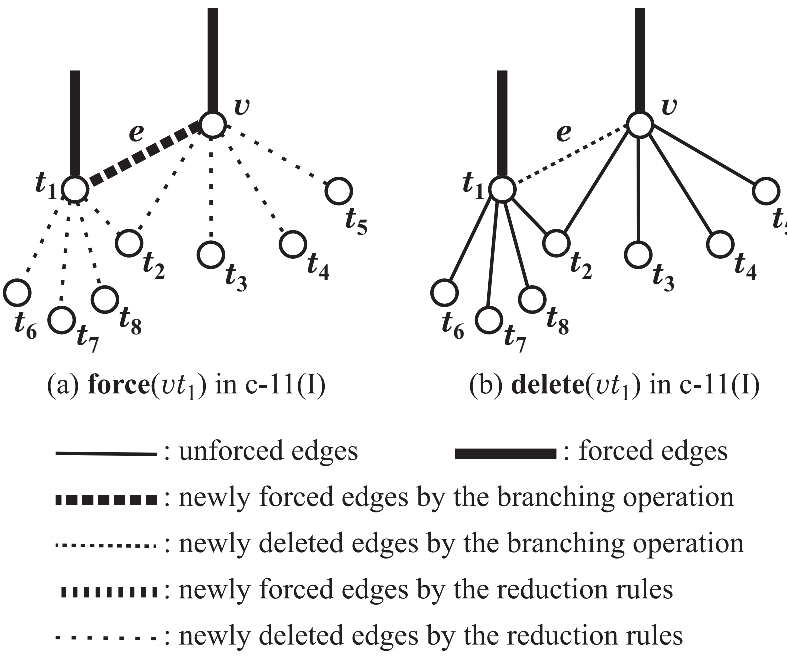



Figure 20: Illustration of branching rule c-11(I), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f6}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2\}$.

In the branch of $\mathbf{force}(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_6$, $t_1t_7$ and $t_1t_8$ will be deleted from $G'$ by the reduction rules. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $w_{6'}$, each. Vertex $t_2$ can be either a type f6, or u6-vertex, and its weight would decreases by at least $m_{11} = \min\{\Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_3$, $t_4$, $t_5$, $t_6$, $t_7$ and $t_8$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{10} = \min\{\Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of $\mathbf{force}(vt_1)$ is at least $(w_{6'} + m_{11} + 6m_{10})$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $\Delta'_{6-5}$. Thus, the total weight decrease in the branch of $\mathbf{delete}(vt_1)$ is at least $(2w_{6'} - 2w_{5'})$.

Then, we get the following branching vector:

$$(w_{6'} + m_{11} + 6m_{10},\ 2w_{6'} - 2w_{5'})\,. \tag{51}$$

**c-11(II).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 21): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7\}$.

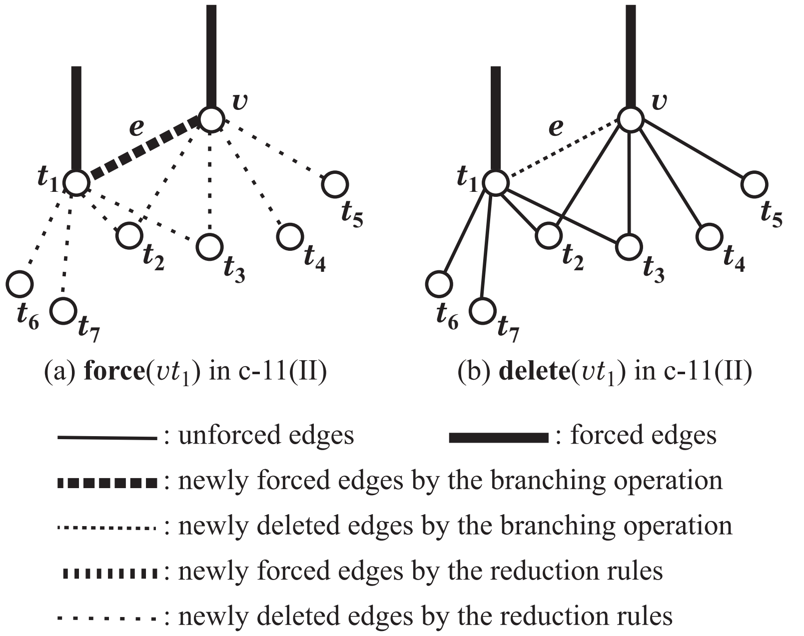



Figure 21: Illustration of branching rule c-11(II), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f6}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_3$, $t_1t_6$ and $t_1t_7$ will be deleted from $G'$ by the reduction rules. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $w_{6'}$, each. Each of vertices $t_2$ and $t_3$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{11} = \min\{\Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_4$, $t_5$, $t_6$ and $t_7$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{10} = \min\{\Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(2w_{6'} + 2m_{11} + 4m_{10})$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $\Delta'_{6-5}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(2w_{6'} - 2w_{5'})$.

