On the identification of degenerate indices in the nonlinear complementarity problem with the proximal point algorithm

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Abstract

In this paper we focus on the problem of identifying the index sets $P(x) := \{i \mid x_i > 0\}$, $N(x) := \{i \mid F_i(x) > 0\}$ and $C(x) := \{i \mid x_i = F_i(x) = 0\}$ for a solution x of the monotone nonlinear complementarity problem NCP(F). The correct identification of these sets is important from both theoretical and practical points of view. Such an identification enables us to remove complementarity conditions from the NCP and locally reduce the NCP to a system which can be dealt with more easily. We present a new technique that utilizes a sequence generated by the proximal point algorithm (PPA). Using the superlinear convergence property of PPA, we show that the proposed technique can identify the correct index sets without assuming the nondegeneracy and the local uniqueness of the solution.

1 Introduction

The nonlinear complementarity problem NCP(F) is to find a vector $x \in \Re^n$ such that

$$x_i \geq 0,\ F_i(x) \geq 0,\ x_i F_i(x) = 0 \text{ for all } i \in J,$$

where F is a mapping from $\Re^n$ to $\Re^n$ and $J := \{1, 2, \ldots, n\}$. When F is affine, the problem is called the linear complementarity problem (LCP). Throughout the paper we assume that F is monotone and continuously differentiable, and that NCP(F) has a nonempty solution set, which we denote X.

The purpose of the paper is to propose a method of identifying three index sets $P(\bar{x}), N(\bar{x})$ and $C(\bar{x})$ for a solution $\bar{x}$ of NCP(F), which are defined by

$$\begin{aligned}
P(\bar{x}) &:= \{i \in J \mid \bar{x}_i > 0, F_i(\bar{x}) = 0\}, \\
N(\bar{x}) &:= \{i \in J \mid \bar{x}_i = 0, F_i(\bar{x}) > 0\}, \\
C(\bar{x}) &:= \{i \in J \mid \bar{x}_i = 0, F_i(\bar{x}) = 0\},
\end{aligned}$$

respectively. When the point $\bar{x}$ under consideration is clear from the context, we shall denote these sets simply by P, N and C. By definition it is easy to verify that $J = P \cup N \cup C$, $P \cap N = \emptyset$, $N \cap C = \emptyset$, and $P \cap C = \emptyset$. Therefore, these sets characterize the solution $\bar{x}$. In particular, we call $\bar{x}$ a degenerate solution if $C \neq \emptyset$, otherwise we call it a nondegenerate solution. Moreover, it is important to identify these sets from a practical point of view. If we can identify the index sets P, N and C before we know $\bar{x}$ exactly, then the original NCP(F) can be reduced to the following system:

$$F_P(x) = 0,\ x_P \geq 0,\ F_N(x) \geq 0,\ x_N = 0,\ F_C(x) = x_C = 0,$$

where, for a vector $z \in \Re^n$ and an index set K, z_K denotes the $|K|$-dimensional vector with components $z_i, i \in K$. This system is more tractable than the original NCP(F).

Several methods have been proposed for identifying the index sets P, N and C at a solution $\bar{x}$ of NCP(F). The case where C is empty and there exists a sequence $\{x^k\}$ converging to $\bar{x}$ is not difficult. In such a case, let index sets P^k and N^k be defined by $P^k := \{i \mid x_i^k > F_i(x^k)\}$ and $N^k := \{i \mid x_i^k < F_i(x^k)\}$, respectively. Then, because of the continuity of F, we have for k sufficiently large, $P(\bar{x}) = P^k$ and $N(\bar{x}) = N^k$. However, this approach cannot be applied when $\bar{x}$ is degenerate. Stoer, Wechs and Mizuno [8] proposed an identification technique based on the central path of the interior point method. The method does not require the local uniqueness of the solution $\bar{x}$. However the method only deals with LCP, and it cannot be directly applied to NCP(F). Recently, Facchinei, Fischer and Kanzow [2] presented a technique based on the error bound results. The method is originally developed for the identification of the active set of an inequality constrained minimization problem, but it is also applicable to NCP(F). However, the technique proposed in [2] requires the local uniqueness of the solution $\bar{x}$. Until now, to the authors' knowledge, no technique has been proposed which can successfully identify the index sets for NCP(F) without assuming the local uniqueness of the solution $\bar{x}$.

In this paper we propose a new identification technique based on the proximal point algorithm (PPA) presented in [9]. The proposed technique generates a sequence of index sets $\{P^k, N^k, C^k\}$ using the information on $\{x^k\}$, and eventually identifies $P(x^*), N(x^*)$ and $C(x^*)$. Moreover, a solution of the equality constrained minimization problem

$$\begin{array}{ll} \min & \frac{1}{2}\|x - x^k\|^2 \\ \text{subject to} & F_{P^k}(x) = 0,\ x_{N^k} = 0 \\ & F_{C^k}(x) = 0,\ x_{C^k} = 0 \end{array}$$

eventually becomes a solution of NCP(F). When F is affine, we can compute a solution of the above problem by solving a system of linear equations derived from its KKT conditions. Therefore, for LCP, we can construct PPA with the finite termination property.

