A New Relaxation Method for Mathematical Programs with Complementarity Constraints

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Abstract. In this paper, we present a new relaxation method for mathematical programs with complementarity constraints. Based on the fact that a variational inequality problem defined on a simplex can be represented by a finite number of inequalities, we use an expansive simplex instead of the nonnegative orthant involved in the complementarity constraints. We then remove some inequalities and obtain a standard nonlinear program. We show that the linear independence constraint qualification or the Mangasarian-Fromovitz constraint qualification holds for the relaxed problem under some mild conditions. We also consider a limiting behavior of the relaxed problem. In particular, we show that any accumulation point of stationary points of the relaxed problems is a weakly stationary point of the original problem and if the function involved in the complementarity constraints does not vanish at this point, it is C-stationary. Furthermore, under some suitable conditions, it is B-stationary.

Key Words. mathematical program with equilibrium constraints, nonlinear complementarity problem, linear independence constraint qualification, nondegeneracy, B-stationary point, weakly stationary point, C-stationary point.

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1 Introduction

The mathematical program with equilibrium constraints (MPEC) is a constrained optimization problem in which the essential constraints are defined by a parametric variational inequality or complementarity system. This problem plays an important role in many fields such as engineering design, economic equilibrium, and multilevel game, see [9], and has attracted much attention in the recent literature, see the attached

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references.

The MPEC considered in this paper is the mathematical program with complementarity constraints:

$$\begin{aligned} \min \quad & f(x,y) \\ \text{s.t.} \quad & g(x) \leq 0, h(x) = 0 \\ & F(x,y) \geq 0, y \geq 0 \\ & y^T F(x,y) = 0, \end{aligned} \tag{1.1}$$

where $f : R^{n+m} \rightarrow R, g : R^n \rightarrow R^p, h : R^n \rightarrow R^q$, and $F : R^{n+m} \rightarrow R^m$ are all continuously differentiable functions. The major difficulty in solving (1.1) is that its constraints fail to satisfy a standard constraint qualification at any feasible point [2], which is necessary for the regularity of a nonlinear program, so that standard methods are likely to fail for this problem. There have been proposed several approaches such as sequential quadratic programming approach, implicit programming approach, penalty function approach, and reformulation approach [1, 4-6, 8-10, 14, 16]. In particular, Fukushima, Luo, and Pang [5] proposed a sequential quadratic programming algorithm and gave their convergence result under nondegeneracy. Fukushima and Pang [6] considered a smoothing continuation method and showed, under the MPEC-linear independence constraint qualification and an additional condition called the asymptotic weak nondegeneracy, that an accumulation point of KKT points satisfying the second-order necessary conditions for the perturbed problems is a B-stationary point of the original problem. In addition, Scholtes [16] presented a regularization scheme and proved, under the MPEC-linear independence constraint qualification and the upper level strict complementarity condition, that an accumulation point of stationary points satisfying the second order necessary conditions for the relaxed problems is a B-stationary point of the original problem.

In this paper, we study problem (1.1) from another point of view. We use an expansive simplex instead of the nonnegative orthant involved in the complementarity constraints. In other words, our method replaces the complementarity constraints by a variational inequality defined on an expansive simplex. It is well known that such a variational inequality problem can be represented by a finite number of inequalities. We remove some inequalities and obtain a standard nonlinear program. We will show that the linear independence constraint qualification (LICQ) or the Mangasarian-Fromovitz constraint qualification (MFCQ) holds for the relaxed problem under some mild conditions. We also consider a limiting behavior of the relaxed problem. In particular, we show that any accumulation point of stationary points of the relaxed problems is a weakly stationary point of the original problem and if the function F does not vanish at this point, it is C-stationary. Furthermore, under some suitable conditions, it is a B-stationary point of the original problem.

The rest of this paper is organized as follows. In the next section, we present the relaxed problem for problem (1.1) and establish the convergence of global optimal solutions of the relaxed problems. In Section 3, we give some results about constraint qualifications and the convergence of stationary points of the relaxed problems. In Section 4, we make some remarks to conclude the paper.

2 The Relaxed Problem and Some Properties

Let $e = (1, 1, \cdots, 1)^T$. For $i = 1, 2, \cdots, m$, $e_i \in R^m$ denotes the i-th column of the $m \times m$ identity matrix. Also we let $e_0 \in R^m$ denote the zero vector, i.e., $e_0 = (0, 0, \cdots, 0)^T$. Then the expansive simplex mentioned in Section 1 is defined by

$$\Omega_k = \mathrm{co}\{e_0^k, e_1^k, \cdots, e_m^k\},$$

where co stands for the convex hull, k is a positive integer, and

$$e_i^k = \frac{1}{k}e + ke_i, \quad i = 0, 1, \cdots, m. \tag{2.1}$$

For a fixed $x \in R^n$, the variational inequality problem VI$(F(x, \cdot), \Omega_k)$ is to find a vector $y \in \Omega_k$ such that

$$(y' - y)^T F(x, y) \geq 0, \quad \forall y' \in \Omega_k,$$

which is equivalent to finding a $y \in R^m$ such that

$$y \in \Omega_k, \quad (e_i^k - y)^T F(x, y) \geq 0, \quad i = 0, 1, \cdots, m$$

or equivalently,

$$\sum_{j=1}^{m} y_j \le \frac{1}{k} + k, \quad y_i \ge \frac{1}{k}, \quad (e_i^k - y)^T F(x,y) \ge 0, \quad i = 0, 1, \cdots, m.$$

In order to simplify the relaxed problem, we replace the condition $y \in \Omega_k$ by $y \ge 0$ and consider the following problem as an approximation of problem (1.1):

$$\begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & g(x) \le 0, \ h(x) = 0, \ y \ge 0 \\ & (e_i^k - y)^T F(x,y) \ge 0, \ i = 0, 1, \cdots, m. \end{array} \tag{2.2}$$

Let $\mathcal{F}$ and $\mathcal{F}_k$ denote the feasible sets of problems (1.1) and (2.2), respectively, and let

$$\phi_i^k(x,y) = (e_i^k - y)^T F(x,y), \quad i = 0, 1, \cdots, m. \tag{2.3}$$

By (2.1), we have

$$\phi_i^k(x,y) = \phi_0^k(x,y) + kF_i(x,y), \quad i = 1, 2, \cdots, m. \tag{2.4}$$

Then we have the following results.

Theorem 2.1 *For problems (1.1) and (2.2), we have*

(i) *for any k, $\mathcal{F} \subseteq \mathcal{F}_{k+1} \subseteq \mathcal{F}_k$;*

(ii) *$\mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k$, which, together with the continuity of the functions involved, implies that any accumulation point of a sequence $\{(x^k, y^k) : (x^k, y^k) \in \mathcal{F}_k\}$ belongs to $\mathcal{F}$.*

Proof: (i) $\mathcal{F} \subseteq \mathcal{F}_{k+1}$ is clear. Let $(x,y) \in \mathcal{F}_{k+1}$. Then, since for each $i = 0, 1, \cdots, m$, e_i^k can be represented as

$$e_i^k = \sum_{j=0}^{m} t_{ij} e_j^{k+1}, \quad \sum_{j=0}^{m} t_{ij} = 1, \quad t_{ij} \ge 0, \quad j = 0, 1, \cdots, m,$$

we have

$$(e_i^k - y)^T F(x,y) = \sum_{j=0}^{m} t_{ij} (e_j^{k+1} - y)^T F(x,y) \ge 0, \quad i = 0, 1, \cdots, m,$$

i.e., $(x,y) \in \mathcal{F}_k$. Hence $\mathcal{F}_{k+1} \subseteq \mathcal{F}_k$.

