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A Non-Standard J-Spectral Factorization of Rational Matrices via Generalized Algebraic Riccati Equation

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Abstract

This paper considers a non-standard J-spectral factorization problem (**NS-J-SFP**) of a rational spectral matrix, which is related to the non-standard H_∞ control problem for a descriptor system. We derive necessary and sufficient conditions for the existence of a semi-stabilizing solution of a generalized algebraic Riccati equation (GARE). We then develop the solvability conditions for the **NS-J-SFP** by adapting the zero compensation technique of Copeland and Safonov [7] to the descriptor system. Thus we can conclude that the **NS-J-SFP** is solvable if and only if the GARE has a semi-stabilizing solution. A numerical result is also included.

Keywords. non-standard J-spectral factorization, generalized algebraic Riccati equation, descriptor system, Hamiltonian matrix pencil

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Contents

1 Introduction

The J-spectral factorization plays a central role in the optimal Hankel norm model reduction [2] and $\mathcal{H}_\infty$ control problems [8, 17, 10, 18, 11]. For this reason, various J-spectral factorization problems have extensively been studied in the literature [13, 5, 19, 20, 1, 21]. Among others, Green *et al.* [10] have derived an algorithm for solving a J-spectral factorization problem in terms of the stabilizing solution of an indefinite algebraic Riccati equation (ARE), by which the two Riccati solution to the standard $\mathcal{H}_\infty$ control problem has been provided. This factorization problem goes under the name of canonical J-spectral factorization problem (**C-J-SFP**), for the derivation of the algorithm is based on the theory of canonical factorization due to Bart *et al.* [3]. Meinsma [20] has shown that the use of equalizing vectors can alleviate the mathematical burden associated with the proof of the canonical J-spectral factorization result. The J-spectral factorization of polynomial matrices has been studied by Kwakernaak and Šebek [19] and Aliev and Larin [1]. In [23], we have obtained a J-spectral factorization based solution to the H_∞ control problem for a descriptor system exploiting a generalized algebraic Riccati equation (GARE). Clements [5] and Oară [21] have dealt with the J-spectral factorization problem in more general framework based on the matrix pencil approach, where the indefinite spectral matrix is allowed to be improper and may have zeros on the extended imaginary axis. Copeland and Sofonov [7] have developed a zero canceling compensator approach to solve singular H_2/H_∞ control problems and related inner-outer factorization problem, and Xin and Kimura [26] have solved a singular J-lossless factorization problem by extending the infinite zero compensation technique of [7]. Also, we have presented a factorization result of the standard J-spectral factorization problem (**S-J-SFP**) for a descriptor system based on the GARE [15], and a numerical method of computing related J-inner-outer factorization is developed in [25].

In this paper we consider the non-standard J-spectral factorization problem (**NS-J-SFP**) that arises in the non-standard $\mathcal{H}_\infty$ control problem for a descriptor system; a more complete definition of it is given in Section 3. The purpose of this paper is to derive an algorithm for solving the **NS-J-SFP** in terms of a semi-stabilizing solution of an indefinite GARE. The process of deriving the algorithm is roughly divided into two parts. In the first part, by extending the technique in our previous work [16], we derive the existence conditions of a semi-stabilizing solution of the GARE and give a parameterization of all the semi-stabilizing solutions. In the second part, adapting the zero canceling compensation technique of Copeland and Safonov [7], improved later by Xin and Kimura [26], for a descriptor system, we derive the solvability conditions for the **NS-J-SFP** and show that they coincide with the existence conditions of a semi-stabilizing solution of the GARE derived in the first part. Thus it can be shown that the **NS-J-SFP** is solvable if and only if the GARE has a semi-stabilizing solution.

For the construction of a semi-stabilizing solution, we need to compute not only a basis of the stable eigenspace of the Hamiltonian matrix pencil but also a basis of the infinite eigenspace of the system matrix (rectangular) pencil. Since the latter basis can be computed, for example, by the algorithm for finding a stair-case form of a singular pencil [24], we can safely say that our algorithm is numerically reliable.

The organization of this paper is as follows. In Section 2 we review the basic facts concerning the eigenstructure of a matrix pencil and introduce a direct sum decomposition of the infinite eigenspace. In Section 3 we formulate the **S-J-SFP** and **NS-J-SFP** and define some notation to be used throughout this paper. In Section 4 we present a $\mathbf{C}_{0e}$ zero compensator ($\mathbf{C}_{0e}$ ZC) for a descriptor system, which is a key tool to derive the solvability conditions of the **NS-J-SFP**. Moreover, using the $\mathbf{C}_{0e}$ ZC, we examine the dimension of the infinite eigenspace of the Hamiltonian matrix pencil. In Section 5, we consider the existence conditions of a semi-stabilizing solution of the GARE and parameterize all the semi-stabilizing solutions. In Section 6, we derive the solvability conditions of the **NS-J-SFP** based on the ZC technique of Section 4. We then show that the **NS-J-SFP** is solvable if and only if the GARE has a semi-stabilizing solution. A numerical example is also included in Section 7. Section 8 gives some concluding remarks. Appendix A briefly states our solution to the **S-J-SFP** [14, 15] and Appendix B gives a lemma which will be used in the proof of Section 6. Appendices C to L include proofs of lemmas and theorems.

The notation of the paper is as follows.

- $(\cdot)^T$, $(\cdot)^H$: transpose, hermitian transpose
- $\mathbf{R}$, $\mathbf{C}$: real and complex number fields
- $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$: open left half plane, imaginary axis, open right half plane
- $\mathbf{C}_{0e} := \mathbf{C}_0 \cup \{\infty\}$: extended imaginary axis
- $\mathcal{R}$: set of real rational matrices
- $\mathcal{RL}_\infty$: set of proper real rational matrices having no poles on $\mathbf{C}_0$
- $\mathcal{RH}_\infty$: set of proper stable real rational matrices
- $G^\sim(s) = G^T(-s)$
- $\lambda\{A\}$: set of eigenvalues of A
- $\lambda\{E, A\}$: set of generalized eigenvalues of a regular pencil $sE - A$
- If D has full column rank, we can define the orthonormal matrix $D^\perp$ satisfying

$$\begin{bmatrix} D^\dagger \\ D^\perp \end{bmatrix} [D \ \ (D^\perp)^T] = I$$

  or equivalently

$$[D \ \ (D^\perp)^T] \begin{bmatrix} D^\dagger \\ D^\perp \end{bmatrix} = I$$

  where $D^\dagger = (D^T D)^{-1} D^T$.

- For matrices A and B, if there exists a nonsingular matrix T such that $T^T AT = B$, then the above equation is expressed in short as $A \approx B$.

- The inertia of a real symmetric matrix A is expressed as $\mathrm{i}\{A\} = (\alpha, \beta, \gamma)$ where α, β and γ are the numbers of negative, zero and positive eigenvalues of A, respectively.

2 Preliminaries

In this section we review some facts related to the eigenstructure of a matrix pencil of full column normal rank. Moreover we provide the direct sum decomposition of the infinite eigenspace determined by the lower infinite eigensubspace.

Let $P(s) = -sM + N$ with $M,\ N \in \mathbf{C}^{m\times n}$ be a matrix pencil of full column normal rank, and $\mathcal{V}\{P(s); \mathbf{S}\}$ denote the generalized eigenspace of $P(s)$ corresponding to the zeros in a set $\mathbf{S} \subseteq \mathbf{C} \cup \{\infty\}$. Also, let the columns of Ξ_f and Ξ_∞ span $\mathcal{V}\{P(s); \mathbf{C}_f\}$ with $\mathbf{C}_f \subseteq \mathbf{C}$ and the infinite eigenspace $\mathcal{V}\{P(s); \infty\}$, respectively. Then, there exist Λ_f and Λ_∞ such that

$$N\Xi_f = M\Xi_f\Lambda_f,\ \lambda\{\Lambda_f\} \subseteq \mathbf{C}_f \tag{1}$$

$$N\Xi_\infty\Lambda_\infty = M\Xi_\infty,\ \lambda\{\Lambda_\infty\} = \{0\}. \tag{2}$$

It should be noted that Algorithm 4.5 in [24] is a reliable numerical method of computing Ξ_f and Ξ_∞.

We now give technical terms on infinite zeros and infinite eigenvectors based on [6]. Without loss of generality, we assume that Λ_∞ of (2) is given by the Jordan canonical form $\Lambda_\infty = \mathrm{diag}\{\Lambda_{\infty,1}, \Lambda_{\infty,2}, \ldots, \Lambda_{\infty,g}\}$, where $\Lambda_{\infty,i}$ is the $k_i \times k_i$ nilpotent Jordan block and $g = \dim \ker M$. Also let $\xi_i^{(j)},\ j = 1, 2, \ldots, k_i$ be the columns of Ξ_∞ corresponding to $\Lambda_{\infty,i}$. Then, for $i = 1, 2, \ldots, g$, the equation (2) can be rewritten as

$$M\xi_i^{(1)} = 0, \quad M\xi_i^{(j+1)} = N\xi_i^{(j)},\ j = 1, 2, \ldots, k_i - 1, \quad N\xi_i^{(k_i)} \notin \mathrm{im}M$$

where $\xi_i^{(j)}, j = 1, \ldots, k_i$ are called the *infinite eigenvectors* of $P(s)$ corresponding to the i-th infinite zero. Also, $k_i - 1$ is called the *order* of the i-th infinite zero. An infinite zero is called *trivial* if its order is zero. Otherwise, it is called *non-trivial.* The space spanned by all the highest grade infinite eigenvector $\xi_i^{(k_i)}$, $i = 1, 2, \ldots, g$ is called the *highest infinite eigensubspace* of $P(s)$ and is denoted by $\mathcal{V}_\infty^{(h)}\{P(s)\}$. The space spanned by all the infinite eigenvectors but the highest grade eigenvectors is called the *lower infinite eigensubspace* of $P(s)$ and is denoted by $\mathcal{V}_\infty^{(l)}\{P(s)\}$.

Let $\mathcal{V}_\infty^{(c)}\{P(s)\}$ be a complement of $\mathcal{V}_\infty^{(l)}\{P(s)\}$ on the infinite eigenspace $\mathcal{V}\{P(s); \infty\}$. Then we have the direct sum decomposition

$$\mathcal{V}\{P(s); \infty\} = \mathcal{V}_\infty^{(l)}\{P(s)\} \oplus \mathcal{V}_\infty^{(c)}\{P(s)\}.$$

It is clear that the highest infinite eigensubspace $\mathcal{V}_\infty^{(h)}\{P(s)\}$ is one of $\mathcal{V}_\infty^{(c)}\{P(s)\}$, and so every $\mathcal{V}_\infty^{(c)}\{P(s)\}$ can be expressed as

$$\mathcal{V}_\infty^{(c)}\{P(s)\} = \mathcal{V}_\infty^{(h)}\{P(s)\} \oplus \mathcal{L} \tag{3}$$

where $\mathcal{L}$ is a subspace of $\mathcal{V}_\infty^{(l)}\{P(s)\}$.

Lemma 2.1 *Let the columns of* $\Xi_\infty^{(l)}$ *and* $\Xi_\infty^{(c)}$ *form bases of* $\mathcal{V}_\infty^{(l)}\{P(s)\}$ *and* $\mathcal{V}_\infty^{(c)}\{P(s)\}$ *respectively. Let the columns of* V *form a basis of* $\ker M^T$*. Then we have*

1. $V^T N[\Xi_f \ \Xi_\infty^{(l)}] = 0$ *holds.*

2. $V^T N \Xi_\infty^{(c)}$ *has full column rank.*

Proof. We first show item 1. Pre-multiply (1) by V^T to get $V^T N \Xi_f = 0$. Similarly, pre-multiplying (2) by V^T, we have $V^T N \Xi_\infty \Lambda_\infty = 0$. Since $\mathrm{im}(\Xi_\infty \Lambda_\infty) = \mathrm{im}\Xi_\infty^{(l)}$, it follows that $V^T N \Xi_\infty^{(l)} = 0$. Next we show item 2. Let the columns of $\Xi_\infty^{(h)}$ form a basis of $\mathcal{V}_\infty^{(h)}\{P(s)\}$. Then we see from (3) that there exists a matrix L such that $\mathrm{im}\Xi_\infty^{(c)} = \mathrm{im}\Xi_\infty^{(h)} + \mathrm{im}(\Xi_\infty^{(l)} L)$. Hence, it follows from item 1 that $V^T N \Xi_\infty^{(c)} = V^T N \Xi_\infty^{(h)}$. By Lemma C.2 of [7], we can observe that $V^T N \Xi_\infty^{(h)}$ has full column rank. ∎

Lemma 2.1 yields a procedure of computing $\Xi_\infty^{(l)}$ and $\Xi_\infty^{(c)}$ for given Ξ_∞.

Step 1. Using an unitary matrix Y, compress the columns of $V^T N \Xi_\infty$ as

$$V^T N \Xi_\infty Y = [0 \ N_c]$$

where N_c has full column rank. The above column compression can be performed by the singular value decomposition (SVD) of $V^T N \Xi_\infty$.

Step 2. In accordance with the right hand side of (2), partition Y as

$$Y = [Y_1 \ \ Y_2].$$

Then a pair of $\Xi_\infty^{(l)}$ and $\Xi_\infty^{(c)}$ is given by

$$\Xi_\infty^{(l)} = \Xi_\infty Y_1, \quad \Xi_\infty^{(c)} = \Xi_\infty Y_2.$$

Remark 2.1 *The computation of a basis of* $\mathcal{V}_\infty^{(h)}\{P(s)\}$ *requires the determination of the infinite eigenstructure of the Kronecker canonical form (see [9]) of* $P(s)$ *and hence is not numerically reliable.*

3 Problem Formulation

In this section we formulate the **S-J-SFP** and **NS-J-SFP**. Let J and J' be $p \times p$ and $m \times m$ nonsingular real symmetric matrices respectively. Let $G(s)$ be an $\mathcal{R}^{p\times m}$ matrix whose poles lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.

Definition 3.1 *1. The* **S-J-SFP** *for $G(s)$ is to find $W(s) \in \mathcal{R}^{m\times m}$ satisfying the conditions (i)-(iv) below. Also, $W(s)$ satisfying (i)-(iv) is called a standard J-spectral factor of $G(s)$.*

2. The **NS-J-SFP** *for $G(s)$ is to find $W(s) \in \mathcal{R}^{m\times m}$ satisfying the conditions (i)-(iii) and (v). Also, $W(s)$ satisfying (i)-(iii) and (v) is called a non-standard J-spectral factor of $G(s)$.*

(i) $G^{\sim}(s)JG(s) = W^{\sim}(s)J'W(s)$.

(ii) $GW^{-1} \in \mathcal{RL}_\infty$.

(iii) All the poles of W lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.

(iv) $W^{-1} \in \mathcal{RH}_\infty$.

(v) All the poles of W^{-1} lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.

Remark 3.1 *Consider the case where J and J' are positive definite. Without loss of generality, we assume that $J = I_p$ and $J' = I_m$. If Condition (i) holds, we see that Condition (ii) is automatically satisfied since*

$$(G(s)W(s)^{-1})^{\sim}G(s)W(s)^{-1} = I_m.$$

*Therefore the present J-**SFP** includes the regular spectral factorization problem as a special case. However, in the case where either J or J' is not positive definite, even if (i) holds, (ii) does not always hold. For example, let*

$$G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix},\ J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J' = 1.$$

It then follows from $G^{\sim}JG = 1$ that the spectral factors satisfying (i) are given by $W(s) = \pm 1$. But, since $\pm G(s)$ are improper, there exists no $W(s)$ satisfying (i) and (ii).

Remark 3.2 *For the* **S-J-SFP** *result, see Appendix A.*

Suppose that a descriptor system realization is given by

$$G(s) = \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right] := C(sE-A)^{-1}B + D \tag{4}$$

where $E,\ A \in \mathbf{R}^{n\times n},\ B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times n},\ D \in \mathbf{R}^{p\times m}$ and $-sE+A$ is regular. Let $r = \mathrm{rank}E$. In this paper we will assume the following conditions.

(A1) All the zeros of $-sE+A$ lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.

(A2) (E, A, B) is finite dynamics stabilizable and impulse controllable.

(A3) The system matrix pencil of $G(s)$

$$P_G(s) = \begin{bmatrix} -sE+A & B \\ C & D \end{bmatrix} =: -sE_G + A_G$$

has full column normal rank.

(A4) (E, A, C) is finite dynamics detectable and impulse observable.

It may be noted that (A1) is not necessary in Section 4, and hence not assumed therein. Since (A4) is required only when the $\mathbf{C}_{0e}$ ZC is used, it disappears from the main results.

In the following, we define some symbols to be used throughout the paper. Let $l_0 = \dim\mathcal{V}\{P_G(s);\mathbf{C}_0\}$ and $l_\infty = \dim\mathcal{V}\{P_G(s);\infty\}$. Let the columns of T_0 and T_∞ form bases of $\mathcal{V}\{P_G(s);\mathbf{C}_0\}$ and $\mathcal{V}\{P_G(s);\infty\}$ respectively. Then there exist $\Lambda_0 \in \mathbf{C}^{l_0\times l_0}$ and $\Lambda_\infty \in \mathbf{R}^{l_\infty \times l_\infty}$ such that

$$A_G T_0 = E_G T_0 \Lambda_0, \quad \lambda\{\Lambda_0\} \subset \mathbf{C}_0 \tag{5}$$

$$A_G T_\infty \Lambda_\infty = E_G T_\infty, \quad \lambda\{\Lambda_\infty\} = \{0\}. \tag{6}$$

It should be noted that T_0 and T_∞ can be chosen as real matrices since the coefficient matrices of $P_G(s)$ are real. Throughout this paper, we assume that T_∞ is a real matrix. However we allow T_0 to be a complex matrix everywhere except Sections 4 and 6.

Next let the columns of $T_\infty^{(l)} \in \mathbf{R}^{(n+m)\times(l_\infty - g)}$ and $T_\infty^{(c)} \in \mathbf{R}^{(n+m)\times g}$ form bases of $\mathcal{V}_\infty^{(l)}\{P_G(s)\}$ and $\mathcal{V}_\infty^{(c)}\{P_G(s)\}$ respectively, where $g := \dim \ker E_G = n - r + m$ is the geometric multiplicity of the infinite zeros of $P_G(s)$. Furthermore let the columns of $V_{G,2} \in \mathbf{R}^{(n+p)\times g}$ form a basis of $\ker E_G^T$. Then by Lemma 2.1 we get the following corollary.

