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Subspace Identification of Closed Loop Systems by Orthogonal Decomposition Method

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Abstract

In this paper, we consider the problem of identifying the closed loop system by applying the stochastic realization technique of (Picci and Katayama, 1996*b*) to the joint input-output process. It is assumed that the exogenous input is feedback-free and persistently exciting (PE). We show that under the preliminary orthogonal decomposition, the identification of a closed loop system is divided into two subproblems: the deterministic identification of plant and controller, and the stochastic identification of noise filter. The subspace method of computing the deterministic and stochastic components of the joint input-output process is presented, and then subspace methods of identifying the deterministic and stochastic parts are derived by adapting the standard subspace methods to the deterministic and stochastic components. In each case, a (balanced) model reduction procedure should be applied for deriving lower order models from obtained higher order models by canceling poles and zeros situated closely. Some numerical results are included to show the applicability of the present technique.

1 Introduction

The identification problem for linear systems operating in closed loop has received much interest in the literature, since closed loop experiments are necessary if the open loop plant is unstable, or the feedback is an inherent mechanism of the system (Söderström and Stoica, 1989; Van den Hof, 1997; Forssell and Ljung, 1999). For example, in chemical process control, the identification of closed loop system has played an essential role in a reconfiguration of control loop with an unstable plant (Lakshminarayanan *et al.*, 2001). Also, safety and maintaining high-quality production may prohibit experiments in open loop. Moreover, it is recognized that closed loop experiments are preferable to open loop ones for the iterative identification and model-based robust control design (Gevers, 1993; Van den Hof, 1997).

Classical closed loop identification methods are based on either the prediction error methods or instrumental variable methods, and are divided into three groups: the direct, indirect and joint input-output approaches as summarized in (Ljung, 1999; Söderström and Stoica, 1989; Van den Hof, 1997). Basic issues associated with the identifiability of stochastic feedback system without exogenous inputs has been studied in (Caines and Chan, 1976; Anderson and Gevers, 1982). In order to circumvent the difficulty of closed loop identification due to the correlation between the plant input and unmeasurable disturbances, two stage least-squares methods have been developed in (Van den Hof and Schrama, 1993; Forssell and Ljung, 2000), for which the first step is the estimation of the sensitivity function of the closed loop system by using ARMA model or (non-causal) FIR model to generate a noise-free input signal to the plant to be identified. Also, an indirect scheme that transforms the closed loop identification problem into an open loop problem has been derived by using the dual Youla parametrization (Hansen *et al.*, 1989; Schrama, 1991). Furthermore, the indirect approach is employed in (Van den Hof and de Callafon, 1996) to analyze some multivariable closed loop identification techniques; it has been shown that the classical indirect approach is a special case of the dual-Youla parametrization approach.

The subspace methods have also been employed in the identification of closed loop systems. Actually, the identification of multivariable stochastic feedback systems without exogenous inputs has been considered based on stochastic realization theory (Van der Klauw *et al.*, 1991), in which the joint input-output approach is used for deriving the state space models of subsystems, followed by a balanced model reduction. Also, based on a subspace method, a related technique of identifying the state-space model of a plant operating in closed loop has been developed by reformulating it as an equivalent open loop problem (Verhaegen, 1993). Moreover, modifying the N4SID method (Van Overschee and Moor, 1994; Van Overschee and Moor, 1996), a closed loop subspace identification method has been derived under the assumption that a finite number of Markov parameters of the controller are known (Van Overschee and Moor, 1997). Recently, we have presented a subspace-based method of identifying the deterministic part of the plant operating in closed loop based on the orthogonal decomposition of the joint input-output process (Katayama *et al.*, 2001).

In this paper, we develop a subspace method of identifying the plant, the controller and the noise filter operating in closed loop by extending the orthogonal decomposition based technique developed in (Katayama *et al.*, 2001). First we decompose the joint input-output process into the stochastic and deterministic components by means of the orthogonal decomposition (Picci and Katayama, 1996*b*; Picci and Katayama, 1996*a*), where the basic assumption is that the exogenous inputs are feedback-free and persistently exciting (PE). It can be shown that by the preliminary orthogonal decomposition, the identification of closed loop system is divided into two subproblems: the identification of the deterministic part, or the plant and controller, and the stochastic part, or the noise filter. Thus the plant and the controller can be obtained from the realization of deterministic component of the joint process, and the noise filter can be retrieved from the realization of stochastic component. Subspace methods of identifying the deterministic and stochastic parts are then derived by adapting some standard subspace methods to the deterministic and stochastic components separately. In each case, a model reduction procedure (Zhou *et al.*, 1999; Varga, 2001) should be employed for deriving lower order models from obtained higher order models by canceling poles and zeros situated very closely.