The paper is organized as follows. In Section 2 we introduce some concepts and results related to the proximal point method, which are used in the subsequent discussion. In Section 3 we present a new identification technique based on PPA, and show that, under mild assumptions, it eventually identifies the sets P, N and C at x^* that is a limit point of the sequence generated by PPA. In Section 4 we report some numerical results which show that the proposed technique can identify the index sets successfully. In Section 5, by using this technique, we construct a minimization problem with equality constraints whose solution is a solution of the original NCP(F).

2 Preliminaries

First, we recall the definition of the monotonicity and the strong monotonicity for the mapping F.

Definition 2.1 *The mapping $F : \Re^n \to \Re^n$ is called*

(i) monotone, if

$$(x - y)^T(F(x) - F(y)) \geq 0 \qquad \forall x, y \in \Re^n,$$

(ii) strongly monotone with modules $\mu > 0$, if there exists $\mu > 0$ such that

$$(x - y)^T(F(x) - F(y)) \geq \mu\|x - y\|^2 \qquad \forall x, y \in \Re^n.$$

Note that, if F is monotone and $c > 0$, then the mapping $x \mapsto F(x) + cx$ is strongly monotone with modulus c.

A function $\phi : \Re^2 \to \Re$ that satisfies the following relation is called an NCP function.

$$\phi(a, b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0$$

Two well known examples of NCP function are the natural residual function ϕ_{NR} and the Fischer-Burmeister function ϕ_{FB} defined by

$$\phi_{NR}(a,b) = \min\{a,b\}$$

and

$$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - a - b,$$

respectively. It is not difficult to see that these two functions satisfy the inequalities

$$(2-\sqrt{2})|\phi_{NR}(a,b)| \le |\phi_{FB}(a,b)| \le (2+\sqrt{2})|\phi_{NR}(a,b)| \quad \forall (a,b)^T \in \Re^2,$$

which implies $|\phi_{NR}(a,b)|$ and $|\phi_{FB}(a,b)|$ are equivalent in a certain sense.

Let $H_F : \Re^n \to \Re^n$ be defined by

$$H_F(x) := \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix},$$

where ϕ is either ϕ_{NR} or ϕ_{FB} or any other NCP function satisfying

$$\nu_1 |\phi_{NR}(a,b)| \le |\phi(a,b)| \le \nu_2 |\phi_{NR}(a,b)| \quad \forall (a,b)^T \in \Re^2$$

for some constants $\nu_2 \ge \nu_1 > 0$. Then it is clear that the system of equations

$$H_F(x) = 0$$

is equivalent to NCP(F). Beside the equivalence, the mapping H_F provides an error bound for NCP(F). The following error bound results play an essential role in ensuring desirable convergence properties of iterative methods based on H_F for solving NCP(F).

Theorem 2.2 **(i)** *[6] Suppose that F is strongly monotone with modulus μ and Lipschitz continuous with constant L. Then $\|H_F(x)\|$ provides a global error bound for NCP(F), that is,*

$$\|x - \hat{x}\| \le \frac{K_1(L+1)}{\mu} \|H_F(x)\| \qquad \forall x \in \Re^n$$

holds, where $\hat{x}$ is the unique solution of NCP(F) and K_1 is a constant independent of F.

(ii) *[5] Suppose that F is affine and there exists a solution of NCP(F). Then $\|H_F(x)\|$ provides a local error bound for NCP(F), that is, there exist positive constants K_2 and K_3 such that*

$$\|H_F(x)\| \leq K_2 \implies \mathrm{dist}\{x, X\} \leq K_3\|H_F(x)\|,$$

where $\mathrm{dist}\{x, X\} = \min\{\|x - \bar{x}\| \mid \bar{x} \in X\}$.

Among various algorithms for solving NCP(F), the proximal point algorithm (PPA) proposed in [9] has a number of desirable convergence properties. Here we state the algorithm in a slightly different form than the original version in [9].

PPA

Step 0: Choose parameters $\beta \in (0,1), c_0 \in (0,1)$ and an initial point $x^0 \in \Re^n$. Set $k := 0$.

Step 1: If x^k satisfies a termination criterion, then stop.

Step 2: Let $F^k : \Re^n \to \Re^n$ be defined by

$$F^k(x) = F(x) + c_k(x - x^k).$$

Find an approximate solution x^{k+1} of NCP(F^k) such that

$$\|H_{F^k}(x^{k+1})\| \leq \beta^k \min\{1, \|x^{k+1} - x^k\|\}. \tag{2.1}$$

Step 3: Update c_k to get $c_{k+1} > 0$. Set $k := k + 1$ and go to Step 2.

It is important from the practical viewpoint to consider which method is used for solving the subproblems in Step 2. Yamashita and Fukushima [9] proposed to adopt the generalized Newton method proposed by De Luca, Facchinei and Kanzow [1]. By using this method, PPA is shown to have a so-called genuine superlinear convergence property. In the present paper, however, we will not focus on this issue, but merely make use of the convergence properties of PPA stated in the following theorem.