Corollary 3.1 *Suppose that (A3) holds. Then* $V_{G,2}^T A_G [T_0 \ T_\infty^{(l)}] = 0$ *holds and* $V_{G,2}^T A_G T_\infty^{(c)}$ *has full column rank.*

In this paper we employ the GARE of the form

$$A_a^T X_a + X_a^T A_a + Q_a + X_a^T R_a X_a = 0, \quad E_a^T X_a = X_a^T E_a \tag{7}$$

where $X \in \mathbf{R}^{(n+m)\times(n+m)}$, and

$$E_a = \begin{bmatrix} E & 0 \\ 0 & 0_{m\times m} \end{bmatrix}, \quad A_a = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ -I_m \end{bmatrix}$$

$$Q_a = \begin{bmatrix} C^T JC & C^T JD \\ D^T JC & D^T JD - J' \end{bmatrix}, \quad R_a = -B_a J'^{-1} B_a^T.$$

Also, define the Hamiltonian matrix pencil associated with the GARE (7) as

$$H_a(s) = \begin{bmatrix} Q_a & sE_a^T + A_a^T \\ -sE_a + A_a & R_a \end{bmatrix} =: -sE_H + A_H. \tag{8}$$

Suppose that $H_a(s)$ is regular. Then the eigenspace of $H_a(s)$ is well-defined. Let $k_- = \dim\mathcal{V}\{H_a(s); \mathbf{C}_-\}$ and let the columns of

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \in \mathbf{C}^{2(n+m)\times k_-}, \quad U_1 \in \mathbf{C}^{(n+m)\times k_-} \tag{9}$$

form a basis of $\mathcal{V}\{H_a(s); \mathbf{C}_-\}$. Then there exists a stable matrix $\Lambda_- \in \mathbf{C}^{k_-\times k_-}$ such that

$$A_H U = E_H U \Lambda_-. \tag{10}$$

Also let the columns of $V_{a,1} \in \mathbf{R}^{(n+m)\times g}$ and $V_{a,2} \in \mathbf{R}^{(n+m)\times g}$ form bases of $\ker E_a$ and $\ker E_a^T$ respectively.

4 $\mathbf{C}_{0e}$ Zero Compensator for Descriptor System

In this section, using the results of Copeland and Safonov [7], we derive a $\mathbf{C}_{0e}$ zero compensator ($\mathbf{C}_{0e}$ ZC) for a descriptor system. The result of this section is mainly used to derive the solvability conditions of the **NS-J-SFP** in Section 6.

Definition 4.1 *Suppose that $G(s) \in \mathcal{R}^{p\times m}$ has full column normal rank. Then $Z(s) \in \mathcal{R}^{m\times m}$ is called a $\mathbf{C}_{0e}$ ZC of $G(s)$ if $Z(s)$ is invertible and $G_c(s) := G(s)Z(s)$ has no zeros on $\mathbf{C}_{0e}$.*

Suppose that a realization of $G(s)$ is given by (4) and satisfies (A2)-(A4). Let $l_{0e} = l_0 + l_\infty$ and let the columns of $T_{0e} \in \mathbf{R}^{(n+m)\times l_{0e}}$ form a basis of $\mathcal{V}\{P_G(s); \mathbf{C}_{0e}\}$. Partition T_{0e} as

$$T_{0e} = \begin{bmatrix} T_{0e}^{(1)} \\ T_{0e}^{(2)} \end{bmatrix} \in \mathbf{R}^{(n+m)\times l_{0e}}, \quad T_{0e}^{(1)} \in \mathbf{R}^{n\times l_{0e}}.$$

Then there exists a matrix of full column rank

$$S_{0e} = \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} \in \mathbf{R}^{(n+p)\times l_{0e}}, \quad S_{0e}^{(1)} \in \mathbf{R}^{n\times l_{0e}}$$

such that

$$\begin{bmatrix} -sE + A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_{0e}^{(1)} \\ T_{0e}^{(2)} \end{bmatrix} = \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} (-sE_{0e} + A_{0e}). \tag{11}$$

where $-sE_{0e} + A_{0e}$ is an $l_{0e} \times l_{0e}$ regular matrix pencil that has only zeros on $\mathbf{C}_{0e}$.

Moreover, without loss of generality, suppose that $-sE + A$ and $-sE_{0e} + A_{0e}$ are in the following SVD forms

$$-sE + A = \begin{bmatrix} -sE_{11} + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{12}$$

$$-sE_{0e} + A_{0e} = \begin{bmatrix} -sE_{0e11} + A_{0e11} & A_{0e12} \\ A_{0e21} & A_{0e22} \end{bmatrix} \tag{13}$$

where $E_{11} \in \mathbf{R}^{r\times r}$ and $E_{0e11} \in \mathbf{R}^{(l_{0e}-g)\times(l_{0e}-g)}$ are nonsingular. In accordance with (12) and (13), we introduce the partitions

$$\begin{aligned}
B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbf{R}^{m\times r}, \qquad C = [C_1 \ C_2], \quad C_1 \in \mathbf{R}^{n\times r} \\
T_{0e}^{(1)} &= \begin{bmatrix} T_{0e,11}^{(1)} & T_{0e,12}^{(1)} \\ T_{0e,21}^{(1)} & T_{0e,22}^{(1)} \end{bmatrix}, \quad T_{0e,11}^{(1)} \in \mathbf{R}^{n\times(l_{0e}-g)} \\
T_{0e}^{(2)} &= [T_{0e,1}^{(2)} \ T_{0e,2}^{(2)}], \quad T_{0e,1}^{(1)} \in \mathbf{R}^{m\times(l_{0e}-g)} \\
S_{0e}^{(1)} &= \begin{bmatrix} S_{0e,11}^{(1)} & S_{0e,12}^{(1)} \\ S_{0e,21}^{(1)} & S_{0e,22}^{(1)} \end{bmatrix}, \quad S_{0e,11}^{(1)} \in \mathbf{R}^{n\times(l_{0e}-g)} \\
S_{0e}^{(2)} &= [S_{0e,1}^{(2)} \ S_{0e,2}^{(2)}], \quad S_{0e,1}^{(1)} \in \mathbf{R}^{p\times(l_{0e}-g)}.
\end{aligned}$$

Then (11) becomes

$$\begin{aligned}
&\begin{bmatrix} -sE_{11}+A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}
\begin{bmatrix} T_{0e,11}^{(1)} & T_{0e,12}^{(1)} \\ T_{0e,21}^{(1)} & T_{0e,22}^{(1)} \\ T_{0e,1}^{(2)} & T_{0e,2}^{(2)} \end{bmatrix} \\
&\qquad = \begin{bmatrix} S_{0e,11}^{(1)} & S_{0e,12}^{(1)} \\ S_{0e,21}^{(1)} & S_{0e,22}^{(1)} \\ S_{0e,1}^{(2)} & S_{0e,2}^{(2)} \end{bmatrix}
\begin{bmatrix} -sE_{0e11}+A_{0e11} & A_{0e12} \\ A_{0e21} & A_{0e22} \end{bmatrix}. \qquad (14)
\end{aligned}$$

Lemma 4.1 *Suppose that (A2)-(A4) hold. Then the following hold.*

1. $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ *is finite dynamics observable and impulse observable.*

2. $T_{0e,12}^{(1)} = 0$ *holds and*

$$\begin{bmatrix} T_{0e,22}^{(1)} \\ T_{0e,2}^{(2)} \end{bmatrix} \in \mathbf{R}^{g\times g} \qquad (15)$$

   is nonsingular.

3. *It follows that*

$$\begin{bmatrix} S_{0e,21}^{(1)} \\ S_{0e,1}^{(2)} \end{bmatrix} = 0 \qquad (16)$$

   holds and

$$\begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} \in \mathbf{R}^{(n-r+p)\times g} \qquad (17)$$

   has full column rank.

Proof. A proof is given in Appendix C. ∎

With arbitrary $B_{0e} \in \mathbf{R}^{l_{0e}\times m}$ and $D_{0e} \in \mathbf{R}^{m\times m}$, we define

$$Z(s) = \left[\begin{array}{c|c} -sE_{0e}+A_{0e} & B_{0e} \\ \hline T_{0e}^{(2)} & D_{0e} \end{array}\right]. \qquad (18)$$

Then, from (11), we have

$$
\begin{aligned}
G_c(s) &:= G(s)Z(s) \\
&= \left[\begin{array}{cc|c} -sE+A & BT_{0e}^{(2)} & BD_{02} \\ 0 & -sE_{0e}+A_{0e} & B_{0e} \\ \hline C & DT_{0e}^{(2)} & DD_{0e} \end{array}\right] \\
&= \left[\begin{array}{cc|c} -sE+A & (-sE+A)T_{0e}^{(1)} + BT_{0e}^{(2)} - S_{0e}^{(1)}(-sE_{0e}+A_{0e}) & BD_{02} - S_{0e}^{(1)}B_{0e} \\ 0 & -sE_{0e}+A_{0e} & B_{0e} \\ \hline C & CT_{0e}^{(1)} + DT_{0e}^{(2)} & DD_{0e} \end{array}\right] \\
&= \left[\begin{array}{cc|c} -sE+A & 0 & BD_{02} - S_{0e}^{(1)}B_{0e} \\ 0 & -sE_{0e}+A_{0e} & B_{0e} \\ \hline C & S_{0e}^{(2)}(-sE_{0e}+A_{0e}) & DD_{0e} \end{array}\right] \qquad (19) \\
&= \left[\begin{array}{c|c} -sE+A & BD_{02} - S_{0e}^{(1)}B_{0e} \\ \hline C & DD_{0e} - S_{0e}^{(2)}B_{0e} \end{array}\right]. \qquad (20)
\end{aligned}
$$

Lemma 4.2 *Suppose that (A2)-(A4) hold. Then there exist* $B_{0e} \in \mathbf{R}^{l_{0e}\times m}$ *and* $D_{0e} \in \mathbf{R}^{m\times m}$ *such that the system matrix pencil of* $Z(s)$

$$
P_Z(s) := \begin{bmatrix} -sE_{0e}+A_{0e} & B_{0e} \\ T_{0e}^{(2)} & D_{0e} \end{bmatrix} \qquad (21)
$$

has no non-trivial infinite zeros and all the finite zeros can be placed arbitrarily in $\mathbf{C}$.

Proof. A proof is deferred in Appendix D. ∎

Lemma 4.3 *Suppose that (A2)-(A4) hold. Furthermore suppose that* $B_{0e} \in \mathbf{R}^{g\times m}$ *and* $D_{0e} \in \mathbf{R}^{m\times m}$ *are chosen so that* $P_Z(s)$ *is stable*[1]*. Then* $G_c(s)$ *of (20) is finite dynamics stabilizable and impulse controllable, and the system matrix pencil of* $G_c(s)$

$$
P_{G_c}(s) = \begin{bmatrix} -sE+A & BD_{0e} - S_{0e}^{(1)}B_{0e} \\ C & DD_{0e} - S_{0e}^{(2)}B_{0e} \end{bmatrix} \qquad (22)
$$

has neither finite zeros on $\mathbf{C}_0$ *nor non-trivial infinite zeros.*

Proof. A proof is given in Appendix E. ∎

Remark 4.1 *Recalling that* $G_c(s) = G(s)Z(s)$ *and that* $Z(s)$ *has no zeros in* $\mathbf{C}_{0e} \cup \mathbf{C}_+$*, we see that no unstable pole of* $G(s)$ *is canceled by the zero of* $Z(s)$ *and hence the first part of Lemma 4.3 holds. In addition, the finite zeros on* $\mathbf{C}_0$ *and the non-trivial infinite zeros of* $P_{G_c}(s)$ *are attributed to those of* $P_G(s)$ *since* $P_Z(s)$ *is a stable pencil. But the pencil* $-sE_{0e} + A_{0e}$*, which contains all the* $\mathbf{C}_{0e}$ *zeros of* $P_G(s)$*, is unobservable in (19) and so we also understand that the second part of the lemma holds.*

[1]A matrix pencil is said to be *stable* if it has no non-trivial infinite zeros and all the finite zeros lie in $\mathbf{C}_-$.

Using the results of the $\mathbf{C}_{0e}$ ZC derived above, we study the lower bounds of $\dim\mathcal{V}\{H_a(s);\mathbf{C}_0\}$ and $\dim\mathcal{V}\{H_a(s);\infty\}$, where $H_a(s)$ is defined in (8).

Suppose that (A2)-(A4) hold. Define the Hamiltonian matrix pencil related to $G_c(s)$ as

$$H_c(s) = \left[\begin{array}{cc} Q_{a,c} & sE_a^T + A_{a,c}^T \\ -sE_a + A_{a,c} & R_a \end{array}\right] =: -sE_H + A_{H,c}$$

where

$$A_{a,c} = \left[\begin{array}{cc} A & \beta \\ 0 & I_m \end{array}\right], \quad Q_{a,c} = \left[\begin{array}{cc} C^TJC & C^TJ\delta \\ \delta^TJC & \delta^TJ\delta - J' \end{array}\right]$$

$$\beta = BD_{0e} - S_{0e}^{(1)}B_{0e}, \quad \delta = DD_{0e} - S_{0e}^{(2)}B_{0e}.$$

Also, define

$$W_a(s) = \left[\begin{array}{ccc} C^TJC & C^TJD & sE^T + A^T \\ D^TJC & D^TJD & B^T \\ -sE + A & B & 0 \end{array}\right]$$

$$W_c(s) = \left[\begin{array}{ccc} C^TJC & C^TJ\delta & sE^T + A^T \\ \delta^TJC & \delta^TJ\delta & \beta^T \\ -sE + A & \beta & 0 \end{array}\right].$$

Then, from

$$\left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & J' & 0 & I_m \end{array}\right]^T H_a(s) \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & J' & 0 & I_m \end{array}\right] = \left[\begin{array}{cc} W_a(s) & 0 \\ 0 & J'^{-1} \end{array}\right]$$

and

$$\left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & J' & 0 & I_m \end{array}\right]^T H_c(s) \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & J' & 0 & I_m \end{array}\right] = \left[\begin{array}{cc} W_c(s) & 0 \\ 0 & J'^{-1} \end{array}\right]$$

we see that $H_a(s)$ and $W_a(s)$ have the same numbers of finite and non-trivial infinite zeros, and that similar statement holds for $H_c(s)$ and $W_c(s)$. Thus, we get

$$\dim\mathcal{V}\{H_a(s);\mathbf{C}_0\} = \dim\mathcal{V}\{W_a(s);\mathbf{C}_0\} \tag{23}$$

$$\dim\mathcal{V}\{H_a(s);\infty\} = \dim\mathcal{V}\{W_a(s);\infty\} + m \tag{24}$$

$$\dim\mathcal{V}\{H_c(s);\mathbf{C}_0\} = \dim\mathcal{V}\{W_c(s);\mathbf{C}_0\} \tag{25}$$

$$\dim\mathcal{V}\{H_c(s);\infty\} = \dim\mathcal{V}\{W_c(s);\infty\} + m. \tag{26}$$

Based on this argument, we examine $\dim\mathcal{V}\{W_a(s);\mathbf{C}_0\}$ and $\dim\mathcal{V}\{W_a(s);\infty\}$.

We now derive a key equation relating the dimensions of $\mathbf{C}_{0e}$ eigenspace of $W_a(s)$ and that of $P_Z(s)$. Defining

$$M_1(s) = \left[\begin{array}{ccccc} I_n & 0 & 0 & 0 & 0 \\ 0 & I_{l_{0e}} & 0 & 0 & 0 \\ 0 & 0 & -sE + A & B & 0 \\ 0 & 0 & 0 & 0 & I_{l_{0e}} \\ 0 & 0 & C & D & 0 \end{array}\right]$$

$$M_2(s) = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_{l_{0e}} & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & T_{0e}^{(2)} & D_{0e} \\ 0 & 0 & 0 & -sE_{0e} + A_{0e} & B_{0e} \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_{l_{0e}} & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & I_{l_{0e}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m \end{bmatrix},$$

we can show that

$$M_1(s)M_2(s) = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_{l_{0e}} & 0 & 0 & 0 \\ 0 & 0 & -sE + A & BT_{0e}^{(2)} & BD_{0e} \\ 0 & 0 & 0 & -sE_{0e} + A_{0e} & B_{0e} \\ 0 & 0 & C & DT_{0e}^{(2)} & DD_{0e} \end{bmatrix}$$

and that

$$\begin{aligned} \Psi(s) &:= M_2^{\sim}(s)M_1^{\sim}(s)\Omega M_1(s)M_2(s) \\ &= \begin{bmatrix} 0 & 0 & -sE + A & BT_{0e}^{(2)} & BD_{0e} \\ 0 & 0 & 0 & -sE_{0e} + A_{0e} & B_{0e} \\ sE^T + A^T & 0 & C^T JC & C^T JDT_{0e}^{(2)} & C^T JD \\ {T_{0e}^{(2)}}^T & sE_{0e}^T + A_{0e}^T & {T_{0e}^{(2)}}^T D^T JC & {T_{0e}^{(2)}}^T D^T JDT_{0e}^{(2)} & {T_{0e}^{(2)}}^T D^T JD \\ D_{0e}^T B^T & B_{0e}^T & D^T JC & D^T JDT_{0e} & D_{0e}^T D^T JDD_{0e} \end{bmatrix}. \end{aligned} \tag{27}$$

Next define

$$N_1 = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ -{S_{0e}^{(1)}}^T & I_{l_{0e}} & 0 & 0 & 0 \\ 0 & 0 & I_n & T_{0e}^{(1)} & 0 \\ 0 & 0 & 0 & I_{l_{0e}} & 0 \\ 0 & 0 & 0 & 0 & I_m \end{bmatrix}$$

$$N_2 = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_{l_{0e}} & -{S_{0e}^{(2)}}^T JC & -\frac{1}{2}{S_{0e}^{(2)}}^T J(CT_{0e}^{(1)} + DT_{0e}^{(2)}) & -{S_{0e}^{(2)}}^T JDD_{0e} \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_{l_{0e}} & 0 \\ 0 & 0 & 0 & 0 & I_m \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{l_{0e}} \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{l_{0e}} & 0 \\ 0 & I_m & 0 & 0 & 0 \end{bmatrix}$$

$$N_4(s) = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & (sE_{0e} - A_{0e})^{-1}B_{0e} & 0 & I_{l_{0e}} & 0 \\ 0 & 0 & 0 & 0 & I_{l_{0e}} \end{bmatrix}.$$

Then some tedious manipulation yields

$$\Psi(s) = N^{\sim}(s)^{-1} \begin{bmatrix} W_c(s) & 0 \\ 0 & \begin{bmatrix} 0 & sE_{0e}^T + A_{0e}^T \\ -sE_{0e} + A_{0e} & 0 \end{bmatrix} \end{bmatrix} N^{-1}(s) \tag{28}$$

where $N(s) = N_1 N_2 N_3 N_4(s)$. From (28), we have

$$\det \Psi(s) = c_1 \det(-sE_{0e} + A_{0e}) \det(sE_{0e} + A_{0e}) \det W_c(s) \tag{29}$$

where $c_1 = (-1)^{l_{0e}} (\det N_3)^{-2} \neq 0$.