(ii) From (i), we only need to prove $\bigcap_{k=1}^{\infty}\mathcal{F}_k \subseteq \mathcal{F}$. Let $(x,y)\in\bigcap_{k=1}^{\infty}\mathcal{F}_k$. Then we have

$$g(x)\le 0,\ \ h(x)=0,\ \ y\ge 0$$

and, for every $i=1,2,\cdots,m$,

$$\phi_i^k(x,y)=(\frac{1}{k}e+ke_i-y)^T F(x,y)\ge 0,\quad \forall k,$$

which implies

$$(\frac{1}{k^2}e+e_i-\frac{1}{k}y)^T F(x,y)\ge 0,\quad \forall k. \tag{2.5}$$

Note that $\bigcap_{k=1}^{\infty}\mathcal{F}_k=\lim_{k\to\infty}\mathcal{F}_k$ by (i). Letting $k\to\infty$ in (2.5), we have

$$F_i(x,y)=e_i^T F(x,y)\ge 0$$

and hence $F(x,y)\ge 0$. On the other hand,

$$(e_0^k-y)^T F(x,y)\ge 0,\quad \forall k$$

implies $-y^T F(x,y)\ge 0$. So, we have $y^T F(x,y)=0$. Therefore, $(x,y)\in\mathcal{F}$ and so $\bigcap_{k=1}^{\infty}\mathcal{F}_k\subseteq\mathcal{F}$. This completes the proof. $\Box$

Theorem 2.2 *Suppose that (x^k,y^k) is a global optimal solution of problem (2.2) and (x^*,y^*) is an accumulation point of the sequence $\{(x^k,y^k)\}$ as $k\to\infty$. Then (x^*,y^*) is a global optimal solution of problem (1.1).*

Proof: Taking a subsequence if necessary, we assume without loss of generality that

$$\lim_{k\to\infty}(x^k,y^k)=(x^*,y^*).$$

By Theorem 2.1, $(x^*,y^*)\in\mathcal{F}$. Since $\mathcal{F}\subseteq\mathcal{F}_k$ for all k, then

$$f(x^k,y^k)\le f(x,y),\quad \forall (x,y)\in\mathcal{F},\quad \forall k.$$

Letting $k\to\infty$, we have from the continuity of f that

$$f(x^*,y^*)\le f(x,y),\quad \forall (x,y)\in\mathcal{F},$$

i.e., (x^*,y^*) is a global optimal solution of problem (1.1). $\Box$

In a similar way, we can prove the next theorem.

Theorem 2.3 *Let $\{\varepsilon_k\} \subseteq (0, +\infty)$ be convergent to 0 and $(x^k, y^k) \in \mathcal{F}_k$ be an approximate solution of problem (2.2) satisfying*

$$f(x^k, y^k) - \varepsilon_k \leq f(x, y), \quad \forall (x, y) \in \mathcal{F}_k.$$

Then any accumulation point of $\{(x^k, y^k)\}$ is a global optimal solution of problem (1.1).

The following result shows that problem (2.2) may satisfy some constraint qualification at its feasible points. This is in contrast with problem (1.1), for which a standard contraint qualification fails to hold at any feasible point.

Theorem 2.4 *For any $(\bar{x}, \bar{y}) \in \mathcal{F}$ with $F(\bar{x}, \bar{y}) \neq 0$, we have*

$$\phi_i^k(\bar{x}, \bar{y}) > 0, \quad i = 0, 1, \cdots, m, \quad \forall k$$

and so they are inactive constraints at $(\bar{x}, \bar{y})$ in problem (2.2). In this case, if the system

$$g(x) \leq 0, \quad h(x) = 0, \quad y \geq 0$$

satisfies some constraint qualification such as LICQ or MFCQ at $(\bar{x}, \bar{y})$, then, for any fixed k, there exists a neighborhood $U_k(\bar{x}, \bar{y})$ of $(\bar{x}, \bar{y})$ such that problem (2.2) satisfies the same constraint qualification at any point $(x, y) \in U_k(\bar{x}, \bar{y})$.

Proof: We obtain the first part from the definition (2.3) of ϕ_i^k and

$$e_i^k > 0, \quad 0 \neq F(\bar{x}, \bar{y}) \geq 0, \quad \bar{y}^T F(\bar{x}, \bar{y}) = 0,$$

immediately. The second part follows from the continuity of g, h, and $\phi_i^k, i = 0, 1, \cdots, m$, directly. $\square$

3 Limiting Behavior of Stationary Points

In this section, we consider the behavior of a stationary point of problem (2.2) as $k \to \infty$. We let $G(x, y) = y$ and

$$\phi^k(x, y) = (\phi_0^k(x, y), \phi_1^k(x, y), \cdots, \phi_m^k(x, y))^T,$$

where ϕ_i^k are defined by (2.3). As in the previous section, we denote the feasible sets of problems (1.1) and (2.2) by $\mathcal{F}$ and $\mathcal{F}_k$, respectively. In addition, for a function $H : R^n \to R^m$ and a fixed vector $x \in R^n$, we let

$$\mathcal{I}_H(x) = \{i : H_i(x) = 0\}$$

denote the active index set of H at x.

Theorem 3.1 *For any $(\bar{x}, \bar{y}) \in \mathcal{F}$, if the set of vectors*

$$\{\nabla_x F_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}), \nabla h_r(\bar{x}) : i = 1, 2, \cdots, m, l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\}$$

is linearly independent, then there exist a neighborhood $U(\bar{x}, \bar{y})$ of $(\bar{x}, \bar{y})$ and a positive integer K such that, for any $(x, y) \in U(\bar{x}, \bar{y})$ and any $k \geq K$, the following conditions hold:

(i) $\mathcal{I}_F(x, y) \subseteq \mathcal{I}_F(\bar{x}, \bar{y})$, $\mathcal{I}_G(x, y) \subseteq \mathcal{I}_G(\bar{x}, \bar{y})$, $\mathcal{I}_g(x) \subseteq \mathcal{I}_g(\bar{x})$;

(ii) $\{\nabla_x F_i(x, y), \nabla g_l(x), \nabla h_r(x) : i = 1, 2, \cdots, m, l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\}$ *is linearly independent;*

(iii) $\mathcal{I}_{\phi^k}(x, y) \subseteq \{0\} \cup \mathcal{I}_F(\bar{x}, \bar{y})$.

Proof: It is obvious that there exists a neighborhood $U_1(\bar{x}, \bar{y})$ such that conditions (i) and (ii) hold for any $(x, y) \in U_1(\bar{x}, \bar{y})$ by the continuity of $F, G, g, \nabla_x F, \nabla g$, and ∇h. Now we show that there exist a neighborhood $U_2(\bar{x}, \bar{y})$ and a positive integer K satisfying condition (iii) for any $(x, y) \in U_2(\bar{x}, \bar{y})$ and any $k \geq K$. Otherwise, there must be an $i_0 \notin \{0\} \cup \mathcal{I}_F(\bar{x}, \bar{y})$, a subsequence $\{k_j\}$ of $\{k\}$, and a sequence $\{(x^j, y^j)\}$ converging to $(\bar{x}, \bar{y})$ such that

$$\phi_{i_0}^{k_j}(x^j, y^j) = 0, \quad \forall j.$$

Since

$$\frac{1}{k_j} \phi_{i_0}^{k_j}(x^j, y^j) = F_{i_0}(x^j, y^j) + \frac{1}{k_j^2} \sum_{i=1}^{m} F_i(x^j, y^j) - \frac{1}{k_j} (y^j)^T F(x^j, y^j),$$

we have

$$\lim_{j \to \infty} \frac{1}{k_j} \phi_{i_0}^{k_j}(x^j, y^j) = F_{i_0}(\bar{x}, \bar{y}) > 0.$$

This implies that

$$\lim_{j\to\infty} \phi_{i_0}^{k_j}(x^j, y^j) = +\infty.$$

This is a contradiction and so the neighborhood $U_2(\bar{x}, \bar{y})$ and positive integer K mentioned above exist. Let

$$U(\bar{x}, \bar{y}) = U_1(\bar{x}, \bar{y}) \cap U_2(\bar{x}, \bar{y}).$$

Then conditions (i)–(iii) hold for any $(x, y) \in U(\bar{x}, \bar{y})$ and any $k \geq K$. $\square$

As we mentioned in Section 1, the nondegeneracy condition has often been assumed in the literature on MPECs. In general, a point $(x, y) \in \mathcal{F}$ is said to be nondegenerate if

$$\mathcal{I}_F(x, y) \cap \mathcal{I}_G(x, y) = \emptyset.$$

Then we have the following result about constraint qualifications.