Theorem 2.3 *[9] Suppose that F is monotone and Lipschitz continuous. Suppose also that there exists a solution of NCP(F). Then the sequence $\{x^k\}$ generated by PPA converges to a solution of NCP(F). Moreover, suppose that $\|H_F(x)\|$ provides a local error bound for NCP(F). Then $\{\mathrm{dist}\{x^k, X\}\}$ converges superlinearly to 0.* □

3 Identification of degenerate indices

The purpose of this section is to present a new technique for identifying the index sets

$$\begin{aligned} P(x^*) &:= \{i \mid x_i^* > 0, F_i(x^*) = 0\}, \\ N(x^*) &:= \{i \mid x_i^* = 0, F_i(x^*) > 0\}, \\ C(x^*) &:= \{i \mid x_i^* = 0, F_i(x^*) = 0\}, \end{aligned}$$

where x^* is the limit point of a sequence generated by PPA.

To begin with, we explain the basic idea of our approach. Suppose that there exists a sequence $\{x^k\}$ converging to a solution x^* of NCP(F). Suppose also that there exists a positive sequence $\{a_k\}$ such that

$$\max\{\|x^k - x^*\|, \|F(x^k) - F(x^*)\|\} \le a_k, \quad \lim_{k\to 0} a_k = 0. \tag{3.1}$$

Using $\{x^k\}$ and $\{a_k\}$, let index sets P^k, N^k and C^k be defined by

$$\begin{aligned} P^k &:= \{i \mid x_i^k > a_k, F_i(x^k) \le a_k\}, \\ N^k &:= \{i \mid x_i^k \le a_k, F_i(x^k) > a_k\}, \\ C^k &:= \{i \mid x_i^k \le a_k, F_i(x^k) \le a_k\}, \end{aligned}$$

respectively. Then, it is not difficult to show that, for k sufficiently large

$$P^k = P^* := P(x^*), \quad N^k = N^* := N(x^*), \quad C^k = C^* := C(x^*).$$

For example, we can show that $C^k = C$ as follows. For each $i \in C$, we have

$$x_i^k \le |x_i^k - x_i^*| \le \|x^k - x^*\| \le a_k,$$

and similarly

$$F_i(x^k) \le a_k.$$

Therefore, we have $C^* \subseteq C^k$. Moreover, noting that $a_k \to 0$ and $x^k \to x^*$, we see that $C^k \subseteq C^*$ for large k. Hence C^k must coincides with C^* for k sufficiently large. In this manner, the idea underlying the approach is easy to understand, but it is not trivial to find an adequate sequence $\{a_k\}$ which satisfies (3.1).

Facchinei, Fischer and Kanzow [2] proposed to use $\|H_F(x^k)\|^{\frac{1}{2}}$ as $\{a_k\}$. In [2], a function $f : \Re^n \to \Re$ satisfying

$$\max\{\|x^k - x^*\|, \|F(x^k) - F(x^*)\|\} \leq f(x^k), \quad \lim_{k \to 0} f(x^k) = 0$$

is called *an identification function.* If x^* is a locally unique solution of NCP(F) and $\|H_F(x)\|$ provides a local error bound, then it is easy to verify that $\|H_F(\cdot)\|^{\frac{1}{2}}$ can serve as an identification function. Here, the local uniqueness of x^* seems crucial, because this local error bound only measures the distance between a point x and the nearest solution of NCP(F) to x, rather than a specific solution. To deal with the case where the local uniqueness assumption fails to hold, we focus on a specific sequence that is generated by the PPA. (Note that in [2] a sequence $\{x^k\}$ need not be generated by a particular algorithm.) The fact that $\{x^k\}$ converges superlinearly to the solution set of NCP(F) will play an essential role in our approach.

Before we present a new technique, we introduce the concept of *an identification function sequence.*

Definition 3.1 *Suppose that a sequence $\{x^k\}$ converges to a solution x^* of NCP(F). Suppose also that P^k, N^k and C^k are defined by*

$$\begin{aligned} P^k &:= \{i \mid x_i^k > f^k(x^k), F_i(x^k) \leq f^k(x^k)\}, \\ N^k &:= \{i \mid x_i^k \leq f^k(x^k), F_i(x^k) > f^k(x^k)\}, \\ C^k &:= \{i \mid x_i^k \leq f^k(x^k), F_i(x^k) \leq f^k(x^k)\}, \end{aligned}$$

where $\{f^k\}$ is a sequence of functions from $\Re^n$ to $\Re$. Then $\{f^k\}$ is called an identification function sequence for $\{x^k\}$, if

$$P^k = P(x^*), \quad N^k = N(x^*), \quad C^k = C(x^*)$$

hold for all k sufficiently large.

Note that if f is an identification function, then the sequence $\{f^k\}$ with $f^k \equiv f$ is an identification function sequence. Moreover, if a sequence $\{a_k\}$ satisfies (3.1), then any sequence $\{f^k\}$ such that $f^k(x^k) = a_k$ can be used as an identification function sequence.