On the other hand, defining

$$L_1 = \begin{bmatrix} 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_{l_{0e}} & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{l_{0e}} \end{bmatrix}, \quad L_2 = \begin{bmatrix} I_{2n+l_{0e}} & 0 & 0 \\ 0 & 0 & I_{l_{0e}} \\ 0 & I_m & 0 \end{bmatrix},$$

we obtain

$$M_1^{\sim}(s) \Omega M_1(s) = L_1^{-T} \begin{bmatrix} W_a(s) & 0 & 0 \\ 0 & 0 & I_{l_{0e}} \\ 0 & I_{l_{0e}} & 0 \end{bmatrix} L_1^{-1} \tag{30}$$

$$M_2(s) = L_2^{-1} \begin{bmatrix} I_{2n+l_{0e}} & 0 \\ 0 & P_{Z(s)} \end{bmatrix}. \tag{31}$$

It thus follows from (27), (30) and (31) that

$$\det \Psi(s) = c_2 \det P_Z(s) \det P_Z(-s) \det W_a(s) \tag{32}$$

where $c_2 = (-1)^{l_{0e}} (\det L_1)^{-2} (\det L_2)^{-2} \neq 0$. Combining (29) and (32) yields the key relation

$$\begin{aligned} &\det P_Z(s) \det P_Z(-s) \det W_a(s) \\ &\qquad = c_1 c_2^{-1} \det(-sE_{0e} + A_{0e}) \det(sE_{0e} + A_{0e}) \det W_c(s). \end{aligned} \tag{33}$$

Lemma 4.4 *Suppose that (A2) and (A3) hold and that $H_a(s)$ is regular. Moreover, let B_{0e} and D_{0e} be chosen such that $P_Z(s)$ is stable. Then*

$$\dim \mathcal{V}\{H_a(s); \mathbf{C}_0\} \geq 2l_0 \tag{34}$$

$$\dim \mathcal{V}\{H_a(s); \infty\} \geq 2l_\infty. \tag{35}$$

Furthermore, the equality in (34) holds if and only if $H_c(s)$ has no finite zeros on $\mathbf{C}_0$, and the equality in (35) holds if and only if $H_c(s)$ has no non-trivial infinite zeros.

Proof. A proof is given in Appendix F. ∎

The following corollary is useful for the spectral factorization, where J, J' are positive definite.

Corollary 4.1 *Suppose that (A2) and (A4) hold and that J and J' are positive definite. Then*

$$\dim \mathcal{V}\{H_a(s); \mathbf{C}_0\} = 2l_0 \tag{36}$$

$$\dim \mathcal{V}\{H_a(s); \infty\} = 2l_\infty. \tag{37}$$

Proof. Take B_{0e} and D_{0e} mentioned in the hypothesis of Lemma 4.4. Then, by Lemma 4.3, (E, A, β) is finite dynamics stabilizable and impulse controllable, and the system matrix pencil

$$P_{G_c}(s) = \begin{bmatrix} -sE + A & \beta \\ C & \delta \end{bmatrix}$$

has neither finite zeros on $\mathbf{C}_0$ nor non-trivial infinite zeros. Using these facts, we can show that $H_c(s)$ in this case has neither finite zeros on $\mathbf{C}_0$ nor non-trivial finite zeros (for instance, see [4, 12]). Therefore, we see from the second part of Lemma 4.4 that (36) and (37) hold. ∎

5 Semi-Stabilizing Solution of GARE

In this section we define the semi-stabilizing solutions of the GARE (7) and derive their existence conditions. Moreover we provide a parameterization of all the semi-stabilizing solutions.

Definition 5.1 *A real solution $X_a \in \mathbf{R}^{(n+m)\times(n+m)}$ of the GARE (7) is called a semi-stabilizing solution if*

1. *$-sE_a + A_a + R_aX_a$ is regular and all the zeros lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.*

2. *$(E_a, A_a + R_aX_a, [C \;\; D])$ has neither observable finite dynamics modes on $\mathbf{C}_0$ nor observable impulsive modes.*

The following lemma shows that a semi-stabilizing solution enables us to solve the **NS-J-SFP**.

Lemma 5.1 *If the GARE (7) has a semi-stabilizing solution, then the* **NS-J-SFP** *is solvable, and a non-standard J-spectral factor is given by*

$$W(s) = \left[\begin{array}{c|c} -sE_a + A_a & B_a \\ \hline J'^{-1}B_a^TX_a & I_m \end{array}\right]. \tag{38}$$

Proof. We show that $W(s)$ of (38) satisfies Conditions (i)-(iii) and (v) of Definition 3.1. Conditions (i) and (iii) hold by the same reason for the standard J-spectral factorization result (see Theorem A.1). Now, from Definition 5.1, we can see that all the poles of

$$W^{-1} = \left[\begin{array}{c|c} -sE_a + A_a + R_aX_a & B_a \\ \hline -J'^{-1}B_a^TX_a & I_m \end{array}\right]$$

lie in $\mathbf{C}_{0e}$ and that

$$GW^{-1} = \left[\begin{array}{c|c} -sE_a + A_a + R_a X_a & B_a \\ \hline [C\ D] & 0 \end{array}\right]$$

has neither observable finite dynamics modes on $\mathbf{C}_0$ nor observable impulsive modes. Thus Conditions (v) and (ii) hold. ∎

Remark 5.1 *As shown in the proof of Lemma 5.1, Condition 2 of Definition 5.1 corresponds to the condition (ii) of Definition 3.1. Thus, by the same reason stated in Remark 3.1, if J and J' are positive definite, then Condition 2 always holds.*

We derive the existence conditions of a semi-stabilizing solution by using the similar technique developed in [16], in which a stabilizing solution of the GARE of general forms is studied.

Theorem 5.1 *Suppose that (A2) and (A3) hold. Then, the GARE (7) has a semi-stabilizing solution if and only if the following two conditions hold.*

1. *The algebraic quadratic equation (AQE)*

$$A_{a,0}^T X_{a,0} + X_{a,0}^T A_{a,0} + Q_{a,0} + X_{a,0}^T R_{a,0} X_{a,0} = 0 \tag{39}$$

has a solution $X_{a,0} \in \mathbf{R}^{g\times g}$ such that $A_{a,0} + R_{a,0}X_{a,0}$ is nonsingular where

$$A_{a,0} = V_{a,2}^T A_a T_\infty^{(c)}, \quad Q_{a,0} = {T_\infty^{(c)}}^T Q_a T_\infty^{(c)}, \quad R_{a,0} = V_{a,2}^T R_a V_{a,2}.$$

2. *The matrix $[U_1\ T_0\ T_\infty]$ is nonsingular.*

Moreover, under these conditions, all the semi-stabilizing solutions of the GARE (7) are given by

$$X_a = [U_2\ M_a T_0\ M_a T_\infty^{(l)}\ V_{a,2}X_{a,0}][U_1\ T_0\ T_\infty^{(l)}\ T_\infty^{(c)}]^{-1}. \tag{40}$$

where $M_a = \mathrm{diag}\{0_{n\times n}, J'\}$.

Proof. A proof is given in Appendix G. ∎

The following lemma provides a necessary and sufficient condition for item 1 of Theorem 5.1.

Lemma 5.2 *Define the $2k \times 2k$ real symmetric matrix as*

$$\hat{\Sigma} = \begin{bmatrix} \hat{Q} & \hat{A}^T \\ \hat{A} & \hat{R} \end{bmatrix}$$

where $\hat{A},\ \hat{Q},\ \hat{R} \in \mathbf{R}^{k\times k}$ with $\hat{Q} = \hat{Q}^T$ and $\hat{R} = \hat{R}^T$. Then the AQE

$$\hat{A}^T\hat{X} + \hat{X}^T\hat{A} + \hat{Q} + \hat{X}^T\hat{R}\hat{X} = 0 \tag{41}$$

has a real solution such that $\hat{A} + \hat{R}\hat{X}$ is nonsingular if and only if

$$\mathrm{i}\{\hat{\Sigma}\} = (k, 0, k). \tag{42}$$

Proof. See [16]. ∎

By Theorem 5.1 and Lemma 5.2, we have the main result of this section.

Theorem 5.2 *Suppose that (A2) and (A3) hold. Then, the GARE (7) has a semi-stabilizing solution if and only if the following two conditions hold.*

(a) The inertia condition

$$\mathrm{i}\{\Sigma_a\} = (g, 0, g) \tag{43}$$

holds where

$$\Sigma_a = \begin{bmatrix} Q_{a,0} & A_{a,0}^T \\ A_{a,0} & R_{a,0} \end{bmatrix}.$$

(b) $[U_1\ T_0\ T_\infty]$ is nonsingular.

Furthermore, under these conditions, all the semi-stabilizing solutions of the GARE (7) are given by (40).

Next we wish to show that if the inertia condition $\mathrm{i}\{D^TJD\} = \mathrm{i}\{J'\}$ holds, then a GARE with reduced dimension similar to the ARE of Green *et al.* [10]

$$A_b^TX + X^TA_b + Q_b + X^TR_bX = 0,\ E^TX = X^TE \tag{44}$$

can be used instead of the GARE (7), where $X \in \mathbf{R}^{n\times n}$, and

$$\begin{aligned} A_b &= A - B(D^TJD)^{-1}D^TJC \\ Q_b &= C^T(J - JD(D^TJD)^{-1}D^TJ)C \\ R_b &= -B(D^TJD)^{-1}B^T. \end{aligned}$$

To begin with, we give the definition of a semi-stabilizing solution for the GARE (44).

Definition 5.2 *A real solution X of the GARE (44) is called a semi-stabilizing solution if*

1. *$-sE + A_b + R_bX$ is regular and all the zeros lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$.*
2. *$(E, A_b + R_bX, D^\perp C)$ has neither observable finite dynamics modes on $\mathbf{C}_0$ nor observable impulsive modes.*

Define the Hamiltonian matrix pencil associated with the GARE (44) as

$$H_b(s) = \begin{bmatrix} Q_b & sE^T + A_b^T \\ -sE + A_b & R_b \end{bmatrix}.$$

And, if $H_b(s)$ is regular, then we denote a matrix whose columns form a basis of $\mathcal{V}\{H_b(s); \mathbf{C}_-\}$ by

$$U^b = \begin{bmatrix} U_1^b \\ U_2^b \end{bmatrix} \in \mathbf{C}^{2n\times k_-},\quad U_1^b \in \mathbf{C}^{n\times k_-}.$$

Next define $\hat{P}_G(s)$ as

$$\hat{P}_G(s) = \begin{bmatrix} -sE + A_b \\ D^{\perp}C \end{bmatrix} =: -s\hat{E}_G + \hat{A}_G$$

and let the columns of $\hat{T}_0 \in \mathbf{C}^{n\times l_0}$, $\hat{T}_\infty \in \mathbf{R}^{n\times l_\infty}$, $\hat{T}_\infty^{(l)} \in \mathbf{R}^{n\times(l_\infty - n + r)}$ and $\hat{T}_\infty^{(c)} \in \mathbf{R}^{n\times(n-r)}$ form bases of $\mathcal{V}\{\hat{P}_G(s); \mathbf{C}_0\}$, $\mathcal{V}\{\hat{P}_G(s); \infty\}$, $\mathcal{V}_\infty^{(l)}\{\hat{P}_G(s)\}$ and $\mathcal{V}_\infty^{(c)}\{\hat{P}_G(s)\}$, respectively. Moreover let the columns of $V_1 \in \mathbf{R}^{n\times(n-r)}$ and $V_2 \in \mathbf{R}^{n\times(n-r)}$ form bases of $\ker E$ and $\ker E^T$, respectively.

Theorem 5.3 *Suppose that (A2) and (A3) hold. Then, the GARE (44) has a semi-stabilizing solution if and only if the following two conditions hold.*

1. *The inertia condition*

$$\mathrm{i}\{\Sigma_b\} = (n-r, 0, n-r) \tag{45}$$

holds where

$$\Sigma_b = \begin{bmatrix} Q_{b,0} & A_{b,0}^T \\ A_{b,0} & R_{b,0} \end{bmatrix}$$

with

$$A_{b,0} = V_2^T A_b \hat{T}_{\infty,c}, \quad Q_{b,0} = \hat{T}_{\infty,c}^T Q_b \hat{T}_{\infty,c}, \quad R_{b,0} = V_2^T R_b V_2.$$

2. $[U_1^b \;\; \hat{T}_0 \;\; \hat{T}_\infty]$ *is nonsingular.*

Furthermore, under these conditions, all the semi-stabilizing solutions of the GARE (44) are given by

$$X = [U_2^b \;\; 0_{n\times(l_0+l_\infty-n+r)} \;\; V_2 X_{b,0}][U_1^b \;\; \hat{T}_0 \;\; \hat{T}_\infty^{(l)} \;\; \hat{T}_\infty^{(c)}]^{-1} \tag{46}$$

where $X_{b,0} \in \mathbf{R}^{(n-r)\times(n-r)}$ is an arbitrary real solution of the AQE

$$A_{b,0}^T X_{b,0} + X_{b,0}^T A_{b,0} + Q_{b,0} + X_{b,0}^T R_{b,0} X_{b,0} = 0. \tag{47}$$

Proof. A proof is omitted, since this is proven similarly to Theorem 5.1. ∎

Lemma 5.3 *If the GARE (44) has a semi-stabilizing solution, then the* **NS-J-SFP** *is solvable, and a non-standard J-spectral factor is given by*

$$W(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline J'^{-1} D_0{}^{-T}(B^T X + D^T J C) & D_0 \end{array}\right] \tag{48}$$

where $D_0 \in \mathbf{R}^{m\times m}$ is a nonsingular matrix satisfying $D^T J D = D_0^T J' D_0$.

Proof. We show that $W(s)$ of (48) satisfies Conditions (i)-(iii) and (v) of Definition 3.1. Conditions (i) and (iii) are easily verified. Now we have

$$W^{-1} = \left[\begin{array}{c|c} -sE + A_b + R_b X & BD_0^{-1} \\ \hline -(D^T J D)^{-1}(B^T X + D^T J C) & D_0^{-1} \end{array}\right] \tag{49}$$

$$GW^{-1} = \left[\begin{array}{c|c} -sE + A_b + R_b X & BD_0^{-1} \\ \hline C - D(D^T J D)^{-1}(B^T X + D^T J C) & DD_0^{-1} \end{array}\right]. \tag{50}$$

Hence (v) is verified by Definition 5.2 and (49). Finally we have to prove (ii). Note that $\begin{bmatrix} D^{\perp} \\ (D^TJD)^{-1}D^TJ \end{bmatrix}$ is nonsingular and that

$$\begin{bmatrix} D^{\perp} \\ (D^TJD)^{-1}D^TJ \end{bmatrix} \left\{C - D(D^TJD)^{-1}(B^TX + D^TJC)\right\} = \begin{bmatrix} D^{\perp}C \\ -(D^TJD)^{-1}B^TX \end{bmatrix}. \tag{51}$$

By Theorem 5.3, we can conclude that the columns of $\hat{T}_0$, $\hat{T}_\infty^{(l)}$ and $\hat{T}_\infty^{(c)}$ form bases of $\mathcal{V}\{-sE + A_b + R_bX; \mathbf{C}_0\}$, $\mathcal{V}_\infty^{(l)}\{-sE + A_b + R_bX\}$ and $\mathcal{V}_\infty^{(c)}\{-sE + A_b + R_bX\}$ respectively and moreover that $X[\hat{T}_0 \;\; \hat{T}_\infty^{(l)}] = 0$. Hence, from (51),

$$\left\{C - D(D^TJD)^{-1}(B^TX + D^TJC)\right\}[\hat{T}_0 \;\; \hat{T}_\infty^{(l)}] = 0,$$

so that all the $\mathbf{C}_0$ finite dynamics modes and impulsive modes of $-sE+A_b+R_bX$ is unobservable in (50), which proves (ii). ∎

By combining Theorems 5.1 and 5.3, we can show the following result.

Theorem 5.4 *Suppose that (A2) and (A3) hold and that* $\mathrm{i}\{D^TJD\} = \mathrm{i}\{J'\}$ *holds. Then the following are equivalent.*

1. *The GARE (7) has a semi-stabilizing solution.*

2. *The GARE (44) has a semi-stabilizing solution.*

Proof. A proof is given in Appendix H. ∎

6 Solution to NS-J-SFP

In this section we derive the solvability conditions of the **NS-J-SFP** based on the ZC developed in Section 4, together with the technique of Xin and Kimura [26]. The basic idea to solve the **NS-J-SFP** is to reduce it to an equivalent **S-J-SFP** by using the $\mathbf{C}_{0e}$ ZC.

Lemma 6.1 *Let $Z(s)$ be a $\mathbf{C}_{0e}$ ZC of $G(s)$ and define $G_c(s) = G(s)Z(s)$. Then the following are equivalent.*

1. *The* **NS-J-SFP** *for $G(s)$ is solvable.*

2. *The* **S-J-SFP** *for $G_c(s)$ is solvable.*

Proof. *(1 ⇒ 2)* Suppose that item 1 holds. Let a non-standard J-spectral factor for $G(s)$ be $W(s)$. We show that $W_c(s) := W(s)Z(s)$ is a standard J-spectral factor for $G_c(s)$ by proving that it satisfies Conditions (i)-(iv) in Definition 3.1. First, pre- and post-multiply $G^{\sim}(s)JG(s) = W^{\sim}(s)J'W(s)$ by $Z^{\sim}(s)$ and $Z(s)$ to get $G_c^{\sim}(s)JG_c(s) = W_c^{\sim}(s)J'W_c(s)$, which proves (i). Next, (ii) is verified by

$$G_c(s)W_c^{-1}(s) = G(s)W(s)^{-1} \in \mathcal{RL}_\infty. \tag{52}$$

Also, since all the poles of W and Z lie in $\mathbf{C}_- \cup \mathbf{C}_{0e}$, (iii) holds. Finally we prove (iv). All the poles of W^{-1} lie in $\mathbf{C}_{0e}$ and $Z^{-1} \in \mathcal{RH}_\infty$, hence it suffices to show that $W_c^{-1} \in \mathcal{RL}_\infty$. It follows from (52) that there exists $S(s) \in \mathcal{RL}_\infty$ such that

$$G_c(s)W_c^{-1}(s) = S(s). \tag{53}$$

Since $G_c(s)$ has no zeros on $\mathbf{C}_{0e}$, the left inverse of $G_c(s)$, namely, $G_c^\dagger(s)$ exists and belongs to $\mathcal{RL}_\infty$. Thus, from (53), $W_c^{-1}(s) = G_c^\dagger(s)S(s) \in \mathcal{RL}_\infty$.
(2 ⇒ 1) This part is proved similarly. ∎

We now construct a matrix whose columns form a basis of $\mathcal{V}\{H_c(s); \mathbf{C}_-\}$ in terms of $U \in \mathbf{C}^{2(n+m)\times k_-}$, $T_{0e} \in \mathbf{R}^{(n+m)\times l_{0e}}$ and the realization data of the $\mathbf{C}_{0e}$ ZC. Suppose that (A1)-(A4) hold. Choose real matrices B_{0e} and D_{0e} so that $P_Z(s)$ of (21) is stable. Moreover, without loss of generality, we adopt the SVD forms of $-sE + A$ and $-sE_{0e} + A_{0e}$ defined by (12) and (13), respectively. Thus it follows from (18) that $Z^{-1}(s)$ can be expressed as the state space form

$$Z^{-1}(s) = \left[\begin{array}{ccc|c} -sE_{0e11} + A_{0e11} & A_{0e12} & B_{0e,1} & 0 \\ A_{0e21} & A_{0e22} & B_{0e,2} & 0 \\ T^{(2)}_{0e,1} & T^{(2)}_{0e,2} & D_{0e} & I_m \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] = \left[\begin{array}{c|c} -sE_{0e11} + A_s & B_s \\ \hline C_s & D_s \end{array}\right]$$

where

$$\begin{aligned}
A_s &= A_{0e11} - [A_{0e12}\ B_{0e,1}] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} A_{0e21} \\ T^{(2)}_{0e,1} \end{bmatrix} \\
B_s &= -[A_{0e12}\ B_{0e,1}] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} 0_{g\times m} \\ I_m \end{bmatrix} \\
C_s &= [0_{m\times g}\ I_m] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} A_{0e21} \\ T^{(2)}_{0e,1} \end{bmatrix} \\
D_s &= [0_{m\times g}\ I_m] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} 0_{g\times m} \\ I_m \end{bmatrix}.
\end{aligned}$$