Theorem 3.2 *Let $(\bar{x}, \bar{y}) \in \mathcal{F}$ be nondegenerate and satisfy*

$$F(\bar{x}, \bar{y}) \neq 0. \tag{3.1}$$

If the set of vectors

$$\{\nabla_x F_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}), \nabla h_r(\bar{x}) : i = 1, 2, \cdots, m, l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\}$$

is linearly independent, then there exists a neighborhood $U(\bar{x}, \bar{y})$ of $(\bar{x}, \bar{y})$ such that problem (2.2) satisfies the standard LICQ at any point $(x, y) \in U(\bar{x}, \bar{y}) \cap \mathcal{F}_k$ for all sufficiently large k.

Proof: By Theorem 3.1, there exist a neighborhood $U(\bar{x}, \bar{y})$ and a positive integer K such that Theorem 3.1 (i)–(iii) hold for any $(x, y) \in U(\bar{x}, \bar{y})$ and any $k \geq K$. Now we let $k \geq K$ and $(x, y) \in U(\bar{x}, \bar{y}) \cap \mathcal{F}_k$.

Suppose that the LICQ does not hold at (x, y) for problem (2.2). This means the set

$$\Bigg\{\nabla \phi_i^k(x, y), \nabla G_j(x, y) = \begin{pmatrix} 0 \\ e_j \end{pmatrix}, \begin{pmatrix} \nabla g_l(x) \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla h_r(x) \\ 0 \end{pmatrix} :$$
$$i \in \mathcal{I}_{\phi^k}(x, y), j \in \mathcal{I}_G(x, y), l \in \mathcal{I}_g(x), r = 1, 2, \cdots, q\Bigg\},$$

which is, by (2.4),

$$\Big\{\nabla\phi_0^k(x,y), k\nabla F_i(x,y)+\nabla\phi_0^k(x,y), \nabla G_j(x,y), \begin{pmatrix}\nabla g_l(x)\\ 0\end{pmatrix}, \begin{pmatrix}\nabla h_r(x)\\ 0\end{pmatrix}:$$
$$0\neq i\in\mathcal{I}_{\phi^k}(x,y), j\in\mathcal{I}_G(x,y), l\in\mathcal{I}_g(x), r=1,2,\cdots,q\Big\}$$

in the case where $0\in\mathcal{I}_{\phi^k}(x,y)$ or

$$\Big\{k\nabla F_i(x,y)+\nabla\phi_0^k(x,y), \nabla G_j(x,y), \begin{pmatrix}\nabla g_l(x)\\ 0\end{pmatrix}, \begin{pmatrix}\nabla h_r(x)\\ 0\end{pmatrix}:$$
$$i\in\mathcal{I}_{\phi^k}(x,y), j\in\mathcal{I}_G(x,y), l\in\mathcal{I}_g(x), r=1,2,\cdots,q\Big\}$$

in the case where $0\notin\mathcal{I}_{\phi^k}(x,y)$, is linearly dependent. Hence, $\nabla\phi_0^k(x,y)$ can be represented as a linear combination of the vectors

$$\Big\{\nabla F_i(x,y), \nabla G_j(x,y), \begin{pmatrix}\nabla g_l(x)\\ 0\end{pmatrix}, \begin{pmatrix}\nabla h_r(x)\\ 0\end{pmatrix}:$$
$$i\in\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}, j\in\mathcal{I}_G(x,y), l\in\mathcal{I}_g(x), r=1,2,\cdots,q\Big\},$$

because, by Theorem 3.1 (ii), the set

$$\Big\{\nabla F_i(x,y)=\begin{pmatrix}\nabla_x F_i(x,y)\\ \nabla_y F_i(x,y)\end{pmatrix}, \nabla G_j(x,y), \begin{pmatrix}\nabla g_l(x)\\ 0\end{pmatrix}, \begin{pmatrix}\nabla h_r(x)\\ 0\end{pmatrix}:$$
$$i\in\mathcal{I}_F(\bar{x},\bar{y}), j\in\mathcal{I}_G(x,y), l\in\mathcal{I}_g(x), r=1,2,\cdots,q\Big\}$$

is linearly independent and

$$\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}\subseteq\mathcal{I}_F(\bar{x},\bar{y}).$$

Therefore, there exist numbers

$$\{\lambda_i,\mu_j,u_l,v_r : i\in\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}, j\in\mathcal{I}_G(x,y), l\in\mathcal{I}_g(x), r=1,2,\cdots,q\}$$

such that

$$\begin{aligned}\nabla\phi_0^k(x,y) \;=\;& \sum_{i\in\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}}\lambda_i\nabla F_i(x,y)+\sum_{j\in\mathcal{I}_G(x,y)}\mu_j\nabla G_j(x,y)\\ &+\sum_{l\in\mathcal{I}_g(x)}u_l\begin{pmatrix}\nabla g_l(x)\\ 0\end{pmatrix}+\sum_{r=1}^{q}v_r\begin{pmatrix}\nabla h_r(x)\\ 0\end{pmatrix}.\end{aligned}$$

Since $\phi_0^k(x,y) = (\frac{1}{k}e - y)^T F(x,y)$, we then have

$$\begin{pmatrix} \sum_{i=1}^m (\frac{1}{k} - y_i)\nabla_x F_i(x,y) \\ \sum_{i=1}^m (\frac{1}{k} - y_i)\nabla_y F_i(x,y) \end{pmatrix} - \begin{pmatrix} 0 \\ F(x,y) \end{pmatrix} = \begin{pmatrix} \sum_{i\in\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}} \lambda_i \nabla_x F_i(x,y) \\ \sum_{i\in\mathcal{I}_{\phi^k}(x,y)\setminus\{0\}} \lambda_i \nabla_y F_i(x,y) \end{pmatrix}$$
$$+ \sum_{j\in\mathcal{I}_G(x,y)} \mu_j \begin{pmatrix} 0 \\ e_j \end{pmatrix} + \sum_{l\in\mathcal{I}_g(x)} u_l \begin{pmatrix} \nabla g_l(x) \\ 0 \end{pmatrix} + \sum_{r=1}^q v_r \begin{pmatrix} \nabla h_r(x) \\ 0 \end{pmatrix}.$$

By Theorem 3.1 (ii), we have

$$y_i = \begin{cases} \frac{1}{k} - \lambda_i, & i \in \mathcal{I}_{\phi^k}(x,y) \setminus \{0\} \\ \frac{1}{k}, & i \notin \mathcal{I}_{\phi^k}(x,y) \setminus \{0\} \end{cases} \tag{3.2}$$

and

$$F_i(x,y) = 0, \quad i \notin \mathcal{I}_G(x,y). \tag{3.3}$$

Suppose that $\phi_0^k(x,y) = 0$. We then have by (3.3) that

$$\phi_i^k(x,y) = kF_i(x,y) + \phi_0^k(x,y) = 0, \quad i \notin \mathcal{I}_G(x,y). \tag{3.4}$$