Then, we get the following branching vector:

$$(2w_{6'} + 2m_{11} + 4m_{10},\ 2w_{6'} - 2w_{5'})\,. \tag{52}$$

**c-11(III).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 22): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_3$, $t_1t_4$ and $t_1t_6$ will be deleted from $G'$ by the reduction rules. Hence, both weights of vertex $v$ and vertex $t_1$ decreases by $w_{6'}$, each. Each of vertices

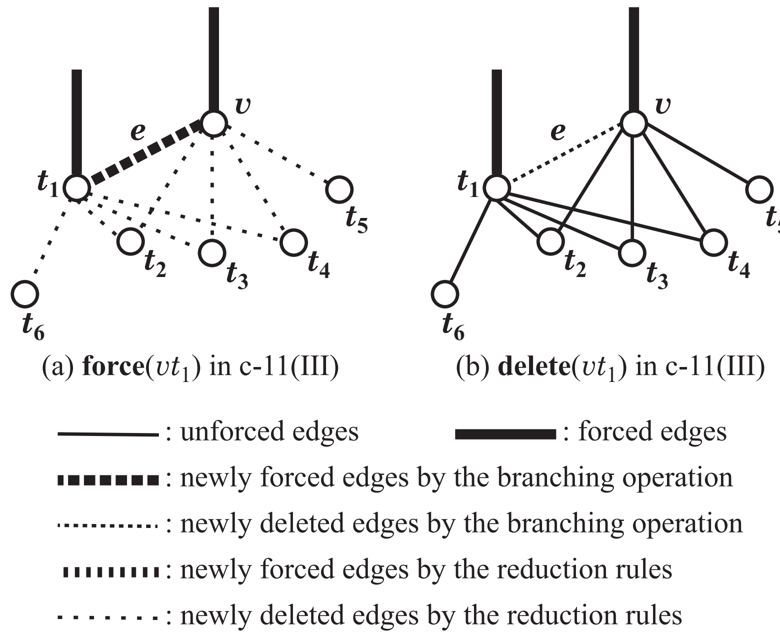



Figure 22: Illustration of branching rule c-11(III), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f6}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$.

$t_2$, $t_3$ and $t_4$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{11} = \min\{\Delta'_{6-4}, \Delta_{6-4}\}$. Each of vertices $t_5$ and $t_6$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{10} = \min\{\Delta'_{6-5}, \Delta_{6-5}\}$. Thus, the total weight decrease in the branch of **force**($vt_1$) is at least $(2w_{6'} + 3m_{11} + 2m_{10})$.

In the branch of **delete**($vt_1$), edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $\Delta'_{6-5}$. Thus, the total weight decrease in the branch of **delete**($vt_1$) is at least $(2w_{6'} - 2w_{5'})$.

Then, we get the following branching vector:

$$(2w_{6'} + 3m_{11} + 2m_{10},\ 2w_{6'} - 2w_{5'})\,. \tag{53}$$

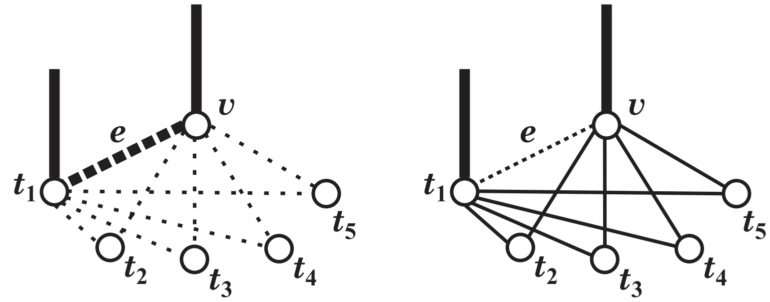
**c-11(IV).** Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5\}$ (see Figure 23): We branch on edge $vt_1$.

In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$, $vt_5$, $t_1t_2$, $t_1t_3$, $t_1t_4$ and $t_1t_5$ will be deleted from $G'$ by the reduction rules. Hence, both weights of vertex $v$ and vertex $t_1$ decreases by $w_{6'}$, each. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can be either a type f6, or u6-vertex, and each of their weights would decrease by at least $m_{11} = \min\{\Delta'_{6-4}, \Delta_{6-4}\}$. Thus, the total weight decrease in the branch of **force**($vt_1$) is at least $(2w_{6'} + 4m_{11})$.