Also note that β is given by

$$\beta = \begin{bmatrix} B_1 D_{0e} - S^{(1)}_{0e,11} B_{0e,1} - S^{(1)}_{0e,12} B_{0e,2} \\ B_2 D_{0e} - S^{(1)}_{0e,22} B_{0e,2} \end{bmatrix} =: \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Lemma 6.2 *The following relation holds.*

$$\begin{bmatrix} -sE_{11} + A_{11} & A_{12} & \beta_1 \\ A_{21} & A_{22} & \beta_2 \\ C_1 & C_2 & \delta \end{bmatrix} \begin{bmatrix} T^{(1)}_{0e,11} \\ T^{(1)}_{0e,21} - K \\ C_s \end{bmatrix} = \begin{bmatrix} E_{11} T^{(1)}_{0e,11} \\ 0 \\ 0 \end{bmatrix} (-sI + E_{0e11}^{-1} A_s) \tag{54}$$

where

$$K = -[T^{(1)}_{0e,22}\ \ 0_{(n-r)\times m}] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} A_{0e21} \\ T^{(2)}_{0e,1} \end{bmatrix}.$$

Proof. A proof is given in Appendix I. ∎

Now introduce the partitions of U_1 and U_2

$$U_1 = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad P_1 \in \mathbf{C}^{n\times k_-},\ P_2 \in \mathbf{C}^{m\times k_-}$$
$$U_2 = \begin{bmatrix} P_3 \\ P_4 \end{bmatrix}, \quad P_3 \in \mathbf{C}^{n\times k_-},\ P_4 \in \mathbf{C}^{m\times k_-}.$$

Then (10) becomes

$$\begin{bmatrix} C^TJC & C^TJD & sE^T+A^T & 0 \\ D^TJC & D^TJD-J' & B^T & I_m \\ -sE+A & B & 0 & 0 \\ 0 & I_m & 0 & -J'^{-1} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} -E^TP_3 \\ 0 \\ EP_1 \\ 0 \end{bmatrix} (-sI+\Lambda_-). \qquad (55)$$

Lemma 6.3 *Define* $T_s = \begin{bmatrix} T^{(1)}_{0e,11} \\ T^{(2)}_{0e,21} - K \end{bmatrix}$. *Then, it follows that*

$$\begin{bmatrix} C^TJC & C^TJ\delta & sE^T+A^T & 0 \\ \delta^TJC & \delta^TJ\delta-J' & \beta^T & I_m \\ -sE+A & \beta & 0 & 0 \\ 0 & I_m & 0 & -J'^{-1} \end{bmatrix} \begin{bmatrix} P_1-LP_2 & T_s \\ D_sP_2 & C_s \\ P_3 & 0 \\ J'D_sP_2 & J'C_s \end{bmatrix}$$
$$= \begin{bmatrix} -E^TP_2 & 0 \\ 0 & 0 \\ EP_1 & ET_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -sI+\Lambda_- & 0 \\ E^{-1}_{0e11}B_sP_2 & -sI+E^{-1}_{0e11}A_s \end{bmatrix} \qquad (56)$$

where

$$L = \begin{bmatrix} 0_{r\times m} \\ L_2 \end{bmatrix}, \qquad L_2 = [T^{(2)}_{0e,2}\ \ 0_{(n-r)\times m}] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Proof. A proof is given in Appendix J. ∎

Lemma 6.4 *Suppose that* $H_c(s)$ *is regular. Then*

$$\hat{U} := \begin{bmatrix} P_1-LP_2 & T_s \\ D_sP_2 & C_s \\ P_3 & 0 \\ J'D_sP_2 & J'C_s \end{bmatrix} \in \mathbf{C}^{2(n+m)\times(k_-+l_{0e}-g)} \qquad (57)$$

has full column rank.

Proof. A proof is given in Appendix K.

Lemma 6.5 *Suppose that* $H_c(s)$ *is regular. Then,* $\dim\mathcal{V}\{H_c(s);\mathbf{C}_-\} = r$ *if and only if* $\hat{U}$ *of (57) has* r *columns, namely,* $k_- + l_{0e} - g = r$. *Furthermore, under this condition, the columns of* $\hat{U}$ *form a basis of* $\mathcal{V}\{H_c(s);\mathbf{C}_-\}$.

Proof. Clearly, $\dim\mathcal{V}\{H_c(s);\mathbf{C}_-\} = r$ if and only if $H_c(s)$ has neither finite zeros on $\mathbf{C}_0$ nor non-trivial infinite zeros. It hence follows by Lemma 4.4 that $\dim\mathcal{V}\{H_c(s);\mathbf{C}_-\} = r$ if and only if $k_- + l_0 + l_\infty = n + m$, which proves the first statement of the lemma. Next we prove the last statement. Suppose that $\dim\{H_c(s);\mathbf{C}_-\} = r$. Then we immediately see from (56) and Lemma 6.4 that the columns of $\hat{U}$ form a basis of $\mathcal{V}\{H_c(s);\mathbf{C}_-\}$. ∎

Theorem 6.1 *Suppose that (A1)-(A3) hold. Then the* **NS-J-SFP** *for $G(s)$ is solvable if and only if the conditions (a) and (b) of Theorem 5.2 hold.*

Proof. A proof is given in Appendix L. ∎

The following theorem is the main result of this paper.

Theorem 6.2 *Suppose that (A1)-(A3) hold. Then, the following are equivalent.*

1. *The* **NS-J-SFP** *for $G(s)$ is solvable.*

2. *The GARE (7) has a semi-stabilizing solution.*

Furthermore, under this condition, a non-standard J-spectral factor of $G(s)$ is given by (38).

Proof. Combining Theorems 5.2 and 6.1 readily gives the equivalence between items 1 and 2. Also, the second part of the theorem follows from Lemma 5.1. ∎

Corollary 6.1 *Suppose that (A1)-(A3) hold and that $\mathrm{i}\{D^TJD\} = \mathrm{i}\{J'\}$ holds. Then the following are equivalent.*

1. *The* **NS-J-SFP** *for $G(s)$ is solvable.*

2. *The GARE (44) has a semi-stabilizing solution.*

Furthermore, under this condition, a non-standard J-spectral factor of $G(s)$ is given by (48).

Proof. The equivalence between items 1 and 2 immediate from Theorems 6.2 and 5.4. The second part of the theorem follows from Lemma 5.3. ∎

7 Numerical Example

Consider

$$G(s) = \begin{bmatrix} \dfrac{s}{(s+3)^2} & \dfrac{1}{s} \\ 0 & \dfrac{(s-1)(s^2+2)}{s} \\ 0 & 0 \end{bmatrix}, \quad J = \mathrm{diag}\{-1, 1, 1\}, \quad J' = \mathrm{diag}\{1, -1\}.$$

Clearly, $G(s)$ has its poles at 0, -3 and ∞ and has its zeros at 0, 1 and ∞. A descriptor realization of $G(s)$ satisfying Assumptions (A1)-(A3) is

$$G(s) = \left[\begin{array}{cccccc|cc} -s & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -s-6 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right].$$

Using Algorithm 4.5 of [24] and the algorithm for computing $T_\infty^{(l)}$ and $T_\infty^{(c)}$, we obtain

$$T_0 = \begin{bmatrix} 0.0565+0.0560i & -0.0734-0.1397i & -0.2231+0.0162i \\ 0.0792-0.0799i & 0.1423+0.1488i & 0.3158-0.0230i \\ -0.0792+0.0799i & -0.1423-0.1488i & -0.3158+0.0230i \\ 0.2259-0.2241i & 0.2938+0.5589i & -0.4056+0.0295i \\ 0.1585-0.1597i & 0.2846+0.2977i & 0.6316-0.0460i \\ 0.1129+0.1121i & -0.1469-0.2795i & 0.2028-0.0148i \\ 0.8708-0.0870i & 0.3396-0.0851i & -0.3158+0.0230i \\ 0.1129+0.1121i & -0.1469-0.2795i & 0.2028-0.0148i \end{bmatrix}$$

$$\Lambda_0 = \begin{bmatrix} 0.0000-1.4142i & 1.7437+2.7448i & 1.1818-0.5416i \\ 0.0000+0.0000i & 0.0000+1.1412i & -0.3159+1.3992i \\ 0.0000+0.0000i & 0.0000+0.0000i & 0.0000+0.0000i \end{bmatrix}$$

$$T_\infty^{(l)} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 0.0000 \end{bmatrix}, \quad T_\infty^{(c)} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.8361 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.1145 & 0.0000 & -0.9239 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.2764 & 0.0000 & 0.3827 \end{bmatrix}$$

$$\Lambda_\infty = \begin{bmatrix} 0.0000 & 0.0000 & -0.8361 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

where Λ_∞ satisfies $A_G \left[\, T_\infty^{(l)} \;\; T_\infty^{(c)} \,\right] \Lambda_\infty = E_G \left[\, T_\infty^{(l)} \;\; T_\infty^{(c)} \,\right]$. Choose

$$V_{a,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

Since $\mathrm{i}\{\Sigma_a\} = (3,0,3)$ holds, the AQE (39) has real solutions, and one of them is

$$X_0 = \begin{bmatrix} -0.4663 & -0.8361 & 0.0442 \\ -0.5138 & 0.0000 & -0.3711 \\ 0.0000 & -0.8361 & 0.0000 \end{bmatrix}.$$

Now, since Σ_a is nonsingular, $H_a(s)$ is regular and U_1 and U_2 are obtained as

$$U_1 = \begin{bmatrix} 0.0514 + 0.0436i \\ -0.0514 - 0.0436i \\ 0.0514 + 0.0436i \\ -0.0514 - 0.0436i \\ 0.0514 + 0.0436i \\ -0.0514 - 0.0436i \\ 0.2057 + 0.1744i \\ -0.0514 - 0.0436i \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.0000 + 0.0000i \\ 0.0000 + 0.0000i \\ 0.6171 + 0.5232i \\ 0.3086 + 0.2616i \\ 0.0000 + 0.0000i \\ 0.0000 + 0.0000i \\ 0.2057 + 0.1744i \\ 0.0514 + 0.0436i \end{bmatrix}, \quad \Lambda_- = -1.0000.$$

It is easy to check that $[U_1\ T_0\ T_\infty^{(l)}\ T_\infty^{(c)}]$ is nonsingular, so that a semi-stabilizing solution of the GARE exists and is given by

$$\begin{aligned} X_a &= \begin{bmatrix} U_2 & M_aT_0 & M_aT_\infty^{(l)} & V_{a,2}X_0 \end{bmatrix} \begin{bmatrix} U_1 & T_0 & T_\infty^{(l)} & T_\infty^{(c)} \end{bmatrix}^{-1} \\ &= \left[\begin{array}{cccccc|cc} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 8.0000 & 0.0000 & 4.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 4.0000 & 0.0000 & 2.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 2.1112 & 0.5556 & 0.5556 & -0.3457 & 0.0000 & 1.4568 \\ \hline 0.0000 & 0.0000 & 2.0000 & 1.0000 & 1.0000 & 0.5556 & 1.0000 & 1.4444 \\ 0.0000 & 1.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 & 0.0000 & 0.0000 \end{array}\right] \\ &=: \left[\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right]. \end{aligned}$$

Therefore, a desired J-spectral factor is

$$W(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline -J'^{-1}X_{21} & I_2 - J'^{-1}X_{22} \end{array}\right] = \begin{bmatrix} 0 & -\dfrac{(s+1)(s^2+2)}{s} \\ \dfrac{s}{(s+3)^2} & \dfrac{1}{s} \end{bmatrix}.$$

Remark 7.1 *Consider the transfer matrix* $G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix}$ *in Remark 3.1. Take a realization*

$$G(s) = \left[\begin{array}{cc|c} 1 & -s & 0 \\ 0 & 1 & -1 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} -sE + A & B \\ \hline C & D \end{array}\right] \quad (J = \mathrm{diag}(1, -1), \quad J' = 1)$$

Then it is not too hard to verify that all the solutions of the GARE (44) are given by

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

For these two solutions,

$$-sE + A_b + R_bX = \begin{bmatrix} 1 & -s \\ 0 & \pm 1 \end{bmatrix}.$$

*Hence we see that Condition 1 of Definition 5.2 are satisfied. However, Condition 2 is not satisfied because $D^{\perp}C = [1\ 0]$. From this example we can observe that the GAREs of this paper have a potential applicability to solve the general J-**SFP**, namely, the problem without Condition (ii). At the present time, the authors do not know under what conditions there exists a solution of the GARE satisfying only Condition 1 and how such solutions are parameterized. The only one thing we can say here is that imposing Condition (ii) causes that the $\mathbf{C}_{0e}$ eigensubspace of $H_a(s)$ ($H_b(s)$) constructing X_a (X) must be attributed from the $\mathbf{C}_{0e}$ eigenspace of $P_G(s)$.*

8 Conclusions

In this paper we have derived a solution to the non-standard J-spectral factorization problem (**NS-J-SFP**) for a rational matrix in terms of a semi-stabilizing solution of an indefinite GARE. Although we can not deny that the process of the derivation is rather complicated, the end result itself has a simple form and does not have any difficulty in numerical computation. It is possible to derive the result of the non-standard spectral factorization problem (**NS-SFP**) without using the zero compensator, which is considered to be a cause of the complication of the derivation (see [14]).

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A Solution to S-J-SFP

Theorem A.1 *Suppose that (A1), (A2) and*

(A3') $P_G(s)$ has full column normal rank and has neither finite zeros on $\mathbf{C}_{0e}$ nor non-trivial infinite zeros.

hold. Then the following are equivalent.

1. *The* **S-J-SFP** *for $G(s)$ is solvable.*
2. *The GARE (7) has a stabilizing solution, namely, a real solution such that $-sE_a + A_a + R_a X_a$ is stable.*
3. (α) *The inertia condition*
$$\mathrm{i}\left\{\begin{bmatrix} Q_{s,0} & A_{s,0}^T \\ A_{s,0} & R_{s,0} \end{bmatrix}\right\} = (g, 0, g)$$
holds where $A_{s,0} = V_{a,2}^T A_a V_{a,1}$, $Q_{s,0} = V_{a,1}^T Q_a V_{a,1}$ and $R_{s,0} = V_{a,2}^T R_a V_{a,2}$.

   (β) $[U_1 \;\; V_{a,1}]$ *is nonsingular.*

In this case, all stabilizing solution are given by

$$X_a = [U_2 \;\; V_{a,2}X_0][U_1 \;\; V_{a,1}]^{-1}$$

where X_0 is an arbitrary real solution of the AQE

$$A_{s,0}^T X_0 + X_0^T A_{s,0} + Q_{s,0} + X_0^T R_{s,0} X_0 = 0.$$

Moreover, by a stabilizing solution of the GARE (7), a standard J-spectral factor is given by (38).

Proof. See [14, 15]. ∎

B Construction of Infinite Eigensubspace of $H_a(s)$

Lemma B.1 *For an arbitrary $\Xi \in \mathbf{R}^{g\times g}$, define U_∞ as*

$$U_\infty = \begin{bmatrix} T_\infty^{(l)} & T_\infty^{(c)} \\ M_a T_\infty^{(l)} & V_{a,2}\Xi \end{bmatrix}.$$

Then there exists a nilpotent matrix $\Lambda_\infty \in \mathbf{R}^{l_\infty \times l_\infty}$ such that

$$A_H U_\infty \Lambda_\infty = E_H U_\infty. \tag{B.1}$$

Proof. Let a real eigenvector chain corresponding to the i-th infinite zero of $P_G(s)$ be $\{\tau_i^{(j)};\ j = 1, \ldots, k_i\}$, where $i = 1, \cdots, g$. Define

$$\begin{aligned} T_1 &= [\tau_1^{(1)} \ldots \tau_1^{(k_1-1)} \tau_2^{(1)} \ldots \tau_2^{(k_2-1)} \ldots \tau_g^{(1)} \ldots \tau_g^{(k_g-1)}] \\ T_2 &= [\tau_1^{(k_1)} \tau_2^{(k_2)} \ldots \tau_g^{(k_g)}]. \end{aligned}$$

Then, by the definitions of $T_\infty^{(l)}$ and $T_\infty^{(c)}$, we can see that there exists a nonsingular matrix

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ 0 & \Delta_{22} \end{bmatrix} \in \mathbf{R}^{l_\infty \times l_\infty}, \quad \Delta_{22} \in \mathbf{R}^{g \times g}$$

such that

$$[T_\infty^{(l)}\ T_\infty^{(c)}] = [T_1\ T_2]\Delta.$$

Suppose that $Y \in \mathbf{R}^{g \times g}$ is an arbitrary solution of the equation $V_{a,2}\Xi = M_a T_1 \Delta_{12} + V_{a,2} Y \Delta_{22}$. Then

$$\tilde{U}_\infty := \begin{bmatrix} T_1 & T_2 \\ M_a T_1 & V_{a,2} Y \end{bmatrix}$$

satisfies

$$U_\infty = \tilde{U}_\infty \Delta. \tag{B.2}$$

Now define $u_i^{(j)}$ $(i = 1, 2, \ldots, g,\ j = 1, 2, \ldots, k_i)$ as

$$u_i^{(j)} = \begin{cases} \begin{bmatrix} \tau_i^{(j)} \\ M_a \tau_i^{(j)} \end{bmatrix}, & \text{if} \;\; j = 1, 2, \ldots, k_i - 1 \\ \begin{bmatrix} \tau_i^{(k_i)} \\ V_{a,2} y_i \end{bmatrix}, & \text{if} \;\; j = k_i \end{cases}$$

where y_i denotes the i-th column vector of Y, so that

$$\tilde{U}_\infty = [u_1^{(1)} \cdots u_1^{(k_1)} u_2^{(1)} \cdots u_2^{(k_2)} \ldots u_g^{(1)} \cdots u_g^{(k_g)}].$$

Also we can easily show that, for $i = 1, 2, \ldots, g$,

$$\begin{aligned} E_H u_i^{(1)} &= 0 \\ E_H u_i^{(j)} &= A_H u_i^{(j-1)}, \quad j = 1, 2, \ldots, k_i. \end{aligned}$$

This implies that there exists a nilpotent matrix in Jordan form $\tilde{\Lambda}_\infty$ such that

$$A_H \tilde{U}_\infty \tilde{\Lambda}_\infty = E_H \tilde{U}_\infty.$$

We therefore see from (B.2) that $\Lambda_\infty := \Delta^{-1} \tilde{\Lambda}_\infty \Delta$ is a nilpotent matrix satisfying (B.1). ∎

C Proof of Lemma 4.1

(*Proof of item 1*) Because all the zeros of $-sE_{0e}+A_{0e}$ lie in $\mathbf{C}_{0e}$, it suffices to show that $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ is finite dynamics observable on $\mathbf{C}_0$ and impulse observable.

