On the Maximization and Minimization of a Quasiconvex Function

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Abstract. This paper is devoted to the problems of minimizing and maximizing a quasiconvex function over an arbitrary set. First, we formulate global optimality conditions and then, based on them, propose algorithms for the case in which the objective function is convex and the feasible set is nonconvex and compact. The proposed algoritms are shown to be globally convergent. Some computational results are presented.

1 Introduction

In this paper we consider two problems of maximizing and minimizing a quasiconvex function over an arbitrary set $D \subset R^n$:

$$f(x) \longrightarrow \max, \quad x \in D, \tag{1.1}$$

$$f(x) \longrightarrow \min, \quad x \in D. \tag{1.2}$$

Both problems (1.1) and (1.2) have many applications in economics and engineering. Economic analysis sometimes requires a maximization and sometimes a minimization problem.

Quasi-convexity was pioneered by Kantorovich [37], De Finetti [12], Fenchel [19], Arrow and Enthoven [3]. Due to their importance in applications, both convex and concave functions, along with their generalizations to quasiconvex and quasiconcave functions, are discussed frequently in Economics. There are many papers [1, 14–16, 34, 42, 48, 53, 60] devoted to optimization problems in microeconomic theory, most of which are concerned with convex minimization or, equivalently, concave maximization problems. It occurs in consumer theory however, where under reasonable assumptions, consumer's preferences can be represented by a quasiconcave utility function [16]. The quasiconcave case in microeconomics was studied in [16]. There is the work [31] for the discrete case related to economics. Under some changes of economic environment of consumer's preference or markets, we face sometimes quasiconvex maximization or, equivalently, quasiconcave minimization problems. Also, the following special problem of the convex multiplicative programming problem [5, 21, 40] can be reduced to a quasiconcave minimization problem [5, 21]:

$$f(x) = \prod_{i=1}^{p} f_i \longrightarrow \min, \quad x \in D, \tag{1.3}$$

where $f_i : R^n \to R$ are finite, convex functions for $i = 1, 2, ..., p$ $(p \geq 2)$, D is a compact convex set in R^n, and $f_i(x) \geq 0$ for all $x \in D$ and $j = 1, 2, ..., p$. The so-called D.C. programming problem, where D.C. is an abbreviation for the difference of two convex functions [30], is another example of a quasiconcave minimization problem:

$$\begin{aligned} & g(x) = g_1(x) - g_2(x) \longrightarrow \min \\ & \text{subject to } \; h_i(x) \leq 0, \; i = 1, 2, ..., m, \end{aligned} \tag{1.4}$$

where g_1, g_2 and h_i $(i = 1, 2, ..., m)$ are all finite convex functions on R^n. The D.C. programming problem can be also formulated as the quasiconvex minimization problem with a nonconvex feasible set [30].

There is a relationship between problems (1.1) and (1.2) via their generalized duality under certain assumptions on f and D ([49]). For example, when f is quasiconvex satisfying $f(0) = \inf\limits_{R^n \setminus \{0\}} f$ and D is a compact convex set containing 0 then

$$\sup_{x \in D} f(x) = -\inf_{v \notin D^o} f^H(v)$$

holds, where

$$f^H(v) = \begin{cases} -\inf\limits_{\langle v, x \rangle \leq 1} f(x) & \text{if } v \neq 0 \\ -\sup\limits_{x \in R^n} f(x) & \text{if } v = 0 \end{cases}$$

is quasinconvex and D^o is the polar of D defined as $D^o = \{v \mid \langle v, x \rangle \leq 1 \text{ for all } x \in D\}$ in ([49]).

Although nonconvex optimization problems play an important role in applications, it has been noted [35] that non-convexity creates great mathematical difficulties. There are a number of results [3, 7–16, 19, 35, 52] on mathematical properties of quasiconvex functions, and quasiconvex (resp., quasiconcave) minimization (resp., maximization) problems. Convex maximization (equivalently, concave minimization) problem as a special case of problem (1.1), where f is convex and D is a polytope, is originated from Tuy [58]. There are many iterative algorithms [6, 25–30, 46, 47, 58, 59] for solving the convex maximization problem by resorting to the cutting plane and/ or branch and bound techniques.

The global optimality condition for the problem of maximizing a convex function over an arbitrary set was first obtained by Strekalovsky [56, 57]. An attempt to construct an algorithm based on these conditions was done in [18]. There is another type of global optimality condition for the problem of maximizing a convex function over a convex set in [23] based on the so-called $\epsilon-$ subdifferentials. On the other hand, the problem of minimizing a quasiconvex function over a convex set is originated from Kantorovich [37], where an global optimality condition was obtained.

From a mathematical perspective, it is well-known that standard nonlinear programming techniques have not been successful for solving nonconvex optimization problems. Moreover, the general Lagrangean method provides only a stationary point without guarantee of local optimality. In addition to these deficiencies, classical methods do not yield conditions for global optimality. For these reasons, global optimization appears to require new methods different from the standard nonlinear programming techniques. We will focus in this paper mainly on the quasiconvex maximization and quasiconvex minization problems with a nonconvex feasible set. The paper is organised as follows: In sections 2 and 3, we consider problems (1.1) and (1.2), and derive new global optimality conditions with applications in certain models of economics. Then we construct some algorithms for the convex case based on those conditions. We show also some connections between the derived conditions and the classical local conditions. We show also that the fractional programming problems such as

$$\frac{f_1(x)}{f_2(x)} \longrightarrow \max, \quad x \in D$$

and

$$\frac{f_1(x)}{f_2(x)} \longrightarrow \min, \quad x \in D,$$

, under certain assumptions on $f_1,\ f_2$ and D, can be reduced to problems (1.1) and (1.2), respectively. Some computational results for problem (1.1) are provided.

2 QUASICONVEX MAXIMIZATION PROBLEM

2.1 Definition and Properties of a Quasiconvex Function

We start this section by recalling the definition of a quasiconvex function and some of its well-known properties [3,11,13] which we use in the subsequent sections.

Definition 2.1 A function $f:\ R^n \to R$ is said to be quasiconvex if

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in R^n$ and $\alpha \in [0,1]$.

Definition 2.2 A function $f:\ R^n \to R$ is said strictly quasiconvex if the inequality

$$f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\}$$

holds for all $x, y \in R^n,\ x \neq y$ and $\alpha \in (0,1)$.

It is clear that any convex function is a quasiconvex, but the converse is not necessarily true. If f is quasiconvex then $-f$ is called quasiconcave.

Definition 2.3 A differentiable function $f:\ R^n \to R$ is said to be pseudoconvex if the inequality $\langle f'(x), y-x\rangle \geq 0$ for $x, y \in R^n$ implies that $f(x) \leq f(y)$.

Note again that any convex function is pseudoconvex, but the converse does not follow. It is also true that a pseudoconvex function is quasiconvex but not conversely.

Lemma 2.1 A function $f:\ R^n \to R$ is quasiconvex if and only if the set

$$L_c(f) = \{x \in R^n \mid f(x) \leq c\}$$

is convex for all $c \in R$.
Proof. Necessity. Suppose that $c \in R$ is an arbitrary number and $x, y \in L_c(f)$. By the definition of quasiconvexity, we have

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} \leq c \quad \text{for all } \alpha \in [0,1],$$

which means that the set $L_c(f)$ is convex.
Sufficiency. Let $L_c(f)$ be a convex set for all $c \in R$. For arbitrary $x, y \in R^n$, define $c^o = \max\{f(x), f(y)\}$. Then $x \in L_{c^o}(f)$ and $y \in L_{c^o}(f)$. Consequently, $\alpha x + (1 - \alpha y) \in L_{c^o}(f)$, for any $\alpha \in [0,1]$. This completes the proof. ■

Lemma 2.2 Let $f: R^n \to R$ be a quasiconvex and differentiable function. Then the inequality $f(x) \leq f(y)$ for $x, y \in R^n$ implies that

$$\langle f'(y), x - y \rangle \leq 0,$$

where f' denotes the gradient and $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors.
Proof. Since f is quasiconvex,

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} = f(y)$$

for all $\alpha \in [0,1]$ and $x, y \in R^n$ such that $f(x) \leq f(y)$. By Taylor's formula, there is a neighborhood of the point y on which:

$$f(y + \alpha(x-y)) - f(y) = \alpha \left(\langle f'(y), x - y \rangle + \frac{o(\alpha\|x-y\|)}{\alpha} \right) \leq 0, \quad \alpha > 0.$$

From the fact that $\frac{o(\alpha\|x-y\|)}{\alpha} \xrightarrow{\alpha \to o} 0$, we obtain $\langle f'(y), x - y \rangle \leq 0$ which completes the proof. ■

2.2 Problem Statement and Global Maximality Condition

Consider the quasiconvex maximization problem,

$$f(x) \longrightarrow \max, \quad x \in D, \tag{2.1}$$

where $f: R^n \to R$ is a quasiconvex and differentiable function and $D \subset R^n$ is a nonempty arbitrary subset of R^n. Then an optimality condition for this problem is given by the following theorem:

Theorem 2.1 Let z be a solution of problem (2.1), and let

$$E_c(f) = \{y \in R^n \mid f(y) = c\}.$$

Then

$$\langle f'(y), x - y \rangle \leq 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D. \tag{2.2}$$

If, in addition, $f'(y) \neq 0$ holds for all $y \in E_{f(z)}(f)$, then condition (2.2) is sufficient for $z \in D$ being a solution to problem (2.1).

Proof. Necessity. Assume that z is a solution of problem (2.1) and let $y \in E_{f(z)}(f)$ and $x \in D$. Then we have $f(x) \leq f(y)$. Applying lemma 2.2, we obtain $\langle f'(y), x - y \rangle \leq 0$.

Sufficiency. Suppose, on the contrary, that z is not a solution to problem (2.1); i.e., there exists an $u \in D$ such that $f(u) > f(z)$. The closed set $L_{f(z)}(f) = \{x \in R^n \mid f(x) \le f(z)\}$ is convex by Lemma 2.1. Let y be the projection of u on $L_{f(z)}(f)$ such that

$$\|y-u\| = \min_{x \in L_{f(z)}(f)} \|x-u\|.$$

Clearly,

$$\|y-u\| > 0 \tag{2.3}$$

holds because $u \notin L_{f(z)}(f)$. Moreover, this y can be considered as a solution of the following convex minimization problem:

$$g(x) = \frac{1}{2}\|x-u\|^2 \longrightarrow \min, \quad x \in L_{f(z)}(f).$$

Applying the Lagrange method to this latter problem, we obtain the following optimality condition at the point y:

$$\begin{cases} \lambda_o \ge 0, \ \lambda \ge 0, \ \lambda_o + \lambda > 0 \\ \lambda_o g'(y) + \lambda f'(y) = 0 \\ \lambda(f(y) - f(z)) = 0 \end{cases} \tag{2.4}$$

If $\lambda_o = 0$, then (2.4) implies that $\lambda > 0$, $f(y) = f(z)$ and $f'(y) = 0$ which contradicts the assumption in the theorem. If $\lambda = 0$, then we have $\lambda_o > 0$ and $g'(y) = y - u = 0$, which also contradicts (2.3). So, without loss of generality, we can set $\lambda_o = 1$ and $\lambda > 0$ in (2.4) to obtain

$$\begin{aligned} &y - u + \lambda f'(y) = 0, \quad \lambda > 0, \\ &f(y) = f(z). \end{aligned}$$

From this we conclude that $\langle f'(y), u - y\rangle > 0$, which contradicts (2.2). This last contradiction implies that the assumption that z is not a solution of problem (2.1) must be false. This completes the proof. ■

REMARK **1**. If $f(x)$ is convex in problem (1.1), then Strekalovsky [56] proved in 1987 the following theorem.