The main objective of the paper is to show that the function sequence $\{\rho^k\}$ given by

$$\rho^k(x) = \sqrt{\frac{\|H_F(x)\|}{c_k}} \tag{3.2}$$

can serve as an identification function sequence for the sequence $\{x^k\}$ generated by the PPA with parameter c_k. To this end, we need the following assumptions.

Assumption 3.2

(i) F is Lipschitz continuous.

(ii) $\|H_F(x)\|$ provides a local error bound for NCP(F), that is, there exist positive constants b_1 and b_2 such that

$$\|H_F(x)\| \leq b_1 \Longrightarrow \mathrm{dist}\{x, X\} \leq b_2\|H_F(x)\|. \tag{3.3}$$

(iii) In the PPA, $\{c_k\}$ is determined by $c_k = \alpha^k$ where $\alpha \in (\beta, 1)$.

Note that the conditions (i) and (ii) hold when F is affine, that is, NCP(F) is LCP.

We recall that the following statements hold under Assumption 3.2.

Remark 3.3

(i) Any sequence $\{x^k\}$ generated by the PPA converges to a solution x^* of NCP(F) by Assumption 3.2 (i), (iii) [10] . Moreover, by Assumption 3.2 (ii), $\{\mathrm{dist}\{x^k, X\}\}$ converges superlinearly to 0 [9], that is,

$$\lim_{k\to\infty} \frac{\mathrm{dist}\{x^{k+1}, X\}}{\mathrm{dist}\{x^k, X\}} = 0. \tag{3.4}$$

(ii) Since F^k is strongly monotone with modulus $c_k = \alpha^k$ and Lipschitz continuous with a constant independent of k, $\dfrac{\|H_{F^k}(x)\|}{\alpha^k}$ provides a global error bound for NCP(F^k) by Theorem 2.2(i), that is, there exists a positive constant b_3 independent of k such that

$$\|x - P_k(x^k)\| \leq b_3\frac{\|H_{F^k}(x)\|}{\alpha^k} \qquad \forall x \in \Re^n, \tag{3.5}$$

where $P_k(x^k)$ is the unique solution of NCP(F^k).

Under Assumption 3.2, we will establish the main result.

Theorem 3.4 *The function sequence $\{\rho^k\}$ defined by (3.2) is an identification function sequence for $\{x^k\}$ generated by the PPA.*

The proof of this theorem will be given after showing some lemmas.

Lemma 3.5 *There exist a positive integer k_1 and a positive constant B_1 such that*

$$\|x^{k+1} - x^k\| \le B_1 \frac{\mathrm{dist}\{x^k, X\}}{\alpha^k} \qquad \forall k \ge k_1. \tag{3.6}$$

Proof. For each k, we have

$$\|x^{k+1} - x^k\| \le \|x^{k+1} - P_k(x^k)\| + \|P_k(x^k) - x^k\|, \tag{3.7}$$

where $P_k(x^k)$ is the unique solution of NCP(F^k). Since $\frac{\|H_{F^k}(x)\|}{\alpha^k}$ provides a global error bound for NCP(F^k) as mentioned in Remark 3.3 (ii), we have

$$\|x^{k+1} - P_k(x^k)\| \le b_3 \frac{\|H_{F^k}(x^{k+1})\|}{\alpha^k},$$

where b_3 is a positive constant independent of k. It then follows from Step 2 in PPA that

$$\|x^{k+1} - P_k(x^k)\| \le b_3 \left(\frac{\beta}{\alpha}\right)^k \|x^{k+1} - x^k\|. \tag{3.8}$$

By the global error bound property of $\frac{\|H_{F^k}(x)\|}{\alpha^k}$, we also have

$$\|P_k(x^k) - x^k\| \le b_3 \frac{\|H_{F^k}(x^k)\|}{\alpha^k},$$

which together with (3.7) and (3.8) implies

$$\|x^{k+1} - x^k\| \le b_3 \left(\frac{\beta}{\alpha}\right)^k \|x^{k+1} - x^k\| + b_3 \frac{\|H_{F^k}(x^k)\|}{\alpha^k}.$$

Since $\beta < \alpha$ by Assumption 3.2 (iii), there exists an integer k_1 such that $b_3 \left(\beta/\alpha\right)^k < \frac{1}{2}$ for all $k \ge k_1$. It then follows from $\|H_{F^k}(x^k)\| = \|H_F(x^k)\|$ that the above inequality can be rewritten as

$$\|x^{k+1} - x^k\| \le \frac{b_3}{1 - b_3 \left(\beta/\alpha\right)^k} \frac{\|H_F(x^k)\|}{\alpha^k} \le 2b_3 \frac{\|H_F(x^k)\|}{\alpha^k} \qquad \forall k \ge k_1. \tag{3.9}$$

Since F is Lipschitz continuous, H_F is also Lipschitz continuous, and hence there exists a constant $\hat{L} > 0$ such that

$$\|H_F(x^k)\| = \|H_F(x^k) - H_F(\bar{x}^k)\| \le \hat{L}\|x^k - \bar{x}^k\| = \hat{L}\text{dist}\{x^k, X\},$$

where $\bar{x}^k$ is the nearest solution of NCP(F) from x^k. It follows from (3.9) that

$$\|x^{k+1} - x^k\| \le 2\hat{L}b_3 \frac{\text{dist}\{x^k, X\}}{\alpha^k} \qquad \forall k \ge k_1.$$

Therefore, setting $B_1 = 2\hat{L}b_3$ yields the desired inequality (3.6). $\square$

From Lemma 3.5, we can deduce the key property for our main theorem.

