A Modified Relaxation Scheme for Mathematical Programs with Complementarity Constraints

Gui-Hua Lin* and Masao Fukushima†

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Abstract. In this paper, we consider a mathematical program with complementarity constraints. We present a modified relaxed program for this problem, which involves less constraints than the relaxation scheme studied by Scholtes (2000). We show that the linear independence constraint qualification holds for the new relaxed problem under some mild conditions. We also consider a limiting behavior of the relaxed problem. We prove that any accumulation point of stationary points of the relaxed problems is C-stationary to the original problem under the MPEC linear independence constraint qualification and, if the Hessian matrices of the Lagrangian functions of the relaxed problems are uniformly bounded below on the corresponding tangent space, it is M-stationary. We also obtain some sufficient conditions of B-stationarity for a feasible point of the original problem. In particular, some conditions described by the eigenvalues of the Hessian matrices mentioned above are new and can be verified easily.

Key Words. mathematical program with complementarity constraints, (MPEC-) linear independence constraint qualification, nondegeneracy, (B-, M-, C-) stationarity, second-order necessary conditions, upper level strict complementarity.

AMS subject classifications. 90C30, 90C33.

1 Introduction

We consider the following mathematical program with complementarity constraints, which constitutes an important subclass of the mathematical program with equilibrium constraints (MPEC):

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g(z) \le 0, \;\; h(z) = 0 \\ & G(z) \ge 0, \;\; H(z) \ge 0 \\ & G(z)^T H(z) = 0, \end{aligned} \tag{1.1}$$

*Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan./Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China. E-mail Address: ghlin@amp.i.kyoto-u.ac.jp.

†Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan. E-mail Address: fuku@amp.i.kyoto-u.ac.jp.

where $f: R^n \to R, g: R^n \to R^p, h: R^n \to R^q$, and $G, H: R^n \to R^m$ are all twice continuously differentiable functions. This problem plays an important role in many fields such as engineering design, economic equilibrium, and multilevel game, see [12], and has attracted much attention in the recent literature. The major difficulty in solving problem (1.1) is that its constraints fail to satisfy a standard constraint qualification at any feasible point so that standard methods are likely to fail for this problem. There have been proposed several approaches such as sequential quadratic programming approach, implicit programming approach, penalty function approach, and reformulation approach [1, 4-6, 8-13, 17, 19]. In particular, Fukushima and Pang [6] considered a smoothing continuation method and showed, under the MPEC-linear independence constraint qualification (MPEC-LICQ) and an additional condition called the asymptotic weak nondegeneracy, that an accumulation point of KKT points satisfying the second-order necessary conditions for the perturbed problems is a B-stationary point of the original problem. Subsequently, Scholtes [19] presented a regularization scheme

$$
\begin{array}{ll}
\min & f(z) \\
\text{s.t.} & g(z) \leq 0, \;\; h(z) = 0 \\
& G(z) \geq 0, \;\; H(z) \geq 0 \\
& G_i(z) H_i(z) \leq \varepsilon, \;\; i = 1, 2, \cdots, m,
\end{array}
\tag{1.2}
$$

where ε is a positive parameter, as an approximation of problem (1.1) and proved, under the MPEC-LICQ and the upper level strict complementarity condition, that an accumulation point of stationary points satisfying the second order necessary conditions for the relaxed problems is a B-stationary point of the original problem.

In this paper, we consider the following scheme as an approximation of problem (1.1):

$$
\begin{array}{ll}
\min & f(z) \\
\text{s.t.} & g(z) \leq 0, \;\; h(z) = 0 \\
& G_i(z) H_i(z) \leq \varepsilon^2 \\
& (G_i(z) + \varepsilon)(H_i(z) + \varepsilon) \geq \varepsilon^2 \\
& i = 1, 2, \cdots, m,
\end{array}
\tag{1.3}
$$

in which there are less constraints than problem (1.2). In the next section, we will show that the standard linear independence constraint qualification (LICQ) holds for the new relaxed problem under some mild conditions. In Section 3, we consider the convergence of global optimal solutions and stationary points of the relaxed problem. We obtain some sufficient conditions of B-stationarity for a feasible point of the original problem. In particular, we show that, under the MPEC-LICQ, an accumulation point of stationary points of the relaxed problems is B-stationary for problem (1.1) if the sequence generalized by the smallest eigenvalues of the Hessian matrices of the corresponding Lagrangian functions of the relaxed problems is bounded below. These new conditions can be verified easily in practice.

2 Some Results on Constraint Qualifications

In this section, we discuss constraint qualifications for problem (1.3). We let $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ denote the feasible sets of problems (1.1) and (1.3), respectively, and let, for $i = 1, 2, \cdots, m$ and $z \in R^n$,

$$\begin{aligned} \phi_{\varepsilon,i}(z) &= (G_i(z)+\varepsilon)(H_i(z)+\varepsilon)-\varepsilon^2, \\ \psi_{\varepsilon,i}(z) &= G_i(z)H_i(z)-\varepsilon^2, \end{aligned}$$

and

$$\begin{aligned} \Phi_\varepsilon(z) &= (\phi_{\varepsilon,1}(z), \phi_{\varepsilon,2}(z), \cdots, \phi_{\varepsilon,m}(z))^T, \\ \Psi_\varepsilon(z) &= (\psi_{\varepsilon,1}(z), \psi_{\varepsilon,2}(z), \cdots, \psi_{\varepsilon,m}(z))^T. \end{aligned}$$

Then we have

$$\nabla\phi_{\varepsilon,i}(z) = (G_i(z)+\varepsilon)\nabla H_i(z) + (H_i(z)+\varepsilon)\nabla G_i(z), \tag{2.1}$$

$$\nabla\psi_{\varepsilon,i}(z) = H_i(z)\nabla G_i(z) + G_i(z)\nabla H_i(z) \tag{2.2}$$

for $i = 1, 2, \cdots, m$ and

$$\begin{aligned} \nabla\Phi_\varepsilon(z) &= (\nabla\phi_{\varepsilon,1}(z), \cdots, \nabla\phi_{\varepsilon,m}(z))^T, \\ \nabla\Psi_\varepsilon(z) &= (\nabla\psi_{\varepsilon,1}(z), \cdots, \nabla\psi_{\varepsilon,m}(z))^T. \end{aligned}$$

For a function $F : R^n \to R^m$ and a given vector $z \in R^n$, we denote by

$$\mathcal{I}_F(z) = \{i : F_i(z) = 0\}$$

the active index set of F at z.

Theorem 2.1 *We have $\mathcal{F} = \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$ and, for any $\varepsilon > 0$,*

$$\mathcal{I}_{\Phi_\varepsilon}(z) \cap \mathcal{I}_{\Psi_\varepsilon}(z) = \emptyset. \tag{2.3}$$

Proof: First of all, $\mathcal{F} \subseteq \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$ is evident. Let $z \in \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$. Then for any $\varepsilon > 0$,

$$\begin{aligned} &G_i(z)H_i(z) \leq \varepsilon^2, \\ &G_i(z)H_i(z) + \varepsilon(G_i(z) + H_i(z)) \geq 0, \end{aligned}$$

and so

$$\varepsilon + (G_i(z) + H_i(z)) \geq 0.$$

Letting $\varepsilon \to 0$, we have

$$G_i(z)H_i(z) = 0, \;\; G_i(z) + H_i(z) \geq 0, \quad i = 1, 2, \cdots, m.$$

This means that $z \in \mathcal{F}$ and hence $\mathcal{F} = \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$.

Next we prove (2.3). Suppose that for some $\varepsilon > 0$ and some $z \in \mathcal{F}_\varepsilon$, $i \in \mathcal{I}_{\Phi_\varepsilon}(z) \cap \mathcal{I}_{\Psi_\varepsilon}(z)$. Then

$$\begin{aligned} &G_i(z)H_i(z) = \varepsilon^2, \\ &G_i(z)H_i(z) + \varepsilon(G_i(z) + H_i(z)) = 0. \end{aligned}$$

Combining these equalities, we have

$$G_i(z) + H_i(z) + \varepsilon = 0.$$

It then follows that

$$0 = \varepsilon^2 - G_i(z)H_i(z) = \varepsilon^2 + H_i(z)^2 + \varepsilon H_i(z) = \left(H_i(z) + \frac{\varepsilon}{2}\right)^2 + \frac{3}{4}\varepsilon^2,$$

which is a contradiction and so (2.3) holds. □

Next we show that, in contrast with problem (1.1), problem (1.3) satisfies the standard LICQ at a feasible point under some conditions.