Theorem 2.2 ([56]). Suppose that D is an arbitrary set. Let a point $z \in D$ satisfy

$$-\infty < \inf_{R^n} f < f(z) < +\infty$$

and $L_{f(z)}(f) = \{x \in R^n \mid f(x) \le f(z)\} \subset intdom\ f$ is compact. Then z is a global maximizer of problem (1.1) if and only if

$$\begin{aligned} &\langle c, x - z\rangle \le 0 \ \text{ for all } x \in \overline{conv}\ D \text{ and } c \in \partial f(z), \\ &\langle v, x - z\rangle \le 1 \ \text{ for all } x \in \overline{conv}\ D \text{ and } v \in S(f,z), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} S(f,z) &= \left\{ v \in R^n \;\middle|\; \begin{array}{l} \exists y \in R^n: \ y \ne z, \ y \in E_{f(z)}(f), \\ \exists \alpha > 0: \ \alpha v \in \partial f(y), \ \langle v, y - z\rangle = 1. \end{array} \right\}, \\ \partial f(z) &= \{c \in R^n \mid f(x) - f(z) \ge \langle c, x - z\rangle, \ \ x \in R^n\}, \\ intdom\ f &= \{x \in R^n \mid f(x) < +\infty\}, \end{aligned}$$

and $\overline{conv}\ D$ is the closed convex hull of D. If f is differentiable as well, then it can be easily shown that condition (2.2) for problem (2.1) is the same as condition (2.5). Thus Theorem 2.1 generalizes Strekalovsky's result to the quasiconvex differentiable case.

Note also that Theorem 2.1 can be easily applied to the pseudoconvex case for problem (2.1).

REMARK **2**. There is another related result obtained by Hiriart-Urruty and Ledyaev in 1996 [24].

Theorem 2.3 ([24]). Suppose f is convex and D is closed convex in problem (1.1). Let a point $z \in D$ satisfy $-\infty \leq \inf_D f < f(z)$. Then z is a global maximizer of (1.1) if and only if

$$\partial f(x) \subset N(x|D) \text{ for all } x \in D \text{ and } x \in E_{f(z)}(f),$$

where $N(x|D)$ is the normal cone to D at x defined as:

$$N(x|D) = \{c \in R^n \mid \langle c, y - x\rangle \leq 0, \quad y \in D\}.$$

REMARK **3**. If D is convex, we obtain from (2.2) with $y = z$ the well-known local optimality condition [51] :

$$\langle f'(z), x - z\rangle \leq 0, \quad \forall x \in D.$$

Thus it is clear that the global optimality condition (2.2) is connected to the classical extremum theory. Note, however, that condition (2.2) does not require the convexity of D at all.

REMARK **4**. Let us consider the variational inequality problem [55] of finding an $x \in D$ such that a vector valued function $g:\ R^n \to R$ becomes inward normal to the given closed convex subset $D \subset R^n$ at x, that is,

$$\langle g(x), y - x\rangle \geq 0, \quad \forall y \in D.$$

This problem can be solved, in some cases, by a componentwise projection method [55]. If D is closed and convex in problem (2.1), then its solution is given by a point $z \in D$ such that

$$\langle g(y), x - y\rangle \geq 0$$

holds for all $y \in E_{f(z)}(f) \cap D$ and $x \in D$ with $g(y) = -f'(y)$. This fact suggests some relationship between the quasiconvex maximization and the variational inequality problems. Detailed investigation in this direction is beyond the scope of this paper.

REMARK **5**. In order to conclude that a point z' in D is not a solution to problem (2.1), Theorem 2.1 tells that we needed to find a pair $x, y \in R^n$ such that

$$\langle f'(y), x - y\rangle > 0, \quad f(y) = f(z'), \quad x \in D.$$

The following examples illustrate the use of this property.

Example 2.1

$$f(x) = \frac{x_1^2 + x_2^2}{x_1 + x_2 - 1}\ ,$$
$$D = \{x \in R^2 \mid 0.6 \leq x_1 \leq 7;\ 0.6 \leq x_2 \leq 2\}.$$

We can easily evaluate the gradient of f as:

$$f'(x) = \left(\frac{x_1^2 + 2x_1x_2 - 2x_1 - x_2^2}{(x_1 + x_2 - 1)^2} \ , \ \frac{x_2^2 + 2x_1x_2 - 2x_2 - x_1^2}{(x_1 + x_2 - 1)^2} \right).$$

Also, by using lemma 2.1, we can show that f is quasiconvex on D. Now we want to check whether a point $x^o = (0.6, 0.6) \in D$, which is clearly locally maximal, is globally optimal or not. Then consider a pair $u = (5, 2) \in D$ and $y = (3, 3)$ satisfying $f(y) = f(x^o) = 3.6$. We have $\langle f'(y), u - y \rangle = \frac{12}{25} > 0$ and it follows that x^o is not a global solution. In fact, we can show that the global solution is $x^* = (7, 0.6)$.

Example 2.2 The model of utility maximization [16]. Consider the utility maximization problem in an oligopsonistic market, where the consumer influences the market price through the amount that he purchases. That is, the price $p_i = p_i(x_i)$ of good i is a function of the amount x_i purchased. If markets of all goods are oligopsonistic, then the utility maximization problem is:

$$u(x) \longrightarrow \max, \quad x \in D,$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector of quantites of n goods to purchase, $u(x)$ a utility function representing the preferences of the individual, and

$$D = \{x \in R^n \mid \sum_{i=1}^{n} p_i(x_i)x_i \leq m, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n\},$$

for a given amount of income m. The consumer chooses the vector x that maximizes his utility function over the budget constrained set D. Assuming that the utility function u is quasiconvex and differentiable on D, we obtain from the global optimality condition (2.2) that

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial u(y)}{\partial x_i}(x_i - y_i) \leq 0 \\ u(y_1, y_2, \ldots, y_n) = u(z) \\ \sum_{i=1}^{n} p_i(x_i)x_i \leq m \\ x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

If in this problem, we have $u(x) = \prod_{i=1}^{p} f_i$, where $f_i : R^n \to R$ is a finite, convex function for each $i = 1, 2, ..., p$ ($p \geq 2$), and $f_i(x) \geq 0$ for all $x \in D$ and $i = 1, 2, ..., p$, then utility maximization problem is represented as the multiplicative maximization problem. In general, the preference ordering theory [15] tells that utility functions exist but might be even discontinuous. The quasiconvex case better fits to the case of substitute goods [60].

Example 2.3 Unlike the D.C.programming problem (1.4), consider the problem of maximizing the difference of two convex functions on a convex subset. (We shall call this as D.C.maximization problem.)

$$\begin{aligned} &g(x) = g_1(x) - g_2(x) \longrightarrow \ \max \\ &\text{subject to } \ h_i(x) \leq 0, \ i = 1, 2, ..., m, \end{aligned}$$

where g_1, g_2 and $h_i,\ i = 1, 2, ..., m$ are convex differentiable functions on R^n. This is equivalent to its counterpart D.C. minimization problem:

$$\begin{aligned}
&-g(x) = g_2(x) - g_1(x) \longrightarrow \ \max \\
&\text{subject to } \ h_i(x) \le 0,\ i = 1, 2, ..., m,
\end{aligned}$$

It has been shown that this problem can be transformed into the equivalent quasiconvex maximization problem [30] as follows:

$$\begin{aligned}
&g_1(x) - x_{n+1} \longrightarrow \ \max \\
&\text{subject to } \ g_2(x) \le x_{n+1},\ h_i(x) \le 0,\ i = 1, 2, ..., m,
\end{aligned}$$

where $(x, x_{n+1}) \in R^n \times R$. Now we can apply condition (2.2) to this problem to obtain

$$\begin{cases}
\sum\limits_{i=1}^{n} \frac{\partial g_1(y)}{\partial x_i}(x_i - y_i) + y_{n+1} - x_{n+1} \le 0 \\
g_1(y) - y_{n+1} = g_1(z) \\
g_2(x) \le x_{n+1} \\
h_i(x) \le 0, \quad i = 1, 2, \ldots, m.
\end{cases}$$

Similarly, we can also have the global optimality condition for the convex multiplicative problem (1.3) for its equivalent quaisiconvex maximization problem in the outcome space which is as follows ([21]):

$$g(x) = -\prod_{i=1}^{p} x_i \ \longrightarrow \ \max, \quad x \in Y,$$

where the convex and compact set Y is constructed for a point in the feasible set D of problem (1.3). Then condition (2.2) for this problem becomes as follows.

$$\sum_{i=1}^{n} \prod_{j \ne i}^{p} y_j(x_i - y_i) \ge 0 \ \text{ for all } y \in E_{g(z)}(g) \text{ and } x \in Y.$$

Example 2.4 Consider a fractional programming problem

$$f(x) = \frac{f_1(x)}{f_2(x)} \ \longrightarrow \ \max, \quad x \in D,$$

where f_1 is convex and differentiable on R^n, and f_2 is concave and differentiable on R^n. Suppose that f_1 and f_2 are defined positively on a ball B containing a subset $D \subset R^n$, that is,

$$f_1(x) > 0, \quad f_2(x) > 0 \text{ for all } x \in D \subset B.$$

We shall call this problem as the mixed fractional maximization problem. By lemma 2.1, we can easily show that $f(x)$ is quasiconvex. Hence, we can apply condition (2.2) at a global solution z for the above problem as follows:

$$\sum_{i=1}^{n} \left(\frac{\partial f_1(y)}{\partial x_i} f_2(y) - \frac{\partial f_2(y)}{\partial x_i} f_1(y) \right) \frac{(x_i - y_i)}{f_2^2(y)} \le 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D.$$

Example 2.5 Money Capital and the Firm [16].
For given output price p and input price vector r, denote the average money capital requirement function by $f_2 = f_2(x, y)$, where x is a quantity of output, and $y = (y_1, y_2, \ldots, y_n)$ is a vector of input quantities. This function indicates the average money capital that the firm needs to support the production of output x with inputs y at prices p and r. The firm's rate of return earned on the money capital invested in it is given by:

$$f(x, y) = \frac{f_1(x, y)}{f_2(x, y)},$$

where $f_1(x, y) = px - \langle r, y\rangle$ is regarded as profit earned by the firm and $f_1(x, y) > 0,\ f_2(x, y) > 0$. In this context, if f_2 is concave then the maximization problem (actually a rate-of-return maximization problem) is formulated as the mixed fractional maximization problem:

$$f(x, y) \longrightarrow \max$$

subject to

$$0 \le x \le \bar{x},\ 0 \le y_i \le \bar{y}_i, \quad i = 1, 2, ..., n,$$

where the maximum output $\bar{x}$ and the maximum input $\bar{y}_i$ are given. Optimality condition (2.2) for this problem is the same as in the previous example.

2.3 Preliminary Results for Maximization

First of all, we introduce the following definition.

Definition 2.4 A set D in R^n is called a simple set if the following conditions hold:

a) D is compact.

b) The problem of maximizing a linear function over D is solvable by a "simple" method.

We say that condition b) holds, for example, if it can be solved as a linear programming problem (i.e., if D is a convex polyhedron) or if an analytical form of the solution is explicitly given.