It should be noted that accuracy of the estimation result hinges on that of the open-loop subspace identification method based on the orthogonal decomposition (Picci and Katayama, 1996*a*; Chiuso and Picci, 1999), for which some numerical issues are explored, showing that it is more robust than the 4SID like methods when the exogenous inputs are colored. Also, the present orthogonal decomposition based approach may be helpful for understanding the structure of the closed loop identification problem as shown in Section 3.

The organization of the paper is as follows. In Section 2, the problem is stated along with the underlying assumptions. In Section 3, we formulate a joint input-output approach to the closed loop identification and derive the deterministic and stochastic components of the joint input-output process by the preliminary orthogonal decomposition. Section 4 derives the state space realizations of deterministic and stochastic components by means of the realization technique. Section 5 gives a subspace identification method for computing the state space models of plant, controller based on deterministic realization. Also, a stochastic subspace method is employed for identifying the noise filter by employing the realization of stochastic component. In Section 6, a model reduction procedure that can be applied to unstable plants is briefly summarized. Section 7 gives some numerical results and Section 8 concludes the paper.

2 Problem Statement

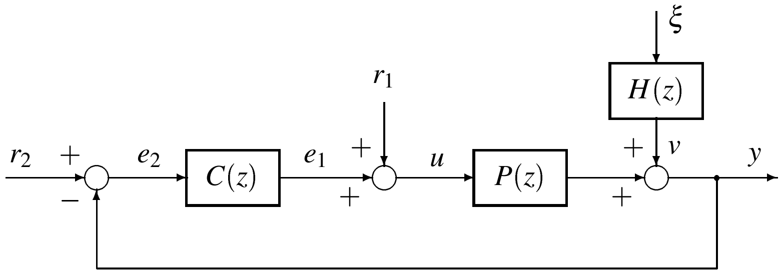

Fig. 1. Closed-loop system.

We consider the problem of identifying a closed loop system given input-output measurements. The configuration of the system is shown in Fig. 1, where $y \in \mathbb{R}^p$ is the output vector of the plant, and $u \in \mathbb{R}^m$ is the input vector. An output disturbance $v \in \mathbb{R}^p$ is described by $v(t) = H(z)\xi(t)$, where $\xi \in \mathbb{R}^p$ is a white noise with mean zero and a covariance matrix $\Lambda > 0$, and $H(z)$ is a minimum phase, square transfer matrix. It is also assumed that $H(\infty) = I_p$, where I_p is the $p \times p$ unit matrix. The signals $r_1 \in \mathbb{R}^m$ and $r_2 \in \mathbb{R}^p$ are exogenous reference vectors, or some probing inputs. We assume that $\{r_1(t),\, r_2(t),\, u(t),\, y(t),\, t = 0, \pm 1, \cdots\}$ are jointly stationary second-order processes with mean zero.

Let the plant be a finite dimensional, linear, time-invariant (FDLTI) system described by

$$y(t) = P(z)u(t) + H(z)\xi(t) \tag{1}$$

where $P(z)$ and $H(z)$ are the transfer matrices of the plant and the noise filter, respectively. The control input is generated by

$$u(t) = r_1(t) + C(z)[r_2(t) - y(t)] \tag{2}$$

where $C(z)$ denotes the transfer matrix of the FDLTI controller.

We introduce the following assumptions on the closed loop system and exogenous inputs and noise.

A1: The closed loop system is well-posed in the sense that (u, y) are uniquely determined by the states of the plant and controller and by the exogenous inputs and noise. This generic condition is satisfied if $I_p + P(\infty)C(\infty)$ and $I_m + C(\infty)P(\infty)$ are non-singular. For the sake of simplicity, it is assumed throughout the paper that the plant is strictly proper, namely, $P(\infty) = 0$.

A2: The controller internally stabilizes the closed loop system.

A3: The exogenous input vector $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in \mathbb{R}^l \;\; (l = m + p)$ is PE, and r_1 and r_2 are mutually uncorrelated. Moreover, the exogenous input r is uncorrelated with the white noise ξ.

