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Local Stability Analysis of Saturating Feedback System Based on LPV Descriptor Represntation

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Abstract

This paper considers the local stability and the quadratic performance of a feedback system with multiple saturation nonlinearities. By combining the linear parameter-varying technique and the descriptor system representation, we develop a new local stability condition that is not only numerically tractable but also less conservative than the circle criterion. We also derive a sufficient condition for the quadratic performance based on this local stability condition. A numerical example is included to show the effectiveness of the present stability condition.

Keywords: saturating system, local stability, quadratic performance, LPV descriptor system representation, nonlinear matrix inequality

1. Introduction

A number of physical systems are subject to input saturations. As well known, the saturation of control inputs has a significant influence on the stability and the performance of the closed-loop systems. Note that this kind of issue has been disscussed in the context of anti-windup control problems. Therefore, it is very important to study the feedback control systems with saturations.

There have been many research works on the stability analysis of the systems with saturation nonlinearity (Tyan and Bernstein 1995, Hindi and Boyd 1998, Pittet *et al.* 1997, Kothare and Morari 1999) and the references therein). Most of them are based on the absolute stability theory (Khalil 1996) or its local versions. That is, we describe the saturating control system as a feedback interconnection of a sector-bounded nonlinearity and a stable LTI system, and then apply the circle or Popov criterion to this feedback system. The approach based on the circle and Popov criteria is conservative since the structure of saturation nonlinearity is neglected by regarding it as sector-bounded uncertainty.

Recently, another approach has been proposed based on the linear parameter-varying (LPV) system representation or gain scheduling framework. Wu and his co-workers (1998,1999) proposed the design methods of an anti-windup LPV controller. For a feedback system with radial amplitude saturation (or single saturation), Takaba and Tomida (2000) and Tomida and Takaba (2000) derived parameter-dependent matrix inequality conditions for local stability, quadratic performance and L_2 gain analysis. Although the use of LPV system representation has a potential of improving conservativeness in both analysis and synthesis for saturating systems, the resulting conditions are given in terms of parameter-dependent matrix inequalities which are difficult to solve numerically.

In this paper, we consider the analysis of local stability and quadratic performance of feedback systems with componentwise saturation (or multiple saturations). By combining the descriptor system representation (Chen *et al.* 1999, Takaba 2000) and the technique by Takaba and Tomida (2000), we derive matrix inequality conditions for local stability and quadratic performance, which are numerically more tractable than the previous conditions by Takaba and Tomida (2000). Moreover, we will show that the present local stability condition is less conservative than the (local) circle criterion.

Notation: Throughout this paper, we use the following standard notations. $\mathrm{Tr}(\,\cdot\,)$ denote the trace of a matrix, respectively. The ellipsoid associated with a positive definite matrix $P \in \mathbb{R}^{n\times n}$ is defined by

$$\mathcal{E}(P) := \{x \in \mathbb{R}^n \mid x^T P x \leq 1\}$$

We also define

$$\mathrm{He}(A) = A^T + A$$

$$\mathrm{col}(x_1, x_2, \cdots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

2. Feedback Control System with Saturation

We consider the feedback control system depicted in Fig. 1, where $\boldsymbol{G}$ and $\boldsymbol{K}$ are the plant and the controller, respectively. We assume that both of the plant and the controller are linear time-invariant (LTI) systems, and that they are expressed by the following state-space realizations:

$$\dot{x}_p(t) = Ax_p(t) + Bu(t) \tag{1a}$$
$$e(t) = C_1x_p(t) + Du(t) \tag{1b}$$
$$y(t) = C_2x_p(t) \tag{1c}$$

$$\dot{x}_k(t) = A_kx_k(t) + B_{k1}y(t) + B_{k2}w(t) \tag{2a}$$
$$z(t) = C_kx_k(t) + D_{k1}y(t) + D_{k2}w(t) \tag{2b}$$

The signals z, u and y denote the output of the controller, the input to the plant and the measurement output.

Moreover, σ represents the *componentwise* saturation nonlinearity consisting of multiple saturating elements $\sigma_1, \cdots, \sigma_m$.

$$\sigma(z) = \begin{bmatrix} \sigma_1(z_1) \\ \vdots \\ \sigma_m(z_m) \end{bmatrix}, \quad \text{for } z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{R}^m \tag{3a}$$

$$\sigma_i(z_i) = \begin{cases} -1, & \text{if } \ z_i < -1 \\ z_i, & \text{if } \ |z_i| \leq 1 \\ 1, & \text{if } \ z_i > 1 \end{cases} \quad \text{for } z_i \in \mathbb{R},\ i = 1, \cdots, m \tag{3b}$$

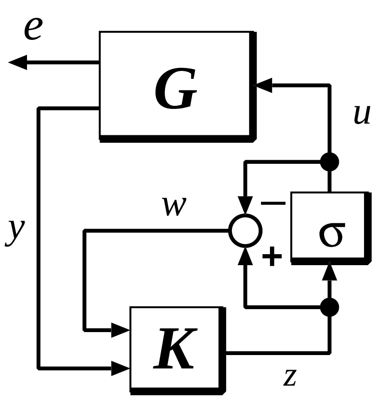

Fig. 1: Feedback control system with saturation

By eliminating z and u from (1)–(3), the closed-loop system is expressed as in Fig. 2, where

$$\Sigma : \begin{aligned} \dot{x}(t) &= \mathsf{A}x(t) + \mathsf{B}w(t) && (4a) \\ z(t) &= \mathsf{C}_0 x(t) + \mathsf{D}_0 w(t) && (4b) \\ e(t) &= \mathsf{C}_1 x(t) + \mathsf{D}_1 w(t) && (4c) \end{aligned}$$

$$w(t) = \phi(z(t)) \tag{5}$$

and

$$\begin{aligned} x(t) &= \mathrm{col}(\, x_p(t),\ x_k(t)\,) \in \mathbb{R}^n \\ \mathsf{A} &= \begin{bmatrix} A + BD_{k1}C_2 & BC_k \\ B_{k1}C_2 & A_k \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} B(D_{k2} - I) \\ B_{k2} \end{bmatrix} \\ \mathsf{C}_0 &= \begin{bmatrix} D_{k1}C_2 & C_k \end{bmatrix}, \quad \mathsf{D}_0 = D_{k2} \\ \mathsf{C}_1 &= \begin{bmatrix} C_1 + D_{12}D_{k1}C_2 & D_{12}C_k \end{bmatrix}, \quad \mathsf{D}_1 = D_{12}(D_{k2} - I) \end{aligned}$$