Consider the convex maximization problem as a particular case of problem (2.1),

$$f(x) \longrightarrow \max, \quad x \in D, \tag{2.6}$$

where $f: R^n \to R$ is convex and $D \subset R^n$ is a simple set. In addition, in problem (2.6) and throughout the rest of section 2, we assume that f is strongly convex and twice continuously differentiable. Then the global optimality condition of Theorem 2.1 can be stated without the assumption

$$f'(y) \neq 0 \text{ for all } y \in E_{f(z)}(f).$$

Theorem 2.4 A point $z \in D$ with $f'(z) \neq 0$ is a solution of problem (2.6) if and only if the following condition hold:

$$\langle f'(y), x - y\rangle \le 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D,$$

where and $E_c(f) = \{y \in R^n \mid f(y) = c\}$.
The proof is completely analogous to the proof of theorem 2.1. ■

Define the auxiliary function $\pi(y)$ as follows:

$$\pi(y) = \max_{x \in D} \langle f'(y), x - y \rangle, \quad y \in R^n. \tag{2.7}$$

For subsequent discussion, it is useful to derive some properties of $\pi(y)$.

Lemma 2.3 The function $\pi(y)$ is continuous on R^n.
Proof. Note that $\pi(y)$ can be written in the form:

$$\pi(y) = \varphi(y) - \langle f'(y), y \rangle, \tag{2.8}$$

where $\varphi(y) = \max_{x \in D} \langle f'(y), x \rangle$. It is clear that the function $\langle f'(y), y \rangle$ is continuous on R^n under the above assumptions on f. Therefore, we shall only show the continuity of φ. Since D is compact, for any $y \in R^n$ there is a point $x(y) \in D$ such that

$$\varphi(y) = \max_{x \in D} \langle f'(y), x \rangle = \langle f'(y), x(y) \rangle. \tag{2.9}$$

Hence, function $\varphi(y)$ is defined on R^n. Let $\{y^k,\ k = 1, 2, ...\}$ is a convergent sequence in R^n, and let

$$\lim_{k \to \infty} y^k = u.$$

Clearly, $\{x(y^k)\} \subset D$. Since D is compact, we can always select the convergent subsequencec $\{x(y^{k_l})\}$ from $\{x(y^k)\}$ with a limit $v \in D$, i.e.,

$$\lim_{l \to \infty} x(y^{k_l}) = v.$$

Therefore, we may without loss of generality suppose that

$$\lim_{k \to \infty} x(y^k) = v.$$

Write down the following obvious inequalities

$$\varphi(u) = \langle f'(u), x(u) \rangle \geq \langle f'(u), x(y^k) \rangle,$$

$$\varphi(y^k) = \langle f'(y^k), x(y^k) \rangle \geq \langle f'(y^k), x(u) \rangle.$$

Thus, we have

$$\langle f'(y^k), x(u) \rangle - \langle f'(u), x(u) \rangle \leq \varphi(y^k) - \varphi(u),$$

$$\varphi(y^k) - \varphi(u) \leq \langle f'(y^k), x(y^k) \rangle - \langle f'(u), x(y^k) \rangle.$$

Now, letting k tend to infinity in these inequalities yeilds

$$\lim_{k \to \infty} \varphi(y^k) = \varphi(u).$$

Taking into account that $\{y^k\}$ is an arbitrary convergent sequence, we conclude that $\varphi(y)$ is continuous on R^n, proving the lemma. ■

Denote by $D(y)$ the set:

$$D(y) = \{x \in D : \langle f'(y), x \rangle = \varphi(y)\}, \quad y \in R^n.$$

Let $y, h \in R^n$ be arbitrary vectors and $\alpha \in R$ $(\alpha \geq 0)$ be a parameter. For fixed y and h, we define the function $\mu : \ R \rightarrow R$ as follows:

$$\mu(\alpha) = \max_{x \in D(y+\alpha h)} \langle f''(y)h, x \rangle, \tag{2.10}$$

where $f''(y)$ is the matrix of second order derivatives of the function f.
Lemma 2.4 Let $\mu(\alpha)$ be defined as above for given $y, h \in R^n$. Let $\{\alpha_k\}$ be a sequence such that $\alpha_k \geq 0$ and $\alpha_k \rightarrow +0$ as $k \rightarrow \infty$. Then

$$\lim_{k\rightarrow\infty} sup\ \mu(\alpha_k) \leq \mu(0).$$

Proof. Let a point $x^k \in D$ provide the maximum in (2.10) at $\alpha = \alpha_k$, that is,

$$\mu(\alpha_k) = \langle f''(y)h, x^k \rangle.$$

Then the sequence $\{\mu(\alpha_k)\}$ is bounded from above, for example, by $\max\limits_{x\in D}\langle f''(y)h, x \rangle$, and hence has an upper limit μ^*:

$$\lim_{k\rightarrow\infty} sup\ \mu(\alpha_k) = \mu^*.$$

We can always select a subsequence $\{\alpha_{k_l}\}, (l = 1, 2, ...)$ such that

$$\lim_{l\rightarrow\infty} \mu(\alpha_{k_l}) = \mu^*.$$

Consider now the corresponding sequence $\{x^{k_l}\}, (l = 1, 2, ...)$. Since $x^{k_l} \in D$ for all l and D is compact, we may without loss of generality suppose that x^{k_l} converges to a pont x^o:

$$\lim_{l\rightarrow\infty} x^{k_l} = x^o.$$

Clearly, this x^o satisfies $x^o \in D$. By the definition of the set $D(y)$ and the fact that $x^{k_l} \in D(y + \alpha_{k_l} h)$, we have

$$\langle f'(y + \alpha_{k_l} h), x^{k_l} \rangle = \varphi(y + \alpha_{k_l} h).$$

Hence, taking the limit in these equalities for $l \rightarrow \infty$, we obtain

$$\langle f'(y), x^o \rangle = \varphi(y).$$

Thus, we conclude that $x^o \in D(y)$. Therefore,

$$\lim_{k\rightarrow\infty} sup\ \mu(\alpha_k) = \mu^* = \lim_{l\rightarrow\infty} \mu(\alpha_{k_l}) = \lim_{l\rightarrow\infty} \langle f''(y)h, x^{k_l} \rangle =$$

$$\langle f''(y)h, x^o \rangle \leq \max_{x\in D(y)} \langle f''(y)h, x \rangle = \mu(0).$$

Consequently, we have

$$\lim_{k\rightarrow\infty} sup\ \mu(\alpha_k) \leq \mu(0).$$

■

We shall now investigate the derivative of function $\pi(y)$.

Theorem 2.5 The directional derivative of $\pi(y)$ at any point $y \in R^n$ in any direction $h \in R^n$ of norm 1 exists, and is given by

$$\frac{\partial \pi(y)}{\partial h} = \max_{x \in D(y)} \langle f''(y)h, x\rangle - \langle f''(y)y + f'(y), h\rangle.$$

Proof. We write $\pi(y)$ of (2.8) in the form:

$$\pi(y) = \varphi(y) - \phi(y),$$

where $\varphi(y)$ was defined in (2.9) and $\phi(y) = \langle f'(y), y\rangle$. Since f is twice continuously differentiable, we can easily show that the function $\phi(y)$ is continuously differentiable on R^n, and for any vector h of norm 1, we have

$$\frac{\partial \phi(y)}{\partial h} = \langle f''(y)y + f'(y), h\rangle.$$

Thus we need only to show that $\varphi(y)$ has the directional derivative at a point $y \in R^n$ in a direction $h \in R^n$. After fixing y and h so that $\|h\| = 1$, consider the point y^α, for $\alpha > 0$, defined by

$$y^\alpha = y + \alpha h.$$

Now write down Taylor's formula for the function $f'(y)$ at y:

$$f'(y^\alpha) - f'(y) = \alpha f''(y)h + o(\alpha), \tag{2.11}$$

where $\|o(\alpha)\| \to 0$ as $\alpha \to +0$. Recall that $x(y^\alpha) \in D(y^\alpha)$ and $x(y) \in D(y)$. Then we can derive the following inequalities:

$$\langle f'(y^\alpha), x(y^\alpha)\rangle = \varphi(y^\alpha) \geq \langle f'(y^\alpha), x(y)\rangle,$$

$$\langle f'(y), x(y^\alpha)\rangle \leq \varphi(y) = \langle f'(y), x(y)\rangle.$$

Hence, we have

$$\langle f'(y^\alpha) - f'(y), x(y)\rangle \leq \varphi(y^\alpha) - \varphi(y) \leq \langle f'(y^\alpha) - f'(y), x(y^\alpha)\rangle. \tag{2.12}$$

Using notations,

$$p^1(\alpha) = \langle o(\alpha), x(y)\rangle,\ p^2(\alpha) = \langle o(\alpha), x(y^\alpha)\rangle,$$

and the compactness of D, we get

$$\lim_{\alpha \to +0} \frac{p^1(\alpha)}{\alpha} = 0, \qquad \lim_{\alpha \to +0} \frac{p^2(\alpha)}{\alpha} = 0.$$

Now taking into account (2.11) and (2.12), we obtain

$$\langle f''(y)h, x(y)\rangle + p^1(\alpha) \leq \frac{\varphi(y^\alpha) - \varphi(y)}{\alpha} \leq \langle f''(y)h, x(y^\alpha)\rangle + p^2(\alpha).$$

Since this holds for all points $z \in D(y)$ and $z^\alpha \in D(y^\alpha)$, we have

$$\langle f''(y)h, z\rangle + q^1(\alpha) \leq \frac{\varphi(y^\alpha) - \varphi(y)}{\alpha} \leq \langle f''(y)h, z^\alpha\rangle + q^2(\alpha), \tag{2.13}$$

where

$$\langle f''(y)h, z\rangle = \max_{x \in D(y)} \langle f''(y)h, x\rangle, \quad \langle f''(y)h, z^\alpha\rangle = \max_{x \in D(y^\alpha)} \langle f''(y)h, x\rangle,$$

and $q^1(\alpha) = \langle o(\alpha), z\rangle, \quad q^2(\alpha) = \langle o(\alpha), z^\alpha\rangle$. It is clear that

$$\lim_{\alpha \to +0} \frac{q^1(\alpha)}{\alpha} = 0 \text{ and } \lim_{\alpha \to +0} \frac{q^2(\alpha)}{\alpha} = 0. \tag{2.14}$$

Also, using the definition of $\mu(\alpha)$ given in (2.10), we can rewrite (2.13) into

$$\mu(0) + q^1(\alpha) \leq \frac{\varphi(y^\alpha) - \varphi(y)}{\alpha} \leq \mu(\alpha) + q^2(\alpha). \tag{2.15}$$

Let $\{\alpha_k\}$ be an arbitrary sequence such that $\alpha_k \geq 0$ and $\alpha_k \to +0$ as $k \to \infty$. Then from (2.14) and (2.15) it follows that

$$\mu(0) \leq \lim_{k \to \infty} inf\ \mu(\alpha_k).$$

On the other hand, by lemma 2.4, we have

$$\lim_{k \to \infty} sup\ \mu(\alpha_k) \leq \mu(0).$$

Hence,

$$\mu(0) \leq \lim_{k \to \infty} inf\ \mu(\alpha_k) \leq \lim_{k \to \infty} sup\ \mu(\alpha_k) \leq \mu(0),$$

that is,

$$\lim_{k \to \infty} \mu(\alpha_k) = \mu(0).$$

Now substituting $\alpha = \alpha_k$ in (2.15), we see that

$$\frac{\partial \varphi(y)}{\partial h} = \lim_{k \to \infty} \frac{\varphi(y + \alpha_k h) - \varphi(y)}{\alpha_k} = \mu(0) = \max_{x \in D(y)} \langle f''(y)h, x\rangle.$$