The objective of this paper is to obtain realizations of the plant $P(z)$, the controller $C(z)$ and the noise filter $H(z)$ based on finite measurement data $\{r_1(t),\, r_2(t),\, u(t),\, y(t)\}$ by using a subspace identification method.

It may be observed that the identification of the controller is not needed, since often we know the dynamics of the controller. However, if the estimate of the controller agrees reasonably well with the actual controller implemented, this will be a confirmation of the validity of the identification procedure employed. Moreover, the process systems with material and energy recycles in chemical industry are inherently of closed loop (Lakshminarayanan *et al.*, 2001). Therefore, to identify all the components in the closed loop system is quite interesting from both practical and theoretical points of view.

If the controller is known, we can define $r = r_1 + C(z)r_2$ combining two exogenous signals, thereby (2) takes the form $u = r - C(z)y$ as in (Van den Hof and de Callafon, 1996; Van den Hof, 1997). But since the controller is not known in the present situation, we could not follow this approach.

The present problem is virtually the same as the one treated in (Verhaegen, 1993), but the approach is quite different. For, as shown below, we employ the approach based on orthogonal decomposition (Picci and Katayama, 1996*b*; Picci and Katayama, 1996*a*) in order to decompose the joint input-output process into the deterministic and stochastic components; subspace methods are then applied to each component to obtain the desired state space models. Further results in (Chiuso and Picci, 1999; Chiuso and Picci, 2000; Kawauchi *et al.*, 2000) show that subspace identification algorithms based on the orthogonal decomposition are more robust than the N4SID like algorithms in the case where the exogenous input is colored.

Before proceeding to the main problem, we define some geometric operations. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be subspaces of a real Hilbert space $\mathcal{H}$. It is assumed that these spaces are closed with respect to the mean-square-norm induced by the scalar product $\langle \xi, \eta \rangle = E\{\xi\eta\}$, where $E\{\cdot\}$ denotes the mathematical expectation. We write $\mathcal{A} \vee \mathcal{B}$ to denote the vector sum, i.e. the closure of $\{\alpha + \beta \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$ and $\mathcal{A} + \mathcal{B}$ to denote the direct sum; $\mathcal{A}^\perp$ is the orthogonal complement of $\mathcal{A}$ in $\mathcal{H}$. The transpose is denoted by $(\cdot)^T$, and the pseudo-inverse by $(\cdot)^\dagger$ (Golub and Loan, 1989).

The orthogonal projection and oblique projection are defined as follows.

Definition 1 *(Orthogonal Projection) Let subspaces $\mathcal{A}$ and $\mathcal{B}$ be spanned by random vectors α and β, respectively. Then the orthogonal projection of $\alpha \in \mathcal{A}$ onto $\mathcal{B}$ is defined as*

$$\hat{E}\{\alpha \mid \mathcal{B}\} = E\left\{\alpha\beta^T\right\} E\left\{\beta\beta^T\right\}^\dagger \beta$$

Definition 2 *(Oblique Projection) Let subspace $\mathcal{C}$ be spanned by a random vector γ. Then the oblique projection of $\alpha \in \mathcal{A}$ onto $\mathcal{B}$ along $\mathcal{C}$ is defined as*

$$\hat{E}_{||\mathcal{C}}\{\alpha|\mathcal{B}\} = \left[E\left\{\alpha\beta^T\right\}\ E\left\{\alpha\gamma^T\right\}\right] \begin{bmatrix} E\left\{\beta\beta^T\right\} & E\left\{\beta\gamma^T\right\} \\ E\left\{\gamma\beta^T\right\} & E\left\{\gamma\gamma^T\right\} \end{bmatrix}^\dagger \begin{bmatrix} \beta \\ 0 \end{bmatrix}$$

The notation $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$ means that two subspaces $\mathcal{A}$ and $\mathcal{B}$ are conditionally orthogonal given a third subspace $\mathcal{C}$, namely, for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$

$$\langle \alpha - \hat{E}\{\alpha \mid \mathcal{C}\}, \beta - \hat{E}\{\beta \mid \mathcal{C}\} \rangle = 0$$

If $\mathcal{C} = 0$, the conditional orthogonality reduces to the usual orthogonality $\mathcal{A} \perp \mathcal{B}$.