Moreover, ϕ is the dead-zone nonlinearity given by

$$\phi(z) = z - \sigma(z) \tag{6}$$

That is,

$$\phi(z) = \begin{bmatrix} \phi_1(z_1) \\ \vdots \\ \phi_m(z_m) \end{bmatrix} \text{ for } z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{R}^m \tag{7a}$$

$$\phi_i(z_i) = z_i - \sigma_i(z_i), \quad \text{for } z_i \in \mathbb{R},\ i = 1, \cdots, m \tag{7b}$$

Note also that $w_i = \phi_i(z_i)$ satisfies the sector condition

$$w_i(w_i - z_i) \le 0,\ \forall z_i \in \mathbb{R}, \quad i = 1, \cdots, m \tag{8}$$

In addition, if z_i is restricted into the local region $|z_i| \le 1/s_i$ for some $0 < s_i \le 1$, then a tighter sector bound can be obtained:

$$w_i[w_i - (1 - s_i)z_i] \le 0, \quad \forall z_i \in \mathbb{R} \text{ s.t. } |z_i| \le 1/s_i \quad , i = 1, \cdots, m \tag{9}$$

3. Stability Analysis

In the following, we consider the general setup of nonlinear feedback system given in Fig. 2. The domain of attraction for the feedback system in Fig. 2 is defined as follows.

Definition 1 The domain of attraction $\mathcal{D}$ is the set of initial state vectors for which the state trajectory converges to the origin as time goes to infinity, namely,

$$\mathcal{D} = \left\{ x_0 \in \mathbb{R}^n \ : \ \lim_{t \to \infty} x(t) = 0 \ \text{ for } x(0) = x_0 \right\}$$

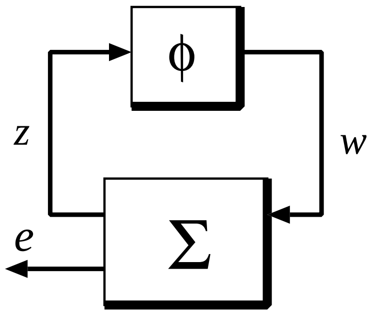

Fig. 2: Diagram for analysis

3.1 Local circle criterion

A typical approach to the analysis of a feedback system with saturation is to employ the circle or Popov criteria (Tyan and Bernstein 1995, Hindi and Boyd 1998, Pittet *et al.* 1997, Kothare and Morari 1999). That is, we rewrite the block diagram of Fig. 1 to obtain the feedback interconnection of the dead-zone nonlinearity ϕ and the LTI sub-system Σ (Fig. 2), and then apply the circle or Popov criterion to this feedback interconnection.

We here present a local stability condition based on the circle criterion. Note that we consider the local stability in the case where $\mathsf{D}_0 \neq 0$, while the stability condition in the case of $\mathsf{D}_0 = 0$ was given by Pittet, Tarbouriech and Burgat (1997), Hindi and Boyd (1998) and Kiyama and Iwasaki (2000).

Lemma 1 *Suppose that there exist a positive definite symmetric matrix* P *and a positive semdefinite diagonal matrix* Λ *such that*

$$\begin{bmatrix} \mathsf{P} & -\mathsf{C}_0^T\Lambda & \mathsf{C}_{0i}^T \\ -\Lambda\mathsf{C}_0 & (I-\mathsf{D}_0)^T\Lambda + \Lambda(I-\mathsf{D}_0) & \mathsf{D}_{0i}^T \\ \mathsf{C}_{0i} & D_{0i} & 1/s_i^2 \end{bmatrix} \geq 0 \tag{10}$$

where C_{0i} *and* D_{0i} *are the* i*-th rows of* C_0 *and* D_0*, respectively. Then,* $|z_i| \leq 1/s_i$ *with* $z = \mathsf{C}_0 x + \mathsf{D}_0\phi(z)$ *holds for every* $x \in \mathcal{E}(\mathsf{P})$*.*

Proof: Let $w = \phi(z)$. Since Λ is positive definite, we see from the sector condition that

$$2w^T\Lambda^{-1}(w-z) = [\, x^T \;\; w^T \,] \begin{bmatrix} 0 & -\mathsf{C}_0^T\Lambda \\ -\Lambda\mathsf{C}_0 & \Lambda(I-\mathsf{D}_0) + (I-\mathsf{D}_0)^T\Lambda \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \tag{11}$$

By applying the Schur complement formula to (10), we get

$$\begin{bmatrix} \mathsf{P} & -\mathsf{C}_0^T\Lambda \\ -\Lambda\mathsf{C}_0 & \Lambda(I-\mathsf{D}_0) + (I-\mathsf{D}_0)^T\Lambda \end{bmatrix} \geq s_i^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} [\, \mathsf{C}_{0i} \;\; \mathsf{D}_{0i} \,] \tag{12}$$

Thus, pre-multiplying (12) by $\mathrm{col}(x,w)^T$ and post-multiplying by $\mathrm{col}(x,w)$ yield

$$x^T \mathsf{P} x + 2w^T \Lambda (w - z) \geq s_i^2 z_i^2 \tag{13}$$

Since $x \in \mathcal{E}(\mathsf{P})$ and (11) are satisfied, the above equation implies $|z_i| \leq 1/s_i$.