Consequently, combining the above derivatives of functions $\phi(y)$ and $\varphi(y)$ in direction h, we obtain

$$\frac{\partial \pi(y)}{\partial h} = \frac{\partial \varphi(y)}{\partial h} - \frac{\partial \phi(y)}{\partial h},$$

or equivalently

$$\frac{\partial \pi(y)}{\partial h} = \max_{x \in D(y)} \langle f''(y)h, x\rangle - \langle f''(y)y + f'(y), h\rangle.$$

■

Corollary. If the set D is strictly convex, then the function $\pi(y)$ becomes differentiable and the gradient is given by

$$\pi'(y) = f''(y)(x(y) - y) - f'(y),$$

where

$$\langle f'(y), x(y)\rangle = \max_{x\in D}\langle f'(y), x\rangle.$$

Lemma 2.5 If there is a point $y \in R^n$ such that $\pi(y) > 0$ and $f(y) = f(z)$ for a feasible point $z \in D$, then

$$f(x(y)) > f(z)$$

holds, where $x(y) \in D$ satisfies $\langle f'(y), x(y)\rangle = \max_{x\in D}\langle f'(y), x\rangle$.
Proof. By the definition of $\pi(y)$, we have

$$\pi(y) = \max_{x\in D}\langle f'(y), x - y\rangle = \langle f'(y), x(y) - y\rangle.$$

Since f is strongly convex by the assumption for (2.6), we have

$$f(u) - f(v) \geq \langle f'(v), u - v\rangle$$

for all $u, v \in R^n$ [38, 61]. Therefore, the assumption in the lemma implies that

$$f(x(y)) - f(z) = f(x(y)) - f(y) \geq \langle f'(y), x(y) - y\rangle = \pi(y) > 0.$$

■

Theorem 2.6 Suppose that $f: R^n \to R$ be strongly convex and differentiable. Let a sequence $\{x^k,\ k = 0, 1, ...\} \subset R^n$ be such that

$$f(x^k) > f(x^{k-1}) > ... > f(x^0) \quad \text{and} \quad x^0 \neq arg\min_{x\in R^n} f(x).$$

Then there exists a positive constant $\delta > 0$ satisfying

$$\|f'(x^k)\| \geq \delta, \quad k = 0, 1, ...$$

Proof. By the definition of strong convexity of f, there exists a positive constant γ such that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \alpha(1-\alpha)\gamma\|x - y\|^2 \tag{2.16}$$

holds for all $x, y \in R^n$ and $\alpha \in [0, 1]$. Denote by x_* the unique global minimizer of $f(x)$ over R^n which also exists due to the strong convexity of f [38]:

$$f(x_*) = \min_{x\in R^n} f(x).$$

By assumption, $x^k \neq x_*$ for all k. Moreover, by the convexity of f, we get

$$f(x) - f(y) \leq \langle f'(x), x - y\rangle. \tag{2.17}$$

Substituting $x = x^k, y = x_*$ and a fixed $\alpha = \alpha_o \in (0,1)$ into (2.16) and (2.17), we obtain

$$\begin{aligned}
&\alpha_o(1-\alpha_o)\gamma\|x^k - x_*\|^2 \\
&\le \alpha_o[f(x^k) - f(\alpha_o x^k + (1-\alpha_o)x_*)] + (1-\alpha_o)[f(x_*) - f(\alpha_o x^k + (1-\alpha_o)x_*)] \\
&\le \alpha_o(1-\alpha_o)\langle f'(x^k), x^k - x_*\rangle + \alpha_o(1-\alpha_o)\langle f'(x_*), x_* - x^k\rangle \\
&\le \alpha_o(1-\alpha_o)\langle f'(x^k) - f'(x_*), x^k - x_*\rangle.
\end{aligned}$$

Thus, taking into account that $f'(x_*) = 0$, we have

$$\gamma\|x^k - x_*\|^2 \le \langle f'(x^k), x^k - x_*\rangle \le \|f'(x^k)\|\|x^k - x_*\|.$$

Consequently,

$$\gamma\|x^k - x_*\| \le \|f'(x^k)\|. \tag{2.18}$$

Now using (2.17) and (2.18), we find that

$$0 < f(x^k) - f(x_*) \le \langle f'(x^k), x^k - x_*\rangle \le \|f'(x^k)\|\|x^k - x_*\| \le \frac{1}{\gamma}\|f'(x^k)\|^2$$

for all k. Since the sequence $\{f(x^k)\}$ is assumed to be strictly increasing, we have

$$0 < \gamma(f(x^0) - f(x_*)) \le \|f'(x^k)\|^2$$

for all k. Thus, setting $\delta = (\gamma(f(x^0) - f(x_*)))^{\frac{1}{2}}$ in this inequality concludes the proof. ■

2.4 Algorithm and its Convergence

Before presenting an algorithm for solving problem (2.6), it is useful to restate Theorem 2.4 in a convenient way via the following function $\theta(z)$ defined for $z \in D$.

$$\theta(z) = \max_{y \in E_{f(z)}(f)} \pi(y),$$

where $\pi(y) = \max_{x \in D}\langle f'(y), x - y\rangle$. Since f is strongly convex, the set $E_{f(z)}(f)$ is compact [38]. Thus, clearly, $\theta(z) < +\infty$. Also, we can note that

$$\pi(y) \le \theta(z) \text{ for all } y \in E_{f(z)}(f).$$

Theorem 2.7 Let $z \in D$ and $f'(z) \neq 0$. If $\theta(z) = 0$ then the point z is a solution to problem (2.6).
Proof. Immediate from the following inequalities:

$$\langle f'(y), x - y\rangle \le \pi(y) = \max_{x \in D}\langle f'(y), x - y\rangle \le \theta(z) = 0$$

which hold for all $x \in D$ and $y \in E_{f(z)}(f)$. ■

Algorithm MAX

Input: A strongly convex function f and a simple set D.
Output: A solution x to problem (2.6); i.e., a global maximizer of f over D.

Step 1. Choose a point $x^0 \in D$ such that $f'(x^0) \neq 0$. Set $k := 0$.
Step 2. Solve the following problem

$$\pi(y) \longrightarrow \max, \quad y \in E_{f(x^k)}(f).$$

Let y^k be a solution of this problem, i.e.,

$$\pi(y^k) = \max_{y \in E_{f(x^k)}(f)} \max_{x \in D} \langle f'(y), x - y \rangle.$$

Let $\theta(x^k) := \pi(y^k)$, and let x^{k+1} be a solution satisfying

$$\pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle.$$

Step 3. If $\theta(x^k) = 0$ then output $x = x^k$ and terminate. Otherwise set $k := k + 1$ and return to Step 2. ■

The convergence of this Algorithm is given by the following theorem.

Theorem 2.8 Assume that $f : R^n \to R$ be strongly convex and continuously differentiable. Then the sequence $\{x^k\}, (k = 0, 1, ...)$ generated by Algorithm MAX is a maximizing sequence for problem (2.6), that is,

$$\lim_{k \to \infty} f(x^k) = \max_{x \in D} f(x),$$

and every accumulation point of the sequence $\{x^k\}$ is a global maximizer of (2.6).
Proof. Note that Algorithm MAX generates points $x^k \in D$ and $f(x^k) \leq f^*$, where $f^* = f(x^*) = \max_{x \in D} f(x)$. Clearly, for all $y \in E_{f(x^k)}(f)$ and $x \in D$ we have

$$\langle f'(y), x - y \rangle \leq \pi(y) \leq \theta(x^k).$$

If there exists a k such that $\theta(x^k) = 0$ then, by theorem 2.7, x^k is a solution to problem (2.6) and, consequently, the desired result follows. Therefore, without loss of generality, we can suppose that $\theta(x^k) > 0$ for all $k = 0, 1, ...$, and prove the theorem by contradiction.

If the assertion is false, that is, x^k is not a maximizing sequence for problem (2.6), the following inequality holds.

$$\lim_{k \to \infty} sup\ f(x^k) < f^*. \tag{2.19}$$

First, we show that the sequence $\{f(x^k)\}$ is strictly increasing. From the construction of the Algorithm and the definition of $\theta(x^k)$, we have

$$\theta(x^k) = \pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle > 0,$$

where $y^k \in E_{f(x^k)}(f)$. Then this fact and the convexity of f [33,38] imply that

$$f(x^{k+1}) - f(x^k) = f(x^{k+1}) - f(y^k) \geq \langle f'(y^k), x^{k+1} - y^k \rangle > 0.$$

Therefore, $f(x^{k+1}) > f(x^k)$ for all k. Furthermore, as the sequence $\{f(x^k)\}$ is bounded from above by f^*, it has a limit:

$$\lim_{k \to \infty} f(x^k) = A < +\infty,$$

and hence we have

$$\lim_{k\to\infty}(f(x^{k+1})-f(x^k))=0.$$

Then from the above results, we can easily conclude that

$$\lim_{k\to\infty}\theta(x^k)=0. \tag{2.20}$$

Now introduce the following sets which are closed and convex.

$$C_k=\{x\in R^n : f(x)\le f(x^k)\},\quad k=0,1,2,\ldots.$$

It is clear that $x^*\notin C_k$ and *int* $C_k\neq\emptyset$. Then take the projection $u^k\in C_k$ of the point x^* on C_k such that

$$\|u^k-x^*\|=\min_{x\in C_k}\|x-x^*\|. \tag{2.21}$$

Note that

$$\|u^k-x^*\|>0 \tag{2.22}$$

holds because $x^*\notin C_k$. The optimality conditions at the solution u^k for the convex minimization problem (2.21) is given as follows [61, 62],

$$\begin{cases} u^k-x^*+\lambda_k f'(u^k)=0\\ f(u^k)=f(x^k),\end{cases} \tag{2.23}$$

where λ_k is the lagrange multiplier. Hence, we have

$$\lambda_k=\frac{\|u^k-x^*\|}{\|f'(u^k)\|}. \tag{2.24}$$

Then condition $f(u^k)=f(x^k)$ of (2.23) and $\theta(x^k)$ imply

$$\theta(x^k)=\max_{y\in E_{f(x^k)}(f)}\pi(y)\ge\max_{x\in D}\langle f'(u^k),x-u^k\rangle\ge\langle f'(u^k),x^*-u^k\rangle. \tag{2.25}$$

Using (2.23), (2.24) and (2.25), we have

$$\langle f'(u^k),x^*-u^k\rangle=\|f'(u^k)\|\|x^*-u^k\|\le\theta(x^k). \tag{2.26}$$

Also by Theorem 2.6, there exists a positive number δ which satisfies

$$\|f'(u^k)\|\ge\delta \tag{2.27}$$

for all k. Therefore, from (2.27) and (2.26), we conclude that

$$0\le\delta\|x^*-u^k\|\le\theta(x^k).$$

Taking into account (2.20), we have

$$\lim_{k\to\infty}u^k=x^*.$$

Hence, by continuity of f on R^n,

$$\lim_{k\to\infty}f(x^k)=\lim_{k\to\infty}f(u^k)=f(x^*), \tag{2.28}$$

which yields a contradiction to (2.19).