First we prove that $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ is finite dynamics observable on $\mathbf{C}_0$. Hoping to show a contradiction, we assume that $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ is unobservable at some $s = j\omega_0$ ($\omega_0 \in \mathbf{R}$). Then there exists a nonzero vector v such that

$$\begin{bmatrix} -j\omega_0 E_{0e} + A_{0e} \\ T_{0e}^{(2)} \end{bmatrix} v = 0.$$

Note that $T_{0e}^{(1)} v \neq 0$ since T_{0e} has full column rank. From (11), we have

$$\begin{bmatrix} -j\omega_0 E + A \\ C \end{bmatrix} T_{0e}^{(1)} v = \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} (-j\omega_0 E_{0e} + A_{0e}) v = 0.$$

This implies that (E, A, C) is unobservable at $s = j\omega_0$, a contradiction to (A4).

Similarly, suppose that $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ is impulse unobservable. Then there exist nonzero vectors v_1 and v_2 such that

$$\begin{bmatrix} E_{0e} \\ 0 \end{bmatrix} v_1 = 0, \quad \begin{bmatrix} E_{0e} \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} A_{0e} \\ T_{0e}^{(2)} \end{bmatrix} v_1. \tag{C.1}$$

From (11), we have

$$\begin{bmatrix} E \\ 0 \end{bmatrix} T_{0e}^{(1)} v_1 = \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} E_{0e} v_1 = 0 \tag{C.2}$$

$$\begin{aligned} \begin{bmatrix} E \\ 0 \end{bmatrix} T_{0e}^{(1)} v_2 &= E_G \begin{bmatrix} T_{0e}^{(1)} \\ T_{0e}^{(2)} \end{bmatrix} v_2 = \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} E_{0e} v_2 \\ &= \begin{bmatrix} S_{0e}^{(1)} \\ S_{0e}^{(2)} \end{bmatrix} A_{0e} v_2 = \begin{bmatrix} A \\ C \end{bmatrix} T_{0e}^{(1)} v_1. \end{aligned} \tag{C.3}$$

We now claim that $E_{0e} v_2 \neq 0$. If $E_{0e} v_2 = 0$, then, from (C.1), $(-sE_{0e} + A_{0e}) v_1 = 0$. Since $-sE_{0e} + A_{0e}$ is regular, $v_1 = 0$, which is a contradiction to $v_1 \neq 0$. Hence $E_{0e} v_2$ must be a nonzero vector. We thus see that the both sides of (C.3) are nonzero and hence that $T_{0e}^{(1)} v_1 \neq 0$ and $T_{0e}^{(1)} v_2 \neq 0$. This means by (C.2) and (C.3) that (E, A, C) is impulse observable, which is a contradiction to (A4).

(*Proof of item 2*) Comparing the $(1, 2)$ blocks of the both sides of (14) yields $T_{0e,12}^{(1)} = 0$. Also since T_{0e} has full column rank, (15) must be nonsingular.

(*Proof of item 3*) Compare the $(2, 1)$ and $(3, 1)$ blocks in the both sides of (14) to get (16). Next we prove that (17) has full column rank. Take $V_{G,2} = \begin{bmatrix} 0_{r\times(n-r+p)} \\ I_{n-r+p} \end{bmatrix}$. Then, from (14),

$$V_{G,2}^T A_G T_{0e} = \begin{bmatrix} A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix} T_{0e} = \begin{bmatrix} S_{0e,22}^{(2)} \\ S_{0e,2}^{(2)} \end{bmatrix} [A_{0e21} \ A_{0e22}]. \tag{C.4}$$

Since $\mathrm{im}T_{0e} = \mathrm{im}[T_0\ T_\infty^{(l)}\ T_\infty^{(c)}]$, it follows from Corollary 3.1 and (C.4) that

$$\mathrm{im}V_{G,2}^T A_G T_\infty^{(c)} = \mathrm{im}V_{G,2}^T A_G T_{0e} = \mathrm{im}\begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix}[A_{0e21}\ A_{0e22}] = \mathrm{im}\begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix}. \tag{C.5}$$

The last line follows since $[A_{0e21}\ A_{0e22}]$ has full row rank. Hence we conclude that there exists a real nonsingular matrix Δ satisfying

$$\begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} = V_{G,2}^T A_G T_\infty^{(c)}\Delta.$$

It thus follows from Corollary 3.1 that (17) has full column rank. ∎

D Proof of Lemma 4.2

Partition B_{0e} as

$$B_{0e} = \begin{bmatrix} B_{0e,1} \\ B_{0e,2} \end{bmatrix}, \quad B_{0e,1} \in \mathbf{C}^{(l_{0e}-g)\times m}.$$

Then (21) is expressed as

$$P_Z(s) = \begin{bmatrix} -sE_{0e11} + A_{0e11} & A_{0e12} & B_{0e,1} \\ A_{0e21} & A_{0e22} & B_{0e,2} \\ T_{0e,1}^{(2)} & T_{0e,2}{}^{(2)} & D_{0e} \end{bmatrix},$$

implying that $P_Z(s)$ has no non-trivial infinite zeros if and only if

$$\begin{bmatrix} A_{0e,22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix} \in \mathbf{R}^{(g+m)\times(g+m)} \tag{D.1}$$

is nonsingular. By item 1 of Lemma 4.1, $(E_{0e}, A_{0e}, T_{0e}^{(2)})$ is impulse observable, so that

$$\begin{bmatrix} A_{0e22} \\ T_{0e,2}^{(2)} \end{bmatrix} \in \mathbf{R}^{(g+m)\times g}$$

has full column rank. We thus see that there exist $B_{0e,2} \in \mathbf{R}^{g\times m}$ and $D_{0e,2} \in \mathbf{R}^{m\times m}$ such that (D.1) is nonsingular.

Choose real matrices $B_{0e,2}$ and D_{0e} so that (D.1) is nonsingular. We now prove that there exists $B_{0e,1} \in \mathbf{R}^{(l_{0e}-g)\times m}$ so that all the finite zeros of $P_Z(s)$, namely, all the zeros of

$$-sE_{0e11} + A_{0e11} - [A_{0e12}\ B_{0e,1}]\begin{bmatrix} A_{0e,22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1}\begin{bmatrix} A_{0e21} \\ T_{0e,1}^{(2)} \end{bmatrix} \tag{D.2}$$

can be placed arbitrarily in $\mathbf{C}$. Define

$$\begin{bmatrix} A_{0e,22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1}\begin{bmatrix} A_{0e21} \\ T_{0e,1}^{(2)} \end{bmatrix} =: \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \qquad \Gamma_1 \in \mathbf{R}^{g\times l_{0e}}.$$

Then (D.2) is written as $-sE_{0e11} + A_{0e11} - A_{0e12}\Gamma_1 - B_{0e,1}\Gamma_2$. We show that $(E_{0e11}, A_{0e11} - A_{0e12}\Gamma_1, \Gamma_2)$ is observable. By item 1 of Lemma 4.1,

$$\begin{bmatrix} -sE_{0e11} + A_{0e11} & A_{0e12} \\ A_{0e21} & A_{0e22} \\ T^{(2)}_{0e,1} & T^{(2)}_{0e,2} \end{bmatrix} \tag{D.3}$$

has full column rank for every $s \in \mathbf{C}$. Moreover, by noting that

$$\begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} A_{0e,22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}^{-1} \end{bmatrix} \times \text{(D.3)} = \begin{bmatrix} -sE_{0e11} + A_{0e11} & A_{0e12} \\ \Gamma_1 & I_g \\ \Gamma_2 & 0 \end{bmatrix},$$

we find that (D.3) is strictly equivalent to

$$\begin{bmatrix} -sE_{0e11} + A_{0e11} - A_{0e12}\Gamma_1 & 0 \\ 0 & I_g \\ \Gamma_2 & 0 \end{bmatrix}.$$

It follows that $(E_{0e11}, A_{0e11} - A_{0e12}\Gamma_1, \Gamma_2)$ is observable and that there exists a real matrix $B_{0e,1}$ so that all the finite zeros of $P_Z(s)$ can be placed arbitrarily in $\mathbf{C}$. ∎

E Proof of Lemma 4.3

First we show that $(E, A, BD_{0e} - S^{(1)}_{0e} B_{0e})$ is finite dynamics stabilizable. Assume that there exists a vector η such that

$$\eta^H(-\lambda E + A) = 0 \tag{E.1}$$

$$\eta^H(BD_{0e} - S^{(1)}_{0e} B_{0e}) = 0 \tag{E.2}$$

for some $\lambda \in \mathbf{C}_0 \cup \mathbf{C}_+$. From (11), we have

$$(-\lambda E + A)T^{(1)}_{0e} + BT^{(2)}_{0e} = S^{(1)}_{0e}(-\lambda E_{0e} + A_{0e}).$$

Pre-multiplying the above equation by η^H yields

$$\eta^H BT^{(2)}_{0e} = \eta^H S^{(1)}_{0e}(-\lambda E_{0e} + A_{0e}). \tag{E.3}$$

It follows from (E.2) and (E.3) that $\eta^H[S^{(1)}_{0e} \;\; -B]P_Z(\lambda) = 0$, which, from (A3), gives

$$\eta^H[S^{(1)}_{0e} \;\; -B] = 0 \tag{E.4}$$

since $P_Z(\lambda)$ is nonsingular. We thus conclude from (E.1) and (E.4) that $\eta^H[-\lambda E + A \;\; B] = 0$, so that $\eta = 0$ by Assumption (A2). Therefore $(E, A, BD_{0e} - DD_{0e})$ is finite dynamics stabilizable.

Similarly we show that $(E, A, BD_{0e} - S^{(1)}_{0e} B_{0e})$ is impulse controllable. It should be noted that $(E, A, BD_{0e} - S^{(1)}_{0e} B_{0e})$ is impulse controllable if and only if $[A_{22} \;\; B_2 D_{0e} - S^{(1)}_{0e,22} B_{0e,2}]$ has full row rank. Assume that there exists a vector η such that

$$\eta^T[A_{22} \;\; B_2 D_{0e} - S^{(1)}_{0e,22} B_{0e,2}] = 0. \tag{E.5}$$

From (11), we have

$$[A_{22} \;\; B_2]\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix} = S^{(1)}_{0e,22}A_{0e22}.$$

Pre-multiply the above equation by η^H to get

$$\eta^H B_2 T^{(2)}_{0e,2} = \eta^H S^{(1)}_{0e,22}A_{0e22}. \tag{E.6}$$

It follows from (E.5) and (E.6) that

$$\eta^H[S^{(1)}_{0e,22} \;\; -B_2]\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix} = 0,$$

which shows

$$\eta^H[S^{(1)}_{0e,22} \;\; -B_2] = 0 \tag{E.7}$$

since $\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}$ is nonsingular. Hence, from (E.5) and (E.7), $\eta^H[A_{22} \; B_2] = 0$, and hence, by Assumption (A2), $\eta = 0$. This means that $(E, A, BD_{0e} - S^{(1)}_{0e}B_{0e})$ is impulse controllable.

Finally we show that $P_{G_c}(s)$ has neither finite zeros on $\mathbf{C}_0$ nor non-trivial infinite zeros. Using the SVD form of $-sE + A$, we write $P_{G_c}(s)$ of (22) as

$$P_{G_c}(s) = \begin{bmatrix} -sE_{11} + A_{11} & A_{12} & B_1D_{0e} - S^{(1)}_{0e,11}B_{0e,1} - S^{(1)}_{0e,12}B_{0e,2} \\ A_{21} & A_{22} & B_2D_{0e} - S^{(1)}_{0e,22}B_{0e,2} \\ C_1 & C_2 & DD_{0e} - S^{(2)}_{0e,2}B_{0e,2} \end{bmatrix}.$$

We then see that $P_{G_c}(s)$ has no non-trivial zeros if and only if

$$\begin{bmatrix} A_{22} & B_2D_{0e} - S^{(1)}_{0e,22}B_{0e,2} \\ C_2 & DD_{0e} - S^{(2)}_{0e,2}B_{0e,2} \end{bmatrix} \tag{E.8}$$

has full column rank. Also, from (14),

$$\begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix} = \begin{bmatrix} S^{(1)}_{0e,22} \\ S^{(2)}_{0e,2} \end{bmatrix} A_{0e22} \begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1}$$

and so (E.8) can be written as

$$\begin{aligned} &\begin{bmatrix} A_{22} & B_2D_{0e} - S^{(1)}_{0e,22}B_{0e,2} \\ C_2 & DD_{0e} - S^{(2)}_{0e,2}B_{0e,2} \end{bmatrix} \\ &= \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}\begin{bmatrix} I_{n-r} & 0 \\ 0 & D_{0e} \end{bmatrix} - \begin{bmatrix} S^{(1)}_{0e,22} \\ S^{(2)}_{0e,2} \end{bmatrix}[0_{g\times(n-r)} \;\; B_{0e,2}] \\ &= \begin{bmatrix} S^{(1)}_{0e,22} \\ S^{(2)}_{0e,2} \end{bmatrix}\left\{ A_{0e22}\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1}\begin{bmatrix} I_{n-r} & 0 \\ 0 & D_{0e} \end{bmatrix} - [0_{g\times(n-r)} \;\; B_{0e,2}] \right\}. \end{aligned}$$

Because $\begin{bmatrix} S^{(1)}_{0e,22} \\ S^{(2)}_{0e,2} \end{bmatrix}$ has full column rank, in order to show that $P_{G_c}(s)$ has no non-trivial infinite zeros, it suffices to prove that

$$A_{0e22}\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1} \begin{bmatrix} I_{n-r} & 0 \\ 0 & D_{0e} \end{bmatrix} - [0_{g\times(n-r)}\ B_{0e,2}] \tag{E.9}$$

is nonsingular. Suppose that there exist $v_1 \in \mathbf{C}^g$ and $v_2 \in \mathbf{C}^m$ such that

$$\left\{A_{0e22}\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1} \begin{bmatrix} I_{n-r} & 0 \\ 0 & D_{0e} \end{bmatrix} - [0_{g\times(n-r)}\ B_{0e,2}]\right\} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

or equivalently

$$A_{0e22}\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ D_{0e}v_2 \end{bmatrix} - B_{0e,2}v_2 = 0. \tag{E.10}$$

Since

$$T^{(2)}_{0e,2}\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ D_{0e}v_2 \end{bmatrix} - D_{0e}v_2 = 0, \tag{E.11}$$

and $\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T^{(2)}_{0e,2} & D_{0e} \end{bmatrix}$ is nonsingular, it follows from (E.10) and (E.11) that $v_1 = 0$ and $v_2 = 0$, which proves that (E.9) is nonsingular.

We now show that $P_{G_c}(s)$ has no finite zeros on $\mathbf{C}_0$. Take K and L so that both $[K\ T_{0e}]$ and $[L\ S_{0e}]$ are nonsingular. Then $P_G(s)$ has the form

$$P_G(s) = [L\ S_{0e}]\begin{bmatrix} -sE_1 + A_1 & 0 \\ -sE_2 + A_2 & -sE_{0e} + A_{0e} \end{bmatrix}[K\ T_{0e}]^{-1}. \tag{E.12}$$

Here, from the definition of $-sE_{0e} + A_{0e}$, it follows that $-sE_1 + A_1$ is a matrix pencil of full column normal rank without $\mathbf{C}_{0e}$ zeros. By using (E.12), we can show that

$$\begin{aligned} P_{G_c}(s) &= P_G(s)\begin{bmatrix} I_n & 0 \\ 0 & D_{0e} \end{bmatrix} - S_{0e}[0_{l_{0e}\times n}\ B_{0e}] \\ &= [L\ \ S_{0e}]\begin{bmatrix} -sE_1 + A_1 & 0 \\ -sE_2 + A_2 & -sE_{0e} + A_{0e} \end{bmatrix}[K\ \ T_{0e}]^{-1}\begin{bmatrix} I_n & 0 \\ 0 & D_{0e} \end{bmatrix} \\ &\quad -S_{0e}[0_{l_{0e}\times n}\ B_{0e}]. \end{aligned} \tag{E.13}$$

Now, suppose that there exists a vector $v = [v_1^H\ v_2^H]^H \in \mathbf{C}^{n+m}$, $v_1 \in \mathbf{C}^n$ satisfying $P_{G_c}(j\omega_0)v = 0$ for some $\omega_0 \in \mathbf{R}$. Then, from (E.13), we have

$$[L\ \ S_{0e}]\begin{bmatrix} -j\omega_0E_1 + A_1 & 0 \\ -j\omega_0E_2 + A_2 & -j\omega_0E_{0e} + A_{0e} \end{bmatrix}[K\ \ T_{0e}]^{-1}\begin{bmatrix} v_1 \\ D_{0e}v_2 \end{bmatrix} - S_{0e}B_{0e}v_2 = 0,$$

namely,

$$\begin{bmatrix} -j\omega_0E_1 + A_1 & 0 \\ -j\omega_0E_2 + A_2 & -j\omega_0E_{0e} + A_{0e} \end{bmatrix}[K\ \ T_{0e}]^{-1}\begin{bmatrix} v_1 \\ D_{0e}v_2 \end{bmatrix} - \begin{bmatrix} 0 \\ I_{l_{0e}} \end{bmatrix}B_{0e}v_2 = 0. \tag{E.14}$$

Introducing the partition

$$[K \;\; T_{0e}]^{-1} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad K_2 \in \mathbf{R}^{l_{0e} \times (n+m)},$$

we can rewrite (E.14) as

$$(-j\omega_0 E_1 + A_1) K_1 \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} = 0 \tag{E.15}$$

$$[-j\omega_0 E_2 + A_2 \quad -j\omega_0 E_{0e} + A_{0e}] \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} - B_{0e} v_2 = 0. \tag{E.16}$$

We see from (E.15) that

$$K_1 \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} = 0 \tag{E.17}$$

since $-j\omega_0 E_1 + A_1$ has full column rank. Thus, from (E.16),

$$(-j\omega_0 E_{0e} + A_{0e}) K_2 \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} - B_{0e} v_2 = 0. \tag{E.18}$$

It is also straightforward to show that

$$T_{0e}^{(2)} K_2 \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} - D_{0e} v_2 = 0. \tag{E.19}$$

From (E.18) and (E.19), we conclude that

$$K_2 \begin{bmatrix} v_1 \\ D_{0e} v_2 \end{bmatrix} = 0, \quad v_2 = 0 \tag{E.20}$$

because $P_Z(s)$ has no finite zeros on $\mathbf{C}_0$. Hence combining (E.17) and (E.20) yields $v_1 = 0$, which implies that $P_{G_c}(s)$ has no finite zeros on $\mathbf{C}_0$. ∎

F Proof of Lemma 4.4

To prove the lemma, we may well assume (A4) in addition to (A2) and (A3). We explain this as follows. Let us consider the case where (A4) is not satisfied. By Assumption (A2), there exists F such that $-sE + A + BF$ is stable. Using this F, we factorize $G(s)$ as

$$G(s) = G_1(s) G_2(s)$$

where

$$G_1(s) = \left[\begin{array}{c|c} -sE + A + BF & B \\ \hline C + DF & D \end{array}\right] =: \left[\begin{array}{c|c} -sE + A_F & B \\ \hline C_F & D \end{array}\right]$$

$$G_2(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline -F & I \end{array}\right].$$

The Hamiltonian matrix pencil related to $G_1(s)$ is

$$H_1(s) := \begin{bmatrix} C_F^T J C_F & C_F^T J D & sE^T + A_F^T & 0 \\ D^T J C_F & D^T J D - J' & B^T & I \\ -sE + A_F & B & 0 & 0 \\ 0 & I & 0 & -J'^{-1} \end{bmatrix}$$
$$=: \begin{bmatrix} Q_{a,F} & sE_a^T + A_{a,F}^T \\ -sE_a + A_{a,F} & R_a \end{bmatrix},$$

and it can be shown that the equation

$$T_1^T H_1(s) T_1 = H_a(s) \tag{F.1}$$

holds for

$$T_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ -F & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -J'F & 0 & 0 & I \end{bmatrix},$$

so that $\dim\mathcal{V}\{H_a(s); \mathbf{S}\} = \dim\mathcal{V}\{H_1(s); \mathbf{S}\}$ for every set $\mathbf{S} \subseteq \mathbf{C}_{0e}$. It should be also noted that the realization of G_1 satisfies (A2)-(A4) since $-sE + A_F$ is stable. Consequently, even if (A4) is not satisfied, we can prove the lemma by using $G_1(s)$ instead of $G(s)$. Hereafter we assume (A2)-(A4), by which the results on the $\mathbf{C}_{0e}$ ZC are available.