Lemma 3.6 *There exist a positive integer k_2 and a positive constant B_2 such that*

$$\|x^k - x^*\| \le B_2 \frac{\|H_F(x^k)\|}{\alpha^k} \qquad \forall k \ge k_2. \tag{3.10}$$

Proof: Since $x^k \to x^*$, we have from the triangle inequality

$$\|x^k - x^*\| \le \sum_{i=0}^{\infty} \|x^{k+i+1} - x^{k+i}\|.$$

It then follows from Lemma 3.5 that for $k \ge k_1$

$$\|x^k - x^*\| \le B_1 \left\{ \frac{\text{dist}\{x^k, X\}}{\alpha^k} + \frac{\text{dist}\{x^{k+1}, X\}}{\alpha^{k+1}} + \cdots \right\}.$$

Since $\{\text{dist}\{x^k, X\}\}$ converges superlinearly to 0 as mentioned in Remark 3.3 (i), there exists k_3 such that

$$\text{dist}\{x^{k+1}, X\} \le \frac{\alpha}{2}\text{dist}\{x^k, X\} \qquad k \ge k_3.$$

Hence, we have for $k \ge \max\{k_1, k_3\}$

$$\begin{aligned} \|x^k - x^*\| &\le B_1 \left\{ \frac{\text{dist}\{x^k, X\}}{\alpha^k} + \frac{1}{2}\frac{\text{dist}\{x^k, X\}}{\alpha^k} + \frac{1}{2^2}\frac{\text{dist}\{x^k, X\}}{\alpha^k} + \cdots \right\} \\ &\le 2B_1 \frac{\text{dist}\{x^k, X\}}{\alpha^k}. \end{aligned} \tag{3.11}$$

Since $\|H_F(x)\|$ provides a local error bound by Assumption 3.2 (ii) and $x^k \to x^* \in X$, there exists a positive constant k_4 such that

$$\text{dist}\{x^k, X\} \le b_2 \|H_F(x^k)\| \qquad \forall k \ge k_4.$$

Let $k_2 = \max\{k_1, k_3, k_4\}$. Then, we have from (3.11)

$$\|x^k - x^*\| \le 2B_1 b_2 \frac{\|H_F(x^k)\|}{\alpha^k} \qquad \forall k \ge k_2,$$

and hence setting $B_2 = 2B_1 b_2$ yields (3.10). $\square$

Lemma 3.7 *The sequence $\{\|H_F(x^k)\|/\alpha^k\}$ converges to 0. Moreover, there exists k_5 such that*

$$\|x^k - x^*\| \le \rho^k(x^k), \ \ \|F(x^k) - F(x^*)\| \le \rho^k(x^k) \qquad \forall k \ge k_5.$$

Proof: By the Lipschitz continuity of H_F, we have $\|H_F(x^{k+1})\| \le \hat{L}\mathrm{dist}\{x^{k+1}, X\}$ for some $\hat{L} > 0$. Since $x^k \to x^* \in X$, Assumption 3.2 (ii) implies $\mathrm{dist}\{x^k, X\} \le b_2\|H_F(x^k)\|$ for all k sufficiently large. Let $\gamma < \frac{\alpha}{\hat{L}b_2}$. Then by (3.4) there exists k_6 such that for all $k > k_6$

$$\frac{\|H_F(x^{k+1})\|}{\alpha^{k+1}} \le \frac{\hat{L}\mathrm{dist}\{x^{k+1}, X\}}{\alpha^{k+1}} \le \frac{\gamma\hat{L}\mathrm{dist}\{x^k, X\}}{\alpha^{k+1}} \le \frac{\gamma\hat{L}b_2}{\alpha}\frac{\|H_F(x^k)\|}{\alpha^k}.$$

This shows that $\{\frac{\|H_F(x^k)\|}{\alpha^k}\}$ converges to 0.

Next we show the last part of the lemma. From Lemma 3.6 and the Lipschitz continuity of F, we have for $k \ge k_2$

$$\begin{aligned}
\max\{\|x^k - x^*\|, \|F(x^k) - F(x^*)\|\} &\le \max\{1, L\}\|x^k - x^*\| \\
&\le \max\{1, L\}B_2\frac{\|H_F(x^k)\|}{\alpha^k} \\
&= \max\{1, L\}B_2\sqrt{\frac{\|H_F(x^k)\|}{\alpha^k}}\rho^k(x^k).
\end{aligned}$$

From the first part of the lemma, there exist k_7 such that $\max\{1, L\}B_2\sqrt{\frac{\|H_F(x^k)\|}{\alpha^k}} \le 1$ for all $k \ge k_7$. Hence setting $k_5 := \max\{k_2, k_7\}$ yields the desired inequality. $\square$

Now we stand at the position to prove our main theorem.