In the branch of **delete**($vt_1$), edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weight of vertex $v$ and vertex $t_1$ decreases by $\Delta'_{6-5}$. Thus, the total weight decrease in the branch of **delete**($vt_1$) is at least $(2w_{6'} - 2w_{5'})$.

Then, we get the following branching vector:

$$(2w_{6'} + 4m_{11},\ 2w_{6'} - 2w_{5'})\,. \tag{54}$$

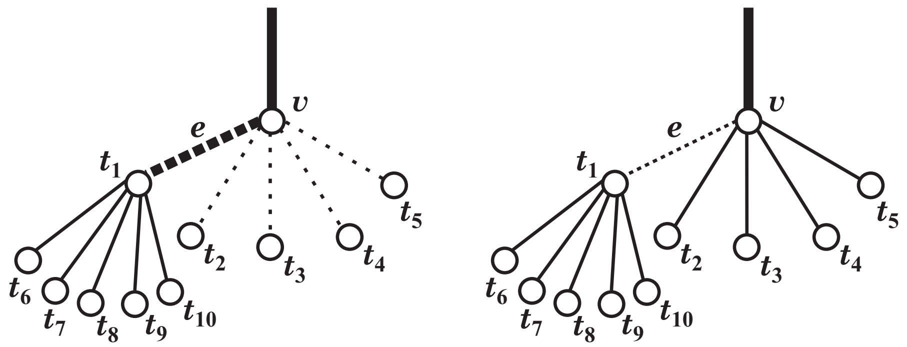



(a) **force**($vt_1$) in c-11(IV) (b) **delete**($vt_1$) in c-11(IV)

: unforced edges : forced edges
: newly forced edges by the branching operation
: newly deleted edges by the branching operation
: newly forced edges by the reduction rules
: newly deleted edges by the reduction rules

Figure 23: Illustration of branching rule c-11(IV), where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{f6}})$ such that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5\}$.

**Case c-12.** There exist vertices $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{u6}})$ (see Figure 24): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_6, t_7, t_8, t_9, t_{10}\}$.




(a) **force**($vt_1$) in c-12 (b) **delete**($vt_1$) in c-12

: unforced edges : forced edges
: newly forced edges by the branching operation
: newly deleted edges by the branching operation
: newly forced edges by the reduction rules
: newly deleted edges by the reduction rules

Figure 24: Illustration of branching rule c-12, where vertex $v \in V_{\mathrm{f6}}$ and $t_1 \in N_U(v; V_{\mathrm{u6}})$.

In the branch of **force**($vt_1$), edge $vt_1$ will be added to $F'$ by the branching operation, and edges $vt_2$, $vt_3$, $vt_4$ and $vt_5$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $w_{6'}$, and the weight of vertex $t_1$ decreases by $\Delta_6$. Each of vertices $t_2$, $t_3$, $t_4$ and $t_5$ can only be a type u6-vertex, and each of their weight decrease by $\Delta_{6-5}$. Thus, the total weight decrease in the branch of **force**($vt_1$) is at least $(w_6 + 4w_6 - 4w_5)$.

In the branch of **delete**($vt_1$), edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta'_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta_{6-5}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_5 + w_{6'} - w_{5'})$.

Then, we get the following branching vector:

$$(1 + 4w_6 - 4w_5,\ 1 - w_5 + w_{6'} - w_{5'})\,. \tag{55}$$

### 4.4 Branching on Edges Around u6-vertices

If none of the first 12 conditions can be executed, this means that the graph has no f6-vertices. Even though the graph has no more f6-vertices, the graph might has u6-vertices, because of two types of vertices. Therefore, this section derives branching vectors for branchings on an optimal edge $e = vt$, incident to a u6-vertex $v$, and there are distinguishing in seven cases (conditions c-13 to c-19).

**Case c-13.** There exist vertices $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{f3}})$ (see Figure 25): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7\}$.