On the other hand, we have

$$\mathcal{I}_G(x,y) = \mathcal{I}_G(\bar{x},\bar{y}). \tag{3.5}$$

Otherwise, since

$$\mathcal{I}_G(x,y) \subseteq \mathcal{I}_G(\bar{x},\bar{y}),$$

there exists an $i_0 \in \mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_G(x,y)$. Then we must have $\phi_{i_0}^k(x,y) = 0$ by (3.4). But, by the nondegenerate property of $(\bar{x},\bar{y})$, $i_0 \in \mathcal{I}_G(\bar{x},\bar{y})$ means $i_0 \notin \mathcal{I}_F(\bar{x},\bar{y})$. So, by Theorem 3.1 (iii), we have $\phi_{i_0}^k(x,y) > 0$. This is a contradiction and so (3.5) holds. This means that

$$y_i = 0, \quad i \in \mathcal{I}_G(\bar{x},\bar{y}). \tag{3.6}$$

Note that

$$\mathcal{I}_G(\bar{x},\bar{y}) \neq \emptyset$$

by (3.1). So, if $i \in \mathcal{I}_G(\bar{x},\bar{y})$, then $i \notin \mathcal{I}_F(\bar{x},\bar{y})$ by the nondegeneracy assumption and so $i \notin \mathcal{I}_{\phi^k}(x,y) \setminus \{0\}$. Hence, from (3.2), we have $y_i = \frac{1}{k}$. This contradicts (3.6). Therefore, we have $\phi_0^k(x,y) \neq 0$ and so

$$\phi_i^k(x,y) = kF_i(x,y) + \phi_0^k(x,y) \neq 0, \quad i \notin \mathcal{I}_G(x,y). \tag{3.7}$$

By Theorem 3.1 (iii) and the fact that $0 \notin \mathcal{I}_{\phi^k}(x,y)$, $\phi_i^k(x,y) = 0$ means that $i \in \mathcal{I}_F(\bar{x},\bar{y})$. On the other hand, for any $i \in \mathcal{I}_F(\bar{x},\bar{y})$, by the nondegeneracy of $(\bar{x},\bar{y})$, we have $i \notin \mathcal{I}_G(\bar{x},\bar{y})$ and so $i \notin \mathcal{I}_G(x,y)$ by Theorem 3.1 (i), which implies $\phi_i^k(x,y) \neq 0$ by (3.7). Hence $\mathcal{I}_{\phi^k}(x,y) = \emptyset$, i.e., all $\phi_i^k, i = 0,1,\cdots,m$, are inactive at (x,y) for problem (2.2) and then, by the fact that the set

$$\left\{\nabla G_j(x,y), \begin{pmatrix} \nabla g_l(x) \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla h_r(x) \\ 0 \end{pmatrix} : j \in \mathcal{I}_G(x,y), l \in \mathcal{I}_g(x), r = 1,2,\cdots,q\right\}$$

is linearly independent, the LICQ holds at (x,y). This is also contradicts our assumption. Therefore, the LICQ holds at (x,y) for problem (2.2). This completes the proof. $\square$

Now we consider the limiting behavior of stationary points of problem (2.2). We will use the standard definition of stationarity for problem (2.2), i.e., $(x^k,y^k) \in \mathcal{F}_k$ is a stationary point of problem (2.2) if there exist Lagrange multiplier vectors $\lambda^k, \mu^k, \delta^k$, and γ^k such that

$$\begin{aligned} \nabla f(x^k,y^k) - \sum_{i \in \mathcal{I}_{\phi^k}(x^k,y^k)} \lambda_i^k \nabla \phi_i^k(x^k,y^k) - \sum_{j \in \mathcal{I}_G(x^k,y^k)} \mu_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ + \sum_{l \in \mathcal{I}_g(x^k)} \delta_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} + \sum_{r=1}^q \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix} = 0 \end{aligned} \tag{3.8}$$

and

$$\lambda^k \geq 0, \quad \mu^k \geq 0, \quad \delta^k \geq 0. \tag{3.9}$$

For problem (1.1), $(\bar{x},\bar{y}) \in \mathcal{F}$ is said to be a B-stationary point if it satisfies

$$d^T \nabla f(\bar{x},\bar{y}) \geq 0, \quad \forall d \in \mathcal{T}((\bar{x},\bar{y}),\mathcal{F}), \tag{3.10}$$

where $\mathcal{T}((\bar{x},\bar{y}),\mathcal{F})$ stands for the tangent cone of $\mathcal{F}$ at $(\bar{x},\bar{y})$. As in [16], a feasible point $(\bar{x},\bar{y})$ is called weakly stationary to problem (1.1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}$, and $\bar{\delta} \geq 0$ such that

$$\begin{aligned} \nabla f(\bar{x},\bar{y}) - \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} \bar{\lambda}_i \nabla F_i(\bar{x},\bar{y}) - \sum_{j \in \mathcal{I}_G(\bar{x},\bar{y})} \bar{\mu}_j \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ + \sum_{l \in \mathcal{I}_g(\bar{x})} \bar{\delta}_l \begin{pmatrix} \nabla g_l(\bar{x}) \\ 0 \end{pmatrix} + \sum_{r=1}^q \bar{\gamma}_r \begin{pmatrix} \nabla h_r(\bar{x}) \\ 0 \end{pmatrix} = 0. \end{aligned} \tag{3.11}$$

If the MPEC-LICQ holds at $(\bar{x}, \bar{y})$, which means

$$\Big\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_j(\bar{x}, \bar{y}), \begin{pmatrix} \nabla g_l(\bar{x}) \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla h_r(\bar{x}) \\ 0 \end{pmatrix} :$$
$$i \in \mathcal{I}_F(\bar{x}, \bar{y}), j \in \mathcal{I}_G(\bar{x}, \bar{y}), l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\Big\}$$

is linearly independent, then the definition (3.10) of B-stationarity can be restated in terms of Lagrange multipliers [9,14,16]: $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1.1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}$, and $\bar{\delta} \geq 0$ such that (3.11) holds with

$$\bar{\lambda}_i \geq 0, \quad \bar{\mu}_i \geq 0, \quad i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y}). \tag{3.12}$$

Another kind of stationarity concepts for MPECs called C-stationarity [16], which is weaker than B-stationarity, is also employed often. We say $(\bar{x}, \bar{y})$ is C-stationary for problem (1.1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}$, and $\bar{\delta} \geq 0$ such that (3.11) holds and

$$\bar{\lambda}_i \bar{\mu}_i \geq 0, \quad i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y}). \tag{3.13}$$

The main result of the paper can be stated as follows.

Theorem 3.3 *Let $(x^k, y^k) \in \mathcal{F}_k$ be a stationary point of problem (2.2) with Lagrange multiplier vectors $\lambda^k, \mu^k, \delta^k$, and γ^k, and $(\bar{x}, \bar{y})$ be an accumulation point of the sequence $\{(x^k, y^k)\}$. If the set of vectors*

$$\{\nabla_x F_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}), \nabla h_r(\bar{x}) : i = 1, 2, \cdots, m, l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\}$$

is linearly independent, then $(\bar{x}, \bar{y})$ is a weakly stationary point of problem (1.1). Furthermore, if $F(\bar{x}, \bar{y}) \neq 0$, $(\bar{x}, \bar{y})$ is C-stationary.