Consequently, $\{x^k\} \subset D$ is a maximizing sequence for problem (2.6). Since D is compact, we can always select the convergent subsequences $\{x^{k_l}\}$ from $\{x^k\}$ such that

$$\lim_{l\to\infty} x^{k_l} = \bar{x} \in D.$$

Then together with (2.28), we obtain

$$\lim_{l\to\infty} f(x^{k_l}) = f(\bar{x}) = f^*,$$

which completes the proof. ■

Algorithm MAX seems to us more theoretical rather than numerical. Note that its efficiency depends on how we can solve successfully the following problem

$$\pi(y) \longrightarrow \max, \quad y \in E_{f(x^k)}(f). \tag{2.29}$$

In general, this problem belongs to a class of nondifferentiable optimization problems. We can use such well known methods as a penalty function method [38,61,62], a method of generalized gradients [43], or a method of nondifferentiable optimization [13]. Moreover, if D is strictly convex, then $\pi(y)$ is differentiable as we noted in corollary of Theorem 2.2. Thus, in principle, we can apply gradient type of methods as well as the Lagrangean method [4]. Hence, we can see that the global optimality condition for problem (2.6) not only is connected to the classical conditions, but also its global algorithm is connected to the classical methods.

Also, there are numerical methods [45, 63] for such problem as (2.29) with the equality constraint, based on the set covering and finite set methods, which find a solution with a given (ε) accuracy. In general, in Algorithm MAX, we can use a numerical method providing ε approximate solution to problem (2.29), in order to find an approximate solution to problem (2.6) with a given accuracy. To make Algorithm MAX numerically implementable for this purpose, we need adapt it as follows.

Algorithm ϵ-MAX

Input: A strongly convex function f and a simple set D, and a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k > 0$ for all k and $\lim_{k\to\infty}\sum_{k=0}^{\infty}\varepsilon_k < +\infty$, and $\varepsilon > 0$.
Output: A global maximizer x of f over D.

Step 1. Choose a feasible solution $x^o \in D$ and a sequence $\{\varepsilon_k\}$ such that $f'(x^o) \neq 0$. Set $k := 0$.
Step 2. Find an ε_k solution $\bar{y}^k$ of the problem

$$\pi(y) \longrightarrow \max, \quad y \in E_{f(x^k)}(f),$$

that is, $\pi(\bar{y}^k) = \max\limits_{x\in D}\langle f'(\bar{y}^k), x - \bar{y}^k\rangle \geq \max\limits_{y\in E_{f(x^k)}(f)} \pi(y) - \varepsilon_k$ and $f(\bar{y}^k) = f(x^k)$.
Let $\bar{x}^{k+1} \in D$ be a solution satisfying $\pi(\bar{y}^k) = \langle f'(\bar{y}^k), \bar{x}^{k+1} - \bar{y}^k\rangle$.
Step 3. If $\pi(\bar{y}^k) \leq \varepsilon - \varepsilon_k$ then output $x = x^k$ and terminate. Otherwise set $k := k+1$ and $x^k := \bar{x}^k$ return to Step 2. ■

In order to ensure the convergence of this algorithm, first we need to prove the following theorems.

Theorem 2.9 Assume that $f : R^n \to R$ be strongly convex and continuously differentiable. Let sequences $\{x^k\} \subset R^n$ and $\{\varepsilon_k\} \subset R$ be such that

$$f(x^k) \geq f(x^{k-1}) - \varepsilon_k, \quad \varepsilon_k > 0 \text{ for } k = 0, 1, ...$$
$$\lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k < +\infty$$

and $\lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k + f_* < f(x^0)$, where $f_* = \min_{x \in R^n} f(x)$. Then there exists a positive constant $\bar{\delta} > 0$ satisfying

$$\lim_{k\to\infty} inf \, \|f'(x^k)\| \geq \bar{\delta}.$$

Proof. The proof is analogous to the proof of Theorem 2.6. ■

Theorem 2.10 Assume that $f : D \to R$ be a continuous function defined on a compact set $D \subset R^n$. Let sequences $\{x^k\} \subset D$ and $\{\varepsilon_k\} \subset R$ be such that

$$f(x^k) \geq f(x^{k-1}) - \varepsilon_k, \quad \varepsilon_k > 0 \text{ for } k = 0, 1, ...,$$
$$\lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k < +\infty.$$

Then there exists a finite limit of the sequence $\{f(x^k)\}$, that is, $\lim_{k\to\infty} f(x^k) = A < +\infty$.

Proof. First, note that $\{f(x^k)\}$ is bounded on a compact set D. From

$$f(x^k) \geq f(x^{k-1}) - \varepsilon_k, \quad \varepsilon_k > 0,$$

we obtain

$$f(x^m) \geq f(x^k) - \sum_{i=k}^{m-1} \varepsilon_i \geq f(x^k) - \sum_{i=k}^{\infty} \varepsilon_i \tag{2.30}$$

for all $m > k \geq 0$. Then there exists a sequence $k_1 < k_2 < ...$ such that

$$\lim_{k\to\infty} sup \; f(x^k) = \lim_{n\to\infty} f(x^{k_n}).$$

Setting $k = k_n$ in (2.30) yields $f(x^m) \geq f(x^{k_n}) - \sum_{i=k_n}^{\infty} \varepsilon_i$ for any $m > k_n$. Hence, we have

$$\lim_{m\to\infty} inf \; f(x^m) \geq f(x^{k_n}) - \sum_{i=k_n}^{\infty} \varepsilon_i$$

for all $n = 1, 2, ...$ Also, we get

$$\lim_{m\to\infty} inf \; f(x^m) \geq \lim_{n\to\infty} f(x^{k_n}) = \lim_{m\to\infty} sup \; f(x^m).$$

Taking into account that $\lim_{m\to\infty} inf \; f(x^m) \leq \lim_{m\to\infty} sup \; f(x^m)$, this shows $\lim_{m\to\infty} inf \; f(x^m) = \lim_{m\to\infty} sup \; f(x^m)$. Therefore, there exists a limit of $\{f(x^k)\}$. Since $\{f(x^k)\}$ is bounded, we have $\lim_{k\to\infty} f(x^k) = A < +\infty$, proving our theorem. ■

Theorem 2.11 Let a sequence $\{\varepsilon_k\} \subset R$ be such that $\varepsilon_k > 0$ and $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$. Then for the the sequence $\{x^k\}$ generated by Algorithm ϵ-MAX, we have

$$\lim_{k\to\infty} \max_{x\in D} \langle f'(\bar{y}^k), x - \bar{y}^k \rangle = 0,$$

where $\max_{x\in D} \langle f'(\bar{y}^k), x - \bar{y}^k \rangle \geq \max_{y\in E_{f(x^k)}(f)} \max_{x\in D} \langle f'(y), x - y \rangle - \varepsilon_k$ and $f(\bar{y}^k) = f(x^k)$.

Proof. Immediate from Theorems 2.8 and 2.10. ■

Theorem 2.12 Assume that $f : R^n \to R$ be strongly convex and continuously differentiable. Let a sequence $\{\varepsilon_k\} \subset R$ be such that $\varepsilon_k > 0$ for $k = 0, 1, ...$ and

$$\lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k + f_* < f(x^0),$$

where $f_* = \min_{x\in R^n} f(x)$. Then the sequence $\{x^k\}$ generated by Algorithm ϵ-MAX is a maximizing sequence for problem (2.6), that is,

$$\lim_{k\to\infty} f(x^k) = \max_{x\in D} f(x),$$

and every accumulation point of the sequence $\{x^k\}$ is a global maximizer of (2.6).

Proof. Immediate from the Theorems 2.9-2.11. ■

2.5 Computational Results

To test the efficiency of the proposed algorithm ϵ-MAX, some numerical experiments have been performed for the quadratic case:

$$f(x) = \frac{1}{2}\langle Ax, x \rangle \longrightarrow \max, \quad x \in D \subset R^n,$$

where A is a symmetric positive definite matrix and the set D is simple. In the experiments, D was defined to be a polytope or a box. In this case its auxiliary problem (2.29) is equivalent to:

$$\begin{aligned} \pi(y) &= \max_{x\in D} \langle Ax, y \rangle \longrightarrow \max \\ \text{subject to:} \quad & \frac{1}{2}\langle Ay, y \rangle = \alpha_k, \end{aligned}$$

where $\alpha_k = \frac{1}{2}\langle Ax^k, x^k \rangle$. As a result of convexity, we may also write this in the form:

$$\begin{aligned} \pi(y) &= \max_{x\in D} \langle Ax, y \rangle \longrightarrow \max \\ \text{subject to:} \quad & \langle Ay, y \rangle \leq 2\alpha_k, \end{aligned}$$

which in Algorithm ϵ-MAX was solved by the finite set method proposed in [45] providing its ε_k-global solution for each ireration k.

Note that after the Algorithm ϵ-MAX finds the first feasible solution, it generates subsequent feasible solutions successively. The first feasible solution was found as follows. After choosing an arbitrary point $u \in R^n$, we found its projection on D by solving the following quadratic programming problem,

$$\|x - u\|^2 \longrightarrow \min, \quad x \in D,$$

using the conditional gradient method [61, 62]. A solution x^0 of this problem is the first feasible solution. If D is a box; i.e.,

$$D = \{x \in R^n \mid \alpha_i \le x_i \le \beta_i, \quad i = 1, 2, ..., n\},$$

then x^0 can be obtained analytically as follows.

$$x_i^0 = \begin{cases} \alpha_i & \text{if } u_i \le \alpha_i, \\ u_i & \text{if } \alpha_i < u_i < \beta_i, \\ \beta_i & \text{if } u_i \ge \beta_i, \quad i = 1, 2, ..., n. \end{cases}$$

We chose the following list of test problems (2.31)-(2.35), where problems (2.31) and (2.33) were taken from [6, 30].