Theorem 1 *Suppose that there exist a constant symmetric matrix* $\hat{\mathsf{P}}$*, diagonal matrices* Π, Λ_i *and scalar constants* s_i $(i = 1, \cdots, m)$ *satisfying*

$$\hat{\mathsf{P}} = \hat{\mathsf{P}}^T > 0, \quad \Pi > 0 \tag{14}$$

$$\hat{\Lambda}_i \geq 0, \; 0 < s_i \leq 1, \quad \forall i \in [1, \, m] \tag{15}$$

$$\begin{bmatrix} \mathsf{A}^T \hat{\mathsf{P}} + \hat{\mathsf{P}} \mathsf{A} & \hat{\mathsf{P}} \mathsf{B} + \mathsf{C}_0^T S \Pi \\ \mathsf{B}^T \hat{\mathsf{P}} + \Pi S \mathsf{C}_0 & -\Pi (I - S\mathsf{D}_0) - (I - S\mathsf{D}_0)^T \Pi \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} \hat{\mathsf{P}} & -\mathsf{C}_0^T \hat{\Lambda}_i & \mathsf{C}_{0i}^T \\ -\hat{\Lambda}_i \mathsf{C}_0 & (I - \mathsf{D}_0)^T \hat{\Lambda}_i + \hat{\Lambda}_i (I - \mathsf{D}_0) & \mathsf{D}_{0i}^T \\ \mathsf{C}_{0i} & D_{0i} & 1/s_i^2 \end{bmatrix} \geq 0 \quad \forall i \in [\,1, \, m\,] \tag{17}$$

where $S = I - \mathrm{diag}(s_1, \cdots, s_m)$*, and where* C_{0i} *and* D_{0i} *respectively denote the i-th rows of* C_0 *and* D_0*. Then, the feedback system is locally asymptotically stable, and*

$$\mathcal{E}(\hat{\mathsf{P}}) \subseteq \mathcal{D} \tag{18}$$

Proof: We define $V(x) = x^T \hat{\mathsf{P}} x \, : \, \mathbb{R}^n \to \mathbb{R}$. In order to prove the local asymptotic stability, we need to show that the time differentiation of $V(x(t))$ along the trajectory is negative for all state $x(t)$ in $\mathcal{E}(\hat{\mathsf{P}}) \backslash \{0\}$.

Let $W : \mathbb{R}^n \to \mathbb{R}$ be given by

$$W(x) = 2x^T \hat{\mathsf{P}} (\mathsf{A}x + \mathsf{B}w) \;\text{ with } w = \phi(z), \; z = \mathsf{C}_0 x + \mathsf{D}_0 w \tag{19}$$

Since $W(x(t)) = \frac{d}{dt} V(x(t)) = 2x^T(t) \hat{\mathsf{P}} [\mathsf{A}x(t) + \mathsf{B}w(t)]$ from (4), we have only to show that

$$W(x) < 0 \;\; \forall x \in \mathcal{E}(\hat{\mathsf{P}}) \backslash \{0\} \tag{20}$$

Post-multiplying (16) by $\mathrm{col}(x,w)$ and pre-multiplying by its transpose yield

$$W(x) + 2w^T \Pi (w - Sz) < 0 \;\; \forall x \neq 0 \tag{21}$$

Moreover, it follows from Lemma 1 and (17) that $x \in \mathcal{E}(\hat{\mathsf{P}}) \Rightarrow |z_i| \leq 1/s_i$ for $i = 1, 2, \ldots, m$. This implies that the local sector condition (8) is satisfied for all $x \in \mathcal{E}(\hat{\mathsf{P}})$. Since Π is positive definite, (8) implies

$$w^T \Pi (w - Sz) \leq 0 \;\; \forall x \in \mathcal{E}(\hat{\mathsf{P}}) \tag{22}$$

We conclude from (21) and (22) that $W(x) < 0 \; \forall x \in \mathcal{E}(\hat{\mathsf{P}}) \backslash \{0\}$. This completes the proof. □

The approach based on the circle or Popov criterion often leads to conservative results since it utilizes only the sector conditions (8) and (9) to prove the stability: in other words, the detailed structure of the nonlinearity is neglected by regarding it as sector-bounded uncertainty. In fact, Kiyama and Iwasaki (2000) showed that the circle criterion does not improve conservativeness in comparison with a linear method in the local stabilization problem.

In the remainder of this paper, we will derive a less conservative condition for the local stability and the quadratic performance.

3.2 LPV descriptor system representation

We introduce a continuous function $\Theta(z)$ such that

$$\sigma(z) = z - \Theta(z)z \quad (\text{equivalently, } \phi(z) = \Theta(z)z\,)$$

It then follows from (3) that

$$\Theta(z) = \mathrm{diag}(\theta_1(z_1),\ \theta_2(z_2), \cdots, \theta_m(z_m)) \quad \text{for} \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{R}^m \tag{23}$$

$$\theta_i(z_i) = \begin{cases} 1 - \dfrac{1}{|z_i|}, & \text{if } |z_i| > 1 \\ 0, & \text{if } |z_i| \le 1 \end{cases} \quad \text{for } z_i \in \mathbb{R} \tag{24}$$

It should be noted that, from (24), we have

$$0 \le \theta_i(z_i) < 1, \quad \forall z_i \in \mathbb{R}^m \tag{25}$$

Since $w(t) = \phi(z(t)) = \Theta(z(t))z(t)$ by the definitions of ϕ and Θ, we see from (4b) that the feedback system of (4) is expressed as

$$\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta(z)\mathsf{C}_0 & -I + \Theta(z)\mathsf{D}_0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{26a}$$

$$e(t) = \mathsf{C}_1 x(t) + \mathsf{D}_1 w(t) \tag{26b}$$

This representation is called *the LPV descriptor system representation* in which Θ and $(x,\ w)$ are the variation parameter and the descriptor variable vector, respectively.

We also define

$$\begin{aligned} \boldsymbol{\Theta}_s &= \{\Theta = \mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_m) \ :\ \ \theta_i \in [0,\ 1 - s_i],\ \forall i\} \\ \bar{\boldsymbol{\Theta}}_s &= \{\Theta = \mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_m) \ :\ \ \theta_i \in \{0,\ 1 - s_i\},\ \forall i\} \end{aligned}$$

for a given set of nonnegative constants $s := (s_1, s_2, \cdots, s_m)$. The set $\bar{\boldsymbol{\Theta}}_s$ consists of the vertices of the parameter box defined by $\boldsymbol{\Theta}_s$.