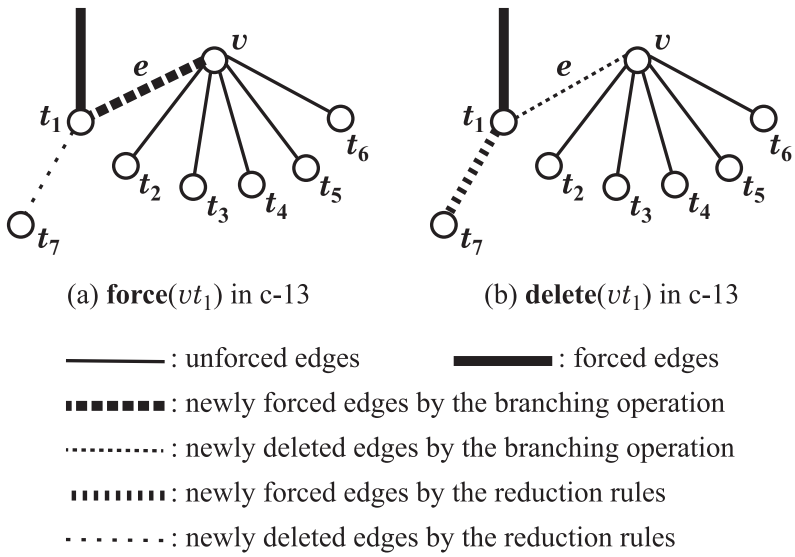



Figure 25: Illustration of branching rule c-13, where vertex $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{f3}})$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edge $t_1t_7$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $w_{3'}$.

There are two sub-cases for the vertex type of vertex $t_7$. First, if vertex $t_7$ is an f3-vertex (see Figure 6), then the weight of vertex $t_7$ decreases by $w_{3'}$. If vertex $x$ is an f3-vertex (resp., u3, f4, u4, f5, u5, or a u6-vertex), then the weight decrease $\beta_2$ of vertex $x$ would be $w_{3'}$ (resp., $\Delta_3$, $w_{4'}$, $\Delta_4$, $w_{5'}$, $\Delta_5$, and $\Delta_6$). Thus the total weight decrease for the first sub-case in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_{3'} + w_{3'} + \beta_2)$.

Second, if vertex $t_7$ is a u3-vertex (resp., f4, u4, f5, u5, or a u6-vertex), then the weight decrease $\alpha_1$ of vertex $t_7$ would be $w_3$ (resp., $\Delta'_{4-3}$, $\Delta_{4-3}$, $\Delta'_{5-4}$, $\Delta_{5-4}$, and $\Delta_{6-5}$). Thus, the total weight decrease for the second sub-case in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_{3'} + \alpha_1)$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation, and edge $t_1t_7$ will be added to $F'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $w_{3'}$.

First, if vertex $t_7$ is an f3-vertex, then the weight of vertex $t_7$ decreases by $w_{3'}$. If vertex $x$ is an f3-vertex (resp., u3, f4, u4, f5, u5, or a u6-vertex), then the weight decrease $\alpha_2$ of vertex $x$ would be $w_{3'}$ (resp., $w_3$, $\Delta'_{4-3}$, $\Delta_{4-3}$, $\Delta'_{5-4}$, $\Delta_{5-4}$, and $\Delta_{6-5}$). Thus, the total weight decrease for the first sub-case in the branch of $\mathbf{delete}(vt_1)$ is at least $(w_6 - w_5 + w_{3'} + w_{3'} + \alpha_2)$.

Second, if vertex $t_7$ is a u3-vertex (resp., f4, u4, f5, u5, or a u6-vertex), then the weight decrease $\beta_1$ of vertex $t_7$ would be $\Delta_3$ (resp., $w_{4'}$, $\Delta_4$, $w_{5'}$, $\Delta_5$, and $\Delta_6$). Thus, the total weight decrease for the second sub-case in the branch of $\mathbf{delete}(vt_1)$ is at least $(w_6 - w_5 + w_{3'} + \beta_1)$.

As a result, we get the following seven branching vectors in the first sub-case, and six branching vectors in the second sub-case, respectively:

$$(w_6 - w_{6'} + w_{3'} + w_{3'} + \beta_2,\ w_6 - w_5 + w_{3'} + w_{3'} + \alpha_2) \tag{56}$$

$$(w_6 - w_{6'} + w_{3'} + \alpha_1,\ w_6 - w_5 + w_{3'} + \beta_1) \tag{57}$$

for $(\alpha_1, \beta_1) \in \{(w_3, \Delta_3), (\Delta'_{4-3}, w_{4'}), (\Delta_{4-3}, \Delta_4), (\Delta'_{5-4}, w_{5'}), (\Delta_{5-4}, \Delta_5), (\Delta_{6-5}, \Delta_6)\}$, and $(\beta_2, \alpha_2) \in \{(w_{3'}, w_{3'}), (\Delta_3, w_3), (w_{4'}, \Delta'_{4-3}), (\Delta_4, \Delta_{4-3}), (w_{5'}, \Delta'_{5-4}), (\Delta_5, \Delta_{5-4}), (\Delta_6, \Delta_{6-5})\}$.