Proof: Without loss of generality, we assume that

$$\lim_{k \to \infty} (x^k, y^k) = (\bar{x}, \bar{y}). \tag{3.14}$$

Then by Theorem 2.1, we have $(\bar{x}, \bar{y}) \in \mathcal{F}$. By Theorem 3.1, we can suppose that k is sufficiently large so that for any k,

$$\mathcal{I}_F(x^k, y^k) \subseteq \mathcal{I}_F(\bar{x}, \bar{y}), \quad \mathcal{I}_G(x^k, y^k) \subseteq \mathcal{I}_G(\bar{x}, \bar{y}),$$
$$\mathcal{I}_g(x^k) \subseteq \mathcal{I}_g(\bar{x}), \quad \mathcal{I}_{\phi^k}(x^k, y^k) \subseteq \{0\} \cup \mathcal{I}_F(\bar{x}, \bar{y}),$$

and

$$\{\nabla_x F_i(x^k, y^k), \nabla g_l(x^k), \nabla h_r(x^k) : i = 1, 2, \cdots, m, l \in \mathcal{I}_g(\bar{x}), r = 1, 2, \cdots, q\}$$

is linearly independent. Note that the MPEC-LICQ holds at $(\bar{x}, \bar{y})$ for problem (1.1).

Since

$$\nabla \phi_i^k(x^k, y^k) = \nabla \phi_0^k(x^k, y^k) + k \begin{pmatrix} \nabla_x F_i(x^k, y^k) \\ \nabla_y F_i(x^k, y^k) \end{pmatrix}, \quad i = 1, 2, \cdots, m$$

and

$$\begin{aligned} \nabla \phi_0^k(x^k, y^k) &= \begin{pmatrix} \nabla_x F(x^k, y^k)(\frac{1}{k}e - y^k) \\ \nabla_y F(x^k, y^k)(\frac{1}{k}e - y^k) - F(x^k, y^k) \end{pmatrix} \\ &= \sum_{i=1}^{m} (\frac{1}{k} - y_i^k) \begin{pmatrix} \nabla_x F_i(x^k, y^k) \\ \nabla_y F_i(x^k, y^k) \end{pmatrix} - \sum_{j=1}^{m} F_j(x^k, y^k) \begin{pmatrix} 0 \\ e_j \end{pmatrix}, \end{aligned} \tag{3.15}$$

we have from (3.8) that

$$\begin{aligned} \nabla f(x^k, y^k) &= \sum_{i \in \mathcal{I}_{\phi^k}(x^k, y^k)} \lambda_i^k \nabla \phi_i^k(x^k, y^k) + \sum_{j \in \mathcal{I}_G(x^k, y^k)} \mu_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ &\quad - \sum_{l \in \mathcal{I}_g(x^k)} \delta_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} - \sum_{r=1}^{q} \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix} \\ &= \sum_{0 \neq i \in \mathcal{I}_{\phi^k}(x^k, y^k)} k \lambda_i^k \begin{pmatrix} \nabla_x F_i(x^k, y^k) \\ \nabla_y F_i(x^k, y^k) \end{pmatrix} + a_k \nabla \phi_0^k(x^k, y^k) + \sum_{j \in \mathcal{I}_G(x^k, y^k)} \mu_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ &\quad - \sum_{l \in \mathcal{I}_g(x^k)} \delta_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} - \sum_{r=1}^{q} \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix} \qquad (3.16) \\ &= \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} u_i^k \begin{pmatrix} \nabla_x F_i(x^k, y^k) \\ \nabla_y F_i(x^k, y^k) \end{pmatrix} + \sum_{i \notin \mathcal{I}_F(\bar{x}, \bar{y})} a_k (\frac{1}{k} - y_i^k) \begin{pmatrix} \nabla_x F_i(x^k, y^k) \\ \nabla_y F_i(x^k, y^k) \end{pmatrix} \\ &\quad + \sum_{j \in \mathcal{I}_G(\bar{x}, \bar{y})} v_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} - \sum_{j \notin \mathcal{I}_G(\bar{x}, \bar{y})} a_k F_j(x^k, y^k) \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ &\quad - \sum_{l \in \mathcal{I}_g(\bar{x})} w_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} - \sum_{r=1}^{q} \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix}, \qquad (3.17) \end{aligned}$$

where

$$a_k = \sum_{i \in \mathcal{I}_{\phi^k}(x^k, y^k)} \lambda_i^k, \tag{3.18}$$

$$u_i^k = \begin{cases} k\lambda_i^k + a_k(\frac{1}{k} - y_i^k), & 0 \neq i \in \mathcal{I}_{\phi^k}(x^k, y^k) \\ a_k(\frac{1}{k} - y_i^k), & i \in \mathcal{I}_F(\bar{x}, \bar{y}) \setminus \mathcal{I}_{\phi^k}(x^k, y^k), \end{cases} \tag{3.19}$$

$$v_j^k = \begin{cases} \mu_j^k - a_k F_j(x^k, y^k), & j \in \mathcal{I}_G(x^k, y^k) \\ -a_k F_j(x^k, y^k), & j \in \mathcal{I}_G(\bar{x}, \bar{y}) \setminus \mathcal{I}_G(x^k, y^k), \end{cases} \tag{3.20}$$

$$w_l^k = \begin{cases} \delta_l^k, & l \in \mathcal{I}_g(x^k) \\ 0, & l \in \mathcal{I}_g(\bar{x}) \setminus \mathcal{I}_g(x^k). \end{cases}$$

Since

$$\left\{\nabla F_i(\bar{x},\bar{y}), \nabla G_j(\bar{x},\bar{y}), \begin{pmatrix} \nabla g_l(\bar{x}) \\ 0 \end{pmatrix}, \begin{pmatrix} \nabla h_r(\bar{x}) \\ 0 \end{pmatrix} : \right.$$
$$\left. i,j = 1,2,\cdots,m,\, l \in \mathcal{I}_g(\bar{x}),\, r = 1,2,\cdots,q \right\}$$

is linearly independent, the multiplier sequences

$$\{u_i^k : i \in \mathcal{I}_F(\bar{x},\bar{y})\},\quad \{v_j^k : j \in \mathcal{I}_G(\bar{x},\bar{y})\},\quad \{-a_k F_j(x^k,y^k) : j \notin \mathcal{I}_G(\bar{x},\bar{y})\},$$
$$\{a_k(\frac{1}{k} - y_i^k) : i \notin \mathcal{I}_F(\bar{x},\bar{y})\},\quad \{w_l^k : l \in \mathcal{I}_g(\bar{x})\},\quad \{\gamma_r^k : r = 1,2,\cdots,q\}$$

are convergent.

(I) Suppose that $(x^k,y^k) \in \mathcal{F}$ for infinitely many k. First we show that if $(x^k,y^k) \in \mathcal{F}$, then it is a B-stationary point of problem (1.1). In fact, if $F(x^k,y^k) \neq 0$, then, from Theorem 2.4, $\mathcal{I}_{\phi^k}(x^k,y^k) = \emptyset$ and so (3.8)–(3.9) means that (x^k,y^k) is a B-stationary point of problem (1.1). If $F(x^k,y^k) = 0$, then $\mathcal{I}_{\phi^k}(x^k,y^k) = \{0,1,\cdots,m\}$ and so, it follows from (3.15) and (3.16) that

$$\begin{aligned}
0 &= \nabla f(x^k,y^k) - \sum_{i=1}^m k\lambda_i^k \nabla F_i(x^k,y^k) - a_k \nabla \phi_0^k(x^k,y^k) \\
&\quad - \sum_{j\in\mathcal{I}_G(x^k,y^k)} \mu_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} + \sum_{l\in\mathcal{I}_g(x^k)} \delta_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} + \sum_{r=1}^q \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix} \\
&= \nabla f(x^k,y^k) - \sum_{i\in\mathcal{I}_F(x^k,y^k)} (a_k(\frac{1}{k} - y_i^k) + k\lambda_i^k)\nabla F_i(x^k,y^k) - \sum_{j\in\mathcal{I}_G(x^k,y^k)} \mu_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\
&\quad + \sum_{l\in\mathcal{I}_g(x^k)} \delta_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} + \sum_{r=1}^q \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix}.
\end{aligned}$$

For $i \in \mathcal{I}_F(x^k,y^k) \cap \mathcal{I}_G(x^k,y^k)$, we have from (3.9) and (3.18) that

$$a_k(\frac{1}{k} - y_i^k) + k\lambda_i^k = \frac{1}{k}a_k + k\lambda_i^k \geq 0,\quad \mu_i^k \geq 0$$

and hence, comparing with (3.11) and (3.12), we see that (x^k,y^k) is a B-stationary point of problem (1.1). Then, taking a limit, we obtain the B-stationarity of $(\bar{x},\bar{y})$ for problem (1.1).