Theorem 2.2 *For any $\bar{z} \in \mathcal{F}$, if the set of vectors*

$$\left\{\nabla g_l(\bar{z}), \nabla h_r(\bar{z}), \nabla G_i(\bar{z}), \nabla H_i(\bar{z}) : \quad l \in \mathcal{I}_g(\bar{z}), \; r = 1, \cdots, q, \; i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\right\}$$

is linearly independent, then, for any fixed $\varepsilon > 0$, there exists a neighborhood $U_\varepsilon(\bar{z})$ of $\bar{z}$ such that problem (1.3) satisfies the LICQ at any point $z \in U_\varepsilon(\bar{z}) \cap \mathcal{F}_\varepsilon$.

Proof: For any $\bar{z} \in \mathcal{F}$, it is obvious that

$$\psi_{\varepsilon,i}(\bar{z}) < 0, \quad i = 1, 2, \cdots, m$$

and

$$\phi_{\varepsilon,i}(\bar{z}) = 0 \iff i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}).$$

On the other hand, it follows from the continuity of the functions g, Φ_ε, and Ψ_ε that, for any fixed $\varepsilon > 0$, there exists a neighborhood $U_\varepsilon(\bar{z})$ of $\bar{z}$ such that, for any point $z \in U_\varepsilon(\bar{z}) \cap \mathcal{F}_\varepsilon$,

$$\mathcal{I}_g(z) \subseteq \mathcal{I}_g(\bar{z}), \quad \mathcal{I}_{\Phi_\varepsilon}(z) \subseteq \mathcal{I}_{\Phi_\varepsilon}(\bar{z}), \quad \mathcal{I}_{\Psi_\varepsilon}(z) \subseteq \mathcal{I}_{\Psi_\varepsilon}(\bar{z}).$$

This means that all the functions

$$\phi_{\varepsilon,i}, \quad \psi_{\varepsilon,j}, \quad i \notin \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}), \quad j = 1, 2, \cdots, m$$

are inactive at z in problem (1.3). In addition, we have that

$$H_i(z) + \varepsilon \neq 0, \quad G_i(z) + \varepsilon \neq 0, \quad i \in \mathcal{I}_{\Phi_\varepsilon}(z).$$

From (2.1), we obtain the conclusion immediately. $\Box$

Remark: If $\bar{z} \in \mathcal{F}$ is *nondegenerate* or *lower level strictly complementary*, which means

$$\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}) = \emptyset,$$

then the condition in Theorem 2.2 becomes very simple.

Under the MPEC-LICQ, we have the following stronger result in which the neighborhood is independent of ε.

Theorem 2.3 *For any $\bar{z} \in \mathcal{F}$, if the MPEC-LICQ holds at $\bar{z}$, which means*

$$\left\{\nabla g_l(\bar{z}), \nabla h_r(\bar{z}), \nabla G_i(\bar{z}), \nabla H_j(\bar{z}) : \ l \in \mathcal{I}_g(\bar{z}), \ r = 1, 2, \cdots, q, \ i \in \mathcal{I}_G(\bar{z}), \ j \in \mathcal{I}_H(\bar{z})\right\}$$

is linearly independent, then there exist a neighborhood $U(\bar{z})$ of $\bar{z}$ and a positive constant $\bar{\varepsilon}$ such that problem (1.3) satisfies the LICQ at any point $z \in U(\bar{z}) \cap \mathcal{F}_\varepsilon$ for any $\varepsilon \in (0, \bar{\varepsilon})$.

Proof: We first consider matrix functions whose columns consist of the vectors

$$\begin{aligned}
&\nabla g_l(z): \quad l \in \mathcal{I}_g(\bar{z}),\\
&\nabla h_r(z): \quad r = 1, 2, \cdots, q,\\
&\nabla G_i(z): \quad i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}),\\
&\nabla G_i(z) + \frac{G_i(z)+\varepsilon}{H_i(z)+\varepsilon}\nabla H_i(z) \ \text{ or } \ \nabla G_i(z) + \frac{G_i(z)}{H_i(z)}\nabla H_i(z): \quad i \in \mathcal{I}_G(\bar{z}) \setminus \mathcal{I}_H(\bar{z}),\\
&\nabla H_j(z): \quad j \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}),\\
&\nabla H_j(z) + \frac{H_j(z)+\varepsilon}{G_j(z)+\varepsilon}\nabla G_j(z) \ \text{ or } \ \nabla H_j(z) + \frac{H_j(z)}{G_j(z)}\nabla G_j(z): \quad j \in \mathcal{I}_H(\bar{z}) \setminus \mathcal{I}_G(\bar{z}).
\end{aligned}$$

Note that there are finitely many such matrix functions, which are denoted by

$$A_1(z,\varepsilon),\ A_2(z,\varepsilon), \cdots,\ A_N(z,\varepsilon). \tag{2.4}$$

Rearranging components if necessary, we may suppose that all these matrices are convergent to the same matrix $A(\bar{z})$ with columns

$$\nabla g_l(\bar{z}): \quad l \in \mathcal{I}_g(\bar{z}), \tag{2.5}$$

$$\nabla h_r(\bar{z}): \quad r = 1, 2, \cdots, q, \tag{2.6}$$

$$\nabla G_i(\bar{z}): \quad i \in \mathcal{I}_G(\bar{z}), \tag{2.7}$$

$$\nabla H_j(\bar{z}): \quad j \in \mathcal{I}_H(\bar{z}), \tag{2.8}$$

respectively, as $z \to \bar{z}$ and $\varepsilon \to 0$. It follows from the MPEC-LICQ assumption of the theorem that $A(\bar{z})$ has full column rank. Since the functions $G, H,$ and g are continuous, there exist a neighborhood $U(\bar{z})$ of $\bar{z}$ and a positive constant $\bar{\varepsilon}$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and any point $z \in U(\bar{z}) \cap \mathcal{F}_\varepsilon$, all the matrices in (2.4) have full column rank and

$$\mathcal{I}_G(z) \subseteq \mathcal{I}_G(\bar{z}),\ \ \mathcal{I}_H(z) \subseteq \mathcal{I}_H(\bar{z}),\ \ \mathcal{I}_g(z) \subseteq \mathcal{I}_g(\bar{z}). \tag{2.9}$$

Now we let $\varepsilon \in (0, \bar{\varepsilon})$ and $z \in U(\bar{z}) \cap \mathcal{F}_\varepsilon$ and show that problem (1.3) satisfies the LICQ at z. We suppose that the multiplier vectors $\lambda, \mu, \delta,$ and γ satisfy

$$\sum_{l \in \mathcal{I}_g(z)} \lambda_l \nabla g_l(z) + \sum_{r=1}^{q} \mu_r \nabla h_r(z) + \sum_{i \in \mathcal{I}_{\Phi_\varepsilon}(z)} \delta_i \nabla \phi_{\varepsilon,i}(z) + \sum_{j \in \mathcal{I}_{\Psi_\varepsilon}(z)} \gamma_j \nabla \psi_{\varepsilon,j}(z) = 0. \tag{2.10}$$