**Case c-14.** There exist vertices $v \in V_{u6}$ and $t_1 \in N_U(v; V_{u3})$ (see Figure 26): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8\}$.

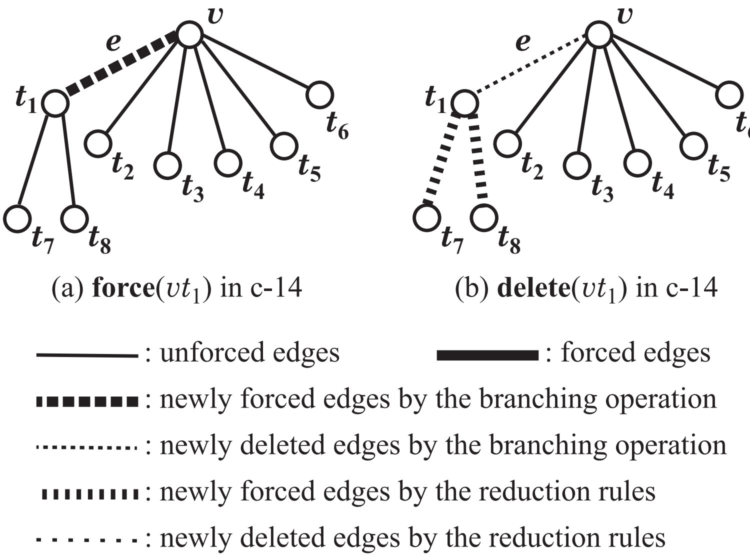



Figure 26: Illustration of branching rule c-14, where vertex $v \in V_{u6}$ and $t_1 \in N_U(v; V_{u3})$.

In the branch of $\mathbf{force}(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $\Delta_3$. Thus, the total weight decrease in the branch of $\mathbf{force}(vt_1)$ is at least $(w_6 - w_{6'} + w_3 - w_{3'})$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation, and edges $t_1t_7$ and $t_1t_8$ will be added to $F'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $w_3$. Each of vertices

$t_7$ and $t_8$ can be either a type f3, u3, f4, u4, f5, u5, or u6-vertex, and each of their weights would decrease by at least $m_{12} = \min\{w_{3'}, w_3 - w_{3'}, w_{4'}, \Delta_4, w_{5'}, \Delta_5, \Delta_6\}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_5 + w_3 + 2m_{12})$.

Then, we get the following branching vector:

$$(1 - w_{6'} + w_3 - w_{3'},\ 1 - w_5 + w_3 + 2m_{12})\,. \tag{58}$$

**Case c-15.** There exist vertices $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{f4}})$ (see Figure 27): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8\}$.

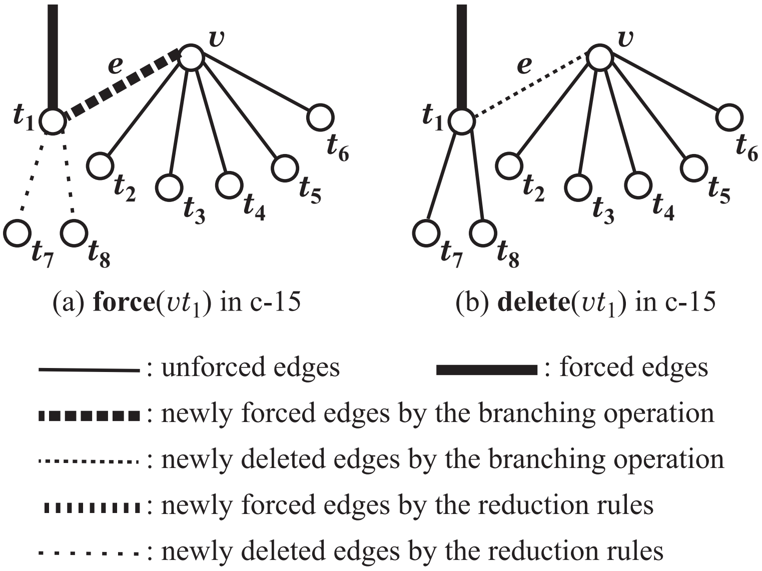



Figure 27: Illustration of branching rule c-15, where vertex $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{f4}})$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $t_1t_7$ and $t_1t_8$ will be deleted from $G$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $w_{4'}$. Each of vertices $t_7$ and $t_8$ can be either a type f3, u3, f4, u4, f5, u5, or u6-vertex, and each of their weights would decrease by at least $m_{13} = \min\{w_{3'}, w_3, \Delta_{4'}, \Delta_4, \Delta_{5'}, \Delta_5, \Delta_6\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_{4'} + 2m_{13})$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_5 + w_{4'} - w_{3'})$.