First we prove the inequality (34), namely,

$$\dim \mathcal{V}\{W_a(s); \mathbf{C}_0\} \geq 2l_0. \tag{F.2}$$

Since $\det W_a(s) \not\equiv 0$, it follows from (33) that $\det W_c(s) \not\equiv 0$. Also note that $\det(-sE_{0e} + A_{0e})$ has $2l_0$ roots on $\mathbf{C}_0$ and $\det P_Z(s)$ has no roots on $\mathbf{C}_0$. We thus see from (33) that $\det W_a(s)$ has at least $2l_0$ roots on $\mathbf{C}_0$, which proves (F.2). Moreover we see that $\dim \mathcal{V}\{W_a(s); \mathbf{C}_0\} = 2l_0$ if and only if $\det W_c(s)$ has no roots on $\mathbf{C}_0$, that is, $H_c(s)$ has no finite zeros on $\mathbf{C}_0$.

Next we prove the inequality (35), namely,

$$\dim \mathcal{V}\{W_a(s); \infty\} \geq 2l_\infty - m. \tag{F.3}$$

Recall that $\dim \mathcal{V}\{A(s); \infty\} = \dim \mathcal{V}\{sA(\frac{1}{s}); s = 0\}$ for an arbitrary regular pencil $A(s)$. From (33), we have

$$\begin{aligned}
&\det\{sP_Z(\frac{1}{s})\}\det\{-sP_Z(\frac{-1}{s})\}\det\{sW(\frac{1}{s})\} \\
&\quad = c_3 s^{2n+2l_{0e}+3m} \det P_Z(\frac{1}{s}) \det P_Z(\frac{-1}{s}) \det W(\frac{1}{s}) \\
&\quad = c_1 c_2^{-1} c_3 s^{2n+2l_{0e}+3m} \det(-\frac{1}{s}E_{0e} + A_{0e}) \det(\frac{1}{s}E_{0e} + A_{0e}) \det W_c(\frac{1}{s}) \\
&\quad = c_1 c_2^{-1} c_3 s^{2m} \det(-E_{0e} + sA_{0e}) \det(E_{0e} + sA_{0e}) \det\{sW_c(\frac{1}{s})\}
\end{aligned} \tag{F.4}$$

where $c_3 = (-1)^{l_{0e}+m}$. Because $P_Z(s)$ has no non-trivial infinite zeros, $\dim\mathcal{V}\{P_Z(s); \infty\} = g + m = n - r + 2m$. Hence we can write $\det\{sP_Z(\frac{1}{s})\}$ as

$$\det\{sP_Z(\frac{1}{s})\} = s^{n-r+2m} d_1(s) \tag{F.5}$$

where $d_1(s)$ is a polynomial satisfying $d_1(0) \neq 0$. Similarly, because $\dim \mathcal{V}\{P_G(s); \infty\} = l_\infty$, we can write $\det(-E_{0e} + sA_{0e})$ as

$$\det(-E_{0e} + sA_{0e}) = s^{l_\infty} d_2(s) \tag{F.6}$$

where $d_2(s)$ is a polynomial satisfying $d_2(0) \neq 0$. Furthermore, noting that $W_c(s)$ has at least $2(n-r)+m$, we see that $\det\{sW_c(\frac{1}{s})\}$ can be expressed in the form

$$\det\{sW_c(\frac{1}{s})\} = s^p d_3(s). \tag{F.7}$$

Here p is an integer that is greater than or equal to $2(n-r)+m$, and $d_3(s)$ is a polynomial satisfying $d_3(0) \neq 0$. It therefore follows from (F.4)-(F.7) that

$$d_1(s)d_1(-s)\det\{sW(\frac{1}{s})\} = c_1 c_2^{-1} c_3 s^{2l_\infty + p - 2(n-r) - 2m} d_2(s) d_2(-s) d_3(s).$$

Consequently

$$\dim \mathcal{V}\{W_a(s); \infty\} = 2l_\infty + p - 2(n-r) - 2m \geq 2l_\infty - m. \tag{F.8}$$

Also the equality in (F.8) holds if and only if $p = 2(n-r)+m$; that is, $W_c(s)$ or $H_c(s)$ has no non-trivial infinite zeros. ∎

G Proof of Theorem 5.1

(*Necessity*) Suppose that the GARE (7) has a semi-stabilizing solution X_a. Then, from (8), we have

$$\begin{bmatrix} I & 0 \\ X_a & I \end{bmatrix}^T H_a(s) \begin{bmatrix} I & 0 \\ X_a & I \end{bmatrix} = \begin{bmatrix} 0 & sE_a^T + (A_a + R_a X_a)^T \\ -sE_a + A_a + R_a X_a & R_a \end{bmatrix}. \tag{G.1}$$

This implies that $H_a(s)$ is regular since so is $-sE_a + A_a + R_a X_a$. Let

$$\begin{aligned} q_- &= \dim \mathcal{V}\{-sE_a + A_a + R_a X_a; \mathbf{C}_-\} \\ q_0 &= \dim \mathcal{V}\{-sE_a + A_a + R_a X_a; \mathbf{C}_0\} \\ q_\infty &= \dim \mathcal{V}\{-sE_a + A_a + R_a X_a; \infty\}. \end{aligned}$$

Obviously, $q_- + q_0 + q_\infty = n + m$. The equation (G.1) also gives

$$q_- = \dim \mathcal{V}\{H_a(s); \mathbf{C}_-\} = k_- \tag{G.2}$$

$$q_- = \dim \mathcal{V}\{H_a(s); \mathbf{C}_+\} = k_- \tag{G.3}$$

$$2q_0 = \dim \mathcal{V}\{H_a(s); \mathbf{C}_0\} \tag{G.4}$$

$$2q_\infty = \dim \mathcal{V}\{H_a(s); \infty\}. \tag{G.5}$$

Let the columns of $V_- \in \mathbf{C}^{(n+m)\times q_-}$, $V_0 \in \mathbf{C}^{(n+m)\times q_0}$ and $V_\infty \in \mathbf{R}^{(n+m)\times q_\infty}$ form bases of $\mathcal{V}\{-sE_a + A_a + R_a X_a; \mathbf{C}_-\}$, $\mathcal{V}\{-sE_a + A_a + R_a X_a; \mathbf{C}_0\}$ and $\mathcal{V}\{-sE_a + A_a + R_a X_a; \infty\}$ respectively.

Then there exist $\Lambda_- \in \mathbf{C}^{q_- \times q_-}$, $\Lambda_0 \in \mathbf{C}^{q_0 \times q_0}$ and $\Lambda_\infty \in \mathbf{R}^{q_\infty \times q_\infty}$ such that

$$(A_a + R_a X_a)V_- = E_a V_- \Lambda_-, \quad \lambda\{\Lambda_-\} \subset \mathbf{C}_- \tag{G.6}$$

$$(A_a + R_a X_a)V_0 = E_a V_0 \Lambda_0, \quad \lambda\{\Lambda_0\} \subset \mathbf{C}_0 \tag{G.7}$$

$$(A_a + R_a X_a)V_\infty \Lambda_\infty = E_a V_\infty, \quad \lambda\{\Lambda_\infty\} = \{0\}. \tag{G.8}$$

Using (G.1) and (G.6)-(G.8), we can show that

$$A_H \begin{bmatrix} V_- \\ X_a V_- \end{bmatrix} = E_H \begin{bmatrix} V_- \\ X_a V_- \end{bmatrix} \Lambda_- \tag{G.9}$$

$$A_H \begin{bmatrix} V_0 \\ X_a V_0 \end{bmatrix} = E_H \begin{bmatrix} V_0 \\ X_a V_0 \end{bmatrix} \Lambda_0 \tag{G.10}$$

$$A_H \begin{bmatrix} V_\infty \\ X_a V_\infty \end{bmatrix} \Lambda_\infty = E_H \begin{bmatrix} V_\infty \\ X_a V_\infty \end{bmatrix}. \tag{G.11}$$

where A_H, E_H are defined by (8). It follows from (G.2) and (G.9) that the columns of

$$\begin{bmatrix} V_- \\ X_a V_- \end{bmatrix} \in \mathbf{C}^{2(n+m) \times k_-}$$

form a basis of $\mathcal{V}\{H_a(s); \mathbf{C}_-\}$. Hence, by the uniqueness of eigenspace,

$$\mathrm{im} \begin{bmatrix} V_- \\ X_a V_- \end{bmatrix} = \mathrm{im} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

which provides

$$X_a U_1 = U_2. \tag{G.12}$$

where U_1, U_2 are defined in (9).

Now, by using item 2 of Definition 5.1, (G.7) and (G.8), we can conclude that

$$[C \quad D]\, V_0 = 0, \quad [C \quad D]\, V_\infty \Lambda_\infty = 0. \tag{G.13}$$

Hence we obtain

$$A_G V_0 = E_G V_0 \Lambda_0, \quad A_G V_\infty \Lambda_\infty = E_G V_\infty.$$

From the above equations, $\mathrm{im} V_0 \subseteq \mathcal{V}\{P_G(s); \mathbf{C}_0\}$ and $\mathrm{im} V_\infty \subseteq \mathcal{V}\{P_G(s); \infty\}$, so that $q_0 \le l_0$ and $q_\infty \le l_\infty$. On the other hand, by Lemma 4.4, $l_0 \le q_0$ and $l_\infty \le q_\infty$. Thus we see that $l_0 = q_0$ and $l_\infty = q_\infty$ and that the columns of V_0 and V_∞ form bases of $\mathcal{V}\{P_G(s); \mathbf{C}_0\}$ and $\mathcal{V}\{P_G(s); \infty\}$ respectively. Since

$$\mathrm{im} U_1 = \mathrm{im} V_-, \quad \mathrm{im} T_0 = \mathrm{im} V_0, \quad \mathrm{im} T_\infty = \mathrm{im} V_\infty \tag{G.14}$$

for every U, T_0 and T_∞, it follows that $\mathrm{im}[U_1 \;\; T_0 \;\; T_\infty] = \mathrm{im}[V_- \; V_0 \; V_\infty]$. This proves item 2 of the theorem.

By (G.14), henceforth, we identify V_0 and V_∞ with T_0 and T_∞, respectively. From (G.10) and (G.13), we have

$$\begin{bmatrix} 0 & 0 \\ 0 & -J' \end{bmatrix} T_0 + \begin{bmatrix} A^T & 0 \\ B^T & I_m \end{bmatrix} X_a T_0 = -E_a^T X_a T_0 \Lambda_0 \tag{G.15}$$

$$\begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} T_0 + \begin{bmatrix} 0 & 0 \\ 0 & -J'^{-1} \end{bmatrix} X_a T_0 = E_a T_0 \Lambda_0. \tag{G.16}$$

The second block row of (G.16) is equivalent to

$$[0_{m\times n} \;\; I_m] X_a T_0 = [0_{m\times n} \;\; J'] T_0. \tag{G.17}$$

Using the above equation, we rewrite (G.15) as

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} [I_n \;\; 0_{n\times m}] X_a T_0 = -\begin{bmatrix} E^T \\ 0 \end{bmatrix} [I_n \;\; 0_{n\times m}] X_a T_0 \Lambda_0.$$

It follows from (A2) that $[I_n \;\; 0_{n\times m}] X_a T_0 = 0$, which together with (G.17) yields

$$X_a T_0 = M_a T_0. \tag{G.18}$$

Similarly, from (G.11) and (G.13), we have

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & -J' \end{bmatrix} T_\infty + \begin{bmatrix} A^T & 0 \\ B^T & I_m \end{bmatrix} X_a T_\infty \right\} \Lambda_\infty = -\begin{bmatrix} E^T & 0 \\ 0 & 0 \end{bmatrix} X_a T_\infty \tag{G.19}$$

$$\left\{ \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} T_\infty + \begin{bmatrix} 0 & 0 \\ 0 & -J'^{-1} \end{bmatrix} X_a T_\infty \right\} \Lambda_\infty = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} T_\infty. \tag{G.20}$$

The second block row of (G.20) is equivalent to

$$[0_{m\times n} \;\; I_m] T_\infty \Lambda_\infty = [0_{m\times n} \;\; J'] T_\infty \Lambda_\infty \tag{G.21}$$

and so (G.19) is rewritten as

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} [I_n \;\; 0_{n\times m}] X_a T_\infty \Lambda_\infty = -\begin{bmatrix} E^T \\ 0 \end{bmatrix} [I_n \;\; 0_{n\times m}] X_a T_\infty.$$

By Assumption (A2), this implies that $[I_n \;\; 0_{n\times m}] X_a T_\infty \Lambda_\infty = 0$, which together with (G.21) gives $X_a T_\infty \Lambda_\infty = M_a T_\infty \Lambda_\infty$, or equivalently,

$$X_a T_\infty^{(l)} = M_a T_\infty^{(l)} \tag{G.22}$$

since $\mathrm{im}(T_\infty \Lambda_\infty) = \mathrm{im} T_\infty^{(l)}$.

Next, from (G.11), we have

$$\mathrm{im} E_H \begin{bmatrix} T_\infty^{(c)} \\ X_a T_\infty^{(c)} \end{bmatrix} \subseteq \mathrm{im} E_H \begin{bmatrix} T_\infty \\ X_a T_\infty \end{bmatrix} = \mathrm{im} A_H \begin{bmatrix} T_\infty \\ X_a T_\infty \end{bmatrix} \Lambda_\infty = \mathrm{im} A_H \begin{bmatrix} T_\infty^{(l)} \\ X_a T_\infty^{(l)} \end{bmatrix}. \tag{G.23}$$

Since

$$E_H \begin{bmatrix} T_\infty^{(c)} \\ X_a T_\infty^{(c)} \end{bmatrix} = \begin{bmatrix} -E_a^T X_a T_\infty^{(c)} \\ E_a T_\infty^{(c)} \end{bmatrix}, \text{ and } A_H \begin{bmatrix} T_\infty^{(l)} \\ X_a T_\infty^{(l)} \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}$$

it follows that $E_a^T X_a T_\infty^{(c)} = 0$, implying $\mathrm{im}(X_a T_\infty^{(c)}) \subseteq \ker E_a^T$. Hence there exists $X_{a,0} \in \mathbf{R}^{g\times g}$ such that

$$X_a T_\infty^{(c)} = V_{a,2} X_{a,0}. \tag{G.24}$$

Pre- and post-multiplying the first equation of the GARE (7) by $T_\infty^{(c)^T}$ and $T_\infty^{(c)}$, we see that $X_{a,0}$ is a real solution of the AQE (39). Moreover, applying Lemma 2.1 to $-sE_a + A_a + R_a X_a$ shows that $V_{a,2}^T (A_a + R_a X_a) T_\infty^{(c)} = A_{a,0} + R_{a,0} X_{a,0}$ is nonsingular. Therefore item 1 of the theorem holds.

We have proved necessity of the theorem. Finally, we show that all the semi-stabilizing solutions are given by (40). Recall that, for any semi-stabilizing solution, the equations (G.12), (G.18), (G.22) and (G.24) hold. Hence we have

$$X_a [U_1 \;\; T_0 \;\; T_\infty^{(l)} \;\; T_\infty^{(c)}] = [U_2 \;\; M_a T_0 \;\; M_a T_\infty^{(l)} \;\; V_{a,2} X_{a,0}].$$

It is clear that $[U_1 \; T_0 \; T_\infty]$ is nonsingular if and only if $[U_1 \; T_0 \; T_\infty^{(l)} \; T_\infty^{(c)}]$ is nonsingular, so we get (40).

(*Sufficiency*) Suppose that items 1 and 2 of the theorem hold. It suffices to verify that X_a defined by (40) is a semi-stabilizing solution of the GARE (7). A proof that (40) is a real matrix can be obtained in the similar way as Mårtenson [22], so is omitted.