By (2.1) and (2.2), we have

$$\sum_{i \in \mathcal{I}_{\Phi_\varepsilon}(z)} \delta_i \nabla \phi_{\varepsilon,i}(z) \ = \sum_{i \in \mathcal{I}_{\Phi_\varepsilon}(z) \cap \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})} \delta_i \Big((H_i(z)+\varepsilon)\nabla G_i(z) + (G_i(z)+\varepsilon)\nabla H_i(z) \Big)$$

$$
\begin{aligned}
&+ \sum_{i \in \mathcal{I}_{\Phi_\varepsilon}(z) \setminus \mathcal{I}_H(\bar{z})} \delta_i (H_i(z) + \varepsilon) \Big(\nabla G_i(z) + \frac{G_i(z) + \varepsilon}{H_i(z) + \varepsilon} \nabla H_i(z) \Big) \\
&+ \sum_{i \in \mathcal{I}_{\Phi_\varepsilon}(z) \setminus \mathcal{I}_G(\bar{z})} \delta_i (G_i(z) + \varepsilon) \Big(\nabla H_i(z) + \frac{H_i(z) + \varepsilon}{G_i(z) + \varepsilon} \nabla G_i(z) \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j \in \mathcal{I}_{\Psi_\varepsilon}(z)} \gamma_j \nabla \psi_{\varepsilon, j}(z) \;\; = & \sum_{j \in \mathcal{I}_{\Psi_\varepsilon}(z) \cap \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})} \gamma_j \Big(H_j(z) \nabla G_j(z) + G_j(z) \nabla H_j(z) \Big) \\
& + \sum_{j \in \mathcal{I}_{\Psi_\varepsilon}(z) \setminus \mathcal{I}_H(\bar{z})} \gamma_j H_j(z) \Big(\nabla G_j(z) + \frac{G_j(z)}{H_j(z)} \nabla H_j(z) \Big) \\
& + \sum_{j \in \mathcal{I}_{\Psi_\varepsilon}(z) \setminus \mathcal{I}_G(\bar{z})} \gamma_j G_j(z) \Big(\nabla H_j(z) + \frac{H_j(z)}{G_j(z)} \nabla G_j(z) \Big).
\end{aligned}
$$

Note that (2.3) and (2.9) hold. Then, renumbering terms if necessary, we can choose a matrix $A_k(z, \varepsilon)$, $1 \leq k \leq N$, in (2.4) so that (2.10) can be rewritten as

$$
A_k(z, \varepsilon) \begin{pmatrix} \lambda \\ 0 \\ \mu \\ \delta_I (H_I(z) + \varepsilon e_I) \\ \gamma_{II} H_{II}(z) \\ 0 \\ \delta_{III} (H_{III}(z) + \varepsilon e_{III}) \\ \gamma_{IV} H_{IV}(z) \\ 0 \\ \delta_I (G_I(z) + \varepsilon e_I) \\ \gamma_{II} G_{II}(z) \\ 0 \\ \delta_V (G_V(z) + \varepsilon e_V) \\ \gamma_{VI} G_{VI}(z) \\ 0 \end{pmatrix} = 0, \tag{2.11}
$$

where

$$
\begin{aligned}
I &= \mathcal{I}_{\Phi_\varepsilon}(z) \cap \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}), \\
II &= \mathcal{I}_{\Psi_\varepsilon}(z) \cap \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}), \\
III &= \mathcal{I}_{\Phi_\varepsilon}(z) \setminus \mathcal{I}_H(\bar{z}), \\
IV &= \mathcal{I}_{\Psi_\varepsilon}(z) \setminus \mathcal{I}_H(\bar{z}), \\
V &= \mathcal{I}_{\Phi_\varepsilon}(z) \setminus \mathcal{I}_G(\bar{z}), \\
VI &= \mathcal{I}_{\Psi_\varepsilon}(z) \setminus \mathcal{I}_G(\bar{z}),
\end{aligned}
$$

and $e_{\mathcal{I}} = (1, 1, \cdots, 1)^T \in R^{|\mathcal{I}|}$. Since $A_k(z, \varepsilon)$ has full column rank, it follows from

(2.11) that the multiplier vector in (2.11) is zero. Noticing that

$$\begin{aligned} H_i(z) + \varepsilon \neq 0, \quad G_i(z) + \varepsilon \neq 0, \qquad & i \in \mathcal{I}_{\Phi_\varepsilon}(z), \\ H_i(z) \neq 0, \quad G_i(z) \neq 0, \qquad & i \in \mathcal{I}_{\Psi_\varepsilon}(z), \end{aligned}$$

and

$$\delta = \begin{pmatrix} \delta_{I} \\ \delta_{III} \\ \delta_{V} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_{II} \\ \gamma_{IV} \\ \delta_{VI} \end{pmatrix},$$

we have from (2.11) that

$$(\lambda^T, \mu^T, \delta^T, \gamma^T) = 0,$$

which implies that problem (1.3) satisfies the LICQ at z. This completes the proof. □

3 Convergence Analysis

In this section, we consider the limiting behavior of problem (1.3) as $\varepsilon \to 0$. First we give the convergence of global optimal solutions.

Theorem 3.1 *Let $\{\varepsilon_k\} \subseteq (0, +\infty)$ be convergent to 0 and suppose that z^k is a global optimal solution of problem (1.3) with $\varepsilon = \varepsilon_k$. If z^* is an accumulation point of the sequence $\{z^k\}$ as $k \to \infty$, then z^* is a global optimal solution of problem (1.1).*

Proof: Taking a subsequence if necessary, we assume without loss of generality that $\lim_{k\to\infty} z^k = z^*$. By Theorem 2.1, $z^* \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{F}_{\varepsilon_k}$ for all k, we have

$$f(z^k) \leq f(z), \quad \forall z \in \mathcal{F}, \quad \forall k.$$

Letting $k \to \infty$, we have from the continuity of f that

$$f(z^*) \leq f(z), \quad \forall z \in \mathcal{F},$$

i.e., z^* is a global optimal solution of problem (1.1). □

In a similar way, we can prove the next theorem.

Theorem 3.2 *Let both $\{\varepsilon_k\} \subseteq (0, +\infty)$ and $\{\bar{\varepsilon}_k\} \subseteq (0, +\infty)$ be convergent to 0 and $z^k \in \mathcal{F}_{\varepsilon_k}$ be an $\bar{\varepsilon}_k$-approximate solution of problem (1.3) with $\varepsilon = \varepsilon_k$, i.e.,*

$$f(z^k) - \bar{\varepsilon}_k \leq f(z), \quad \forall z \in \mathcal{F}_{\varepsilon_k}.$$

Then any accumulation point of $\{z^k\}$ is a global optimal solution of problem (1.1).

Now we consider the limiting behavior of stationary points of problem (1.3). We will use the standard definition of stationarity for problem (1.3), i.e., z is a *stationary* point of problem (1.3) if there exist Lagrange multiplier vectors $\lambda \in R^p, \mu \in R^q$, and $\delta, \gamma \in R^m$ such that

$$\nabla f(z) + \nabla g(z)^T \lambda + \nabla h(z)^T \mu - \nabla \Phi_\varepsilon(z)^T \delta + \nabla \Psi_\varepsilon(z)^T \gamma = 0, \tag{3.1}$$

$$\lambda \geq 0, \quad \delta \geq 0, \quad \gamma \geq 0, \tag{3.2}$$

$$g(z) \leq 0, \quad h(z) = 0, \quad \Phi_\varepsilon(z) \geq 0, \quad \Psi_\varepsilon(z) \leq 0, \tag{3.3}$$

$$g(z)^T \lambda = 0, \quad \Phi_\varepsilon(z)^T \delta = 0, \quad \Psi_\varepsilon(z)^T \gamma = 0. \tag{3.4}$$

For problem (1.1), $\bar{z} \in \mathcal{F}$ is said to be a *B-stationary* point if it satisfies

$$d^T \nabla f(\bar{z}) \geq 0, \quad \forall d \in \mathcal{T}(\bar{z}, \mathcal{F}), \tag{3.5}$$

where $\mathcal{T}(\bar{z}, \mathcal{F})$ stands for the tangent cone of $\mathcal{F}$ at $\bar{z}$. As in [19], a feasible point $\bar{z}$ is called *weakly stationary* to problem (1.1) if there exist multiplier vectors $\bar{\lambda} \in R^p, \bar{\mu} \in R^q$, and $\bar{u}, \bar{v} \in R^m$ such that

$$\nabla f(\bar{z}) + \nabla g(\bar{z})^T \bar{\lambda} + \nabla h(\bar{z})^T \bar{\mu} - \nabla G(\bar{z})^T \bar{u} - \nabla H(\bar{z})^T \bar{v} = 0, \tag{3.6}$$

$$\bar{\lambda} \geq 0, \quad \bar{z} \in \mathcal{F}, \quad \bar{\lambda}^T g(\bar{z}) = 0, \tag{3.7}$$

$$\bar{u}_i = 0, \quad i \notin \mathcal{I}_G(\bar{z}), \tag{3.8}$$

$$\bar{v}_i = 0, \quad i \notin \mathcal{I}_H(\bar{z}). \tag{3.9}$$

If the MPEC-LICQ holds at $\bar{z}$, then the definition (3.5) of B-stationarity can be restated in terms of Lagrange multipliers [12,17,19]: $\bar{z}$ is a B-stationary point of problem (1.1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{u}$, and $\bar{v}$ such that (3.6)–(3.9) hold with

$$\bar{u}_i \geq 0, \quad \bar{v}_i \geq 0, \quad i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}). \tag{3.10}$$