Proof of Theorem 3.4. It suffices to show that there exists a positive integer $\bar{k}$ such that for all $k \ge \bar{k}$

$$x_i^* > 0 \iff x_i^k > \rho^k(x^k), \tag{3.12}$$

$$x_i^* = 0 \iff x_i^k \le \rho^k(x^k), \tag{3.13}$$

$$F_i(x^*) > 0 \iff F_i(x^k) > \rho^k(x^k), \tag{3.14}$$

$$F_i(x^*) = 0 \iff F_i(x^k) \le \rho^k(x^k). \tag{3.15}$$

Notice that $\{\|H_F(x^k)\|/\alpha^k\}$ converges to 0 by Lemma 3.7, and hence $\{\rho^k(x^k)\}$ converges to 0. It then follows from $x^k \to x^*$ and the continuity of F that there exists a positive integer k_8 such that

$$\begin{aligned} x_i^* > 0 &\quad\Rightarrow\quad x_i^k > \rho^k(x^k), \\ x_i^* = 0 &\quad\Leftarrow\quad x_i^k \le \rho^k(x^k), \\ F_i(x^*) > 0 &\quad\Rightarrow\quad F_i(x^k) > \rho^k(x^k), \\ F_i(x^*) = 0 &\quad\Leftarrow\quad F_i(x^k) \le \rho^k(x^k), \end{aligned}$$

for $k \ge k_8$. Next, we show that the converse of each relation also holds for $k \ge k_5$, where k_5 is the integer indicated in Lemma 3.7. First we show $x_i^* = 0 \Rightarrow x_i^k \le \rho^k(x^k)$ for $k \ge k_5$. Suppose that $x_i^* = 0$. Then we have from Lemma 3.7

$$x_i^k = x_i^k - x_i^* \le \|x^k - x^*\| \le \rho^k(x^k) \qquad k \ge k_5.$$

In a similar way, we can show $F_i(x^*) = 0 \Rightarrow F_i(x^k) \le \rho^k(x^k)$. Next we show $x_i^k > \rho^k(x^k) \Rightarrow x_i^* > 0$ for all $k \ge k_5$. Suppose the contrary, that is, there exists $k \ge k_5$ such that $x_i^k > \rho^k(x^k)$ and $x_i^* = 0$. This immediately leads to a contradiction, because $x_i^* = 0$ implies $x_i^k \le \rho^k(x^k)$ for $k \ge k_5$ as shown above. In a similar way, we can show $F_i(x^k) > \rho^k(x^k) \Rightarrow F_i(x^*) > 0$. Consequently we have (3.12)-(3.15) for all $k \ge \bar{k} := \max\{k_5, k_8\}$. □

4 Numerical results

In this section, we present some numerical results for the identification technique proposed in the previous section. We employed the Fischer-Burmeister function ϕ_{FB} to define the identification function sequence $\{\rho^k\}$ and solved the following test problems.

Problem 1 (LCP):

$$F(x) = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The solution $\bar{x}$ of the problem is $\left(0, \frac{1}{15}, \frac{4}{15}\right)^T$ and $F(\bar{x}) = \left(\frac{14}{15}, 0, 0\right)^T$. The solution is unique and nondegenerate.

Problem 2 (LCP):

$$F(x) = \begin{pmatrix} 1 & -4 & -4 & -4 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

The solution $\bar{x}$ of this problem is $(19, 4, 1, 0)^T$ and $F(\bar{x}) = (0, 0, 0, 0)^T$. The solution is unique and degenerate.

Problem 3 (LCP):

$$F(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

The solutions of this problem are given by $\left(a, \frac{4}{15}, \frac{1}{15}\right)^T$ for $a \geq 0$, and hence the solution is not locally unique. Moreover, since $F(x) = 0$ for all solutions, the particular solution $\left(0, \frac{4}{15}, \frac{1}{15}\right)^T$ is degenerate.

Problem 4 (NCP(F)):

$$F(x) = \begin{pmatrix} x_1 - 2 \\ x_2^3 + x_2 - x_3 + 3 \\ x_2 + 2x_3^3 + x_3 - 3 \end{pmatrix}$$

The solution $\bar{x}$ of this problem is $(2, 0, 1)^T$ and $F(\bar{x}) = (0, 2, 0)^T$. The solution is unique and nondegenerate.

Problem 5 (NCP(F):

$$F(x) = \begin{pmatrix} x_1 - 2 \\ x_2^3 + x_2 - x_3 + 1 \\ x_2 + 2x_3^3 + x_3 - 3 \end{pmatrix}$$

The solution $\bar{x}$ of this problem is $(2,0,1)^T$ and $F(\bar{x}) = (0,0,0)^T$. The solution is unique and degenerate.