3 The Joint Input-Output Approach

In order to obtain the state space models of the plant and controller in closed loop, we use the joint input-output approach. In this and next sections, we shall be concerned with the stationary case where infinite data are available.

3.1 Joint Input-Output Process

Define the joint input-output vector process as $w := \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbb{R}^l$. From Fig. 1, the closed loop system is described by

$$w(t) = \begin{bmatrix} T_{ur_1}(z) & T_{ur_2}(z) \\ T_{yr_1}(z) & T_{yr_2}(z) \end{bmatrix} r(t) + \begin{bmatrix} T_{u\xi}(z) \\ T_{y\xi}(z) \end{bmatrix} \xi(t) \tag{3}$$

where $T_{ab}(z)$ is the transfer matrix from b to a. It is easy to see that

$$\begin{bmatrix} T_{ur_1}(z) & T_{ur_2}(z) \\ T_{yr_1}(z) & T_{yr_2}(z) \end{bmatrix} = \begin{bmatrix} S_i(z) & C(z)S_o(z) \\ P(z)S_i(z) & P(z)C(z)S_o(z) \end{bmatrix} \tag{4a}$$

$$= \begin{bmatrix} S_i(z) & S_i(z)C(z) \\ P(z)S_i(z) & P(z)S_i(z)C(z) \end{bmatrix} \tag{4b}$$

and

$$\begin{bmatrix} T_{u\xi}(z) \\ T_{y\xi}(z) \end{bmatrix} = \begin{bmatrix} -C(z)S_o(z) \\ S_o(z) \end{bmatrix} H(z) \tag{5}$$

where $S_i(z) = (I_m + C(z)P(z))^{-1}$, $S_o(z) = (I_p + P(z)C(z))^{-1}$ are the input and output sensitivity matrices, respectively. It should be noted that the feedback system is internally stable if and only if four transfer matrices in (4) are stable. Moreover, the transfer matrices from r to $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ are extressed as

$$\begin{bmatrix} T_{e_1r_1}(z) & T_{e_1r_2}(z) \\ T_{e_2r_1}(z) & T_{e_2r_2}(z) \end{bmatrix} = \begin{bmatrix} -C(z)S_o(z)P(z) & C(z)S_o(z) \\ -S_o(z)P(z) & S_o(z) \end{bmatrix}$$

Since there is no feedback from w to r, we can employ the stochastic realization method of (Picci and Katayama, 1996*b*) to get a state space model of the transfer matrix $T_{wr}(z)$ with input r and output w. Once the auxiliary transfer matrices in $T_{wr}(z)$ of (4) are obtained, we can get the transfer matrices of plant and controller, say, from $P(z) = T_{yr_1}(z)T_{ur_1}^{-1}(z)$ and $C(z) = T_{ur_1}^{-1}(z)T_{ur_2}(z)$. This can be viewed as a version of classical instrumental variable method (Ljung, 1999; Söderström and Stoica, 1989) that identifies the plant transfer function by cross-correlating the exogenous signal with the input and the output.

There are different combinations of transfer matrices in (4a), (4b), which can be used to recover the plant and the controller; see an SISO case treated in Section 4.1. Thus, unlike the indirect methods (Van den Hof and de Callafon, 1996; Van Overschee and Moor, 1997), no knowledge of the controller is employed and no auxiliary signals are generated.

3.2 Decomposition of Joint Input-Output Process

According to (Picci and Katayama, 1996*b*), the Hilbert spaces generated by second-order random variables of exgonenous inputs and the joint input-output are respectively expressed as $\mathcal{R} = \overline{\text{span}}\{r(t) \mid t \in \mathbb{Z}\}$ and $\mathcal{W} = \overline{\text{span}}\{w(t) \mid t \in \mathbb{Z}\}$. We also define the Hibert spaces of second-order random variables spanned by the infinite past and future of the input and output vectors at the present time t as

$$\mathcal{R}_t^- := \overline{\text{span}}\{r(\tau) \mid \tau < t\}, \qquad \mathcal{W}_t^- := \overline{\text{span}}\{w(\tau) \mid \tau < t\}$$

and

$$\mathcal{R}_t^+ := \overline{\text{span}}\{r(\tau) \mid \tau \geq t\}, \qquad \mathcal{W}_t^+ := \overline{\text{span}}\{w(\tau) \mid \tau \geq t\}$$

Thus they are regarded as subspaces of an ambient Hilbert space $\mathcal{H} := \mathcal{R} \vee \mathcal{W}$ containing all the linear functionals of the process (r, w).