3.3 New stability condition

Theorem 2 *Suppose that there exist a constant symmetric matrix* P*, constant matrices* Q_i*,* R_i*, diagonal matrices* Λ_i *and scalar constants* α_i*,* s_i $(i = 1, \cdots, m)$ *satisfying*

$$\mathsf{P} = \mathsf{P}^T > 0 \tag{27}$$

$$\Lambda_i \geq 0,\ \ \alpha_i > 0,\ \ 0 \leq s_i \leq 1, \quad \forall i \in [1,\, m] \tag{28}$$

$$\mathsf{A}^T\mathsf{P} + \mathsf{P}\mathsf{A} < 0 \tag{29}$$

$$\begin{aligned}
&\mathrm{He}\left\{ \begin{bmatrix} \mathsf{P} & 0 \\ \mathsf{Q}_i & \mathsf{R}_i \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta\mathsf{C}_0 & -I + \Theta\mathsf{D}_0 \end{bmatrix} \right\} \\
&+ \alpha_i \left\{ (1-\theta_i)^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & \mathsf{D}_{0i} \end{bmatrix} - \begin{bmatrix} \mathsf{P} & 0 \\ 0 & 0 \end{bmatrix} \right\} < 0 \\
&\qquad\qquad \forall \Theta \in \bar{\boldsymbol{\Theta}}_s,\ \forall i \in [1,\, m]
\end{aligned} \tag{30}$$

$$\begin{bmatrix} \mathsf{P} & -\mathsf{C}_0^T\Lambda_i & \mathsf{C}_{0i}^T \\ -\Lambda_i\mathsf{C}_0 & (I - \mathsf{D}_0)^T\Lambda_i + \Lambda_i(I - \mathsf{D}_0) & \mathsf{D}_{0i}^T \\ \mathsf{C}_{0i} & D_{0i} & 1/s_i^2 \end{bmatrix} \geq 0 \quad \forall i \in [\,1,\, m\,] \tag{31}$$

where C_{0i} *and* D_{0i} *respectively denote the* i*-th rows of* C_0 *and* D_0*. Then, the feedback system is locally asymptotically stable, and*

$$\mathcal{E}(\mathsf{P}) \subseteq \mathcal{D} \tag{32}$$

Proof: We define $W : \mathbb{R}^n \to \mathbb{R}$ as

$$W(x) = 2x^T\mathsf{P}(\mathsf{A}x + \mathsf{B}w) < 0, \text{ with } w = \phi(z) = \Theta(z)z,\ z = \mathsf{C}_0x + \mathsf{D}_0w \tag{33}$$

Similarly to the proof of Theorem 1, it suffices to show that $W(x) < 0$ holds for all $x \in \mathcal{E}(\mathsf{P})\backslash\{0\}$.

Since $w = \phi(z)$, we see from Lemma 1 and (31) that $x \in \mathcal{E}(\mathsf{P})$ implies $|z_i| \leq 1/s_i$ for all $i \in [1, m]$. Hence, by the definition of θ_i, we obtain $0 \leq \theta_i(z_i) \leq 1 - s_i,\ i \in [1, m]$ for all $x \in \mathcal{E}(\mathsf{P})$, namely

$$\Theta(z) \in \boldsymbol{\Theta}_s \quad \forall x \in \mathcal{E}(\mathsf{P}) \tag{34}$$

We first assume that the saturation does not occur ($|z_i| \leq 1$ for all i). Since $\Theta(z) = 0$ holds in this case, we get from (29) $W(x) = x^T(\mathsf{A}^T\mathsf{P} + \mathsf{P}\mathsf{A})x < 0$ for $x \neq 0$.

Next, we consider the case where the saturation takes place. Note that the matrix inequality in (30) is satisfied for all $\Theta \in \boldsymbol{\Theta}_s$ since it can be easily rewritten as a matrix inequality that is affine in Θ by using the Schur complement formula. We here assume that the i-th element of $z(t)$ is saturated, namely $|z_i| > 1$. We can easily prove the stability in the same way even if the other elements of z are saturated as well. It follows from (24) and $|z_i| > 1$ that

$$[1 - \theta_i(z_i)]^2 = \frac{1}{z_i^2} \tag{35}$$

Post-multiplying (30) by $\mathrm{col}(x,\ w) \neq 0$, pre-multiplying by its transpose, and substituting $\Theta = \Theta(z) \in \Theta_s$, (35) into (30) yield

$$2x^T\mathsf{P}(\mathsf{A}x + \mathsf{B}w) + \alpha_i\left\{(1-\theta_i(z_i))^2 z_i^2 - x^T\mathsf{P}x\right\} = W(x) + \alpha_i(1 - x^T\mathsf{P}x) < 0, \quad \forall x \in \mathcal{E}(\mathsf{P})\backslash\{0\}$$

Hence, we obtain

$$W(x) < \alpha_i(x^T\mathsf{P}x - 1) \leq 0$$

In the last inequality, we have used the conditions of $\alpha_i > 0$ and $x \in \mathcal{E}(\mathsf{P})$.

Consequently, we have proved that the feedback system is locally asymptotically stable since $V(x) = x^T\mathsf{P}x$ is a quadratic Lyapunov function which satisfies $dV(x(t))/dt = W(x(t)) < 0$ for $x(t) \neq 0$ in the ellipsoidal domain $\mathcal{E}(\mathsf{P})$. □

The stability condition in Theorem 2 is substantially simplified in the case where $\mathsf{D}_0 = 0$.