First we show that (40) is a solution of the GARE (7). By the definition (5) of T_0, there exists a matrix $\Lambda_0 \in \mathbf{C}^{l_0 \times l_0}$ such that

$$A_H \begin{bmatrix} T_0 \\ M_a T_0 \end{bmatrix} = E_H \begin{bmatrix} T_0 \\ M_a T_0 \end{bmatrix} \Lambda_0, \quad \lambda\{\Lambda_0\} \subset \mathbf{C}_0. \tag{G.25}$$

According to Lemma B.1 in Appendix B, there exists a nilpotent matrix $\Lambda_\infty \in \mathbf{R}^{l_\infty \times l_\infty}$ such that

$$A_H \begin{bmatrix} T_\infty^{(l)} & T_\infty^{(c)} \\ M_a T_\infty^{(l)} & V_{a,2} X_{a,0} \end{bmatrix} \Lambda_\infty = E_H \begin{bmatrix} T_\infty^{(l)} & T_\infty^{(c)} \\ M_a T_\infty^{(l)} & V_{a,2} X_{a,0} \end{bmatrix} \tag{G.26}$$

where $X_{a,0}$ is a real solution of the AQE (39) such that $A_{a,0} + R_{a,0} X_{a,0}$ is nonsingular. Define

$$U_0 = \begin{bmatrix} T_0 \\ M_a T_0 \end{bmatrix}, \quad U_\infty = [U_\infty^{(1)} \;\; U_\infty^{(2)}], \quad U_\infty^{(1)} = \begin{bmatrix} T_\infty^{(l)} \\ M_a T_\infty^{(l)} \end{bmatrix}, \; U_\infty^{(2)} = \begin{bmatrix} T_\infty^{(c)} \\ V_{a,2} X_{a,0} \end{bmatrix}$$

and

$$-sE_{ss} + A_{ss} = \mathrm{diag}\{-sI + \Lambda_-, -sI + \Lambda_0, -s\Lambda_\infty + I\}.$$

Then we have

$$H_a(s)[U \;\; U_0 \;\; U_\infty] = [E_H U \;\; E_H U_0 \;\; A_H U_\infty](-sE_{ss} + A_{ss}). \tag{G.27}$$

Also, define

$$X_{a,1} = [U_1 \;\; T_0 \;\; T_\infty^{(l)} \;\; T_\infty^{(c)}], \quad X_{a,2} = U_2 \; M_a T_0 \; M_a T_\infty^{(l)} \; V_{a,2} X_{a,0}].$$

Then, since $X_{a,1}$ is nonsingular, we see that (40) is a solution of the GARE (7) if and only if

$$\begin{bmatrix} X_{a,1} \\ X_{a,2} \end{bmatrix}^H H_a(s) \begin{bmatrix} X_{a,1} \\ X_{a,2} \end{bmatrix} = 0$$

or equivalently

$$[U \ \ U_0 \ \ U_\infty]^H H_a(s)[U \ \ U_0 \ \ U_\infty] = 0. \tag{G.28}$$

Pre-multiply (G.27) by U^H to get

$$\begin{aligned} U^H[E_H U \ \ E_H U_0 \ \ A_H U_\infty](-sE_{ss} + A_{ss}) &= U^H H_a(s)[U \ \ U_0 \ \ U_\infty] \\ &= (sI + \Lambda_-^H)U^H E_H^T[U \ \ U_0 \ \ U_\infty] \end{aligned}$$

which implies that, since $\lambda\{E_{ss}, A_{ss}\} \cap \lambda\{-I, \Lambda_-^H\} = \emptyset$,

$$U^H[E_H U \ \ E_H U_0 \ \ A_H U_\infty] = U^H E_H^T[U \ \ U_0 \ \ U_\infty] = 0 \tag{G.29}$$

Hence we have $U^H H_a(s)[U \ \ U_0 \ \ U_\infty] = 0$. Also, noting that $U_0^H E_H U_0 = 0$, we get $U_0^H H_a(s)U_0 = U_0^H E_H U_0(-sI + \Lambda_0) = 0$. Next, pre-multiply (G.27) by U_0^H to get

$$U_0^H A_H U_\infty(-s\Lambda_\infty + I) = U_0^H H_a(s)U_\infty = (sI + \Lambda_0^H)U_0^H E_H U_\infty,$$

which implies that $U_0^H E_H U_\infty = U_0^H A_H U_\infty = 0$, since $\lambda\{\Lambda_\infty, I\} \cap \lambda\{-I, \Lambda_0\} = \emptyset$. Thus $U_0^H H_a(s)U_\infty = 0$. Furthermore, we can easily show that

$$U_\infty^T A_H U_\infty = \begin{bmatrix} {U_\infty^{(1)}}^T A_H U_\infty^{(1)} & {U_\infty^{(1)}}^T A_H U_\infty^{(2)} \\ {U_\infty^{(2)}}^T A_H U_\infty^{(1)} & {U_\infty^{(2)}}^T A_H U_\infty^{(2)} \end{bmatrix} = 0.$$

where it may be noted that the $(2,2)$ block is exactly the AQE (39). Hence, $U_\infty^T H_a(s)U_\infty = U_\infty^T A_H U_\infty(-s\Lambda_\infty + I) = 0$. Consequently, (G.28) holds. In other words, X_a of (40) is a real solution of the GARE (7).

Finally we have to show that X_a of (40) satisfies the two conditions of Definition 5.1. From (10), we have

$$(A_a + R_a X_a)U_1 = E_a U_1 \Lambda_-. \tag{G.30}$$

Also, from (G.25) and (G.26), we have

$$(A_a + R_a X_a)T_0 = E_a T_0 \Lambda_0 \tag{G.31}$$

$$(A_a + R_a X_a)[T_\infty^{(l)} \ \ T_\infty^{(c)}]\Lambda_\infty = E_a[T_\infty^{(l)} \ \ T_\infty^{(c)}]. \tag{G.32}$$

Combining (G.30)-(G.32) yields

$$(-sE_a + A_a + R_a X_a)X_{a,1} = \Big[E_a U_1 \ \ E_a T_0 \ \ (A_a + R_a X_a)[T_\infty^{(l)} \ \ T_\infty^{(c)}]\Big](-sE_{ss} + A_{ss}). \tag{G.33}$$

We assume that $-sE_a + A_a + R_a X_a$ is not regular, hoping to show a contradiction. Then there exists a polynomial vector

$$\eta(s) = \eta_e s^e + \eta_{e-1}s^{e-1} + \cdots + \eta_0 \ \ (e \geq 0, \ \eta_e \neq 0)$$

such that

$$\eta^T(s)(-sE_a + A_a + R_aX_a) = 0. \tag{G.34}$$

Because $-sE_{ss} + A_{ss}$ is regular, by pre-multiplying (G.33) by $\eta^T(s)$, we obtain

$$\eta^T(s)\Big[E_aU_1 \;\; E_aT_0 \;\; (A_a + R_aX_a)[T_\infty^{(l)} \;\; T_\infty^{(c)}]\Big] = 0,$$

implying

$$\eta_e^T(A_a + R_aX_a)T_\infty^{(c)} = 0. \tag{G.35}$$

On the other hand, (G.34) means that $\eta_eE_a = 0$, so that η_e can be expressed as $\eta_e = V_{a,2}\alpha$ for some $\alpha \neq 0$. We thus see that (G.35) is expressed as $\alpha^T(A_{a,0} + R_{a,0}X_{a,0}) = 0$. Since $A_{a,0} + R_{a,0}X_{a,0}$ is nonsingular, we get $\alpha = 0$. This is a contradiction to $\alpha \neq 0$. Therefore, by (G.33), $-sE_a + A_a + R_aX_a$ is strictly equivalent to $-sE_{ss} + A_{ss}$, and thus X_a of (40) satisfies the condition 1 of Definition 5.1. Furthermore, it follows from (G.33) and the definitions of T_0 and $[T_\infty^{(l)} \;\; T_\infty^{(c)}]$ that

$$\begin{bmatrix} A_a + R_aX_a \\ [C \;\; D] \end{bmatrix} T_0 = \begin{bmatrix} E_a \\ 0 \end{bmatrix} \Lambda_0$$

$$\begin{bmatrix} A_a + R_aX_a \\ [C \;\; D] \end{bmatrix} [T_\infty^{(l)} \;\; T_\infty^{(c)}]\Lambda_\infty = \begin{bmatrix} E_a \\ 0 \end{bmatrix} [T_\infty^{(l)} \;\; T_\infty^{(c)}].$$

Hence the condition 2 of Definition 5.1 is satisfied. This completes the proof. ∎

H Proof of Theorem 5.4

By Theorems 5.1 and 5.3, in order to prove the equivalence between items 1 and 2 of the theorem, it suffices to show the following (α) and (β).

(α) $\mathrm{i}\{\Sigma_a\} = (g, 0, g) \Leftrightarrow \mathrm{i}\{\Sigma_b\} = (n - r, 0, n - r)$.

(β) $[U_1 \; T_0 \; T_\infty]$ is nonsingular $\Leftrightarrow$ $[U_1^b \; \hat{T}_0 \; \hat{T}_\infty]$ is nonsingular.

Before proving (α) and (β), we require some preliminaries. We can show that the following relation holds between $H_a(s)$ and $H_b(s)$

$$T^T H_a(s) T = \begin{bmatrix} H_b(s) & 0 \\ 0 & \Theta_b \end{bmatrix} \tag{H.1}$$

where

$$\Theta_b = \begin{bmatrix} \Delta_b & 0 \\ 0 & -J'^{-1} \end{bmatrix}, \qquad \Delta_b = D^TJD,$$

$$T = \begin{bmatrix} I_n & 0 & 0 & 0 \\ -\Delta_b^{-1}D^TJC & -\Delta_b^{-1}B^T & I_m & 0 \\ 0 & I_n & 0 & 0 \\ -J'\Delta_b^{-1}D^TJC & -J'\Delta_b^{-1}B^T & J' & I_m \end{bmatrix}.$$

Now we can show that if D^TJD is nonsingular, then $\begin{bmatrix} D^\perp \\ (D^TJD)^{-1}D^TJ \end{bmatrix} \in \mathbf{R}^{p\times p}$ is nonsingular. Hence, noting that

$$\begin{bmatrix} I_n & 0 \\ 0 & D^\perp \\ 0 & (D^TJD)^{-1}D^TJ \end{bmatrix} P_G(s) \begin{bmatrix} I_n & 0 \\ -(D^TJD)^{-1}D^TJC & I_m \end{bmatrix} = \begin{bmatrix} \hat{P}_G(s) & 0 \\ 0 & I_m \end{bmatrix},$$

we see that T_0 and T_∞ can be chosen as

$$T_0 = \begin{bmatrix} \hat{T}_0 \\ -(D^TJD)^{-1}D^TJC\hat{T}_0 \end{bmatrix}, \quad T_\infty = \begin{bmatrix} \hat{T}_\infty & 0 \\ 0 & I_m \end{bmatrix}.$$

By the above argument, take $T_\infty^{(c)} = \begin{bmatrix} \hat{T}_\infty^{(c)} & 0 \\ 0 & I_m \end{bmatrix}$ and $V_{a,2} = \begin{bmatrix} V_2 & 0 \\ 0 & I_m \end{bmatrix}$. Let us begin with (α). It is straightforward to show that

$$TK = \begin{bmatrix} T_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix} Y$$

where

$$K = \begin{bmatrix} \hat{T}_\infty^{(c)} & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}, \quad Y = \begin{bmatrix} I_{n-r} & 0 & 0 & 0 \\ -\Delta_b^{-1}D^TJC\hat{T}_\infty^{(c)} & -\Delta_b^{-1}B^TV_2 & I_m & 0 \\ 0 & I_{n-r} & 0 & 0 \\ -J'\Delta_b^{-1}D^TJC\hat{T}_\infty^{(c)} & -J'\Delta_b^{-1}B^TV_2 & J' & I_m \end{bmatrix}.$$

Pre- and post-multiply (H.1) by K^T and K to get

$$Y^T\Sigma_aY = \begin{bmatrix} \Sigma_b & 0 \\ 0 & \Theta_b \end{bmatrix}.$$

It follows by the Silvester's inertia theorem that

$$\mathrm{i}\{\Sigma_b\} = \mathrm{i}\{\Sigma_a\} - \mathrm{i}\{\Theta_b\} = \mathrm{i}\{\Sigma_a\} - (m, 0, m),$$

which proves (α).

Next, in order to prove (β), suppose that $H_b(s)$ is regular, so that $\mathcal{V}\{H_b(s);\cdot\}$ is well-defined. It then follows from (H.1) that the columns of

$$T\begin{bmatrix} U_1^b \\ U_2^b \\ 0_{2m\times k_-} \end{bmatrix} \in \mathbf{C}^{2(n+m)\times k_-}$$

form a basis of $\mathcal{V}\{H_a(s);\mathbf{C}_-\}$. Hence, by the uniqueness of eigenspace, for an arbitrary U satisfying (10), we get

$$\mathrm{im}\left[\frac{U_1}{U_2}\right] = \mathrm{im} T\begin{bmatrix} U_1^b \\ U_2^b \\ 0_{2m\times k_-} \end{bmatrix} = \mathrm{im}\begin{bmatrix} U_1^b \\ * \\ \hline * \end{bmatrix},$$

implying $\mathrm{im}U_1 = \mathrm{im}\begin{bmatrix} U_1^b \\ * \end{bmatrix}$. Also, for arbitrary T_0, T_∞ of (5), (6),

$$\mathrm{im}T_0 = \mathrm{im}\begin{bmatrix} \hat{T}_0 \\ -(D^TJD)^{-1}D^TJC\hat{T}_0 \end{bmatrix}, \quad \mathrm{im}T_\infty = \mathrm{im}\begin{bmatrix} \hat{T}_\infty & 0 \\ 0 & I_m \end{bmatrix}.$$

Consequently,

$$\mathrm{im}[U_1\ T_0\ T_\infty] = \mathrm{im}\begin{bmatrix} U_1^b & \hat{T}_0 & \hat{T}_\infty & 0 \\ * & -(D^TJD)^{-1}D^TJC\hat{T}_0 & 0 & I_m \end{bmatrix},$$

which proves (β). ∎

I Proof of Lemma 6.2

From (14), it can be shown that

$$\begin{bmatrix} -S_{0e,12}^{(1)} & B_1 \\ -S_{0e,22}^{(2)} & B_2 \\ -S_{0e,2}^{(2)} & D \end{bmatrix} = \begin{bmatrix} S_{0e,11}^{(1)}A_{0e12} - A_{12}T_{0e,22}^{(1)} & \beta_1 + S_{0e,11}^{(1)}B_{0e,1} \\ -A_{22}T_{0e,22}^{(1)} & \beta_2 \\ -C_2T_{0e,22}^{(1)} & \delta \end{bmatrix}\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1}.$$

Thus we get

$$\begin{aligned}
&\begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} T_{0e,1}^{(2)} - \begin{bmatrix} S_{0e,12}^{(1)} \\ S_{0e22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} A_{0e21} \\
&= \begin{bmatrix} S_{0e,11}^{(1)}A_{0e12} - A_{12}T_{0e,22}^{(1)} & \beta_1 + S_{0e,11}^{(1)}B_{0e,1} \\ -A_{22}T_{0e,22}^{(1)} & \beta_2 \\ -C_2T_{0e,22}^{(1)} & \delta \end{bmatrix}\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1}\begin{bmatrix} A_{0e21} \\ T_{0e,1}^{(2)} \end{bmatrix} \\
&= \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \\ 0 \end{bmatrix}[A_{0e12}\ \ B_{0e,1}]\begin{bmatrix} A_{0e22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1}\begin{bmatrix} A_{0e21} \\ T_{0e,1}^{(2)} \end{bmatrix} - \begin{bmatrix} A_{12} \\ A_{22} \\ C_2 \end{bmatrix} K + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \delta \end{bmatrix} C_s.
\end{aligned}$$

Using the above equation and (14), we rewrite the left-hand side of (54) as

$$\begin{aligned}
&\begin{bmatrix} -sE_{11} + A_{11} \\ A_{21} \\ C_1 \end{bmatrix} T_{0e,11}^{(1)} + \begin{bmatrix} A_{12} \\ A_{22} \\ C_2 \end{bmatrix}(T_{0e,21}^{(1)} - K) + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \delta \end{bmatrix} C_s \\
&= -\begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} T_{0e,1}^{(2)} + \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \\ 0 \end{bmatrix}(-sE_{0e11} + A_{0e11}) + \begin{bmatrix} S_{0e,12}^{(1)} \\ S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} A_{0e21} \\
&\quad - \begin{bmatrix} A_{12} \\ A_{22} \\ C_2 \end{bmatrix} K + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \delta \end{bmatrix} C_s \\
&= \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \\ 0 \end{bmatrix}(-sE_{0e11} + A_s) = \begin{bmatrix} E_{11}T_{0e,11}^{(1)} \\ 0 \\ 0 \end{bmatrix}(-sI + E_{0e11}^{-1}A_s),
\end{aligned}$$

which gives the desired result. ∎

J Proof of Lemma 6.3

By the definition of T_s, (54) is written as

$$\begin{bmatrix} -sE+A & \beta \\ C & \delta \end{bmatrix} \begin{bmatrix} T_s \\ C_s \end{bmatrix} = \begin{bmatrix} ET_s \\ 0 \end{bmatrix} (-sI + E_{0e11}^{-1} A_s)$$

This verifies the second block column of (56). We prove the first block column. The $(4,1)$ block is obviously true. Now, we write $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and D as

$$\begin{aligned} B &= \begin{bmatrix} S_{0e,11}^{(1)} A_{0e12} - A_{12} T_{0e,22}^{(1)} & \beta_1 + S_{0e,11}^{(1)} B_{0e,1} \\ -A_{22} T_{0e,22}^{(1)} & \beta_2 \end{bmatrix} \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\ &= - \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \end{bmatrix} B_s + \beta D_s - AL \end{aligned} \tag{J.1}$$

$$D = [-C_2 T_{0e,22}^{(1)} \ \ \delta] \begin{bmatrix} A_{0e22} & B_{0e,2} \\ T_{0e,2}^{(2)} & D_{0e} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = -C_2 L_2 + \delta D_s = -CL + \delta D_s. \tag{J.2}$$

By these equations, the left-hand sides of the $(1,1)$ and $(3,1)$ blocks of (56) are, respectively, written as

$$\begin{aligned} & C^T JC(P_1 - LP_2) + C^T J \delta D_s P_2 + (sE^T + A^T) P_3 \\ &\quad = -C^T JDP_2 - E^T P_3(-sI + \Lambda_-) - C^T JCLP_2 + C^T J\delta D_s P_2 \quad \text{(because of (55))} \\ &\quad = -E^T P_3(-sI + \Lambda_-) \end{aligned}$$

and

$$\begin{aligned} (-sE+A)(P_1 - LP_2) + \beta D_s P_2 &= (-sE+A)(P_1 - LP_2) + (B + \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \end{bmatrix} B_s + AL) P_2 \\ &= (-sE+A)P_1 + BP_2 + \begin{bmatrix} S_{0e,11}^{(1)} \\ 0 \end{bmatrix} B_s P_2 \\ &= EP_1(-sI + \Lambda_-) + ET_s E_{0e11}^{-1} B_s P_2. \end{aligned}$$

Hence the $(1,1)$ and $(3,1)$ blocks are true. Finally we prove the $(2,1)$ block. Recall that, in the proof of Theorem 5.1, under Assumptions (A2) and (A3), it was shown that (G.29) holds. Hence we get $U^H E_H [U_0 \ \ U_\infty] = 0$. And, we can show that

$$\begin{aligned} U^H E_H [U_0 \ \ U_\infty] = 0 &\Leftrightarrow U_2^H E_a [T_0 \ \ T_\infty] = 0 \\ &\Leftrightarrow \begin{bmatrix} P_3 \\ P_4 \end{bmatrix}^H E_a \begin{bmatrix} T_{0e,1}^{(1)} & T_{0e,2}^{(1)} \\ T_{0e,1}^{(2)} & T_{0e,2}^{(2)} \end{bmatrix} = 0 \\ &\Leftrightarrow P_3^H E T_{0e,1}^{(1)} = 0 \\ &\Leftrightarrow P_3^H E_{11} T_{0e,11}^{(1)} = 0 \\ &\Leftrightarrow P_3^H S_{0e,11}^{(1)} = 0. \end{aligned} \tag{J.3}$$