Then, we get the following branching vector:

$$(1 - w_{6'} + w_{4'} + 2m_{13},\ 1 - w_5 + w_{4'} - w_{3'})\,. \tag{59}$$

**Case c-16.** There exist vertices $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{u4}})$ (see Figure 28): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8, t_9\}$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $\Delta_4$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_4 - w_{4'})$.

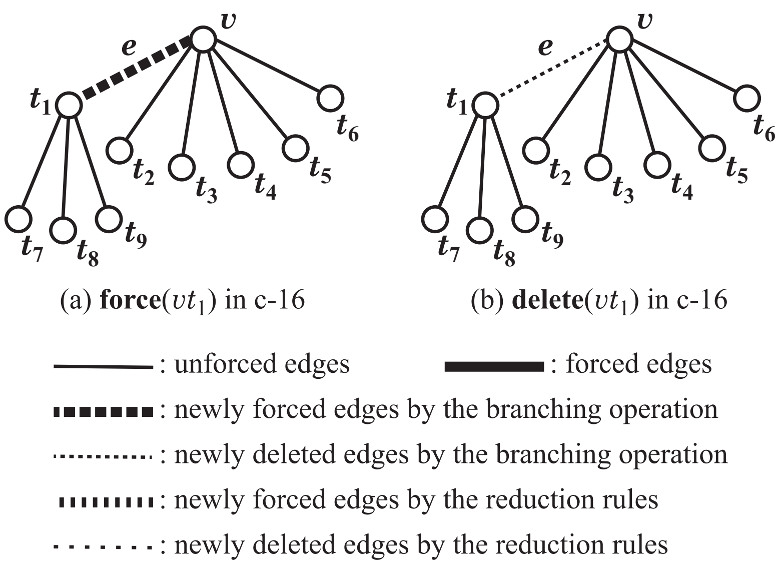



Figure 28: Illustration of branching rule c-16, where vertex $v \in V_{\mathrm{u}6}$ and $t_1 \in N_U(v; V_{\mathrm{u}4})$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta_{4-3}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_5 + w_4 - w_3)$.

Then, we get the following branching vector:

$$(1 - w_{6'} + w_4 - w_{4'},\ 1 - w_5 + w_4 - w_3)\,. \tag{60}$$

**Case c-17.** There exist vertices $v \in V_{\mathrm{u}6}$ and $t_1 \in N_U(v; V_{\mathrm{f}5})$ (see Figure 29): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8, t_9\}$.

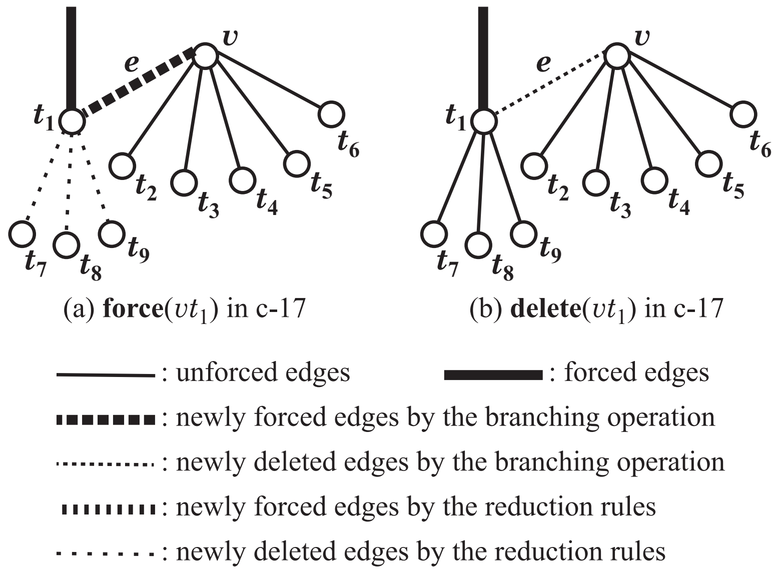



Figure 29: Illustration of branching rule c-17, where vertex $v \in V_{\mathrm{u}6}$ and $t_1 \in N_U(v; V_{\mathrm{f}5})$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation, and edges $t_1t_7$, $t_1t_8$ and $t_1t_9$ will be deleted from $G'$ by the reduction rules. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $w_{5'}$. Each of vertices $t_7$, $t_8$ and $t_9$ can be either a type f3, u3, f4, u4, f5, u5, or u6-vertex, and each of their weights would decrease by at least $m_{13} = \min\{w_{3'}, w_3, \Delta_{4'}, \Delta_4, \Delta_{5'}, \Delta_5, \Delta_6\}$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_{5'} + 3m_{13})$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta'_{5-4}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_5 + w_{5'} - w_{4'})$.