Corollary 1 *Suppose that $\mathsf{D}_0 = 0$. We assume that there exist a constant symmetric matrix P and scalar constants α_i, s_i $(i = 1, \cdots, m)$ satisfying*

$$\mathsf{P} = \mathsf{P}^T > 0 \tag{36}$$

$$\alpha_i > 0,\ 0 \leq s_i \leq 1, \quad \forall i \in [1,\ m] \tag{37}$$

$$\mathsf{A}^T\mathsf{P} + \mathsf{P}\mathsf{A} < 0 \tag{38}$$

$$(\mathsf{A} + \mathsf{B}\Theta\mathsf{C}_0)^T\mathsf{P} + \mathsf{P}(\mathsf{A} + \mathsf{B}\Theta\mathsf{C}_0) + \alpha_i[(1-\theta_i)^2\mathsf{C}_{0i}^T\mathsf{C}_{0i} - \mathsf{P}] < 0$$
$$\forall \Theta \in \bar{\boldsymbol{\Theta}}_s,\ \forall i \in [1,\ m] \tag{39}$$

$$\mathsf{P} \geq s_i^2\mathsf{C}_{0i}^T\mathsf{C}_{0i} \quad \forall i \in [\,1,\ m\,] \tag{40}$$

Then, the feedback system is locally asymptotically stable, and

$$\mathcal{E}(\mathsf{P}) \subseteq \mathcal{D} \tag{41}$$

Proof: It suffices to show that the matrix inequality condition of (27)–(31) is equivalent to the condition of (36)–(40).

Since the equivalence between (27)–(29) and (36)–(38) is obvious except for $\Lambda_i \geq 0$, we shall prove that (30) and (31) are equivalent to (39) and (40), respectively.

We assume that (36)–(40) have a solution $(\mathsf{P}, \alpha_1, \cdots, \alpha_m, s_1, \cdots, s_m)$. Since the set $\bar{\Theta}_s$ is closed and bounded, there exist positive constants ε_i $(i = 1, \cdots, m)$ such that

$$(\mathsf{A} + \mathsf{B}\Theta\mathsf{C}_0)^T\mathsf{P} + \mathsf{P}(\mathsf{A} + \mathsf{B}\Theta\mathsf{C}_0) + \alpha_i[(1-\theta_i)^2\mathsf{C}_{0i}^T\mathsf{C}_{0i} - \mathsf{P}] + \frac{1}{2}\varepsilon_i\mathsf{C}_0^T\Theta\Theta\mathsf{C}_0 < 0$$
$$\forall \Theta \in \bar{\boldsymbol{\Theta}}_s,\ \forall i \in [1,\ m]$$

Then, a direct calculation including the Schur complement formula shows that the matrix inequality condition of (27)–(31) is satisfied for

$$\mathsf{P},\quad \alpha_1, \cdots, \alpha_m,\ s_1, \cdots, s_m$$
$$\mathsf{Q}_i = \mathsf{B}^T\mathsf{P},\quad \mathsf{R}_i = \varepsilon_i,\quad \Lambda_i = 0\ (i = 1, \cdots, m)$$

Conversely, we assume that (27)–(31) have a solution $(\mathsf{P}, \mathsf{Q}_i, \mathsf{R}_i, \alpha_1, \cdots, \alpha_m, s_1, \cdots, s_m)$. Then, since D_0 is equal to zero, (30) is rewritten as

$$\begin{aligned} &\mathrm{He}\left\{ \begin{bmatrix} \mathsf{P} & 0 \\ \mathsf{Q}_i & \mathsf{R}_i \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta \mathsf{C}_0 & -I \end{bmatrix} \right\} \\ &+ \alpha_i \left\{ (1-\theta_i)^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ 0 \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & 0 \end{bmatrix} - \begin{bmatrix} \mathsf{P} & 0 \\ 0 & 0 \end{bmatrix} \right\} < 0 \end{aligned}$$

Pre-multiplying this by $[I \ \ \mathsf{C}_0^T \Theta]$ and post-multiplying by its transpose yield

$$\mathrm{He}[\mathsf{P}(\mathsf{A} + \mathsf{B}\Theta \mathsf{C}_0)] + \alpha_i[(1-\theta_i)^2 \mathsf{C}_{0i}^T \mathsf{C}_{0i} - \mathsf{P}] < 0$$

Hence, the inequality (39) is satisfied. Moreover, by the Schur complement formula, (31) reduces to

$$\begin{bmatrix} \mathsf{P} - s_i^2 \mathsf{C}_{0i}^T \mathsf{C}_{0i} & -\mathsf{C}_0^T \Lambda_i \\ -\Lambda_i \mathsf{C}_0 & 2\Lambda_i \end{bmatrix} \geq 0 \quad \forall i \in [\,1,\, m\,]$$

We then obtain (40) from the $(1,1)$-block of the above inequality. □

The ellipsoid $\mathcal{E}(\mathsf{P})$ represents a guaranteed domain of attraction, or an inner approximation of $\mathcal{D}$. As well known, we can maximize the volume of $\mathcal{E}(\mathsf{P})$ by minimizing $\log\det \mathsf{P}$ (Boyd *et al.* 1994). Another reasonable and computationally cheap optimization for estimating the domain of attraction is to minimize $\mathrm{Tr}(\mathsf{P})$.

$$\textbf{(OP-S)} \quad \inf_{\mathsf{P},\mathsf{Q},\mathsf{R},\Lambda,\alpha,s} \mathrm{Tr}(\mathsf{P}) \text{ subject to (27)–(31)}$$

where

$$\begin{aligned} &\mathsf{Q} := (\mathsf{Q}_1, \cdots, \mathsf{Q}_m), \quad \mathsf{R} := (\mathsf{R}_1, \cdots, \mathsf{R}_m) \\ &\Lambda := (\Lambda_1, \cdots, \Lambda_m) \\ &\alpha := (\alpha_1, \cdots, \alpha_m), \quad s := (s_1, \cdots, s_m) \end{aligned}$$

The matrix inequalities (27)–(31) are reduced to linear matrix inequalities (LMIs), which can be efficiently solved by existing convex programming techniques, by setting α and $s = (s_1, \cdots, s_m)$ to appropriate constants. Hence, we can easily solve **(OP-S)** by the following algorithm.

Algorithm:

Step 1: Set α to a proper positive value, and introduce N grid points $s^k := (s_1^k, \cdots, s_m^k)$ $(k = 1, \cdots, N)$.

Step 2: For each grid point $s = s^k$, solve the convex optimization problem:

$$\textbf{(OP-LMI)} \quad \inf_{\mathsf{P},\mathsf{Q},\mathsf{R},\Lambda} \mathrm{Tr}(\mathsf{P}) \text{ subject to (27)–(31)}$$

and set $(\mathsf{P}^k,\ \mathsf{Q}^k, \mathsf{R}^k, \Lambda^k)$ to the resulting values of $(\mathsf{P}, \mathsf{Q}, \mathsf{R}, \Lambda)$.