Suppose that there is no feedback from w to r in the sense that the future of r is conditionally uncorrelated with the past of w given the past of r itself. Thus we can write the feedback-free condition as $\mathcal{W}_t^- \perp \mathcal{R}_t^+ \mid \mathcal{R}_t^-$. It follows from the definition of conditional orthogonality that this condition can be rewritten as $\mathcal{W}_t^- \perp \mathcal{R} \mid \mathcal{R}_t^-$. Therefore we have the following lemma.

Lemma 1 *(Picci and Katayama, 1996*b*) The absence of feedback is equivalent to the causality of the estimator, i.e.*

$$\hat{E}\{w(t) \mid \mathcal{R}\} = \hat{E}\{w(t) \mid \mathcal{R}_{t+1}^-\}, \quad t \in \mathbb{Z} \tag{6}$$

We see from (6) that

$$\begin{aligned} w_s(t) &:= w(t) - \hat{E}\{w(t) \mid \mathcal{R}_{t+1}^-\} \\ &= w(t) - \hat{E}\{w(t) \mid \mathcal{R}\} = \hat{E}\{w(t) \mid \mathcal{R}^\perp\} \end{aligned} \tag{7}$$

where $\mathcal{R}^\perp$ denotes the orthogonal complement of $\mathcal{R}$ with respect to $\mathcal{H}$. This implies that $w_s(t) \perp \mathcal{R}$ for all $t \in \mathbb{Z}$, namely, the causal estimation error is uncorrelated with the whole history of the exogenous reference input r. The process w_s defined by (7) is called the stochastic component of w. In the same way, the process defined by the orthogonal projection

$$w_d(t) := \hat{E}\{w(t) \mid \mathcal{R}\} \tag{8}$$

is called the deterministic component, the part of w linearly related to the exogenous input.

From above, we have the orthogonal decomposition of the joint process: $w(t) = w_d(t) + w_s(t)$, or

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u_d(t) \\ y_d(t) \end{bmatrix} + \begin{bmatrix} u_s(t) \\ y_s(t) \end{bmatrix} \tag{9}$$

It should be also noted that the deterministic and stochastic components are uncorrelated (Picci and Katayama, 1996*b*), namely

$$E\{w_s(t)w_d(\tau)^T\} = 0, \quad \forall\, t,\, \tau \in \mathbb{Z}$$

From this orthogonal decomposition, we have basic relations satisfied by deterministic and stochastic components in the closed loop system shown in Fig. 1.

Proposition 1 *The deterministic and stochastic components respectively satisfy the following decoupled equations*

$$y_d(t) = P(z)u_d(t), \quad u_d(t) = r_1(t) + C(z)[r_2(t) - y_d(t)] \tag{10}$$

and

$$y_s(t) = P(z)u_s(t) + H(z)\xi(t), \quad u_s(t) = -C(z)y_s(t) \tag{11}$$

Proof: We see from (1), (2) and (9) that

$$\begin{aligned} y_d(t) + y_s(t) &= P(z)[u_d(t) + u_s(t)] + H(z)\xi(t) \\ u_d(t) + u_s(t) &= r_1(t) + C(z)[r_2(t) - y_d(t) - u_s(t)] \end{aligned}$$

Taking the orthogonal projections of the above equations onto the spaces $\mathcal{R}$ and $\mathcal{R}^{\perp}$ immediately yields (10) and (11), respectively. $\square$

Remark 1 *We could obtain a more general feedback configuration by putting another noise at the output of the controller. Then the noise term like $H(z)\xi(t)$ also appears in the second equation of (11). Since the introduction of another noise rather complicates the subsequent treatment of stochastic component, we shall not do so in this paper. But, some results will be presented elsewhere.*

We see from (10) that

$$\begin{bmatrix} u_d(t) \\ y_d(t) \end{bmatrix} = \begin{bmatrix} S_i(z) & S_i(z)C(z) \\ P(z)S_i(z) & P(z)S_i(z)C(z) \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} \tag{13}$$

Since the transfer matrices in the right hand side of (13) are the same as those in (4b), they are completely determined by the realization of deterministic component. Moreover, the noise filter $H(z)$ is obtained from the first equation of (11).