Then, we get the following branching vector:

$$(1 - w_{6'} + w_{5'} + 3m_{13},\ 1 - w_5 + w_{5'} - w_{4'})\,. \tag{61}$$

**Case c-18.** There exist vertices $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{u5}})$ (see Figure 30): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8, t_9, t_{10}\}$.

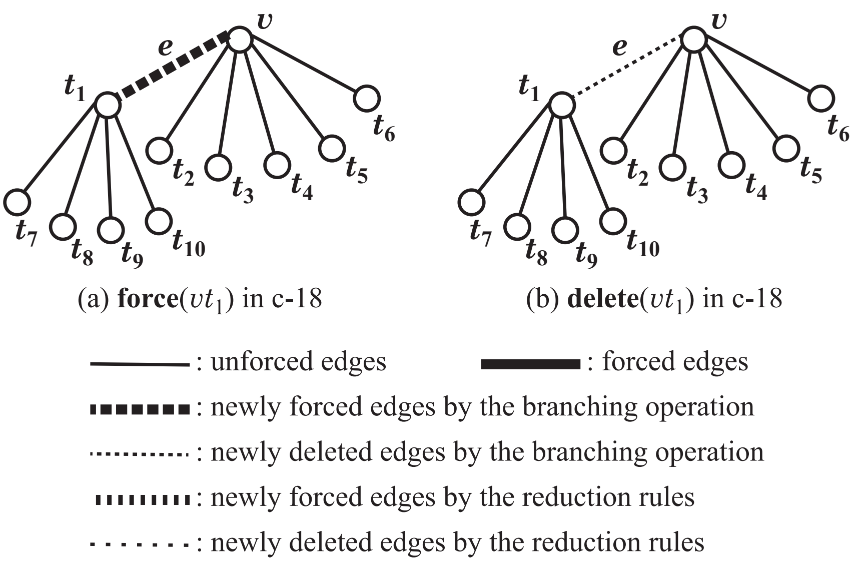



Figure 30: Illustration of branching rule c-18, where vertex $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{u5}})$.

In the branch of **force**$(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_6$, and the weight of vertex $t_1$ decreases by $\Delta_5$. Thus, the total weight decrease in the branch of **force**$(vt_1)$ is at least $(w_6 - w_{6'} + w_5 - w_{5'})$.

In the branch of **delete**$(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, the weight of vertex $v$ decreases by $\Delta_{6-5}$, and the weight of vertex $t_1$ decreases by $\Delta_{5-4}$. Thus, the total weight decrease in the branch of **delete**$(vt_1)$ is at least $(w_6 - w_4)$.

Then, we get the following branching vector:

$$(1 - w_{6'} + w_5 - w_{5'},\ 1 - w_4)\,. \tag{62}$$

**Case c-19.** There exist vertices $v \in V_{\mathrm{u6}}$ and $t_1 \in N_U(v; V_{\mathrm{u6}})$ (see Figure 31): We branch on edge $vt_1$. Note that $N_U(t_1) \setminus \{v\} = \{t_7, t_8, t_9, t_{10}, t_{11}\}$.