Step 3: Let

$$k^* := \arg\min_k \ \mathrm{Tr}(\mathsf{P}^k)$$

Then, the desired near-optimal solution of **(OP-S)** is obtain by $(\mathsf{P}, \mathsf{Q}, \mathsf{R}, \Lambda, q) := (\mathsf{P}^{k^*}, \mathsf{Q}^{k^*}, \mathsf{R}^{k*}, \Lambda^{k^*}, s^{k^*})$.

3.4 Comparison of two stability conditions

In this subsection, we compare the conservativeness of the stability conditions in Theorems 1 and 2.

Lemma 2 *Suppose that a symmetric matrix $\hat{\mathsf{P}}$, a positive definite diagonal matrix Π and scalar constants $(s_1, \cdots, s_m)$ are given. Let $S = I - \operatorname{diag}(s_1, \cdots, s_m)$, $\hat{\mathsf{Q}} = -\Pi S \mathsf{C}_0$ and $\hat{\mathsf{R}} = \Pi(2I - S\mathsf{D}_0)$. Then, we have*

$$\begin{aligned}
&\mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}} & 0 \\ \hat{\mathsf{Q}} & \hat{\mathsf{R}} \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta \mathsf{C}_0 & -I + \Theta \mathsf{D}_0 \end{bmatrix}\right) \\
&\leq \begin{bmatrix} \mathsf{A}^T \hat{\mathsf{P}} + \hat{\mathsf{P}} \mathsf{A} & \hat{\mathsf{P}} \mathsf{B} + \mathsf{C}_0^T S \Pi \\ \mathsf{B}^T \hat{\mathsf{P}} + \Pi S \mathsf{C}_0 & -\Pi(I - S\mathsf{D}_0) - (I - S\mathsf{D}_0)^T \Pi \end{bmatrix} \quad \forall \Theta \in \boldsymbol{\Theta}_s
\end{aligned}$$

Proof: We define $\Delta = 2\Theta - S$. Then, we have $\Delta \Pi \Delta \leq S \Pi S$ since Δ, S and Π are diagonal, $\Pi > 0$ and $\theta_i \in [0,\ 1 - s_i] \forall i$. It follows that

$$\begin{aligned}
\Phi :=& \mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}} & 0 \\ \hat{\mathsf{Q}} & \hat{\mathsf{R}} \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta \mathsf{C}_0 & -I + \Theta \mathsf{D}_0 \end{bmatrix}\right) \\
=& \mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}} & 0 \\ \hat{\mathsf{Q}} & \hat{\mathsf{R}} \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \frac{1}{2}(\Delta + S)\mathsf{C}_0 & -I + \frac{1}{2}(\Delta + S)\mathsf{D}_0 \end{bmatrix}\right) \\
=& \mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}}\mathsf{A} + \frac{1}{2}\hat{\mathsf{Q}}^T S \mathsf{C}_0 & \hat{\mathsf{P}}\mathsf{B} - \hat{\mathsf{Q}}^T (I - \frac{1}{2} S \mathsf{D}_0) \\ \frac{1}{2}\hat{\mathsf{R}}^T S \mathsf{C}_0 & -\hat{\mathsf{R}}(I - \frac{1}{2} S \mathsf{D}_0) \end{bmatrix}\right) + \frac{1}{2}\mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{Q}}^T \\ \hat{\mathsf{R}}^T \end{bmatrix} \Delta \begin{bmatrix} \mathsf{C}_0 & \mathsf{D}_0 \end{bmatrix}\right)
\end{aligned}$$

Applying the inequality

$$X^T Y + Y^T X \leq X^T \Lambda X + Y^T \Lambda^{-1} Y, \quad \forall \Lambda = \Lambda^T > 0,\ \forall X,\ \forall Y$$

to the above equation yields

$$\begin{aligned}
\Phi \leq& \mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}}\mathsf{A} + \frac{1}{2}\hat{\mathsf{Q}}^T S \mathsf{C}_0 & \hat{\mathsf{P}}\mathsf{B} - \hat{\mathsf{Q}}^T (I - \frac{1}{2} S \mathsf{D}_0) \\ \frac{1}{2}\hat{\mathsf{R}}^T S \mathsf{C}_0 & -\hat{\mathsf{R}}(I - \frac{1}{2} S \mathsf{D}_0) \end{bmatrix}\right) \\
&+ \frac{1}{2}\left(\begin{bmatrix} \hat{\mathsf{Q}}^T \\ \hat{\mathsf{R}}^T \end{bmatrix} \Pi^{-1} \begin{bmatrix} \hat{\mathsf{Q}} & \hat{\mathsf{R}} \end{bmatrix} + \begin{bmatrix} \mathsf{C}_0^T \\ \mathsf{D}_0^T \end{bmatrix} \Delta \Pi \Delta \begin{bmatrix} \mathsf{C}_0 & \mathsf{D}_0 \end{bmatrix}\right) \\
\leq& \mathrm{He}\left(\begin{bmatrix} \hat{\mathsf{P}}\mathsf{A} + \frac{1}{2}\hat{\mathsf{Q}}^T S \mathsf{C}_0 & \hat{\mathsf{P}}\mathsf{B} - \hat{\mathsf{Q}}^T (I - \frac{1}{2} S \mathsf{D}_0) \\ \frac{1}{2}\hat{\mathsf{R}}^T S \mathsf{C}_0 & -\hat{\mathsf{R}}(I - \frac{1}{2} S \mathsf{D}_0) \end{bmatrix}\right) \\
&+ \frac{1}{2}\left(\begin{bmatrix} \hat{\mathsf{Q}}^T \\ \hat{\mathsf{R}}^T \end{bmatrix} \Pi^{-1} \begin{bmatrix} \hat{\mathsf{Q}} & \hat{\mathsf{R}} \end{bmatrix} + \begin{bmatrix} \mathsf{C}_0^T \\ \mathsf{D}_0^T \end{bmatrix} S \Pi S \begin{bmatrix} \mathsf{C}_0 & \mathsf{D}_0 \end{bmatrix}\right)
\end{aligned}$$