(II) Next we assume that $(x^k,y^k) \notin \mathcal{F}$ for all sufficiently large k.

(IIa) If $\mathcal{I}_F(\bar{x},\bar{y})=\emptyset$, then $\mathcal{I}_G(\bar{x},\bar{y})=\{1,2,\cdots,m\}$ and so (3.17) becomes

$$\begin{aligned}\nabla f(x^k,y^k) &= \sum_{i=1}^m a_k(\frac{1}{k}-y_i^k)\nabla F_i(x^k,y^k)+\sum_{j=1}^m v_j^k\begin{pmatrix}0\\ e_j\end{pmatrix}\\ &\quad -\sum_{l\in\mathcal{I}_g(\bar{x})} w_l^k\begin{pmatrix}\nabla g_l(x^k)\\ 0\end{pmatrix}-\sum_{r=1}^q \gamma_r^k\begin{pmatrix}\nabla h_r(x^k)\\ 0\end{pmatrix}. \end{aligned}\tag{3.21}$$

For any sufficiently large k, since $(x^k,y^k)\notin\mathcal{F}$, there exists an index $j\in\mathcal{I}_G(\bar{x},\bar{y})\setminus\mathcal{I}_G(x^k,y^k)$. Therefore, we can choose an index j_0 and a subsequence $\{k_l\}$ of $\{k\}$ such that

$$j_0\in\mathcal{I}_G(\bar{x},\bar{y})\setminus\mathcal{I}_G(x^{k_l},y^{k_l}),\quad \forall l,$$

i.e., by (3.20),

$$v_{j_0}^{k_l}=-a_{k_l}F_{j_0}(x^{k_l},y^{k_l}),\quad \forall l.$$

Since $\{v_{j_0}^{k_l}\}$ converges and, by $\mathcal{I}_F(\bar{x},\bar{y})=\emptyset$,

$$\lim_{l\to\infty}F_{j_0}(x^{k_l},y^{k_l})>0,$$

it follows that the sequence $\{a_{k_l}\}$ is convergent. Noticing that $\{y^k\}$ tends to $\bar{y}=0$ as $k\to\infty$, we have that, for each j,

$$\lim_{k\to\infty}a_k(\frac{1}{k}-y_j^k)=\lim_{l\to\infty}a_{k_l}(\frac{1}{k_l}-y_j^{k_l})=0.$$

Letting $k\to\infty$ in (3.21) and denoting

$$\bar{v}_j=\lim_{k\to\infty}v_j^k,\quad \bar{w}_l=\lim_{k\to\infty}w_l^k,\quad \bar{\gamma}_r=\lim_{k\to\infty}\gamma_r^k,$$

we obtain

$$\nabla f(\bar{x},\bar{y})=\sum_{j=1}^m\bar{v}_j\begin{pmatrix}0\\ e_j\end{pmatrix}-\sum_{l\in\mathcal{I}_g(\bar{x})}\bar{w}_l\begin{pmatrix}\nabla g_l(\bar{x})\\ 0\end{pmatrix}-\sum_{r=1}^q\bar{\gamma}_r\begin{pmatrix}\nabla h_r(\bar{x})\\ 0\end{pmatrix}.$$

This, together with

$$\mathcal{I}_F(\bar{x},\bar{y})\cap\mathcal{I}_G(\bar{x},\bar{y})=\emptyset,$$

implies that $(\bar{x},\bar{y})$ is a B-stationary point of problem (1.1).

(IIb) Now we consider the case where $\mathcal{I}_F(\bar{x},\bar{y})\neq\emptyset$.

(i) Suppose that there is a subsequence $\{k_l\}$ of $\{k\}$ such that $0 \notin \mathcal{I}_{\phi^{k_l}}(x^{k_l}, y^{k_l})$ for all l. Then, noting that, by (3.18) and (3.19),

$$\begin{aligned} \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} u_i^{k_l} &= \sum_{i \in \mathcal{I}_{\phi^{k_l}}(x^{k_l},y^{k_l})} (k_l \lambda_i^{k_l} + a_{k_l}(\frac{1}{k_l} - y_i^{k_l})) + \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y}) \setminus \mathcal{I}_{\phi^{k_l}}(x^{k_l},y^{k_l})} a_{k_l}(\frac{1}{k_l} - y_i^{k_l}) \\ &= a_{k_l}(k_l + \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} (\frac{1}{k_l} - y_i^{k_l})) \end{aligned}$$

and

$$\begin{aligned} &\lim_{l\to\infty} \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} u_i^{k_l} \quad \text{exists}, \\ &\lim_{l\to\infty} (k_l + \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} (\frac{1}{k_l} - y_i^{k_l})) = +\infty, \end{aligned}$$

we have that

$$\lim_{l\to\infty} a_{k_l} = 0. \tag{3.22}$$

Therefore, we obtain

$$\lim_{k\to\infty} a_k(\frac{1}{k} - y_i^k) = \lim_{l\to\infty} a_{k_l}(\frac{1}{k_l} - y_i^{k_l}) = 0, \quad i \notin \mathcal{I}_F(\bar{x},\bar{y}) \tag{3.23}$$

and

$$\lim_{k\to\infty} a_k F_j(x^k, y^k) = \lim_{l\to\infty} a_{k_l} F_j(x^{k_l}, y^{k_l}) = 0, \quad j \notin \mathcal{I}_G(\bar{x},\bar{y}). \tag{3.24}$$

On the other hand, by (3.9), (3.22), and (3.19)–(3.20), we have

$$\lim_{k\to\infty} u_i^k \geq 0, \quad \lim_{k\to\infty} v_i^k \geq 0, \quad i \in \mathcal{I}_G(\bar{x},\bar{y}) \cap \mathcal{I}_F(\bar{x},\bar{y}). \tag{3.25}$$

It then follows from (3.17) and (3.23)–(3.25) that conditions (3.11) and (3.12) hold. Therefore, $(\bar{x},\bar{y})$ is a B-stationary point of problem (1.1).

(ii) Suppose that $0 \in \mathcal{I}_{\phi^k}(x^k, y^k)$ for all sufficiently large k. Then it follows from (2.4) that

$$\mathcal{I}_{\phi^k}(x^k, y^k) = \{0\} \cup \mathcal{I}_F(x^k, y^k). \tag{3.26}$$

(iia) If there exist a subsequence $\{k_l\}$ of $\{k\}$ and an index i_0 such that

$$i_0 \notin \mathcal{I}_G(\bar{x},\bar{y}), \quad i_0 \in \mathcal{I}_F(\bar{x},\bar{y}) \setminus \mathcal{I}_F(x^{k_l}, y^{k_l}), \quad \forall l$$

or

$$i_0 \notin \mathcal{I}_F(\bar{x},\bar{y}), \quad i_0 \in \mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_G(x^{k_l},y^{k_l}), \quad \forall l,$$

then, by (3.19) and (3.20),

$$u_{i_0}^{k_l} = a_{k_l}\Big(\frac{1}{k_l} - y_{i_0}^{k_l}\Big), \quad \forall l \tag{3.27}$$

or

$$v_{i_0}^{k_l} = -a_{k_l} F_{i_0}(x^{k_l},y^{k_l}), \quad \forall l \tag{3.28}$$

holds. Since we have

$$\lim_{l\to\infty}\Big(\frac{1}{k_l} - y_{i_0}^{k_l}\Big) = -\bar{y}_{i_0} < 0$$

in the former case, or

$$\lim_{l\to\infty} F_{i_0}(x^{k_l},y^{k_l}) = F_{i_0}(\bar{x},\bar{y}) > 0$$

in the latter case, it follows from (3.27) or (3.28) that $\{a_{k_l}\}$ converges. Then we also have (3.23)–(3.25) and hence $(\bar{x},\bar{y})$ is a B-stationary point of problem (1.1).