Obviously any weakly stationary point $\bar{z}$ is B-stationary whenever $\bar{z}$ satisfies the lower level strictly complementarity condition

$$\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}) = \emptyset.$$

Other two kinds of stationarity concepts for MPECs called C-stationarity and M-stationarity [19], which are stronger than the weak stationarity but weaker than B-stationarity, are also employed often. We say $\bar{z}$ is *C-stationary* to problem (1.1) if there exist multiplier vectors $\bar\lambda, \bar\mu, \bar u$, and $\bar v$ such that (3.6)–(3.9) hold and

$$\bar u_i \bar v_i \geq 0, \quad i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z) \tag{3.11}$$

and we say $\bar z$ is *M-stationary* to problem (1.1) if, furthermore, either $\bar u_i > 0,\ \bar v_i > 0$ or $\bar u_i \bar v_i = 0$ for all $i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z)$. In addition, a weakly stationary point $\bar z \in \mathcal{F}$ of problem (1.1) is said to satisfy the *upper level strict complementarity* condition if there exist multiplier vectors $\bar\lambda, \bar\mu, \bar u$, and $\bar v$ satisfying (3.6)–(3.9) and

$$\bar u_i \bar v_i \neq 0, \quad i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z). \tag{3.12}$$

The upper level strict complementarity is weaker than the lower level strict complementarity. Also, it is obvious that any M-stationary point of problem (1.1) satisfying the upper level strict complementarity condition is B-stationary.

Then we have the following convergence results.

Theorem 3.3 *Let $\{\varepsilon_k\} \subseteq (0, +\infty)$ be convergent to 0 and $z^k \in \mathcal{F}_{\varepsilon_k}$ be a stationary point of problem (1.3) with $\varepsilon = \varepsilon_k$ for each k. Suppose that $\bar z$ is an accumulation point of the sequence $\{z^k\}$. Then, if the MPEC-LICQ holds at $\bar z$, $\bar z$ is a C-stationary point of problem (1.1).*

Proof: Without loss of generality, we assume that

$$\lim_{k\to\infty} z^k = \bar z. \tag{3.13}$$

Since all the functions involved in problem (1.1) are continuous, $\mathcal{F}$ is closed and hence $\bar z \in \mathcal{F}$ by Theorem 2.1. It follows from the MPEC-LICQ assumption, (3.13), and Theorem 2.3 that, for any sufficiently large k, problem (1.3) with $\varepsilon = \varepsilon_k$ satisfies the LICQ at z^k and hence, by the stationarity of z^k, there exist unique Lagrange multiplier vectors $\lambda^k \in R^p, \mu^k \in R^q$, and $\delta^k, \gamma^k \in R^m$ such that

$$\nabla f(z^k) + \nabla g(z^k)^T \lambda^k + \nabla h(z^k)^T \mu^k - \nabla \Phi_{\varepsilon_k}(z^k)^T \delta^k + \nabla \Psi_{\varepsilon_k}(z^k)^T \gamma^k = 0, \tag{3.14}$$

$$\lambda^k \geq 0, \quad \delta^k \geq 0, \quad \gamma^k \geq 0, \tag{3.15}$$

$$g(z^k) \leq 0, \quad h(z^k) = 0, \quad \Phi_{\varepsilon_k}(z^k) \geq 0, \quad \Psi_{\varepsilon_k}(z^k) \leq 0, \tag{3.16}$$

$$g(z^k)^T \lambda^k = 0, \quad \Phi_{\varepsilon_k}(z^k)^T \delta^k = 0, \quad \Psi_{\varepsilon_k}(z^k)^T \gamma^k = 0. \tag{3.17}$$

It follows from (3.15)–(3.17) that

$$\lambda_i^k = 0, \quad i \notin \mathcal{I}_g(z^k), \tag{3.18}$$

$$\delta_i^k = 0, \quad i \notin \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k), \tag{3.19}$$

$$\gamma_i^k = 0, \quad i \notin \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k). \tag{3.20}$$

Now suppose that, for all sufficiently large k, (3.14)–(3.17) hold and, in addition, the conditions

$$\mathcal{I}_G(z^k) \subseteq \mathcal{I}_G(\bar{z}), \quad \mathcal{I}_H(z^k) \subseteq \mathcal{I}_H(\bar{z}), \quad \mathcal{I}_g(z^k) \subseteq \mathcal{I}_g(\bar{z}) \tag{3.21}$$

hold and all the matrix functions $A_i(z, \varepsilon)$, $i = 1, 2, \cdots, N$, in (2.4) defined in the proof of Theorem 2.3 have full column rank at (z^k, ε_k). By (2.1) and (2.2), we have

$$\begin{aligned}
\nabla\Phi_{\varepsilon_k}(z^k)^T\delta^k \;=\; & \sum_{i\in\mathcal{I}_G(\bar{z})\cap\mathcal{I}_H(\bar{z})} \delta_i^k\Big((H_i(z^k)+\varepsilon_k)\nabla G_i(z^k) + (G_i(z^k)+\varepsilon_k)\nabla H_i(z^k)\Big) \\
& + \sum_{i\in\mathcal{I}_G(\bar{z})\setminus\mathcal{I}_H(\bar{z})} \delta_i^k(H_i(z^k)+\varepsilon_k)\Big(\nabla G_i(z^k) + \frac{G_i(z^k)+\varepsilon_k}{H_i(z^k)+\varepsilon_k}\nabla H_i(z^k)\Big) \\
& + \sum_{i\in\mathcal{I}_H(\bar{z})\setminus\mathcal{I}_G(\bar{z})} \delta_i^k(G_i(z^k)+\varepsilon_k)\Big(\nabla H_i(z^k) + \frac{H_i(z^k)+\varepsilon_k}{G_i(z^k)+\varepsilon_k}\nabla G_i(z^k)\Big)
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\nabla\Psi_{\varepsilon_k}(z^k)^T\gamma^k \;=\; & \sum_{j\in\mathcal{I}_G(\bar{z})\cap\mathcal{I}_H(\bar{z})} \gamma_j^k\Big(H_j(z^k)\nabla G_j(z^k) + G_j(z^k)\nabla H_j(z^k)\Big) \\
& + \sum_{j\in\mathcal{I}_G(\bar{z})\setminus\mathcal{I}_H(\bar{z})} \gamma_j^k H_j(z^k)\Big(\nabla G_j(z^k) + \frac{G_j(z^k)}{H_j(z^k)}\nabla H_j(z^k)\Big) \\
& + \sum_{j\in\mathcal{I}_H(\bar{z})\setminus\mathcal{I}_G(\bar{z})} \gamma_j^k G_j(z^k)\Big(\nabla H_j(z^k) + \frac{H_j(z^k)}{G_j(z^k)}\nabla G_j(z^k)\Big).
\end{aligned} \tag{3.23}$$