$$(x_1 - 1.2)^2 + (x_2 - 0.6)^2 \longrightarrow \max, \tag{2.31}$$

subject to

$$\begin{aligned} & -2x_1 + x_2 \le 1, \\ & x_2 \le 2, \\ & x_1 + x_2 \le 4, \\ & x_1 \le 3, \\ & 0.5x_1 - x_2 \le 1, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

$$x_1^2 + x_2^2 \longrightarrow \max, \tag{2.32}$$

subject to

$$\begin{aligned} & -5x_1 + 13x_2 \le 72, \\ & 11x_1 - 7x_2 \le 36, \\ & 5x_1 - 9x_2 \le 28, \\ & -11x_1 + 9x_2 \le 56. \end{aligned}$$

$$4(x_1 - 1)^2 + 25(x_2 - 2)^2 \longrightarrow \max, \tag{2.33}$$

subject to

$$\begin{aligned} & 8.3x_1 + 20.5x_2 \le 170.15, \\ & -7.5x_1 + 18x_2 \le 135, \\ & -10.5x_1 + 7.7x_2 \le 80.85, \\ & -3.7x_1 - 10.2x_2 \le 37.74, \\ & -2.7x_1 - 13x_2 \le 35.1, \\ & 4.5x_1 - 7x_2 \le 31.5, \\ & -20 \le x_1 \le 20, \quad -20 \le x_2 \le 20. \end{aligned}$$

$$f(x) = \langle Cx, x\rangle \longrightarrow \max, \tag{2.34}$$

Table 1: Numerical Experiments

Problem	n	Initial value of the functions	Global value of the functions	Computing time (min:sec)
(2.31)	2	2.45	3.4	0:5.18
(2.32)	2	98	162	0:5.45
(2.33)	2	549.08	871.947	0:5.33
(2.34)	2	12.8	45.5	0:1
(2.34)	5	85	3604	0:81
(2.34)	10	670	109333.5	0:3
(2.34)	30	18010	25766625.5	1:12
(2.34)	40	42680	108196334	2:33
(2.34)	70	228690	1767930209	2:45
(2.34)	80	341360	3444342668	3:50
(2.34)	90	486030	6203290501.5	5:26
(2.34)	99	646899	9986343609	7:26
(2.35)	20	57885	1290389	0:4
(2.35)	40	857230	20204638	0:11
(2.35)	100	31888025	778330545	0:28
(2.35)	150	159648225	3927744505	0:50
(2.35)	200	501752550	12393657590	1:18
(2.35)	250	1220863500	30228632300	1:37
(2.35)	300	2525893575	62641481135	2:25

subject to

$$-(n-i+1) \le x_i \le n+0.5i, \quad i=1,2,...,n,$$

where

$$C = \begin{pmatrix} n & n-1 & n-2 & ... & 2 & 1 \\ n-1 & n & n-1 & ... & 3 & 2 \\ \cdots & \cdots & \cdots & ... & \cdots & \cdots \\ \cdots & \cdots & \cdots & ... & \cdots & \cdots \\ \cdots & \cdots & \cdots & ... & \cdots & \cdots \\ 1 & 2 & 3 & \dots & n-1 & n \end{pmatrix}.$$

$$\sum_{i=1}^{n}(n-1-0.1i)x_i^2 \longrightarrow \max, \quad x \in D, \tag{2.35}$$

subject to

$$D = \{x \in R^n \mid -1-i \le x_i \le 1+5i, \quad i=1,2,...,n\}.$$

The results of the numerical experiments for problems (2.31)-(2.35), implemented on an IBM PC/386, 486 in Pascal, are shown in Table 1 for different values of n. More computational results on the proposed algorithms will be provided in subsequent papers.

3 QUASICONVEX MINIMIZATION PROBLEM

3.1 Problem Statement and Global Minimality Condition

Now we consider the following quasiconvex minimization problem;

$$f(x) \longrightarrow \min, \quad x \in D, \tag{3.1}$$

where $f : R^n \to R$ is a quasiconvex and continuously differentiable function and $D \subset R^n$ is a nonempty arbitrary subset. We can obtain the following global optimality condition, which is the counterpart of Theorem 2.1.

Theorem 3.1 Let z be a solution of problem (3.1). Then

$$\langle f'(x), x-y\rangle \ge 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D, \tag{3.2}$$

where $E_c(f) = \{y \in R^n \mid f(y) = c\}$. If, in addition,

$$\lim_{\|x\|\to\infty} f(x) = +\infty \text{ and } f'(x+\alpha f'(x)) \ne 0 \tag{3.3}$$

holds for all $x \in D$ and $\alpha \ge 0$, then condition (3.2) becomes sufficient.
Proof. **Necessity**. Assume that z is a solution of problem (3.1). Let $x \in D$ and

$y \in E_{f(z)}(f)$. Then we have $0 \geq f(z) - f(x) = f(y) - f(x)$, and Lemma 2.2 implies $\langle f'(x), x - y \rangle \geq 0$.

Sufficiency. Suppose, on the contrary, that z is not a solution of (3.1). Then there exists a $u \in D$ such that $f(u) < f(z)$. Construct a ray y_α for $\alpha > 0$ defined by

$$y_\alpha = u + \alpha f'(u).$$

We claim that $f(y_\alpha) > f(u)$ holds for all positive α. By Taylor's formula, we have

$$f(u + \alpha f'(u)) - f(u) = \alpha \left(\|f'(u)\|^2 + \frac{o(\alpha \|f'(u)\|)}{\alpha} \right)$$

for small $\alpha > 0$, where $\lim\limits_{\alpha \to\ 0+} \frac{o(\alpha\|f'(u)\|)}{\alpha}=0$. Therefore, there exist $\alpha_o > 0$ such that $f(y_\alpha) - f(u) > 0$ holds for all $\alpha \in (0, \alpha_o)$. Hence, by Lemma 2.2, we have $\langle f'(u + \alpha_o f'(u)), f'(u) \rangle \geq 0$ since $f'(u) \neq 0$ and $f'(u + \alpha_o f'(u)) \neq 0$ by the assumption. Note that for all $\gamma > 1$, we also have $f(u + \gamma\alpha_o f'(u)) > f(u + \alpha_o f'(u))$; for otherwise, we would have $f(u + \gamma\alpha_o f'(u)) \leq f(u + \alpha_o f'(u))$, and consequently, by lemma 2.2, $\langle f'(u + \alpha_o f'(u)), \alpha_o(\gamma - 1) f'(u) \rangle \leq 0$, which would imply $\gamma \leq 1$ contradicting to the assumption that $\gamma > 1$. Moreover, we can show that $f(u + \gamma\alpha_o f'(u))$ is increasing in $\gamma > 0$. For if $f(u + \gamma'\alpha_o f'(u)) < f(u + \gamma\alpha_o f'(u))$ holds for some $\gamma' > \gamma$, then $\alpha_o(\gamma' - \gamma)\langle f'(u + \gamma\alpha_o f'(u)), f'(u) \rangle \leq 0$, which would contradict the fact that $\gamma' > \gamma$. These prove our claim $f(y_\alpha) > f(u)$ for all $\alpha > 0$.

Now it is obvious that the function $\varphi : R^+ \to R$ defined as

$$\varphi(\alpha) = f(y_\alpha)$$

is continuous on $[0, \infty)$. Also, by assumption (3.3), $\lim\limits_{\alpha \to \infty} \varphi(\alpha) = +\infty$, and therefore, there exists an $\hat{\alpha}$ such that $\varphi(\hat{\alpha}) > f(z)$. Using the continuity of $\varphi(\alpha)$ and the inequalities $\varphi(\hat{\alpha}) > f(z) > f(u)$, there exists an $\bar{\alpha}$ such that

$$f(y + \bar{\alpha} f'(u)) = f(z),$$

which means that $y_{\bar{\alpha}} \in E_{f(z)}(f)$. On the other hand, we have $f'(u) = \frac{1}{\bar{\alpha}}(y_{\bar{\alpha}} - u)$. Thus we get

$$\langle f'(u), u - y_{\bar{\alpha}} \rangle = \frac{1}{\bar{\alpha}} \langle y_{\bar{\alpha}} - u, u - y_{\bar{\alpha}} \rangle = -\frac{1}{\bar{\alpha}} \|y_{\bar{\alpha}} - y\|^2 < 0,$$

which contradicts (3.2). This means that z must be a solution of (3.1). ■

REMARK **1**. If $f(x)$ is convex in problem (1.2), then the following theorem was obtained in Deitrich [22].

Theorem 3.2 ([22]). Let D be an arbitrary nonempty set and f be a lower-semicontinuous proper convex function. Suppose that the assumptions

$$\begin{aligned} D &\subset M(\partial f), \\ -\infty &\leq \inf_{R^n} f < f(x) \text{ for all } x \in D, \\ L_{f(v)}(f) &= \{x \in R^n \mid f(x) \leq f(v)\} \subset \mathit{intdom}\ f \text{ for all } v \in D, \end{aligned}$$

are fulfilled. Then a point $z \in D$ is a global minimizer of problem (1.2) if and only if

$$\langle x^*, x - y\rangle \geq 0 \text{ for all } y \in E_{f(z)}(f),\ x \in D \text{ and } \ x^* \in \partial f(x), \tag{3.4}$$

where

$$\begin{aligned} \partial f(x) &= \{c \in R^n \mid f(v) - f(x) \geq \langle c, v - x\rangle,\ \ v \in R^n\}, \\ intdom\ f &= \{x \in R^n \mid f(x) < +\infty\}, \end{aligned}$$

and $M(\partial f)$ is the effective domain of the subdifferential mapping $\partial f : R^n \to 2^{R^n}$.

We can easily note that if f is differentiable as well then necessary condition (3.2) for problem (3.1) is the same as condition (3.4). Thus Theorem 3.1 is a generalization of this result to the quasiconvex differentiable case.

REMARK **2**. If D is convex, then under some additional assumptions on f there is a global optimality condition by Kantorovich stated as follows.

Theorem 3.3 ([37]). If D is convex, then a point $z \in D$ is a solution of problem (1.2) if and only if there exists a $c \in R^n$ such that

$$\begin{aligned} &\langle c, x\rangle \leq \langle c, z\rangle \ \text{ for all } x \in D, \\ &\langle c, x\rangle \geq \langle c, z\rangle \ \text{ for all } x \in R^n \text{ with } f(x) \leq f(z). \end{aligned}$$

REMARK **3**. Note that by the assumption of Theorem 3.1, we have $f'(x) \neq 0$ for all $x \in D$. If f is strictly quasiconvex and D is convex, we do not need to use condition (3.2). Instead we have the following result which is immediate from the classical local optimality condition [61, 62].

Theorem 3.4 Assume that f is strictly quasiconvex and differentiable, and D is convex in (3.1). Then a point $z \in D$ is the solution of problem (3.1) if and only if

$$\langle f'(z), x - z\rangle \geq 0 \text{ for all } x \in D.$$

Also, for the pseudoconvex case we have the following result.

Lemma 3.1 let $f : R^n \to R$ be pseudoconvex and $x_* \in D$. If $f'(x_*) = 0$, then x_* is a global minimizer of problem (3.1).

REMARK **4**. There is a strong relationship between the quasiconvex minimization problem (3.1) and the variational inequality problem defined on a convex set [17]. For example, if f is continuously differentiable and convex, and D is convex , then the problem $\min\limits_{x \in D} f(x) = f(x_*)$ and the variational problem

$$\langle f'(x), x - x_*\rangle \geq 0 \text{ for all } x \in D$$

are equivalent [17]. Therefore, condition (3.2) might be regarded as a generalization of this well known problem for the quasiconvex case with an arbitrary set D.

REMARK **5**. In order to conclude that a point $z \in D$ is not a solution of (3.1) it is sufficient to have a pair $u, y \in R^n$ such that $\langle f'(u), u - y\rangle < 0, f(y) = f(z)$ and $u \in D$.

The use of Theorem 3.1 is illustrated in the following examples.

Example 3.1 Consider the following problem:

$$f(x) = x_1^2 + x_2^2 \longrightarrow \min,$$

subject to

$$D = \{x \in R^2 : \ 5x_1^2 + 2x_2 + 3x_1 \le 16, \quad x_1^2 + x_2 \ge 5\}.$$

The Lagrangean condition for this problem provides three stationary points $z^o = (1,4)$, $z^1 = (0,5)$ and $z^2 = (-2,1)$. To see if z^o is a global solution, consider points $u = (-1,4) \in D$ and $y = (0, \sqrt{17})$ satisfying $f(y) = f(z^o) = 17$. Evaluating $\langle f'(u), u - y\rangle$ we have $\langle f'(u), u - y\rangle = 2(16 - 4\sqrt{17}) < 0$. Consequently, the point z^o is not a global solution. In fact, it is not difficult to see that z^2 is the global minimizer with $f(z^2) = 5$.