(iib) Now suppose that

$$\{1,2,\cdots,m\} \setminus \mathcal{I}_F(\bar{x},\bar{y}) \subseteq \mathcal{I}_G(x^k,y^k) \tag{3.29}$$

and

$$\{1,2,\cdots,m\} \setminus \mathcal{I}_G(\bar{x},\bar{y}) \subseteq \mathcal{I}_F(x^k,y^k) \tag{3.30}$$

for all sufficiently large k. Then, since $F_j(x^k,y^k) = 0$ for any $j \notin \mathcal{I}_G(\bar{x},\bar{y})$ and $y_i^k = 0$ for any $i \notin \mathcal{I}_F(\bar{x},\bar{y})$, (3.17) yields

$$\begin{aligned} \nabla f(x^k,y^k) &= \sum_{i\in\mathcal{I}_F(\bar{x},\bar{y})} u_i^k \begin{pmatrix} \nabla_x F_i(x^k,y^k) \\ \nabla_y F_i(x^k,y^k) \end{pmatrix} + \sum_{j\in\mathcal{I}_G(\bar{x},\bar{y})} v_j^k \begin{pmatrix} 0 \\ e_j \end{pmatrix} \\ &\quad + \sum_{i\notin\mathcal{I}_F(\bar{x},\bar{y})} \frac{a_k}{k} \begin{pmatrix} \nabla_x F_i(x^k,y^k) \\ \nabla_y F_i(x^k,y^k) \end{pmatrix} - \sum_{l\in\mathcal{I}_g(\bar{x})} w_l^k \begin{pmatrix} \nabla g_l(x^k) \\ 0 \end{pmatrix} \\ &\quad - \sum_{r=1}^{q} \gamma_r^k \begin{pmatrix} \nabla h_r(x^k) \\ 0 \end{pmatrix} \end{aligned} \tag{3.31}$$

for all sufficiently large k. If $F(\bar{x},\bar{y}) = 0$, i.e., $\mathcal{I}_F(\bar{x},\bar{y}) = \{1,2,\cdots,m\}$, then (3.31) implies that the limit $(\bar{x},\bar{y})$ of $\{(x^k,y^k)\}$ satisfies the weak stationarity condition (3.11) for problem (1.1). If $F(\bar{x},\bar{y}) \neq 0$, then there exists an index i such that $F_i(\bar{x},\bar{y}) > 0$ and $\bar{y}_i = 0$, which implies

$$\mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_F(\bar{x},\bar{y}) \neq \emptyset$$

and

$$\sum_{i \in \mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_F(\bar{x},\bar{y})} F_i(\bar{x},\bar{y}) > 0. \tag{3.32}$$

By (3.29) and (3.30), for all sufficiently large k, we have

$$\begin{aligned} 0 &= \phi_0^k(x^k,y^k) \\ &= \sum_{i=1}^m (\frac{1}{k} - y_i^k) F_i(x^k,y^k) \\ &= \sum_{i \in \mathcal{I}_G(\bar{x},\bar{y})} (\frac{1}{k} - y_i^k) F_i(x^k,y^k) \\ &= \sum_{i \in \mathcal{I}_G(\bar{x},\bar{y}) \cap \mathcal{I}_F(\bar{x},\bar{y})} (\frac{1}{k} - y_i^k) F_i(x^k,y^k) + \sum_{i \in \mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_F(\bar{x},\bar{y})} \frac{1}{k} F_i(x^k,y^k). \end{aligned} \tag{3.33}$$

For any $i \in \mathcal{I}_F(\bar{x},\bar{y}) \cap \mathcal{I}_G(\bar{x},\bar{y})$, it follows from (3.19) and (3.26) that

$$\begin{aligned} |a_k(\frac{1}{k} - y_i^k) F_i(x^k,y^k)| &= \begin{cases} 0, & i \in \mathcal{I}_F(x^k,y^k) \\ |u_i^k F_i(x^k,y^k)|, & i \in \mathcal{I}_F(\bar{x},\bar{y}) \setminus \mathcal{I}_F(x^k,y^k) \end{cases} \\ &\leq |u_i^k F_i(x^k,y^k)| \end{aligned}$$

and so

$$\lim_{k\to\infty} a_k(\frac{1}{k} - y_i^k) F_i(x^k,y^k) = 0. \tag{3.34}$$

Hence, by (3.32), (3.33), and (3.34), we have

$$\lim_{k\to\infty} \frac{a_k}{k} = -\lim_{k\to\infty} \frac{\sum_{i \in \mathcal{I}_G(\bar{x},\bar{y}) \cap \mathcal{I}_F(\bar{x},\bar{y})} a_k(\frac{1}{k} - y_i^k) F_i(x^k,y^k)}{\sum_{i \in \mathcal{I}_G(\bar{x},\bar{y}) \setminus \mathcal{I}_F(\bar{x},\bar{y})} F_i(x^k,y^k)} = 0. \tag{3.35}$$

Therefore, taking a limit in (3.31), we obtain (3.11) from (3.35).

Next we show (3.13) holds. Let $i \in \mathcal{I}_F(\bar{x},\bar{y}) \cap \mathcal{I}_G(\bar{x},\bar{y})$. Note that, by the assumption of (ii),

$$kF_i(x^k,y^k) = \phi_i^k(x^k,y^k) - \phi_0^k(x^k,y^k) = \phi_i^k(x^k,y^k) \geq 0, \quad i = 1,2,\cdots,m,$$

i.e.,

$$F(x^k,y^k) \geq 0 \tag{3.36}$$

for all sufficiently large k. Suppose that there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$y_i^{k_l} F_i(x^{k_l}, y^{k_l}) \neq 0, \quad \forall l.$$

It follows from (3.19) and (3.20) that

$$u_i^{k_l} = a_{k_l}(\frac{1}{k_l} - y_i^{k_l}), \quad v_i^{k_l} = -a_{k_l} F_i(x^{k_l}, y^{k_l}).$$

By (3.35) and (3.36), we have

$$\lim_{k\to\infty} u_i^k v_i^k = \lim_{l\to\infty} u_i^{k_l} v_i^{k_l} = \lim_{l\to\infty} a_{k_l}^2 y_i^{k_l} F_i(x^{k_l}, y^{k_l}) \geq 0. \tag{3.37}$$

Next we suppose that

$$y_i^k F_i(x^k, y^k) = 0 \tag{3.38}$$

for all sufficiently large k. First consider the case where there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$y_i^{k_l} \neq 0, \quad \forall l.$$

Then, by (3.38) and (3.20), $F_i(x^{k_l}, y^{k_l}) = 0$ and hence $v_i^{k_l} = 0$ for any sufficiently large l. So we obtain

$$\lim_{k\to\infty} u_i^k v_i^k = \lim_{l\to\infty} u_i^{k_l} v_i^{k_l} = 0. \tag{3.39}$$

Next consider the case where $y_i^k = 0$ for all sufficiently large k. If there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$F_i(x^{k_l}, y^{k_l}) \neq 0, \quad \forall l,$$

then, by (3.19) and (3.35), we have

$$\lim_{l\to\infty} u_i^{k_l} = \lim_{l\to\infty} \frac{a_{k_l}}{k_l} = 0$$

and so (3.39) also holds. If for any sufficiently large k,

$$y_i^k = 0, \quad F_i(x^k, y^k) = 0,$$

then, by (3.9), (3.19)–(3.20), and (3.35),

$$\lim_{k\to\infty} u_i^k v_i^k = \lim_{k\to\infty} \mu_i^k (k\lambda_i^k + \frac{a_k}{k}) = \lim_{k\to\infty} k\lambda_i^k \mu_i^k \geq 0.$$

Therefore, $\lim_{k\to\infty} u_i^k v_i^k \geq 0$ in any case, i.e., $(\bar{x}, \bar{y})$ is a C-stationary point of problem (1.1).