Then, taking into account (2.3), we have from (3.14) and (3.18)–(3.23) that

$$\begin{aligned}
-\nabla f(z^k) \;=\; & \nabla g(z^k)^T\lambda^k + \nabla h(z^k)^T\mu^k \\
& - \sum_{i\in\mathcal{I}_G(\bar{z})\cap\mathcal{I}_H(\bar{z})} \Big(\delta_i^k(H_i(z^k)+\varepsilon_k) - \gamma_i^k H_i(z^k)\Big)\nabla G_i(z^k) \\
& - \sum_{i\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)\setminus\mathcal{I}_H(\bar{z})} \delta_i^k(H_i(z^k)+\varepsilon_k)\Big(\nabla G_i(z^k) + \frac{G_i(z^k)+\varepsilon_k}{H_i(z^k)+\varepsilon_k}\nabla H_i(z^k)\Big) \\
& - \sum_{i\in\mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\setminus\mathcal{I}_H(\bar{z})} \Big(-\gamma_i^k H_i(z^k)\Big)\Big(\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)}\nabla H_i(z^k)\Big) \\
& - \sum_{i\in\mathcal{I}_G(\bar{z})\cap\mathcal{I}_H(\bar{z})} \Big(\delta_i^k(G_i(z^k)+\varepsilon_k) - \gamma_i^k G_i(z^k)\Big)\nabla H_i(z^k) \\
& - \sum_{i\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)\setminus\mathcal{I}_G(\bar{z})} \delta_i^k(G_i(z^k)+\varepsilon_k)\Big(\nabla H_i(z^k) + \frac{H_i(z^k)+\varepsilon_k}{G_i(z^k)+\varepsilon_k}\nabla G_i(z^k)\Big) \\
& - \sum_{i\in\mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\setminus\mathcal{I}_H(\bar{z})} \Big(-\gamma_i^k G_i(z^k)\Big)\Big(\nabla H_i(z^k) + \frac{H_i(z^k)}{G_i(z^k)}\nabla G_i(z^k)\Big)
\end{aligned}$$

$$= \quad A_{N_k}(z^k, \varepsilon_k) \begin{pmatrix} \lambda^k_{\mathcal{I}_g(\bar{z})} \\ \mu^k \\ u^k \\ v^k \end{pmatrix}, \tag{3.24}$$

where u^k, v^k are given by

$$u_i^k = \begin{cases} \delta_i^k (H_i(z^k) + \varepsilon_k), & i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cap \mathcal{I}_G(\bar{z}) \\ -\gamma_i^k H_i(z^k), & i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \cap \mathcal{I}_G(\bar{z}) \\ 0, & i \in \mathcal{I}_G(\bar{z}) \setminus \left(\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\right), \end{cases} \tag{3.25}$$

$$v_i^k = \begin{cases} \delta_i^k (G_i(z^k) + \varepsilon_k), & i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cap \mathcal{I}_H(\bar{z}) \\ -\gamma_i^k G_i(z^k), & i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \cap \mathcal{I}_H(\bar{z}) \\ 0, & i \in \mathcal{I}_H(\bar{z}) \setminus \left(\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\right), \end{cases} \tag{3.26}$$

respectively, and $A_{N_k}(z, \varepsilon)$ is one of the matrix functions in (2.4). As we assumed above, $A_{N_k}(z^k, \varepsilon_k)$ has full column rank for all sufficiently large k. In consequence, it follows from (3.13) and (3.24) that all the multiplier sequences

$$\{\lambda_l^k : \ l \in \mathcal{I}_g(\bar{z})\}, \quad \{\mu_r^k : \ r = 1, 2, \cdots, q\}, \tag{3.27}$$

$$\{u_i^k : \ i \in \mathcal{I}_G(\bar{z})\}, \quad \{v_j^k : \ j \in \mathcal{I}_H(\bar{z})\} \tag{3.28}$$

are convergent. Define $\bar{\lambda} \in R^p, \bar{\mu} \in R^q$, and $\bar{u}, \bar{v} \in R^m$ as follows:

$$\bar{\lambda}_l = \begin{cases} \lim_{k\to\infty} \ \lambda_l^k & , \quad l \in \mathcal{I}_g(\bar{z}) \\ 0 & , \quad l \notin \mathcal{I}_g(\bar{z}) \end{cases}, \tag{3.29}$$

$$\bar{\mu}_r = \lim_{k\to\infty} \ \mu_r^k, \quad r = 1, 2, \cdots, q, \tag{3.30}$$

$$\bar{u}_i = \begin{cases} \lim_{k\to\infty} \ u_i^k & , \quad i \in \mathcal{I}_G(\bar{z}) \\ 0 & , \quad i \notin \mathcal{I}_G(\bar{z}) \end{cases}, \tag{3.31}$$

$$\bar{v}_j = \begin{cases} \lim_{k\to\infty} \ v_j^k & , \quad j \in \mathcal{I}_H(\bar{z}) \\ 0 & , \quad j \notin \mathcal{I}_H(\bar{z}) \end{cases}. \tag{3.32}$$

Letting $k \to \infty$ in (3.24) and noticing that

$$\lim_{k\to\infty} A_{N_k}(z^k, \varepsilon_k) = A(\bar{z}),$$

where $A(\bar{z})$ is the matrix with the columns (2.5)–(2.8), we have from (3.29)–(3.32) that

$$-\nabla f(\bar{z}) = \nabla g(\bar{z})^T \bar{\lambda} + \nabla h(\bar{z})^T \bar{\mu} - \nabla G(\bar{z})^T \bar{u} - \nabla H(\bar{z})^T \bar{v},$$

i.e., (3.6) holds for the multiplier vectors $\bar\lambda, \bar\mu, \bar u, \bar v$. On the other hand, we have (3.7)–(3.9) immediately from (3.15), (3.16), (3.29), (3.31), and (3.32). Then the rest of the proof is to show (3.11). In fact, for each $i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z)$, we have from (2.3) and (3.25)–(3.26) that

$$u_i^k v_i^k = \begin{cases} (\delta_i^k)^2 (H_i(z^k) + \varepsilon_k)(G_i(z^k) + \varepsilon_k) = (\delta_i^k \varepsilon_k)^2, & i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \\ (\gamma_i^k)^2 H_i(z^k) G_i(z^k) = (\gamma_i^k \varepsilon_k)^2, & i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \\ 0, & i \notin \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k). \end{cases}$$

Letting $k \to \infty$, we obtain (3.11) since the sequences $\{u_i^k\}$ and $\{v_i^k\}$ in (3.28) are convergent. Hence $\bar z$ is a C-stationary point of problem (1.1). This completes the proof. □

From the definitions of B- and C-stationarity, we have the following result immediately.

Corollary 3.1 *Let the assumptions in Theorem 3.3 be satisfied. If, in addition, $\bar z$ is nondegenerate, then it is a B-stationary point of problem (1.1).*

Next we consider some other sufficient conditions on M- and B-stationarity. We say $z \in R^n$ satisfies the *second-order necessary conditions* for problem (1.3) if there exist multiplier vectors $\lambda \in R^p, \mu \in R^q$, and $\delta, \gamma \in R^m$ such that (3.1)–(3.4) hold and, in addition,

$$d^T \nabla_z^2 L_\varepsilon(z, \lambda, \mu, \delta, \gamma) d \ge 0, \quad \forall d \in \mathcal{T}_\varepsilon(z), \tag{3.33}$$

where

$$L_\varepsilon(z, \lambda, \mu, \gamma, \delta) = f(z) + \lambda^T g(z) + \mu^T h(z) - \delta^T \Phi_\varepsilon(z) + \gamma^T \Psi_\varepsilon(z)$$

stands for the Lagrangian of problem (1.3) and

$$\begin{aligned} \mathcal{T}_\varepsilon(z) = \Big\{ d \in R^n : \quad & d^T \nabla \phi_{\varepsilon,i}(z) = 0, \ i \in \mathcal{I}_{\Phi_\varepsilon}(z); \\ & d^T \nabla \psi_{\varepsilon,j}(z) = 0, \ j \in \mathcal{I}_{\Psi_\varepsilon}(z); \\ & d^T \nabla g_l(z) = 0, \ l \in \mathcal{I}_g(z); \\ & d^T \nabla h_r(z) = 0, \ r = 1, 2, \cdots, q \Big\}. \end{aligned}$$

We next introduce a new kind of conditions which are weaker than the second-order necessary conditions for problem (1.3). Suppose that α is a nonnegative number. We say that, at a stationary point z of problem (1.3), the matrix $\nabla_z^2 L_\varepsilon(z, \lambda, \mu, \delta, \gamma)$ is *bounded below* with constant α on the corresponding tangent space $\mathcal{T}_\varepsilon(z)$ if

$$d^T \nabla_z^2 L_\varepsilon(z, \lambda, \mu, \delta, \gamma) d \ge -\alpha ||d||^2, \quad \forall d \in \mathcal{T}_\varepsilon(z). \tag{3.34}$$

A few words about (3.33) and (3.34): The condition (3.34) is clearly weaker than (3.33). In fact, for the matrix $\nabla_z^2 L_\varepsilon(z,\lambda,\mu,\delta,\gamma)$, there must exist a number α such that (3.34) hold. For example, any α such that $-\alpha$ is less than the smallest eigenvalue of $\nabla_z^2 L_\varepsilon(z,\lambda,\mu,\delta,\gamma)$ must satisfy (3.34). However, the condition (3.33) means that the matrix $\nabla_z^2 L_\varepsilon(z,\lambda,\mu,\delta,\gamma)$ should have some kind of semi-definiteness on the tangent space $\mathcal{T}_\varepsilon(z)$. Note that, in (3.34), the constant $-\alpha$ may be larger than the smallest eigenvalue mentioned above.