Example 3.2 ([16]). The Classical Utility Maximization. Going back to example 2.2 in section 2, consider the classical utility maximization problem in an oligopsonistic market with a quasiconcave utility function:

$$u(x) \longrightarrow max, \quad x \in D,$$

where D is defined by,

$$D = \{x \in R^n \mid \sum_{i=1}^{n} p_i(x_i)x_i \le m, \quad x_i \ge 0, \quad i = 1, 2, \ldots, n\}.$$

This can be equivalently presented as the following quasiconvex minimization problem:

$$-u(x) \longrightarrow \min, \quad x \in D.$$

Then global optimality conditions (3.2) for this problem are as follows:

$$\begin{cases} \sum\limits_{i=1}^{n} \frac{\partial u(x)}{\partial x_i}(x_i - y_i) \le 0 \\ u(y_1, y_2, \ldots, y_n) = u(z) \\ \sum\limits_{i=1}^{n} p_i(x_i)x_i \le m \\ x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

Note that if all goods are discrete [60] in this problem (or equivalently all x_i are discrete variables), then the problem can be formulated as a resource allocation problem [31] and then solved by methods as described in [31].

Example 3.3 Consider the D.C. programming problem (1.4). Assume additionally that the functions g_1 and g_2 are differentiable. This problem can be converted into an equivalent convex minimization problem with an nonconvex feasible set, which is sometimes called reverse convex constraint [30].

$$g_1(x) - x_{n+1} \longrightarrow \min,$$
$$\text{subject to: } g_2(x) \ge x_{n+1}, \ h_i(x) \le 0, \ i = 1, 2, ..., m,$$

where $(x, x_{n+1}) \in R^n \times R$. Now the condition (3.2) at a solution $z \in D$ becomes as follows.

$$\begin{cases} \sum\limits_{i=1}^{n} \frac{\partial g_1(y)}{\partial x_i}(x_i - y_i) + y_{n+1} - x_{n+1} \geq 0 \\ g_1(y) - y_{n+1} = g_1(z) \\ g_2(x) \geq x_{n+1} \\ h_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \end{cases}$$

Example 3.4 Consider the fractional programming problem

$$f(x) = \frac{f_1(x)}{f_2(x)} \longrightarrow \min, \quad x \in D, \tag{3.5}$$

where f_1 is convex and differentiable on R^n, and f_2 is concave and differentiable on R^n. Suppose that f_1 and f_2 are defined positively on a ball B containing a subset $D \subset R^n$, that is,

$$f_1(x) > 0, \quad f_2(x) > 0 \text{ for all } x \in D \subset B.$$

We shall call this problem as the mixed fractional minimization problem. By lemma 2.1, we can easily show that $f(x)$ is quasiconvex. Hence, the optimality condition (3.2) at a solution z of (3.5) is as follows:

$$\sum_{i=1}^{n} \left(\frac{\partial f_1(x)}{\partial x_i} f_2(x) - \frac{\partial f_2(x)}{\partial x_i} f_1(x) \right) \frac{(x_i - y_i)}{f_2^2(x)} \geq 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D.$$

When f_1, f_2 and D were all convex, then problem (3.5) was studied in [32, 54].

3.2 Preliminary Results for Minimization

Assume that $f: R^n \to R$ be a continuously differentiable function and $D \subset R^n$ be an arbitrary compact set. For the purpose of describing an algorithm for solving (3.1), we define the axiliary functions $\alpha(y)$ and $\beta(y)$ for $y \in R^n$,

$$\alpha(y) = \min_{x \in D} \langle f'(x), y \rangle, \quad \beta(y) = \max_{x \in D} \langle f'(x), y \rangle.$$

Lemma 3.2 The above functions $\alpha(y)$ and $\beta(y)$ are continuous at point 0.
Proof. We show that $\alpha(y)$ is continuous, as the case of $\beta(y)$ is analogous. It is obvious that $\|f'(x)\|$ is continuous on R^n because $f(x)$ is assumed to be continuously differentiable. Let M denote the global maximum of $\|f'(x)\|$ on D:

$$M = \max_{x \in D} \|f'(x)\|.$$

Let $\Delta y \in R^n$ be a small increment at 0. Then the corresponding increase of $\alpha(y)$ at 0 is:

$$\begin{aligned} \Delta\alpha(0) &= \alpha(\Delta y) - \alpha(0) = \min_{x \in D} \langle f'(x), \Delta y \rangle \\ &\leq \langle f'(x), \Delta y \rangle \leq \|f'(x)\| \|\Delta y\| \leq M \|\Delta y\|. \end{aligned} \tag{3.6}$$

On the other hand, we have

$$\Delta\alpha(0) = \min_{x \in D}\langle f'(x), \Delta y\rangle \geq \min_{x \in D}(-\|f'(x)\|\|\Delta y\|) =$$
$$\|\Delta y\| \min_{x \in D}(-\|f'(x)\|) = -\|\Delta y\| \max_{x \in D}\|f'(x)\| = -M\|\Delta y\|. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$|\Delta\alpha(0)| \leq M\|\Delta y\|,$$

and hence

$$\lim_{\Delta y \to 0} \Delta\alpha(0) = \lim_{\Delta y \to 0} \alpha(\Delta y) = 0,$$

which complete the proof. ■

Next we define another auxiliary functions $p(y)$:

$$p(y) = \min_{x \in D}\langle f'(x), x - y\rangle, \quad y \in R^n. \tag{3.8}$$

Lemma 3.3 Function $p(y)$ is continuous on R^n.
Proof. Let $y \in R^n$ be an arbitrary point and $\Delta y \in R^n$ be a small increase at y. Then we have

$$\begin{aligned}
\Delta p(y) &= p(y + \Delta y) - p(y) \\
&= \min_{x \in D}\langle f'(x), x - (y + \Delta y)\rangle - \min_{x \in D}\langle f'(x), x - y\rangle \\
&\leq \max_{x \in D}(\langle f'(x), x - (y + \Delta y)\rangle - \langle f'(x), x - y\rangle) \\
&= \max_{x \in D}\langle f'(x), -\Delta y)\rangle = -\alpha(\Delta y).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-\Delta p(y) &= p(y) - p(y + \Delta y) \\
&= \min_{x \in D}\langle f'(x), x - y\rangle - \min_{x \in D}\langle f'(x), x - (y + \Delta y)\rangle \\
&\leq \max_{x \in D}\langle f'(x), \Delta y)\rangle) = \beta(\Delta y).
\end{aligned}$$

Consequently, we have

$$-\beta(\Delta y) \leq \Delta p(y) \leq -\alpha(\Delta y),$$

and hence

$$\lim_{\Delta y \to 0} \Delta p(y) = 0.$$

■

3.3 An Algorithm for the Convex Minimization Problem

We consider the convex minimization problem as a special case of problem (3.1):

$$f(x) \longrightarrow \min, \quad x \in D, \tag{3.9}$$

where $f: R^n \to R$ is strongly convex and continuously differentiable, and D is an arbitrary compact set in R^n. In this case, then we can weaken condition (3.3) as shown in the next theorem.

Theorem 3.5 Let z be a solution of problem (3.9). Then

$$\langle f'(x), x - y\rangle \geq 0 \text{ for all } y \in E_{f(z)}(f) \text{ and } x \in D. \tag{3.10}$$

If, in addition,

$$\min_{x \in D} \|f'(x)\| > 0 \tag{3.11}$$

holds, then condition (3.10) is also sufficient.
Proof. Necessity. Assume that z is a solution of problem (3.9). Consider $x \in D$ and $y \in E_{f(z)}(f)$. Then by the convexity of f, we have

$$0 \geq f(z) - f(x) = f(y) - f(x) \geq \langle f'(x), y - x\rangle.$$

Sufficiency. Let us prove the assertion by contradiction. Assume that (3.10) holds and there exists a point $u \in D$ such that

$$f(u) < f(z).$$

Clearly, $f'(u) \neq 0$ by assumption (3.11). Now define u_α as follows for $\alpha > 0$:

$$u_\alpha = u + \alpha f'(u).$$

Then, by the convexity of f, we have

$$f(u_\alpha) - f(u) \geq \langle f'(u), u_\alpha - u\rangle = \alpha \|f'(u)\|^2,$$

which implies

$$f(u_\alpha) \geq f(u) + \alpha \|f'(u)\|^2 > f(u).$$

Then find $\alpha = \bar{\alpha}$ such that

$$f(u) + \bar{\alpha}\|f'(u)\|^2 = f(z),$$

that is,

$$\bar{\alpha} = \frac{f(z) - f(u)}{\|f'(u)\|^2} > 0.$$

Thus we get

$$f(u_{\bar{\alpha}}) \geq f(u) + \bar{\alpha}\|f'(u)\|^2 = f(z) > f(u).$$

Define a function $h : R^+ \rightarrow \; R$ as

$$h(\alpha) = f(u + \alpha f'(u)) - f(z),$$

where $R^+ = \{\alpha \in R \mid \alpha \geq 0\}$. It is clear that h is continuous on $[0, +\infty)$. Note that $h(\bar{\alpha}) \geq 0$ and $h(0) < 0$. There are two cases with respect to the values of $h(\bar{\alpha})$ which we should consider.

Case a: $h(\bar{\alpha}) = 0$ (or $f(u + \bar{\alpha} f'(u)) = f(z)$), then

$$\langle f'(u), u - u_{\bar{\alpha}} \rangle = -\langle f'(u), \bar{\alpha} f'(u) \rangle = -\bar{\alpha} \|f'(u)\|^2 < 0,$$

contradicting condition (3.10).
Case b: $h(\bar{\alpha}) > 0$ and $h(0) < 0$. Since h is continuous, there exists a point $\alpha_o \in (0, \bar{\alpha})$ such that $h(\alpha_o) = 0$ (or $f(u + \alpha_o f'(u)) = f(z)$). Then we have

$$\langle f'(u), u - u_{\alpha_o} \rangle = -\alpha_o \|f'(u)\|^2 < 0,$$

again contradicting (3.10).

Thus, in both cases, we find contradictions, proving the theorem. ■

Now using the function $p(y)$ of (3.8), we reformulate Theorem 3.5 in terms of function $\psi(z)$ defined as follows.

$$\psi(z) = \min_{y \in E_{f(z)}(f)} p(y), \quad z \in D.$$

Theorem 3.6 Assume that $f : R^n \rightarrow R$ is strongly convex and continuously differentiable, and D is a compact set in R^n. Let $\min_{x \in D} \|f'(x)\| > 0$. If $\psi(z) = 0$, then the point z is a solution to problem (3.9).
Proof. This is an obvious consequence of the following relations.

$$0 = \psi(z) \leq p(y) \leq \langle f'(x), x - y \rangle,$$

which are fulfilled for all $y \in E_{f(z)}(f)$ and $x \in D$. ■

Now we are ready to present an algorithm for solving problem (3.9). We also suppose that one can efficiently solve the problem of computing $\min_{x \in D} \langle f'(x), x - y \rangle$ for any given $y \in R^n$. For example, this is possible in the following cases:

1. f is linear and D is a simple set defined as in Section 2.3.

2. D is convex and f is such that $s''(x)$ is a nonnegative definite matrix, where $s''(x)$ is the matrix of second order derivatives of the function $s = \langle f'(x), x - y \rangle$ defined for $x \in D$ and $y \in R^n$.

Algorithm MIN

Input: A strongly convex function f and a compact set D.
Output: A solution x to the minimization problem (3.9).