Several interesting observations for decoupled deterministic and stochastic components may be derived from Proposition 1.

(1) We see that the realization of deterministic and stochastic components can be decoupled, since the two components are uncorrelated each other. It should be, however, noted that though true for ideal case with infinite data, the above observation is not true practically. This is because, in case of finite input-output data, the estimate of the stochastic component w_s is influenced by the unknown initial condition associated with the estimate of the deterministic component w_d as discussed in (Picci and Katayama, 1996*a*). But the effect due to unknown initial conditions surely decreases for sufficiently long data.

(2) Suppose that $P(z)$ and $C(z)$ are stable. Then, we could apply some deterministic subspace identification method to (10) for obtaining their state space realizations. Moreover, a stochastic subspace identification method (Picci and Katayama, 1996*a*; Katayama and Picci, 1999) can be applied to (11) in order to obtain realizations of $P(z)$ and $H(z)$. If both realizations of $P(z)$ obtained from (10) and (11) show good agreement, we may safely say that the present realization approach to the closed loop identification is valid.

(3) If $P(z)$ and/or $C(z)$ is unstable or marginally stable, we cannot follow the above procedure, since the deterministic (or stochastic) subspace method applied to the first equation of (10) or (11) yields erroneous results. For, it is impossible to connect the second-order processes u_d and y_d (or u_s and y_s) by an unstable matrix $P(z)$. Moreover, controllers in practical control systems are often marginally stable due to the existence of integrators implemented. In order to deal with both stable and unstable $P(z)$ and/or $C(z)$, we develop a subspace closed loop identification method based on the realization of the joint input-output process w given the exogenous input r.

(4) From (11), we can identify the noise filter $H(z)$ together with $P(z)$ by some stochastic subspace method like (Van Overschee and Moor, 1993; Katayama and Picci, 1999). Alternatively, solving (11),

$$u_s(t) = -C(z)S_o(z)H(z)\xi(t), \quad y_s(t) = S_o(z)H(z)\xi(t)$$

Given the estimates of $P(z)$ and $C(z)$, we obtain $S_o(z)$, so that the above relations can be employed to identify the noise filter $H(z)$, say, by using the spectral factorization technique. But, we do not pursue the latter approach in this paper.

4 Realization of Closed Loop System

In this section, we derive state space realizations for the deterministic and stochastic components based on the technique of (Picci and Katayama, 1996*b*) and then compute the state space models of the plant, the controller and the noise filter.

Suppose that for each t the input space $\mathcal{R}$ admits the direct sum decomposition $\mathcal{R} = \mathcal{R}_t^+ + \mathcal{R}_t^-$, that is, $\mathcal{R}_t^+ \cap \mathcal{R}_t^- = 0$. An analogous condition is that the spectral density matrix of r is strictly positive definite on the unit circle, i.e. $\Phi_r(e^{j\omega}) > cI_l$, $c > 0$ or all canonical angles between the past and future subspaces of r are strictly positive.

Let $\hat{\mathcal{W}}$ be spanned by deterministic component w_d. Let $\hat{\mathcal{W}}_t^+$ denote the subspace generated by the future $w_d(\tau)$, $\tau \geq t$. According to Section 4 of (Picci and Katayama, 1996*b*), we define the oblique predictor subspace as

$$\mathcal{X}_t^{+/-} := \hat{E}_{\|\mathcal{R}_t^+}\{\hat{\mathcal{W}}_t^+ \mid \mathcal{R}_t^-\} \tag{14}$$

This is the oblique projection of $\hat{\mathcal{W}}_t^+$ on the past $\mathcal{R}_t^-$ along the future $\mathcal{R}_t^+$, and $\mathcal{X}_t^{+/-}$ is the state space for the deterministic component. Clearly, if r is a white noise process, (14) reduces to the ordinary orthogonal projection onto $\mathcal{R}_t^-$.

Let the dimension of the state space $\mathcal{X}_t^{+/-}$ be n. In general, n equals the sum of the orders of the plant and the controller. Any basis vector of $x_d(t) \in \mathcal{X}_t^{+/-}$ yields a state space representation of $w_d(t)$, so that

$$x_d(t+1) = Ax_d(t) + [B_1\ B_2]r(t) \tag{15a}$$

$$\begin{bmatrix} u_d(t) \\ y_d(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x_d(t) + \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} r(t) \tag{15b}$$

where $A \in \mathbb{R}^{n\times n}$, and where $D_{21} = 0$, $D_{22} = 0$ due to the fact that the plant $P(z)$ is strictly proper. Furthermore we have $D_{11} = I_m$ and $D_{12} = D_c$.