In the last inequality, we have used the fact that $\Delta \Pi \Delta \leq S \Pi S$. Finally, by substituting $\hat{\mathsf{Q}} = -\Pi S \mathsf{C}_0$ and $\hat{\mathsf{R}} = \Pi(2I - S\mathsf{D}_0)$ into the above inequality, we complete the proof. □

Theorem 3 *Suppose that the matrix inequalities in Theorem 1 has a solution* $(\hat{\mathsf{P}}, \Pi, \hat{\Lambda})$ *for given* $s = (s_1, \cdots, s_m)$*. Then, there exist* Q_i*,* R_i*,* Λ_i *and* α_i $(i = 1, \cdots, m)$ *such that* $(\hat{\mathsf{P}}, \mathsf{Q}, \mathsf{R}, \Lambda, \alpha)$ *satisfies the matrix inequalities in Theorem 2.*

Proof: We have only to prove the relation between (16) and (30) since the conditions (27)–(29) and (31) are obvious.

Since $\bar{\mathbf{\Theta}}_s$ is a bounded set, there exist positive constants α_i $(i = 1, \ldots, m)$ such that

$$\begin{bmatrix} \mathsf{A}^T\hat{\mathsf{P}} + \hat{\mathsf{P}}\mathsf{A} & \hat{\mathsf{P}}\mathsf{B} + \mathsf{C}_0^T S\Pi \\ \mathsf{B}\hat{\mathsf{P}} + \Pi S\mathsf{C}_0 & -\Pi(I - S\mathsf{D}_0) - (I - S\mathsf{D}_0)^T\Pi \end{bmatrix}$$
$$+ \alpha_i \left\{ (1-\theta_i)^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & \mathsf{D}_{0i} \end{bmatrix} - \begin{bmatrix} \hat{\mathsf{P}} & 0 \\ 0 & 0 \end{bmatrix} \right\} < 0, \quad \forall \Theta \in \bar{\mathbf{\Theta}}_s, \ \forall i \in [1, \ m]$$

Hence, by Lemma 2, if the inequality (16) is satisfied, then

$$\mathrm{He}\left\{ \begin{bmatrix} \mathsf{P} & 0 \\ \mathsf{Q}_i & \mathsf{R}_i \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta\mathsf{C}_0 & -I + \Theta\mathsf{D}_0 \end{bmatrix} \right\}$$
$$+ \alpha_i \left\{ (1-\theta_i)^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & \mathsf{D}_{0i} \end{bmatrix} - \begin{bmatrix} \mathsf{P} & 0 \\ 0 & 0 \end{bmatrix} \right\} < 0$$
$$\forall \Theta \in \bar{\mathbf{\Theta}}_s, \ \forall i \in [1, \ m]$$

holds for

$$\begin{aligned} &\mathsf{P} := \hat{\mathsf{P}} \\ &\mathsf{Q}_i := \Pi(I - S\mathsf{D}_0), \quad \mathsf{S}_i := -\Pi S\mathsf{C}_0, \quad \Lambda_i := \hat{\Lambda}_i, \quad (i = 1, \cdots, m) \end{aligned}$$

This completes the proof. □

The above theorem shows that, if the stability condition in Theorem 1 (circle criterion) is satisfied, then the condition in Theorem 2 is also satisfied. This implies that the present stability condition is less conservative than the circle criterion.

4. Quadratic Performance

We next consider the quadratic performance

$$J := \int_0^\infty e^T(t)e(t)\,dt \tag{42}$$

Theorem 4 *Let* $(\mathsf{P}, \mathsf{R}_i, \mathsf{Q}_i, \Lambda_i, \alpha_i, s_i)$ $(i = 1, \cdots, m)$ *be a solution to the matrix inequalities in Theorem 2. Suppose that there exist a constant symmetric matrix* X *and constant matrices*

Y_i, Z_i *and scalar constants* μ_i $(i = 1, \cdots, m)$ *and* γ *satisfying*

$$\mathsf{X} = \mathsf{X}^T > 0, \ \gamma > 0 \tag{43}$$

$$\mu_i > 0, \quad \forall i \in [1, m] \tag{44}$$

$$\mathsf{A}^T\mathsf{X} + \mathsf{X}\mathsf{A} + \mathsf{C}_1^T\mathsf{C}_1 < 0 \tag{45}$$

$$\begin{aligned} &\mathrm{He}\left\{ \begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y}_i & \mathsf{Z}_i \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta\mathsf{C}_0 & -I + \Theta\mathsf{D}_0 \end{bmatrix} \right\} + \begin{bmatrix} \mathsf{C}_1^T \\ \mathsf{D}_1^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_1 \end{bmatrix} \\ &+ \mu_i \left\{ (1-\theta_i)^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & \mathsf{D}_{0i} \end{bmatrix} - \begin{bmatrix} \mathsf{P} & 0 \\ 0 & 0 \end{bmatrix} \right\} < 0 \\ &\qquad\qquad \forall \Theta \in \bar{\boldsymbol{\Theta}}_s, \ \forall i \in [1, m] \end{aligned} \tag{46}$$

$$\mathsf{X} \le \gamma \mathsf{P} \tag{47}$$

Then, the feedback system satisfies the quadratic performance bound

$$J \le \gamma \tag{48}$$

for any initial state $x_0 \in \mathcal{E}(\mathsf{P})$.

Proof: We see from Theorem 2 that, if the state trajectory starts inside $\mathcal{E}(\mathsf{P})$, then $x(t) \in \mathcal{E}(\mathsf{P})$ holds for all $t \ge 0$. Hence, we have $|z_i(t)| \le 1/s_i$ and $\Theta(z(t)) \in \boldsymbol{\Theta}_s$ and for all $t \ge 0$.