Step 1. Choose a feasible point $x^0 \in D$. Set $k := 0$.
Step 2. Solve the following problem

$$p(y) \;\longrightarrow\; \min, \quad y \in E_{f(x^k)}(f).$$

Let y^k be a solution of this problem (i.e., $p(y^k) = \min_{x \in D} \langle f'(x), x - y^k \rangle = \min_{y \in E_{f(x^k)}(f)} p(y)$), and let x^{k+1} realizes $p(y^k)$ (i.e., $\psi(x^k) = p(y^k) = \langle f'(x^{k+1}), x^{k+1} - y^k \rangle$).
Step 3. If $\psi(x^k) = 0$ then output $x = x^k$ and terminate. Otherwise, let $k := k + 1$ and return Step 2. ■

The convergence of this Algotithm is based on the following theorem.

Theorem 3.7 Assume that $f : R^n \to R$ be strongly convex and continuously differentiable, and D is a compact set in R^n. Let $\min_{x \in D} \|f'(x)\| > 0$. Then the sequence $\{x^k, \ k = 0, 1, ...\}$ generated by Algorithm MIN is a minimizing sequence for problem (3.9), that is,

$$\lim_{k \to \infty} f(x^k) = \min_{x \in D} f(x),$$

and every accumulation point of the sequence $\{x^k\}$ is a global minimizer of (3.9).
Proof. From the construction of $\{x^k\}$, we have $x^k \in D$ and $f(x^k) \geq f_*$ for all k, where $f_* = f(x_*) = \min_{x \in D} f(x)$. Clearly, $f'(x_*) \neq 0$ by assumption. Also, note that for all $y \in E_{f(x^k)}(f)$ and $x \in D$, we have

$$\psi(x^k) = \min_{y \in E_{f(x^k)}(f)} \min_{x \in D} \langle f'(x), x - y \rangle \leq \langle f'(x), x - y \rangle \leq 0.$$

If there exists a k such that $\psi(x^k) = 0$ then, by theorem 3.6, x^k is a solution to problem (3.9) and, in this case the proof is complete. Therefore, without loss of generality, we can assume $\psi(x^k) < 0$ for all k, and prove the theorem by contradiction. If the assertion is false, that is, x^k is not a minimizing sequence for problem (3.9), the following inequality holds.

$$\lim_{k \to \infty} inf \ f(x^k) > f_*. \tag{3.12}$$

By the definition of $\psi(x^k)$ and Algorithm MIN, we have

$$p(y^k) = \psi(x^k) = \min_{y \in E_{f(x^k)}(f)} \min_{x \in D} \langle f'(x), x - y \rangle = \langle f'(x^{k+1}), x^{k+1} - y^k \rangle$$

and $f(y^k) = f(x^k)$. The convexity of f implies that

$$\begin{aligned} f(x^k) - f(x^{k+1}) = f(y^k) - f(x^{k+1}) \geq \\ \langle f'(x^{k+1}), y^k - x^{k+1} \rangle = -\psi(x^k) > 0. \end{aligned} \tag{3.13}$$

Hence, we obtain $f(x^{k+1}) < f(x^k)$ for all k, and the sequence $\{f(x^k)\}$ is strictly decreasing. Since the sequence is bounded from below by f_*, it has a limit and satisfies

$$\lim_{k \to \infty} (f(x^{k+1}) - f(x^k)) = 0. \tag{3.14}$$

Then, from (3.13) and (3.14), we obtain

$$\lim_{k \to \infty} \psi(x^k) = 0.$$

From (3.12) we have $f(x^k) > f(x_*)$ for all k. Now define v_α as follows:

$$v_\alpha = x_* + \alpha f'(x_*), \quad \alpha > 0.$$

Then, by the convexity of f, we have

$$f(v_\alpha) - f(x_*) \geq \langle f'(x_*), v_\alpha - x_* \rangle = \alpha \|f'(x_*)\|^2,$$

which implies

$$f(v_\alpha) \geq f(x_*) + \alpha \|f'(x_*)\|^2 > f(x_*), \quad \alpha > 0.$$

Choose $\alpha = \alpha_k$ such that

$$f(x_*) + \alpha_k \|f'(x_*)\|^2 > f(x_k),$$

that is,

$$\alpha_k > \frac{f(x^k) - f(x_*)}{\|f'(x_*)\|^2} > 0.$$

Define a function $h_k : R^+ \rightarrow \ R$ as

$$h_k(\alpha) = f(x_* + \alpha f'(x_*)) - f(x^k),$$

where $R^+ = \{\alpha \in R \mid \alpha \geq 0\}$. It is clear that h_k is continuous on $[0, +\infty)$. Note that $h_k(\alpha_k) > 0$ and $h_k(0) < 0$. Since h_k is continuous, there exists a point $\bar{\alpha_k} \in (0, \alpha_k)$ such that $h_k(\bar{\alpha_k}) = 0$, that is $f(v_{\bar{\alpha_k}}) = f(x^k)$ and $v_{\bar{\alpha_k}} = x_* + \bar{\alpha_k} f'(x_*)$. Also, note that

$$\psi(x^k) = \min_{y \in E_{f(x^k)}(f)} \min_{x \in D} \langle f'(x), x - y \rangle \leq \langle f'(x_*), x_* - v_{\bar{\alpha_k}} \rangle.$$

Taking into account $v_{\bar{\alpha_k}} = x_* + \bar{\alpha_k} f'(x_*)$, we have

$$-\psi(x^k) \geq \langle f'(x_*), v_{\bar{\alpha_k}} - x_* \rangle = \|f'(x_*)\| \|v_{\bar{\alpha_k}} - x_*\| \geq \min_{x \in D} \|f'(x)\| \|v_{\bar{\alpha_k}} - x_*\| > 0.$$

Since $\lim_{k \rightarrow \infty} \psi(x^k) = 0$, this implies

$$\lim_{k \rightarrow \infty} v_{\bar{\alpha_k}} = x_*.$$

The continuity of f on R^n yields

$$\lim_{k \rightarrow \infty} f(x^k) = \lim_{k \rightarrow \infty} f(v_{\bar{\alpha_k}}) = f(x_*), \tag{3.15}$$

which is a contradiction to (3.12).

Consequently, $\{x^k\}$ is a minimizing sequence for problem (3.9). Since D is compact, we can always select the convergent subsequences $\{x^{k_l}\}$ from $\{x^k\}$ such that

$$\lim_{l \rightarrow \infty} x^{k_l} = \bar{x} \in D.$$

Then together with (3.15), we obtain

$$\lim_{l \rightarrow \infty} f(x^{k_l}) = f(\bar{x}) = f_*,$$

which completes the proof. ■

Remark 1. If D is convex, there are a number of well established convex minimization algorithms; e.g., [4, 33, 38, 50, 61, 62, 64]. Thus, the proposed algorithms mainly intended for the case with a nonconvex feasible set.

Remark 2. In some case, computing $\psi(x^k)$, in Step 2 of Algorithm MIN, computing $\psi(x^k)$ may not be easier than the original problem. But if we can find a point $v \in D$ such that $\langle f'(v), v - y^k \rangle < 0$ during this computation, it implies $f(v) < f(x^k)$, improving the initial value of $f(x^k)$. Therefore this v may be used as x^{k+1} in Step 2. A modification of Algorithm MIN to incorporate this type of heuristic rule may turn out to be useful.

Like Algorithm MAX, note that the efficiency of Algorithm MIN depends on how we can solve successfully the following problem

$$p(y) \longrightarrow \min, \quad y \in E_{f(x^k)}(f). \tag{3.16}$$

In general, we can again use in Algorithm MIN the numerical methods [45, 63] for such problem as (3.16) which find an approximate solution to problem (3.16) at each iteration k. Thus, in order to make Algorithm MIN numerically implementable based on approximate solutions of problem (3.16) we need adapt it as follows.

Algorithm ϵ-MIN

Input: A strongly convex function f and a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k > 0$ for all k and $\lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k < +\infty$, and $\varepsilon > 0$.
Output: A global minimizer x of f over D.

Step 1. Choose a feasible solution $x^o \in D$ and a sequence $\{\varepsilon_k\}$. Set $k := 0$.
Step 2. Find an ε_k solution $\bar{y}^k$ of the problem

$$p(y) \longrightarrow \min, \quad y \in E_{f(x^k)}(f),$$

that is, $p(\bar{y}^k) \le \min\limits_{y \in E_{f(x^k)}(f)} p(y) + \varepsilon_k$ and $f(\bar{y}^k) = f(x^k)$. Let $\bar{x}^{k+1} \in D$ be a solution satisfying $p(\bar{y}^k) = \langle f'(\bar{x}^{k+1}), \bar{x}^{k+1} - \bar{y}^k \rangle$.
Step 3. If $p(\bar{y}^k) \ge \varepsilon_k - \varepsilon$ then output $x = x^k$ and terminate. Otherwise set $k := k+1$ and $x^k := \bar{x}^k$, and return to Step 2. ■

The convergence of this algorithm is based on the following theorems which are the counterparts of Theorems 2.10-2.12.

Theorem 3.8 Assume that $f : D \to R$ be a continuous function defined on a compact set $D \subset R^n$. Let sequences $\{x^k\} \subset D$ and $\{\varepsilon_k\} \subset R$ be such that

$$\begin{aligned} & f(x^k) \le f(x^{k-1}) + \varepsilon_k, \quad \varepsilon_k > 0 \text{ for } k = 0, 1, ... \\ & \lim_{k\to\infty} \sum_{k=0}^{\infty} \varepsilon_k < +\infty \end{aligned}$$

Then there exists a finite limit of the sequence $\{f(x^k)\}$, that is, $\lim_{k\to\infty} f(x^k) = B$.
Proof. The proof is analogous to the proof of Theorem 2.10.

Theorem 3.9 Let a sequence $\{\varepsilon_k\} \subset R$ be such that $\varepsilon_k > 0$ and $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$. Then for the the sequence $\{x^k\}$ generated by Algorithm ϵ-MIN, we have

$$\lim_{k\to\infty} \min_{x\in D} \langle f'(x), x - \bar{y}^k \rangle = 0,$$

where $\min\limits_{x\in D} \langle f'(x), x - \bar{y}^k \rangle \le \min\limits_{y \in E_{f(x^k)}(f)} \min\limits_{x\in D} \langle f'(x), x - y \rangle + \varepsilon_k$ and $f(\bar{y}^k) = f(x^k)$.
Proof. The proof is analogous to the proof of Theorem 3.7. ■

Theorem 3.10 Assume that $f: R^n \to R$ be strongly convex and continuously differentiable, and D is a compact set in R^n. Let $\min_{x \in D} \|f'(x)\| > 0$ and a sequence $\{\varepsilon_k\} \subset R$ be such that $\varepsilon_k > 0$ and $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$. Then the sequence $\{x^k,\ k = 0, 1, ...\}$ generated by Algorithm ϵ-MIN is a minimizing sequence for problem (3.9), that is,

$$\lim_{k \to \infty} f(x^k) = \min_{x \in D} f(x),$$

and every accumulation point of the sequence $\{x^k\}$ is a global minimizer of (3.9).
Proof. Immediate from Thereoms 3.8 and 3.9.

Conclusion

To provide a unified view, we considered in this paper both the minimization and maximization of a quasiconvex function over an arbitrary compact set. Global optimality conditions for these cases were developed, based on which an attempt was made to construct algorithms with global convergences. One of the proposed algorithms was shown to be numerically implementable. We explored some connections of these problems with the multiplicative, fractional, and D.C. programmings as well as the variational inequality problem via the global optimality conditions.

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