Problem 6 (NCP(F):

$$F(x) = \begin{pmatrix} 0 \\ x_2^3 + x_2 - x_3 + 3 \\ x_2 + 2x_3^3 + x_3 - 3 \end{pmatrix}$$

The solutions of this problem are given by $(a,0,1)^T$ for $a \geq 0$, and hence the solution is not locally unique. Moreover, since $F(x) = (0,2,0)^T$ for all solutions, the particular solution $(0,0,1)^T$ is degenerate.

Problem 7 (NCP(F))[11]:

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_2 + x_2^3 - x_3 + 3 \\ x_2 + x_3 + 2x_3^3 - 3 \\ x_4 + 2x_4^3 \end{pmatrix}$$

The solution $\bar{x}$ of this problem is $(2,0,1,0)^T$ and $F(\bar{x}) = (0,2,0,0)^T$. The solution is unique and degenerate.

Problem 8 (NCP(F)) [4]:

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_2^2 + x_1 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}$$

The function F is not monotone and the problem has two solutions $\bar{x}^1 = \left(\frac{\sqrt{6}}{2}, 0, 0, \frac{1}{2}\right)^T$ and $\bar{x}^2 = (1,0,3,0)^T$. Moreover, since $F(\bar{x}^1) = \left(0, \frac{\sqrt{6}}{2} + 2, 0, 0\right)^T$, the solution $\bar{x}^1$ is degenerate.

Remark 4.1 *All problems except for Problem 8 are monotone. Problem 8 is not monotone, and hence the PPA may fail to find a solution. As to local error bound, it is easily verified that $\|H_F(x)\|$ provides a local error bound for Problems 1-6 and 8, while it does not for Problem 7. Therefore, Theorem 3.4 guarantees that the proposed technique can identify the index sets P^*, N^* and C^* for Problems 1-6.*

Table 1: Number of iterations for convergence versus index set identification

Problem	1			2			3			4		
Initial point	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$
$\hat{k}$	10	9	10	16	15	15	10	10	9	12	12	12
$\bar{k}$	7	6	5	11	9	9	7	6	5	3	2	3

Problem	5			6			7			8		
Initial point	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$	$x^{0,1}$	$x^{0,2}$	$x^{0,3}$
$\hat{k}$	12	12	12	8	7	8	11	11	8	11	13	F
$\bar{k}$	4	2	3	3	2	3	3	3	3	5	3	–

"$\hat{k}$" denotes the iteration number at which the termination criterion $\|H_F(x^k)\| \leq 10^{-7}$ was satisfied for the first time.

"$\bar{k}$" denotes the iteration number at which P^*, N^* and C^* were correctly identified for the first time.

"F" means that the PPA failed to find a solution.

We set the parameters in the PPA as $\alpha = 0.8$ and $\beta = 0.5$, and used three initial points $x^{0,1} = (10, \cdots, 10)^T, x^{0,2} = (1, \cdots, 1)^T, x^{0,3} = (0, \cdots, 0)^T$.

Table 1 shows two numbers for each problem. One is the iteration number at which the termination criterion $\|H_F(x^k)\| \leq 10^{-7}$ was satisfied for the first time. The other is the iteration number at which the index sets P^*, N^* and C^* were correctly identified for the first time. In Tables 2-6, we also present the detailed results for Problems 2, 3, 5, 6 and 8. Specifically, Tables 2-6 show how the index sets are identified as the iteration proceeds. That is, "P", "N" and "C" mean that index i belongs to P^k, N^k and C^k, respectively, at the kth iteration, and "$-$" means that index i belongs to neither P^k nor N^k nor C^k.

Table 1 shows that the index sets were successfully identified for Problems 1-6, as guaranteed by Theorem 3.4. Moreover, though Problems 7 and 8 do not satisfy the assumptions of Theorem 3.4, it is seen that the proposed technique also succeeded in identifying the index sets for appropriately chosen initial points.

Table 2: Problem 2 with $x^{0,1}$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Index 1	C	P	P	P	-	-	P	P	P	P	P	P	P	P	P	P	P
Index 2	C	C	P	P	P	C	C	C	C	P	P	P	P	P	P	P	P
Index 3	C	C	C	C	C	C	C	C	C	C	C	P	P	P	P	P	P
Index 4	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C

Table 3: Problem 3 with $x^{0,1}$

k	0	1	2	3	4	5	6	7	8	9	10
Index 1	P	P	P	P	P	P	P	P	P	P	P
Index 2	-	-	N	C	C	C	P	P	P	P	P
Index 3	-	-	N	C	C	C	C	P	P	P	P

The generated sequence converged to the solution $x^* = \left(10, \frac{4}{15}, \frac{1}{15}\right)^T$.

Table 4: Problem 5 with $x^{0,2}$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
Index 1	C	P	P	P	P	P	P	P	P	P	P	P	P
Index 2	N	C	C	C	C	C	C	C	C	C	C	C	C
Index 3	C	C	P	P	P	P	P	P	P	P	P	P	P

Table 5: Problem 6 with $x^{0,2}$

k	0	1	2	3	4	5	6	7
Index 1	C	C	P	P	P	P	P	P
Index 2	N	N	N	N	N	N	N	N
Index 3	C	C	P	P	P	P	P	P

The generated sequence converged to the solution $x^* = (1,0,1)^T$.