In the case where the saturation does not take place, we get from (4) $\dot{x}(t) = \mathsf{A}x(t)$. Hence, it follows from (45) that

$$\frac{d}{dt}[x^T(t)\mathsf{X}x(t)] + e^T(t)e(t) = x(t)^T(\mathsf{A}^T\mathsf{X} + \mathsf{X}\mathsf{A} + \mathsf{C}_1^T\mathsf{C}_1)x(t) \le 0$$

Next, we assume that the i-th element of $z(t)$ is saturated. Since $[1-\theta_i(z_i(t))]^2 = 1/z_i^2(t)$ holds in this case, we see from (4) and (46) that

$$\begin{aligned} &\frac{d}{dt}[x^T(t)\mathsf{X}x(t)] + e^T(t)e(t) \\ &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \left(\mathrm{He}\left\{ \begin{bmatrix} \mathsf{X} & 0 \\ \mathsf{Y}_i & \mathsf{Z}_i \end{bmatrix}^T \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \Theta(z(t))\mathsf{C}_0 & -I + \Theta(z(t))\mathsf{D}_0 \end{bmatrix} \right\} \right. \\ &\qquad\qquad \left. + \begin{bmatrix} \mathsf{C}_1^T \\ \mathsf{D}_1^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_1 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &\le \mu_i \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \left\{ [1-\theta_i(z_i(t))]^2 \begin{bmatrix} \mathsf{C}_{0i}^T \\ \mathsf{D}_{0i}^T \end{bmatrix} \begin{bmatrix} \mathsf{C}_{0i} & \mathsf{D}_{0i} \end{bmatrix} - \begin{bmatrix} \mathsf{P} & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &= \mu_i \left\{ [1-\theta_i(z_i(t))]^2 z_i^2(t) - x^T(t)\mathsf{P}x(t) \right\} \\ &= \mu_i [1 - x^T(t)\mathsf{P}x(t)] \le 0 \end{aligned}$$

Consequently,

$$\frac{d}{dt}[x^T(t)\mathsf{X}x(t)] + e^T(t)e(t) \le 0 \tag{49}$$

holds for any trajectory which starts inside $\mathcal{E}(\mathsf{P})$.

It is clear from Theorem 2 that, if $x(0) \in \mathcal{E}(\mathsf{P})$, $x(t)$ tends to zero as t goes to infinity. Hence, integrating the inequality (49) from $t = 0$ to ∞ yields

$$J \le x(0)^T \mathsf{X} x(0) \tag{50}$$

Since $x(0) \in \mathcal{E}(\mathsf{P})$ and $\mathsf{X} \le \gamma \mathsf{P}$, we conclude that $J \le \gamma$. □

In order to obtain a tight upper bound on the quadratic performance index J, we need to solve the optimization problem:

$$\textbf{(OP-Q)} \quad \inf_{\mathsf{X},\mathsf{Y},\mathsf{Z},\mu,\gamma} \gamma \text{ subject to (43)–(47)}$$

where

$$\begin{aligned} \mathsf{Y} &:= (\mathsf{Y}_1, \cdots, \mathsf{Y}_m), \quad \mathsf{Z} := (\mathsf{Z}_1, \cdots, \mathsf{Z}_m) \\ \mu &:= (\mu_1, \cdots, \mu_m) \end{aligned}$$

Since the matrix inequalities (43)–(47) are affine in the variables X, Y, Z, μ and γ, we can efficiently find a global solution to this optimization problem by an existing convex programming algorithm.

5. Numerical Example

We consider the system (4),(5) with

$$\begin{aligned} \mathsf{A} &= \begin{bmatrix} -0.5 & 5 \\ 0 & -2 \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathsf{C}_0 &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathsf{D}_0 = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix} \\ \mathsf{C}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathsf{D}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

In Theorem 2, we set $(\alpha_1,\ \alpha_2) = (1.5,\ 2)$. In this case, by solving **(OP-S)**, we get $\mathrm{Tr}(\mathsf{P}) = 4.6961$ at

$$\mathsf{P} = \begin{bmatrix} 0.9363 & 1.0292 \\ 1.0292 & 3.7598 \end{bmatrix}, \quad (s_1,\ s_2) = (0.2,\ 0.7)$$

The guaranteed domain of attraction obtained from this P is illustrated in Fig. 3, together with the result based on Theorem 1 (circle criterion). We see from the figure that the present method provides a less conservative result than the circle criterion.

Next, by solving the optimization problem **(OP-Q)** for the above values of P and q, we obtained the optimal performance bound $\gamma = 10.5987$. This bound is attained by $x(0) = \mathrm{col}(0.9336\,,\ -0.5936) \in \mathcal{E}(\mathsf{P})$.

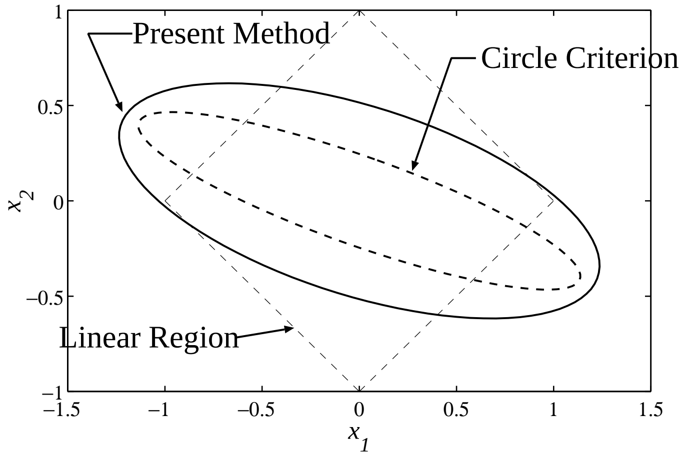

Fig. 3: Guaranteed domain of attraction

6. Conclusion

In this paper, we considered the local stability and the quadratic performance of saturating feedback systems. By combining the LPV technique of Takaba and Tomida (2000) and the descriptor system representation, we have derived a sufficient matrix inequality condition for local stability. It turns out that the present stability condition is less conservative than the local circle criterion. Moreover, we have derived an matrix inequality condition for satisfying the quadratic performance. It may be noted that the approach of this paper can be also applied to the L_2-gain analysis. The design of a stabilizing controller based on Theorem 2 remains as a future research topic.

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