Theorem 3.4 *Let $\{\varepsilon_k\} \subseteq (0,+\infty)$ be convergent to 0 and $z^k \in \mathcal{F}_{\varepsilon_k}$ be a stationary point of problem (1.3) with $\varepsilon = \varepsilon_k$ and multiplier vectors $\lambda^k, \mu^k, \delta^k$, and γ^k. Suppose that, for each k, $\nabla_z^2 L_{\varepsilon_k}(z^k,\lambda^k,\mu^k,\delta^k,\gamma^k)$ is bounded below with constant α_k on the corresponding tangent space $\mathcal{T}_{\varepsilon_k}(z^k)$. Let $\bar{z}$ be an accumulation point of the sequence $\{z^k\}$. If the sequence $\{\alpha_k\}$ is bounded and the MPEC-LICQ holds at $\bar{z}$, then $\bar{z}$ is an M-stationary point of problem (1.1).*

Proof: Assume that $\lim_{k\to\infty} z^k = \bar{z}$ without loss of generality. First of all, we note from Theorem 3.3 that $\bar{z}$ is a C-stationary point of problem (1.1). To prove the theorem, we assume to the contrary that $\bar{z}$ is not M-stationary to problem (1.1). Then, it follows from the definitions of C-stationarity and M-stationarity that there must exist an $i_0 \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$ such that

$$\bar{u}_{i_0} < 0, \quad \bar{v}_{i_0} < 0. \tag{3.35}$$

By (3.25)–(3.26) and (3.31)–(3.32), we have

$$i_0 \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)$$

for every sufficiently large k. First we consider the case where $i_0 \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)$ for infinitely many k. Furthermore, taking a subsequence if necessary, we may assume without loss of generality that

$$i_0 \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \tag{3.36}$$

for all sufficiently large k. Then, by (3.25) and (3.26),

$$\bar{u}_{i_0} = -\lim_{k\to\infty} \gamma_{i_0}^k H_{i_0}(z^k) < 0, \tag{3.37}$$

$$\bar{v}_{i_0} = -\lim_{k\to\infty} \gamma_{i_0}^k G_{i_0}(z^k) < 0, \tag{3.38}$$

and so

$$\lim_{k\to\infty} \frac{H_{i_0}(z^k)}{G_{i_0}(z^k)} = \frac{\bar{u}_{i_0}}{\bar{v}_{i_0}} > 0. \tag{3.39}$$

In what follows, we suppose that, for all sufficiently large k, (3.14)–(3.17), (3.21), and

$$\frac{H_{i_0}(z^k)}{G_{i_0}(z^k)} > 0$$

hold and all the matrix functions $A_i(z,\varepsilon)$, $i = 1, 2, \cdots, N$, in (2.4) have full column rank at (z^k, ε_k). For such k, the matrix $A_{N_k}(z^k, \varepsilon_k)$ whose columns consist of the vectors

$$\begin{aligned}
&\nabla g_l(z^k): \quad l \in \mathcal{I}_g(\bar{z}),\\
&\nabla h_r(z^k): \quad r = 1, 2, \cdots, q,\\
&\nabla G_i(z^k): \quad i \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_G(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big),\\
&\nabla G_i(z^k) + \frac{G_i(z^k)+\varepsilon_k}{H_i(z^k)+\varepsilon_k}\nabla H_i(z^k): \quad i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z}),\\
&\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)}\nabla H_i(z^k): \qquad i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z}),\\
&\nabla H_j(z^k): \quad j \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_H(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big),\\
&\nabla H_j(z^k) + \frac{H_j(z^k)+\varepsilon_k}{G_j(z^k)+\varepsilon_k}\nabla G_j(z^k): \quad j \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z}),\\
&\nabla H_j(z^k) + \frac{H_j(z^k)}{G_j(z^k)}\nabla G_j(z^k): \qquad j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z})
\end{aligned}$$

has full column rank. Therefore, we can choose a vector $d^k \in R^n$ such that

$$(d^k)^T \nabla g_l(z^k) = 0, \qquad l \in \mathcal{I}_g(\bar{z}); \tag{3.40}$$

$$(d^k)^T \nabla h_r(z^k) = 0, \qquad r = 1, 2, \cdots, q; \tag{3.41}$$

$$(d^k)^T \nabla G_i(z^k) = 0, \quad i \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_G(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big), \ i \neq i_0; \tag{3.42}$$

$$(d^k)^T \Big(\nabla G_i(z^k) + \frac{G_i(z^k)+\varepsilon_k}{H_i(z^k)+\varepsilon_k}\nabla H_i(z^k)\Big) = 0, \quad i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z}); \tag{3.43}$$

$$(d^k)^T \Big(\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)}\nabla H_i(z^k)\Big) = 0, \qquad i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z}); \tag{3.44}$$

$$(d^k)^T \nabla H_j(z^k) = 0, \quad j \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_H(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big), \ j \neq i_0; \tag{3.45}$$

$$(d^k)^T \Big(\nabla H_j(z^k) + \frac{H_j(z^k)+\varepsilon_k}{G_j(z^k)+\varepsilon_k}\nabla G_j(z^k)\Big) = 0, \quad j \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z}); \tag{3.46}$$

$$(d^k)^T \Big(\nabla H_j(z^k) + \frac{H_j(z^k)}{G_j(z^k)}\nabla G_j(z^k)\Big) = 0, \qquad j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z}); \tag{3.47}$$

$$(d^k)^T \nabla G_{i_0}(z^k) = 1; \tag{3.48}$$

$$(d^k)^T \nabla H_{i_0}(z^k) = -\frac{H_{i_0}(z^k)}{G_{i_0}(z^k)}.$$

Then for any $i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)$ and any $j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)$, since

$$\begin{aligned}
\nabla \phi_{\varepsilon_k,i}(z^k) &= (G_i(z^k)+\varepsilon_k)\nabla H_i(z^k) + (H_i(z^k)+\varepsilon_k)\nabla G_i(z^k), \\
\nabla \psi_{\varepsilon_k,j}(z^k) &= H_j(z^k)\nabla G_j(z^k) + G_j(z^k)\nabla H_j(z^k),
\end{aligned}$$

we have

$$\begin{aligned}
(d^k)^T \nabla \phi_{\varepsilon_k,i}(z^k) = 0, \quad i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k), \\
(d^k)^T \nabla \psi_{\varepsilon_k,j}(z^k) = 0, \quad j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k),
\end{aligned}$$

and so $d^k \in \mathcal{T}_{\varepsilon_k}(z^k)$. Furthermore, we can choose the sequence $\{d^k\}$ to be bounded. Since $\nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ is bounded below with constant α_k on the corresponding tangent space $\mathcal{T}_{\varepsilon_k}(z^k)$, we have from (3.34) that there exists a constant C such that

$$(d^k)^T \nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \geq -\alpha_k ||d^k||^2 \geq C, \tag{3.49}$$

where the last inequality follows from the boundedness of the sequences $\{\alpha_k\}$ and $\{d^k\}$. Note that

$$\begin{aligned}
\nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \gamma^k, \delta^k) &= \nabla^2 f(z^k) + \sum_{l=1}^{p} \lambda_l^k \nabla^2 g_l(z^k) + \sum_{r=1}^{q} \mu_r^k \nabla^2 h_r(z^k) \\
&\quad - \sum_{i=1}^{m} \delta_i^k \nabla^2 \phi_{\varepsilon_k,i}(z^k) + \sum_{j=1}^{m} \gamma_j^k \nabla^2 \psi_{\varepsilon_k,j}(z^k) \\
&= \nabla^2 f(z^k) + \sum_{l \in \mathcal{I}_g(\bar{z})} \lambda_l^k \nabla^2 g_l(z^k) + \sum_{r=1}^{q} \mu_r^k \nabla^2 h_r(z^k) \\
&\quad - \sum_{i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)} \delta_i^k \nabla^2 \phi_{\varepsilon_k,i}(z^k) + \sum_{j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)} \gamma_j^k \nabla^2 \psi_{\varepsilon_k,j}(z^k)
\end{aligned}$$

by (3.18)–(3.20) and

$$\begin{aligned}
\nabla^2 \phi_{\varepsilon_k,i}(z^k) &= \nabla G_i(z^k) \nabla H_i(z^k)^T + \nabla H_i(z^k) \nabla G_i(z^k)^T \\
&\quad + (G_i(z^k)+\varepsilon_k)\nabla^2 H_i(z^k) + (H_i(z^k)+\varepsilon_k)\nabla^2 G_i(z^k), \\
\nabla^2 \psi_{\varepsilon_k,j}(z^k) &= \nabla G_j(z^k) \nabla H_j(z^k)^T + \nabla H_j(z^k) \nabla G_j(z^k)^T \\
&\quad + G_j(z^k)\nabla^2 H_j(z^k) + H_j(z^k)\nabla^2 G_j(z^k).
\end{aligned}$$