It follows from (55) and (J.3) that

$$\begin{bmatrix} T_{0e,1}^{(1)} & T_{0e,2}^{(1)} \\ T_{0e,1}^{(2)} & T_{0e,2}^{(2)} \end{bmatrix}^T \left\{ [C\ D]^T J [C\ D] \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P_3 \right\} = \begin{bmatrix} -T_{0e,1}^{(1)\,T} E^T P_3 \\ 0 \end{bmatrix} \Lambda_- = 0.$$

Moreover, using (14), we can show that the above equation holds if and only if

$$\begin{bmatrix} A_{0e21} \\ A_{0e22} \end{bmatrix}^T S_{0e,2}^{(2)\,T} J(CP_1 + DP_2) + A_{0e}^T [S_{0e,1}^{(1)}\ S_{0e,2}^{(1)}]^T P_3 = 0.$$

This is equivalent to

$$\begin{aligned} &\begin{bmatrix} A_{0e21} \\ A_{0e22} \end{bmatrix}^T \left\{ S_{0e,2}^{(2)\,T} J(CP_1 + DP_2) + S_{0e,2}^{(1)\,T} P_3 \right\} = 0 \\ \Leftrightarrow \quad & S_{0e,2}^{(2)\,T} J(CP_1 + DP_2) + S_{0e,2}^{(1)\,T} P_3 = 0. \end{aligned} \tag{J.4}$$

The last equivalence follows since $[A_{0e21}\ \ A_{0e22}]$ has full row rank. Hence, from (55), (J.2) and (J.4), we have

$$\begin{aligned} &\delta^T J(CP_1 - LP_2) + \delta^T J \delta D_s P_2 + \beta^T P_3 \\ &\quad = \delta^T J(CP_1 + DP_2) + \beta^T P_3 \\ &\quad = D_{0e}^T (D^T JCP_1 + D^T JDP_2 + B^T P_3) - B_{0e}^T \left[S_{0e,2}^{(2)\,T} J(CP_1 + DP_2) + S_{0e,2}^{(1)\,T} P_3 \right] = 0. \end{aligned}$$

This completes the proof. ∎

K Proof of Lemma 6.4

Because $H_c(s)$ is regular, it should be noted that $\hat{U}$ has full column rank if and only if

$$E_H \hat{U} = \begin{bmatrix} -E^T P_3 & 0 \\ 0 & 0 \\ EP_1 & ET_s \\ 0 & 0 \end{bmatrix}$$

has full column rank. Now, it is simple to show that

$$H_a(s) \begin{bmatrix} T_{0e}^{(1)} \\ T_{0e}^{(2)} \\ 0 \\ J' T_{0e}^{(2)} \end{bmatrix} = \begin{bmatrix} C^T J S_{0e}^{(2)} \\ D^T J S_{0e}^{(2)} \\ S_{0e}^{(1)} \\ 0 \end{bmatrix} (-sE_{0e} + A_{0e}).$$

Also, from (33), $H_c(s)$ is regular if and only if $H_a(s)$ is regular. Noting that U and T_{0e} have full column rank and that $\mathcal{V}\{H_a(s); \mathbf{C}_-\} \cap \mathcal{V}\{H_a(s); \mathbf{C}_{0e}\} = \{0\}$, we conclude that

$$\begin{bmatrix} P_1 & T_{0e}^{(1)} \\ P_2 & T_{0e}^{(2)} \\ P_3 & 0 \\ P_4 & J' T_{0e}^{(2)} \end{bmatrix}$$

and

$$E_H \begin{bmatrix} P_1 & T_{0e}^{(1)} \\ P_2 & T_{0e}^{(2)} \\ P_3 & 0 \\ P_4 & J'T_{0e}^{(2)} \end{bmatrix} = \begin{bmatrix} -E^TP_3 & C^TJS_{0e}^{(2)} \\ 0 & D^TJS_{0e}^{(2)} \\ EP_1 & S_{0e}^{(1)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -E^TP_3 & 0 & C^TJS_{0e,2}^{(2)} \\ 0 & 0 & D^TJS_{0e,2}^{(2)} \\ EP_1 & ET_s & \begin{bmatrix} S_{0e,12}^{(1)} \\ S_{0e,22}^{(1)} \end{bmatrix} \\ 0 & 0 & 0 \end{bmatrix}$$

have full column rank, which implies that $E_H\hat{U}$ has full column rank. ∎

L Proof of Theorem 6.1

We first prove the theorem under (A1)-(A4) and then gives a proof without (A4).

Besides (A1)-(A3), we now assume that (A4) holds. By Lemma 6.1, the **NS-J-SFP** for $G(s)$ is solvable if and only if the **S-J-SFP** for $G_c(s)$ is solvable. We see by Theorem A.1 that the **S-J-SFP** for $G_c(s)$ is solvable if and only if the following conditions (α) and (β) holds.

(α) The inertia condition $\mathrm{i}\{\Sigma_c\} = (g, 0, g)$ holds, where

$$\Sigma_c = \begin{bmatrix} V_{a,1}^TQ_{a,c}V_{a,1} & V_{a,1}^TA_{a,c}^TV_{a,2} \\ V_{a,2}^TA_{a,c}V_{a,1} & V_{a,2}^TR_aV_{a,2} \end{bmatrix}.$$

(β) Let a matrix whose columns form a basis of $\mathcal{V}\{H_c(s); \mathbf{C}_-\}$ be

$$U^c = \begin{bmatrix} U_1^c \\ U_2^c \end{bmatrix} \in \mathbf{C}^{2(n+m)\times h_-}, \quad U_1^c \in \mathbf{C}^{(n+m)\times h_-}.$$

Then $\mathrm{rank}E_aU_1^c = r$.

We show that (α) and (β) are, respectively, equivalent to (a) and (b) of Theorem 5.2.

First we show the equivalence between (α) and (a). Choose $V_{a,1}$ and $V_{a,2}$ as

$$V_{a,1} = V_{a,2} = \begin{bmatrix} 0_{r\times r} & 0 \\ 0 & I_{n-r+m} \end{bmatrix}.$$

Then Σ_c is

$$\Sigma_c = \begin{bmatrix} C_2^TJC_2 & C_2^TJ\delta & A_{22}^T & 0 \\ \delta^TJC_2 & \delta^TJ\delta - J' & \beta_2^T & I_m \\ A_{22} & \beta_2 & 0 & 0 \\ 0 & I_m & 0 & -J'^{-1} \end{bmatrix}.$$

In the proof of Lemma 4.3, we proved that

$$\begin{bmatrix} A_{22} & \beta_2 \\ C_2 & \delta \end{bmatrix} = \begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} \left\{ A_{0e22} \begin{bmatrix} T_{0e,22}^{(1)} \\ T_{0e,2}^{(2)} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & D_{0e} \end{bmatrix} - [0 \ B_{0e,2}] \right\}$$

and that

$$\Delta_1 := A_{0e22} \begin{bmatrix} T_{0e,22}^{(1)} \\ T_{0e,2}^{(2)} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & D_{0e} \end{bmatrix} - [0 \ B_{0e,2}]$$

is nonsingular. Also, in the proof of item 3 of Lemma 4.1, we proved that there exists a real nonsingular matrix Δ_2 such that

$$\begin{bmatrix} S_{0e,22}^{(1)} \\ S_{0e,2}^{(2)} \end{bmatrix} = \begin{bmatrix} V_2 & 0 \\ 0 & I_p \end{bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} T_\infty^{(c)} \Delta_2 = \begin{bmatrix} V_2^T A & V_2^T B \\ C & D \end{bmatrix} \Delta_2.$$

Hence we have

$$\begin{aligned}
\Sigma_c &\approx \begin{bmatrix} C_2^T J C_2 & C_2^T J\delta & A_{22}^T & 0 \\ \delta^T J C_2 & \delta^T J \delta & \beta_2^T & 0 \\ A_{22} & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & -J'^{-1} \end{bmatrix} \\
&\approx \begin{bmatrix} \Delta_1^{-1} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} C_2^T J C_2 & C_2^T J\delta & A_{22}^T & 0 \\ \delta^T J C_2 & \delta^T J \delta & \beta_2^T & 0 \\ A_{22} & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & -J'^{-1} \end{bmatrix} \begin{bmatrix} \Delta_1^{-1} & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} {S_{0e,22}^{(1)}}^T J S_{0e,22}^{(1)} & {S_{0e,22}^{(1)}}^T & 0 \\ S_{0e,22}^{(2)} & 0 & 0 \\ 0 & 0 & -J'^{-1} \end{bmatrix} \\
&\approx \begin{bmatrix} \Delta_2^{-1} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} {S_{0e,22}^{(1)}}^T J S_{0e,22}^{(1)} & {S_{0e,22}^{(1)}}^T & 0 \\ S_{0e,22}^{(2)} & 0 & 0 \\ 0 & 0 & -J'^{-1} \end{bmatrix} \begin{bmatrix} \Delta_2^{-1} & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} {T_\infty^{(c)}}^T [C\ D]^T J [C\ D] T_\infty^{(c)} & {T_\infty^{(c)}}^T \begin{bmatrix} A^T \\ B^T \end{bmatrix} V_2 & 0 \\ V_2^T [A\ B] T_\infty^{(c)} & 0 & 0 \\ 0 & 0 & -J'^{-1} \end{bmatrix} \\
&\approx \begin{bmatrix} {T_\infty^{(c)}}^T Q_a T_\infty^{(c)} & {T_\infty^{(c)}}^T \begin{bmatrix} A^T \\ B^T \end{bmatrix} V_2 & {T_\infty^{(c)}}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\ V_2^T [A\ B] T_\infty^{(c)} & 0 & 0 \\ [0\ I_m] T_\infty^{(c)} & 0 & -J'^{-1} \end{bmatrix} \\
&= \begin{bmatrix} Q_{a,0} & {T_\infty^{(c)}}^T A_a^T \mathrm{diag}\{V_2, I_m\} \\ \mathrm{diag}\{V_2, I_m\}^T A_a T_\infty^{(c)} & \mathrm{diag}\{V_2, I_m\}^T R_a \mathrm{diag}\{V_2, I_m\} \end{bmatrix} \approx \Sigma_a.
\end{aligned}$$

It follows by the Silvester's inertia theorem that (α) and (a) are equivalent.

Secondly we prove the equivalence between (β) and (b). Partition P_1 as

$$P_1 = \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix}, \quad P_{11} \in \mathbf{C}^{r \times k_-}.$$

Define

$$\hat{U}_1 = \begin{bmatrix} P_1 - LP_2 & T_s \\ D_s & C_s \end{bmatrix}, \qquad \hat{U}_2 = \begin{bmatrix} P_3 & 0 \\ J' D_s P_2 & J' C_s \end{bmatrix}.$$

Then $\hat{U}$ of (57) is expressed as $\hat{U} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$, and we have

$$E_a \hat{U}_1 = \begin{bmatrix} E(P_1 - LP_2) & ET_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_{11}[P_{11}\ T_{0e,11}^{(1)}] \\ 0 \end{bmatrix}. \qquad \text{(L.1)}$$

Now, by uniqueness of eigenspace, for arbitrary T_0 and T_∞,

$$\mathrm{im}[T_0\ T_\infty] = \mathrm{im}\begin{bmatrix} T^{(1)}_{0e,11} & 0 \\ T^{(1)}_{0e,21} & T^{(1)}_{0e,22} \\ T^{(2)}_{0e,1} & T^{(2)}_{0e,2} \end{bmatrix}.$$

Consequently,

$$\mathrm{im}[U_1\ T_0\ T_\infty] = \mathrm{im}\begin{bmatrix} P_{11} & T^{(1)}_{0e,11} & 0 \\ P_{12} & \begin{bmatrix} T^{(1)}_{0e,21} \\ T^{(2)}_{0e,1} \end{bmatrix} & \begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix} \end{bmatrix}. \tag{L.2}$$

By item 2 of Lemma 4.1, $\begin{bmatrix} T^{(1)}_{0e,22} \\ T^{(2)}_{0e,2} \end{bmatrix}$ is nonsingular, so we see from (L.1) and (L.2) that (b) and the condition $\mathrm{rank}E_a\hat{U}_1 = r$ are equivalent. It thus suffices to prove the equivalence between $\mathrm{rank}E_a\hat{U}_1 = r$ and (β). Suppose that $\mathrm{rank}E_a\hat{U}_1 = r$. Then $\hat{U}$ of (57) has r columns and hence we see from Lemma 6.5 that the columns of $\hat{U}$ form a basis of $\mathcal{V}\{H_c(s); \mathbf{C}_-\}$. Therefore, by uniqueness of eigenspace, $\mathrm{im}\hat{U} = \mathrm{im}U^c$, so that $\mathrm{rank}E_aU^c_1 = \mathrm{rank}E_a\hat{U}_1 = r$. Conversely, suppose that (β) holds. It then follows from $\dim\{H_c(s); \mathbf{C}_-\} = r$ and Lemma 6.5 that the columns of $\hat{U}$ form a basis of $\mathcal{V}\{H_c(s); \mathbf{C}_-\}$. Hence $\mathrm{rank}E_a\hat{U}_1 = \mathrm{rank}E_aU^c_1 = r$.

We proved that, under Assumptions (A1)-(A4), the **NS-J-SFP** is solvable if and only if (a) and (b) of Theorem 5.2 hold. Next consider the case where (A4) does not hold. Take a real matrix F that makes $-sE + A + BF$ stable. Then we have the following factorization

$$G(s) = G_1(s)G_2(s)$$

where

$$G_1(s) = \left[\begin{array}{c|c} -sE + A + BF & B \\ \hline C + DF & D \end{array}\right] =: \left[\begin{array}{c|c} -sE + A_F & B \\ \hline C_F & D \end{array}\right]$$
$$G_2(s) = \left[\begin{array}{c|c} -sE + A & B \\ \hline -F & I \end{array}\right].$$

Thus it can be shown that the **NS-J-SFP** for $G(s)$ is solvable if and only if the **NS-J-SFP** for $G_1(s)$ is solvable. Since $-sE + A + BF$ is stable, the realization of $G_1(s)$ satisfies (A1)-(A4). Hence we can apply the argument with (A4) to the **NS-J-SFP** for $G_1(s)$.

Let the columns of Ψ_0, $\Psi^{(l)}_\infty$ and $\Psi^{(c)}_\infty$ form bases of the subspaces $\mathcal{V}\{P_{G_1}(s); \mathbf{C}_0\}$, $\mathcal{V}^{(l)}_\infty\{P_{G_1}(s)\}$ and $\mathcal{V}^{(c)}_\infty\{P_{G_1}(s)\}$ respectively. Define the Hamiltonian pencil related to $G_1(s)$ as

$$H_1(s) = \begin{bmatrix} Q_{a,F} & sE_a^T + A_{a,F}^T \\ -sE_a + A_{a,F} & R_a \end{bmatrix}$$

where

$$A_{a,F} = \begin{bmatrix} A_F & B \\ 0 & I_m \end{bmatrix}, \qquad Q_{a,F} = \begin{bmatrix} C_F^TJC_F & C_F^TJD \\ D^TJC_F & D^TJD - J' \end{bmatrix}.$$

If $H_1(s)$ is regular, then let the columns of

$$\Upsilon = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} \in \mathbf{C}^{2(n+m)\times k_-}, \ \Upsilon_1 \in \mathbf{C}^{(n+m)\times k_-}$$

form a basis of $\mathcal{V}\{H_1(s); \mathbf{C}_-\}$. Then we see from the first half of the proof that the **NS-J-SFP** for G_1 is solvable if and only if the following conditions (a') and (b') hold.

(a') The inertia condition

$$\mathrm{i}\left\{\begin{bmatrix} \Psi_\infty^{(c)T} Q_{a,F}\Psi_\infty^{(c)} & \Psi_\infty^{(c)T} A_{a,F}^T V_{a,2} \\ V_{a,2}^T A_{a,F}\Psi_\infty^{(c)} & V_{a,2}^T R_a V_{a,2} \end{bmatrix}\right\} = (g, 0, g)$$

holds.

(b') $[\Upsilon_1 \ \Psi_0 \ \Psi_\infty^{(l)} \ \Psi_\infty^{(c)}]$ is nonsingular.

Now, since $P_{G_1}(s)\begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} = P_G(s)$, it follows that

$$\left[\Psi_0 \ \Psi_\infty^{(l)} \ \Psi_\infty^{(c)}\right] = \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} [T_0 \ T_\infty^{(l)} \ T_\infty^{(c)}]. \tag{L.3}$$

Similarly, noting that

$$T_1^T H_1(s) T_1 = H_a(s)$$

where

$$T_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ -F & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -J'F & 0 & 0 & I \end{bmatrix},$$

we see that Υ is expressed as

$$\Upsilon = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} U_1 \\ \begin{bmatrix} 0 & 0 \\ -J'F & 0 \end{bmatrix} U_1 + U_2 \end{bmatrix}. \tag{L.4}$$

By using (L.3) and (L.4), we wish to show that (a') and (b') are, respectively, equivalent to (a) and (b) of Theorem 5.2. First we show the equivalence between (a') and (a). Since $\mathrm{im}\begin{bmatrix} 0 & 0 \\ J'F & 0 \end{bmatrix} \subseteq \ker E_a^T$, there exists a real matrix Y such that $\begin{bmatrix} 0 & 0 \\ J'F & 0 \end{bmatrix} = V_{a,2}Y$. Hence, from (L.3), we have

$$T_1^{-1}\begin{bmatrix} \Psi_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix} = \begin{bmatrix} T_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix}\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix},$$

so that

$$\begin{bmatrix} {\Psi_\infty^{(c)}}^T Q_a \Psi_\infty^{(c)} & {\Psi_\infty^{(c)}}^T A_{a,F}^T V_{a,2} \\ V_{a,2}^T A_{a,F} \Psi_\infty^{(c)} & V_{a,2}^T R_a V_{a,2} \end{bmatrix}$$

$$= \begin{bmatrix} \Psi_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix}^T \begin{bmatrix} Q_{a,F} & A_{a,F}^T \\ A_{a,F} & R_a \end{bmatrix} \begin{bmatrix} \Psi_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}^T \begin{bmatrix} T_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix}^T \begin{bmatrix} Q_a & A_a^T \\ A_a & R_a \end{bmatrix} \begin{bmatrix} T_\infty^{(c)} & 0 \\ 0 & V_{a,2} \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}^T \begin{bmatrix} {T_\infty^{(c)}}^T Q_a T_\infty^{(c)} & {T_\infty^{(c)}}^T A_a^T V_{a,2} \\ V_{a,2}^T A_a T_\infty^{(c)} & V_{a,2}^T R_a V_{a,2} \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \approx \Sigma_a,$$

which proves the equivalence between (a') and (a). Finally, since

$$[\Upsilon_1 \ \Psi_0 \ \Psi_\infty^{(l)} \ \Psi_\infty^{(c)}] = \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} [U_1 \ T_0 \ T_\infty^{(l)} \ T_\infty^{(c)}],$$

the equivalence between (b') and (b) follows. ∎