Table 6: Problem 8 with $x^{0,1}$. $x^* = \bar{x}^1$

k	0	1	2	3	4	5	6	7	8	9	10	11
Index 0	-	N	C	P	P	P	P	P	P	P	P	P
Index 1	-	N	N	N	N	N	N	N	N	N	N	N
Index 2	-	N	N	C	C	C	C	C	C	C	C	C
Index 3	-	-	C	C	C	P	P	P	P	P	P	P

5 Application

In this section, we give an application of the identification technique presented in Section 3. As mentioned in the Introduction, if we can identify the sets $P(x^*), N(x^*)$ and $C(x^*)$, we may obtain x^* by solving a system of nonlinear equations and inequalities. Here we will further reduce the system into a problem that involves no inequalities. More specifically, we show that, by using Theorem 3.4, a solution of the equality constrained minimization problem

$$\begin{array}{ll} \min & \frac{1}{2}\|x - x^k\|^2 \\ \text{subject to} & F_{P^k}(x) = 0,\ x_{N^k} = 0 \\ & F_{C^k}(x) = 0,\ x_{C^k} = 0 \end{array} \tag{5.1}$$

eventually becomes a solution of the original NCP(F). Note that, if F is affine, then the problem (5.1) is an equality constrained quadratic programming problem, which can be solved by way of the system of linear equations derived from its KKT system.

Theorem 5.1 *Let $\{x^k\}$ be generated by the PPA and $\{\rho^k\}$ be defined by (3.2). If the index sets P^k, N^k and C^k are determined by the identification function sequence $\{\rho^k\}$, then, for k sufficiently large, a solution of the minimization problem (5.1) is a solution of NCP(F).*

Proof. Let S denote the feasible region of (5.1). Since $x^k \to x^*$ and $P^k = P(x^*)$, $N^k = N(x^*)$, $C^k = C(x^*)$ for k sufficiently large by Theorem 3.4, we have $F_{P^k}(x^*) = 0, x^*_{N^k} = 0, F_{C^k}(x^*) = 0, x^*_{C^k} = 0$. Therefore $x^* \in S$, and hence $S \neq \emptyset$. Since the objective function of (5.1) is strongly convex and S is closed by the continuity of F, the minimization problem (5.1) has a solution for all k sufficiently large.

Let $\bar{x}$ be a solution of (5.1). Since $\bar{x} \in S$, $\bar{x}$ is a solution of NCP(F) if $\bar{x}_{P^k} > 0$ and $F_{N^k}(\bar{x}) > 0$. Since $x^*_{P^k} > 0, F_{N^k}(x^*) > 0$ for k sufficiently large and F is continuous, there exists $\epsilon > 0$ such that

$$\|x - x^*\| < \epsilon \implies x_{P^k} > 0, F_{N^k}(x) > 0.$$

Let k be large enough to satisfy $\|x^* - x^k\| < \frac{\epsilon}{2}$. Since $x^* \in S$ and $\bar{x}$ is a solution of (5.1), we have

$$\|\bar{x} - x^k\| \leq \|x^* - x^k\| < \frac{\epsilon}{2}.$$

Hence, we obtain

$$\|\bar{x} - x^*\| \leq \|\bar{x} - x^k\| + \|x^k - x^*\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which means that $\bar{x}_{P^k} > 0$ and $F_{N^k}(\bar{x}) > 0$. Consequently $\bar{x}$ is a solution of NCP(F). □

This theorem enables us to modify the PPA to construct a hybrid method with a finite termination property, which attempts to find a solution of NCP(F) by solving the minimization problem (5.1) with the index sets determined by the identification function sequence $\{\rho^k\}$. The prototype of the hybrid method is described as follows.

Hybrid algorithm

Step 1: Choose parameters $\alpha \in (0,1), \beta \in (0, \alpha)$ and an initial point $x^0 \in \Re^n$. Set $k := 0$.

Step 2: If $\|H_F(x^k)\|$ is sufficiently small, then go to Step 3. Otherwise go to Step 4.

Step 3: Solve the equality constrained minimization problem (5.1). If it is solvable and the solution satisfies $x_{P^k} \geq 0$ and $F_{N^k}(x) \geq 0$, then stop.

Step 4: Let $F^k : \Re^n \to \Re^n$ be defined by

$$F^k(x) = F(x) + \alpha^k(x - x^k).$$

Obtain an approximate solution x^{k+1} of NCP(F^k) which satisfies

$$\|H_{F^k}(x^{k+1})\| \leq \beta^k \min\{1, \|x^k - x^{k+1}\|\}.$$

Step 5: Set $k := k + 1$ and go to Step 2 .

The problem (5.1) solved in Step 3 is a nonlinear programming problem, which in general may not be solved easily. However, when we deal with LCP, the minimization problem (5.1) is reduced to a system of linear equations as mentioned earlier. Therefore, the hybrid algorithm can find a solution of monotone LCP in a finite number of arithmetic operations.

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