We then have

$$\begin{aligned}
&(d^k)^T \nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \\
&= (d^k)^T \nabla^2 f(z^k) d^k + \sum_{l \in \mathcal{I}_g(\bar{z})} \lambda_l^k (d^k)^T \nabla^2 g_l(z^k) d^k + \sum_{r=1}^{q} \mu_r^k (d^k)^T \nabla^2 h_r(z^k) d^k
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)} \delta_i^k\Big((d^k)^T\nabla G_i(z^k)\nabla H_i(z^k)^T d^k + (d^k)^T\nabla H_i(z^k)\nabla G_i(z^k)^T d^k \\
&\qquad + (G_i(z^k)+\varepsilon_k)(d^k)^T\nabla^2 H_i(z^k)d^k + (H_i(z^k)+\varepsilon_k)(d^k)^T\nabla^2 G_i(z^k)d^k\Big) \\
&+ \sum_{j\in\mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)} \gamma_j^k\Big((d^k)^T\nabla G_j(z^k)\nabla H_j(z^k)^T d^k + (d^k)^T\nabla H_j(z^k)\nabla G_j(z^k)^T d^k \\
&\qquad + G_j(z^k)(d^k)^T\nabla^2 H_j(z^k)d^k + H_j(z^k)(d^k)^T\nabla^2 G_j(z^k)d^k\Big).
\end{aligned} \tag{3.50}
$$

By the twice continuous differentiability of the functions, the boundness of the sequence $\{d^k\}$, and the convergence of the sequences $\{z^k\}$, $\{\lambda_l^k\}$ and $\{\mu_r^k\}$ (by (3.29)–(3.30)), the terms

$$
(d^k)^T\nabla^2 f(z^k)d^k, \quad \sum_{l\in\mathcal{I}_g(\bar z)} \lambda_l^k (d^k)^T\nabla^2 g_l(z^k)d^k, \quad \sum_{r=1}^{q}\mu_r^k(d^k)^T\nabla^2 h_r(z^k)d^k
$$

are all bounded. Consider arbitrary indices i and j such that $i\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)$ for infinitely many k and $j\in\mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\setminus\{i_0\}$ for infinitely many k, respectively. If

$$
i\in\mathcal{I}_G(\bar z)\cap\mathcal{I}_H(\bar z) \quad \text{or} \quad j\in\mathcal{I}_G(\bar z)\cap\mathcal{I}_H(\bar z),
$$

then

$$
(d^k)^T\nabla G_i(z^k)=0 \quad \text{or} \quad (d^k)^T\nabla H_j(z^k)=0
$$

and, by (3.25)–(3.26) and (3.31)–(3.32), the sequences

$$
\Big\{\delta_i^k(G_i(z^k)+\varepsilon_k)\Big\}, \quad \Big\{\delta_i^k(H_i(z^k)+\varepsilon_k)\Big\},
$$

and

$$
\Big\{\gamma_j^k G_j(z^k)\Big\}, \quad \Big\{\gamma_j^k H_j(z^k)\Big\}
$$

are all convergent. If

$$
i,j\notin\mathcal{I}_G(\bar z)\cap\mathcal{I}_H(\bar z),
$$

then, also by (3.25)–(3.26) and (3.31)–(3.32), the sequences $\{\delta_i^k\}$ and $\{\gamma_j^k\}$ are convergent. Therefore, we have that the terms

$$
\begin{aligned}
&\sum_{i\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)} \delta_i^k\Big((d^k)^T\nabla G_i(z^k)\nabla H_i(z^k)^T d^k + (d^k)^T\nabla H_i(z^k)\nabla G_i(z^k)^T d^k + \\
&\qquad (G_i(z^k)+\varepsilon_k)(d^k)^T\nabla^2 H_i(z^k)d^k + (H_i(z^k)+\varepsilon_k)(d^k)^T\nabla^2 G_i(z^k)d^k\Big)
\end{aligned}
$$

and

$$\sum_{j\in\mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)\setminus\{i_0\}} \gamma_j^k\Big((d^k)^T\nabla G_j(z^k)\nabla H_j(z^k)^T d^k + (d^k)^T\nabla H_j(z^k)\nabla G_j(z^k)^T d^k + G_j(z^k)(d^k)^T\nabla^2 H_j(z^k)d^k + H_j(z^k)(d^k)^T\nabla^2 G_j(z^k)d^k\Big)$$

are bounded. On the other hand, however, we have (3.36) for all sufficiently large k and

$$\begin{aligned}
&\gamma_{i_0}^k\Big((d^k)^T\nabla G_{i_0}(z^k)\nabla H_{i_0}(z^k)^T d^k + (d^k)^T\nabla H_{i_0}(z^k)\nabla G_{i_0}(z^k)^T d^k \\
&+G_{i_0}(z^k)(d^k)^T\nabla^2 H_{i_0}(z^k)d^k + H_{i_0}(z^k)(d^k)^T\nabla^2 G_{i_0}(z^k)d^k\Big) \qquad (3.51)\\
= &-\frac{2\gamma_{i_0}^k H_{i_0}(z^k)}{G_{i_0}(z^k)} + \gamma_{i_0}^k\Big(G_{i_0}(z^k)(d^k)^T\nabla^2 H_{i_0}(z^k)d^k + H_{i_0}(z^k)(d^k)^T\nabla^2 G_{i_0}(z^k)d^k\Big).
\end{aligned}$$

Since (3.39) holds and $\gamma_{i_0}^k \to +\infty$ as $k\to\infty$ by (3.15) and (3.37), we have

$$-\frac{2\gamma_{i_0}^k H_{i_0}(z^k)}{G_{i_0}(z^k)} \to -\infty$$

as $k\to\infty$. Note that, by (3.37) and (3.38), the sequences

$$\Big\{\gamma_{i_0}^k G_{i_0}(z^k)\Big\}, \quad \Big\{\gamma_{i_0}^k H_{i_0}(z^k)\Big\}$$

are also convergent. We then have that the term (3.51) tends to $-\infty$ as $k\to\infty$. Therefore, it follows from (3.50) that

$$(d^k)^T\nabla_z^2 L_{\varepsilon_k}(z^k,\lambda^k,\mu^k,\delta^k,\gamma^k)d^k \to -\infty$$

as $k\to\infty$. This contradicts (3.49) and hence $\bar z$ is M-stationary to problem (1.1).