Consequently, $(\bar{x}, \bar{y})$ is a B-stationary point or a weakly stationary point of problem (1.1) in any case. Moreover, if $F(\bar{x}, \bar{y}) \neq 0$, then $(\bar{x}, \bar{y})$ is C-stationary. This completes the proof. □

Corollary 3.1 *Let the assumptions in Theorem 3.3 be satisfied. If, in addition, $(\bar{x}, \bar{y})$ is nondegenerate, then it is a B-stationary point of problem (1.1).*

Proof: It is obvious from the definitions of weak stationarity and B-stationarity that, under the nondegeneracy assumption, they are equivalent. So, we have this corollary from Theorem 3.3. □

Corollary 3.2 *Let the assumptions in Theorem 3.3 be satisfied. If $(x^k, y^k) \in \mathcal{F}$ for some k, then it is a B-stationary point of problem (1.1).*

Proof: We can obtain this corollary from Part (I) in the proof of Theorem 3.3 directly. □

We also have the following two corollaries from the proof of Theorem 3.3.

Corollary 3.3 *Let the assumptions in Theorem 3.3 be satisfied. If $0 \notin \mathcal{I}_{\phi^k}(x^k, y^k)$ for infinitely many k, then $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1.1).*

For a sequence $\{(x^k, y^k)\}$ of stationary points of problem (2.2), let us define

$$\begin{aligned} \mathcal{I}_1 &= \{i : y_i^k > 0 \ \text{ for infinitely many } \ k\}, \\ \mathcal{I}_2 &= \{i : F_i(x^k, y^k) \neq 0 \ \text{ for infinitely many } \ k\}. \end{aligned}$$

Then we have

$$\begin{aligned} \{1, 2, \cdots, m\} \setminus \mathcal{I}_G(\bar{x}, \bar{y}) &\subseteq \mathcal{I}_1 \cap \mathcal{I}_F(\bar{x}, \bar{y}), \\ \{1, 2, \cdots, m\} \setminus \mathcal{I}_F(\bar{x}, \bar{y}) &\subseteq \mathcal{I}_2 \cap \mathcal{I}_G(\bar{x}, \bar{y}). \end{aligned}$$

Corollary 3.4 *Let the assumptions in Theorem 3.3 be satisfied. If* $\mathcal{I}_1 \setminus \mathcal{I}_F(\bar{x}, \bar{y}) \neq \emptyset$ *or* $\mathcal{I}_2 \setminus \mathcal{I}_G(\bar{x}, \bar{y}) \neq \emptyset$*, then* $(\bar{x}, \bar{y})$ *is a B-stationary point of problem (1.1).*

4 Concluding Remarks

Suppose that the condition $\sum_{j=1}^m y_j \leq \frac{1}{k} + k$ is retained in the constraints of problem (2.2), i.e., problem (2.2) is replaced by the problem

$$\begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & g(x) \leq 0, \ h(x) = 0 \\ & y \geq 0, \ \displaystyle\sum_{j=1}^m y_j \leq \frac{1}{k} + k \\ & (e_i^k - y)^T F(x,y) \geq 0, \ i = 0, 1, \cdots, m. \end{array} \tag{4.1}$$

Then, since the constraint

$$\sum_{j=1}^m y_j \leq \frac{1}{k} + k$$

eventually becomes inactive at any fixed point as k tends to ∞, all the results established in the previous sections remain ture except that the results in Theorem 2.1 are replaced by $\mathcal{F} = \lim_{k \to \infty} \mathcal{F}_k$. When the set $X = \{x \in R^n : g(x) \leq 0, h(x) = 0\}$ is bounded, problem (4.1) has a compact feasible region and so it is solvable for any k as long as it is feasible.

In addition, we remark that the term $\frac{1}{k}e$ in (2.1) is necessary for problem (2.2) to have desirable properties. In fact, the problem

$$\begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & g(x) \leq 0, \ h(x) = 0, \ y \geq 0 \\ & (ke_i - y)^T F(x,y) \geq 0, \ i = 0, 1, \cdots, m \end{array} \tag{4.2}$$

is difficult to handle because problem (4.2) does not satisfy the MFCQ at any point $(\bar{x}, \bar{y}) \in \mathcal{F}$ for all sufficiently large k. For simplicity, we assume that the constraints $g(x) \leq 0$ and $h(x) = 0$ are absent and let

$$G(x,y) = y, \quad \psi_i^k(x,y) = (ke_i - y)^T F(x,y), \quad i = 0, 1, \cdots, m. \tag{4.3}$$

Note that

$$\psi_i^k(x,y) = kF_i(x,y) + \psi_0^k(x,y), \quad i = 1, 2, \cdots, m. \tag{4.4}$$

At $(\bar{x}, \bar{y}) \in \mathcal{F}$, the set of active constraints is

$$\{\psi_0^k,\ \psi_i^k,\ G_j :\ i \in \mathcal{I}_F(\bar{x},\bar{y}),\ j \in \mathcal{I}_G(\bar{x},\bar{y})\}.$$

Suppose that k is large enough to satisfy

$$1 - \frac{1}{k} \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} \bar{y}_i > 0 \tag{4.5}$$

and the MFCQ holds at $(\bar{x}, \bar{y})$ for problem (4.2). Then there exists a vector $(x, y) \in R^{n+m}$ such that

$$\nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} > 0, \tag{4.6}$$

$$\nabla \psi_i^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} > 0, \quad i \in \mathcal{I}_F(\bar{x},\bar{y}), \tag{4.7}$$

$$y_j = \nabla G_j(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} > 0, \quad j \in \mathcal{I}_G(\bar{x},\bar{y}). \tag{4.8}$$

By (4.4) and (4.7), we have

$$-\nabla F_i(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} < \frac{1}{k} \nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix}, \quad i \in \mathcal{I}_F(\bar{x},\bar{y}). \tag{4.9}$$

It then follows from (4.3), (4.8), and (4.9) that

$$\begin{aligned}
\nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} &= -\sum_{i=1}^m \bar{y}_i \nabla F_i(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} - \sum_{j=1}^m F_j(\bar{x},\bar{y}) y_j \\
&= -\sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} \bar{y}_i \nabla F_i(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} - \sum_{j \in \mathcal{I}_G(\bar{x},\bar{y})} F_j(\bar{x},\bar{y}) y_j \\
&\leq \left(\frac{1}{k} \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} \bar{y}_i\right) \nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix},
\end{aligned}$$

i.e.,

$$\left(1 - \frac{1}{k} \sum_{i \in \mathcal{I}_F(\bar{x},\bar{y})} \bar{y}_i\right) \nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} \leq 0.$$

By (4.5), we have

$$\nabla \psi_0^k(\bar{x},\bar{y})^T \begin{pmatrix} x \\ y \end{pmatrix} \leq 0.$$

This contradicts (4.6) and hence the MFCQ does not hold at $(\bar{x}, \bar{y})$ for problem (4.2) when k is sufficiently large.

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