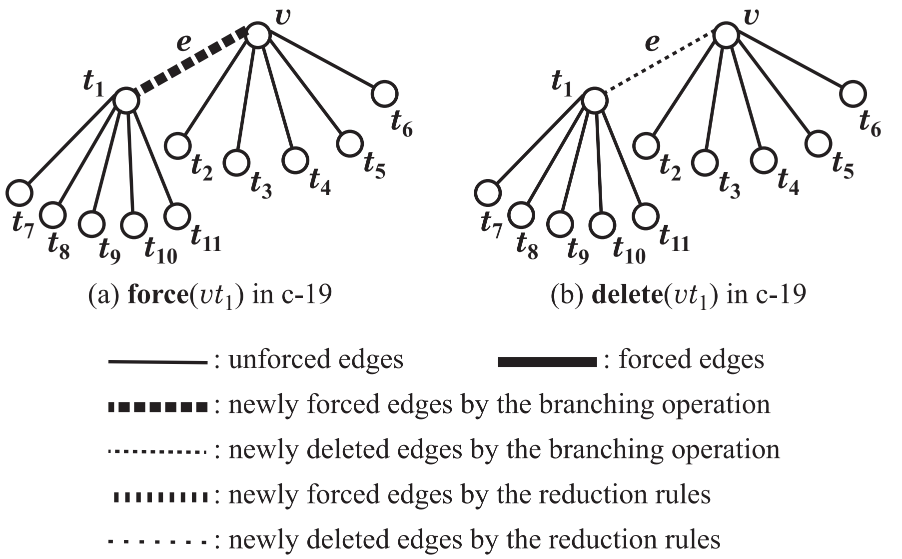



Figure 31: Illustration of branching rule c-19, where vertex $v \in V_{u6}$ and $t_1 \in N_U(v; V_{u6})$.

In the branch of $\mathbf{force}(vt_1)$, edge $vt_1$ will be added to $F'$ by the branching operation. Hence, both weights of vertex $v$ and vertex $t_1$ decreases by $\Delta_6$, each. Thus, the total weight decrease in the branch of $\mathbf{force}(vt_1)$ is at least $(2w_6 - 2w_{6'})$.

In the branch of $\mathbf{delete}(vt_1)$, edge $vt_1$ will be deleted from $G'$ by the branching operation. Hence, both weights of vertex $v$ and vertex $t_1$ decreases by $\Delta_{6-5}$, each. Thus, the total weight decrease in the branch of $\mathbf{delete}(vt_1)$ is at least $(2w_6 - 2w_5)$.

Then, we get the following branching vector:

$$(2 - 2w_{6'},\ 2 - 2w_5)\,. \tag{63}$$

## 4.5 Switching to TSP5

If none of these 19 cases can be executed, this means that the graph has no more degree-6 vertices. In that case, we can switch and use a fast algorithm for TSP in degree5 graphs (tsp5$(G, F)$) to solve the remaining instances. Xiao and Nagamochi [10, Lemma 3] have shown how to leverage results obtained by a measure-and-conquer analysis, and that an algorithm can used as a sub-procedure, given that we know the respective weight setting mechanism. To get a combination of total running time bound of these two algorithms, we can use the maximum branching factor for TSP in degree-4 graphs algorithm and a measure $\mu$ is calculated based on the maximum ratio of vertex weights for TSP in degree-5 graphs and TSP in degree-6 graphs [7].

Here, we use the $O^*(2.4531^n)$-time algorithm by Md Yunos et al. [6], where the weights of vertices in degree-4 graphs are set as follows:

$$\begin{aligned}
w_{3'} &= 0.156687 \\
w_3 &= 0.276915 \\
w_{4'} &= 0.313373 \\
w_4 &= 0.607542 \\
w_{5'} &= 0.470059 \\
w_5 &= 1
\end{aligned}$$

For this step, the running time bound is

$$T(\mu) \leq O\left(2.453051^{\max\left\{\frac{0.156687}{w_{3'}},\frac{0.276915}{w_3},\frac{0.313373}{w_{4'}},\frac{0.607542}{w_4},\frac{0.470059}{w_{5'}},\frac{1}{w_5}\right\}}\right). \tag{64}$$

### 4.6 Overall Analysis

As a result, the branching factor of each of the branching vectors from (37) to (64) does not exceed 2.746706. The tight constraints in the quasiconvex program are in conditions c-4, c-5(I), c-5(II), c-14, c-16 and c-18. This completes a proof of Theorem 1.

## 5 Conclusion

In this paper, we have presented an exact algorithm for TSP in degree-6 graphs. Our algorithm is a simple branching algorithm, imitate the branch-and-reduce paradigm of the TSP in degree-5 graphs, and it operates in space which is polynomial of the size of an input instance. To the best of our knowledge, this is the first polynomial space exact algorithm developed specifically for graphs of maximum degree at most 6.

We used the measure and conquer method for the analysis of the running time of the proposed algorithm, and have obtained an upper bound of $O^*(2.7467^n)$, where $n$ is the number of vertices in a given instance. This result compares favorably with the polynomial-space TSP algorithm for general graphs by Gurevich and Shelah [3], which runs in $O^*(4^n n^{\log n})$-time.

It remains an open question whether this time bound can be further improved by a modified analysis technique, or by a careful re-examination of the branching rules. Indeed, it would be most interesting to obtain a polynomial-space algorithm with a running time of $O^*(2^n)$ or less, or simply show that this cannot be achieved.

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