Finally we consider the case where $i_0\in\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)$ for infinitely many k. By (3.25) and (3.26), we have from (3.35) that

$$\begin{aligned}
\bar u_{i_0} &= \lim_{k\to\infty}\delta_{i_0}^k(H_{i_0}(z^k)+\varepsilon_k) < 0,\\
\bar v_{i_0} &= \lim_{k\to\infty}\delta_{i_0}^k(G_{i_0}(z^k)+\varepsilon_k) < 0,
\end{aligned}$$

and so

$$\lim_{k\to\infty}\frac{H_{i_0}(z^k)+\varepsilon_k}{G_{i_0}(z^k)+\varepsilon_k} = \frac{\bar u_{i_0}}{\bar v_{i_0}} > 0.$$

Therefore, we can also choose a bounded sequence $\{d^k\}$ such that (3.40)–(3.48) and

$$(d^k)^T\nabla H_{i_0}(z^k) = -\frac{H_{i_0}(z^k)+\varepsilon_k}{G_{i_0}(z^k)+\varepsilon_k}$$

hold for each k. In a similar way, we then obtain a contradiction and so $\bar{z}$ is M-stationary to problem (1.1). This completes the proof. □

Corollary 3.2 *Let* $\{\varepsilon_k\}, \{z^k\}$, *and* $\bar{z}$ *be the same as in Theorem 3.4. If* z^k *together with the corresponding multiplier vectors* $\lambda^k, \mu^k, \delta^k$, *and* γ^k *satisfies the second-order necessary conditions for problem (1.3) with* $\varepsilon = \varepsilon_k$ *and the MPEC-LICQ holds at* $\bar{z}$, *then* $\bar{z}$ *is an M-stationary point of problem (1.1).*

Corollary 3.3 *Let the assumptions in Theorem 3.4 be satisfied. If, in addition,* $\bar{z}$ *satisfies the upper level strict complementarity conditions, then it is a B-stationary point of problem (1.1).*

The last corollary establishes convergence to a B-stationary point under the second-order necessary conditions and the upper level strict complementarity. These or similar conditions have also been assumed in [6, 8, 9, 19], but they are somewhat restrictive and may be difficult to verify in practice. The next theorem provides a new condition for convergence to a B-stationary point, which can be dealt with more easily. We note that, unlike [6, 8, 9, 19], it relies on neither the upper level strict complementarity nor the asymptotic weak nondegeneracy.

Theorem 3.5 *Let* $\{\varepsilon_k\}, \{z^k\}$, *and* $\bar{z}$ *be the same as in Theorem 3.4 and* $\lambda^k, \mu^k, \delta^k$, *and* γ^k *be the multiplier vectors corresponding to* z^k. *Let* β_k *be the smallest eigenvalue of the matrix* $\nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$. *If the sequence* $\{\beta_k\}$ *is bounded below and the MPEC-LICQ holds at* $\bar{z}$, *then* $\bar{z}$ *is a B-stationary point of problem (1.1).*

Proof: It is easy to see that the assumptions of Theorem 3.4 are satisfied with $\alpha_k = \max\{-\beta_k, 0\}$ and so $\bar{z}$ is an M-stationary point of problem (1.1). Suppose that $\bar{z}$ is not B-stationary to problem (1.1). Then, by the definitions of B- and M-stationarity, there exists an $i_0 \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$ such that

$$\bar{u}_{i_0} < 0, \quad \bar{v}_{i_0} = 0 \tag{3.52}$$

or

$$\bar{u}_{i_0} = 0, \quad \bar{v}_{i_0} < 0.$$

By (3.25)–(3.26) and (3.31)–(3.32), we have

$$i_0 \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)$$

for every sufficiently large k. Without loss of generality, we assume that (3.52) holds.

First we consider the case where $i_0 \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k)$ for infinitely many k. By taking a subsequence if necessary, we assume

$$i_0 \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \tag{3.53}$$

for all sufficiently large k. Then, it follows from (3.25), (3.26), and (3.52) that

$$\bar{u}_{i_0} = -\lim_{k\to\infty} \gamma_{i_0}^k H_{i_0}(z^k) < 0$$

and so, by (3.15), we have

$$\lim_{k\to\infty} \gamma_{i_0}^k = +\infty. \tag{3.54}$$

Now we suppose that, for all sufficiently large k, (3.14)–(3.17) and (3.21) hold and the matrix $A_{N_k}(z^k, \varepsilon_k)$ defined in the proof of Theorem 3.4 has full column rank. Therefore, we can choose a vector $d^k \in R^n$ such that

$$\begin{aligned}
&(d^k)^T \nabla g_l(z^k) = 0, && l \in \mathcal{I}_g(\bar{z});\\
&(d^k)^T \nabla h_r(z^k) = 0, && r = 1, 2, \cdots, q;\\
&(d^k)^T \nabla G_i(z^k) = 0, \quad i \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_G(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big),\ i \neq i_0; &&\\
&(d^k)^T \Big(\nabla G_i(z^k) + \frac{G_i(z^k) + \varepsilon_k}{H_i(z^k) + \varepsilon_k} \nabla H_i(z^k)\Big) = 0, && i \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z});\\
&(d^k)^T \Big(\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)} \nabla H_i(z^k)\Big) = 0, && i \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_H(\bar{z});\\
&(d^k)^T \nabla H_j(z^k) = 0, \quad j \in \Big(\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big) \cup \Big(\mathcal{I}_H(\bar{z}) \setminus (\mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \cup \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k))\Big),\ j \neq i_0; &&\\
&(d^k)^T \Big(\nabla H_j(z^k) + \frac{H_j(z^k) + \varepsilon_k}{G_j(z^k) + \varepsilon_k} \nabla G_j(z^k)\Big) = 0, && j \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z});\\
&(d^k)^T \Big(\nabla H_j(z^k) + \frac{H_j(z^k)}{G_j(z^k)} \nabla G_j(z^k)\Big) = 0, && j \in \mathcal{I}_{\Psi_{\varepsilon_k}}(z^k) \setminus \mathcal{I}_G(\bar{z});\\
&(d^k)^T \nabla G_{i_0}(z^k) = 1; &&\\
&(d^k)^T \nabla H_{i_0}(z^k) = -1. &&
\end{aligned}$$

Furthermore, we can choose the sequence $\{d^k\}$ to be bounded. By the assumptions of the theorem, there exists a constant C such that

$$(d^k)^T \nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \geq \beta_k ||d^k||^2 \geq C \tag{3.55}$$

holds for all k. In a similar way to the proof of Theorem 3.4, we can show that all the terms on the right-hand side of (3.50) except

$$\begin{aligned}
&\gamma_{i_0}^k \Big((d^k)^T \nabla G_{i_0}(z^k) \nabla H_{i_0}(z^k)^T d^k + (d^k)^T \nabla H_{i_0}(z^k) \nabla G_{i_0}(z^k)^T d^k \\
&+ G_{i_0}(z^k)(d^k)^T \nabla^2 H_{i_0}(z^k) d^k + H_{i_0}(z^k)(d^k)^T \nabla^2 G_{i_0}(z^k) d^k \Big)
\end{aligned}$$

are bounded. On the other hand,

$$\gamma_{i_0}^k \Big((d^k)^T \nabla G_{i_0}(z^k) \nabla H_{i_0}(z^k)^T d^k + (d^k)^T \nabla H_{i_0}(z^k) \nabla G_{i_0}(z^k)^T d^k \Big) = -2\gamma_{i_0}^k \to -\infty$$

by the definition of $\{d^k\}$ and (3.54), and

$$\gamma_{i_0}^k \left(G_{i_0}(z^k)(d^k)^T \nabla^2 H_{i_0}(z^k) d^k + H_{i_0}(z^k)(d^k)^T \nabla^2 G_{i_0}(z^k) d^k \right)$$

is bounded by the convergence of the sequences

$$\left\{ \gamma_{i_0}^k G_{i_0}(z^k) \right\}, \quad \left\{ \gamma_{i_0}^k H_{i_0}(z^k) \right\}.$$

In consequence, we have

$$(d^k)^T \nabla_z^2 L_{\varepsilon_k}(z^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \to -\infty$$

as $k \to \infty$. This contradicts (3.55) and hence $\bar{z}$ is B-stationary to problem (1.1).

For the case where $i_0 \in \mathcal{I}_{\Phi_{\varepsilon_k}}(z^k)$ for infinitely many k, we can show that $\bar{z}$ is B-stationary to problem (1.1) in a similar way as in the proof of Theorem 3.4. This completes the proof. □

4 Concluding Remarks

In this paper, we have proposed a modified relaxation scheme for a mathematical program with complementarity constraints. The new relaxed problem involves less constraints than the one considered by Scholtes [19]. All desirable properties established in [19] remain valid for the new relaxed problem. In addition, we obtain some weaker sufficient conditions for B-stationarity described by the eigenvalues of the Hessian matrix of the Lagrangian of the relaxed problem. From the proof, it is easy to see that, even if the matrix mentioned above is replaced by the Hessian matrix of the simpler function

$$\tilde{L}_\varepsilon(z, \gamma, \delta) = \gamma^T \Psi_\varepsilon(z) - \delta^T \Phi_\varepsilon(z),$$

all the results remain ture. Similar extension is possible for the relaxation schemes presented by Scholtes [19] and Lin and Fukushima [11] as well.

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