We now derive state space models for the plant and controller by using the realization of the joint input-output model (15). It follows from (13) and (15) that the closed loop transfer matries are expressed as

$$\begin{bmatrix} T_{ur_1}(z) & T_{ur_2}(z) \\ T_{yr_1}(z) & T_{yr_2}(z) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right]$$

We see from (4) that $T_{yr_1}(z) = P(z)T_{ur_1}(z)$ and $T_{ur_2}(z) = T_{ur_1}(z)C(z)$, so that the plant and controller are computed by $P(z) = T_{yr_1}(z)T_{ur_1}^{-1}(z)$ and $C(z) = T_{ur_1}^{-1}(z)T_{ur_2}(z)$, respectively. Thus we have the following proposition.

Proposition 2 *Suppose that* $\det D_{11} \neq 0$. *Then (non-minimal) state space representations of* *the plant and the controller are given by*

$$P(z) = \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_1 D_{11}^{-1} \\ \hline C_2 & 0 \end{array}\right] \tag{16a}$$

$$C(z) = \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_2 - B_1 D_{11}^{-1} D_{12} \\ \hline D_{11}^{-1} C_1 & D_{11}^{-1} D_{12} \end{array}\right] \tag{16b}$$

Proof: The state space realization of the plant can be obtained by

$$\begin{aligned} P(z) = T_{yr_1}(z) T_{ur_1}^{-1}(z) &= \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & 0 \end{array}\right] \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & 0 \end{array}\right] \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & -B_1 D_{11}^{-1} \\ \hline D_{11}^{-1} C_1 & D_{11}^{-1} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A - B_1 D_{11}^{-1} C_1 & 0 & -B_1 D_{11}^{-1} \\ B_1 D_{11}^{-1} C_1 & A & B_1 D_{11}^{-1} \\ \hline 0 & C_2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A - B_1 D_{11}^{-1} C_1 & 0 & -B_1 D_{11}^{-1} \\ 0 & A & 0 \\ \hline -C_2 & C_2 & 0 \end{array}\right] \end{aligned}$$

Deleting uncontrollable modes from above equation, we arrive at (16a). The state space representation of the controller (16b) can be obtained similarly. □

The following proposition shows that the models in (16) are not minimal (Verhaegen, 1993).

Proposition 3 *Let* $\{A_p, B_p, C_p, 0\}$, $A_p \in \mathbb{R}^{n_p \times n_p}$ *and* $\{A_c, B_c, C_c, D_c\}$, $A_c \in \mathbb{R}^{n_c \times n_c}$ *be (minimal) realizations of the plant and controller, respectively. Then the realizations*

$$P(z) = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ -B_c C_p & A_c & 0 \\ \hline C_p & 0 & 0 \end{array}\right] \tag{17a}$$

$$C(z) = \left[\begin{array}{cc|c} A_p & 0 & 0 \\ -B_p C_p & A_c & B_c \\ \hline -D_c C_p & C_c & D_c \end{array}\right] \tag{17b}$$

are input-output equivalent to (16a) and (16b), respectively. Hence the observable part of each non-minimal state space model is the original plant and controller state space model, respectively.

Proof: Let $x_p(t) \in \mathbb{R}^{n_p}$ and $x_c(t) \in \mathbb{R}^{n_c}$ be the state vectors of the plant and controller, respectively. Let the state space models of the plant and controller be respectively given by

$$\begin{aligned} x_p(t+1) &= A_p x_p(t) + B_p u(t) \\ y(t) &= C_p x_p(t) \end{aligned}$$

and

$$\begin{aligned} x_c(t+1) &= A_c x_c(t) + B_c \{r_2(t) - y(t)\} \\ u(t) &= r_1(t) + C_c x_c(t) + D_c \{r_2(t) - y(t)\} \end{aligned}$$

Then combining the above two subsystems yields

$$\begin{bmatrix} x_p(t+1) \\ x_c(t+1) \end{bmatrix} = \begin{bmatrix} A_p - B_p D_c C_p & B_p C_c \\ -B_c C_p & A_c \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B_p & B_p D_c \\ 0 & B_c \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -D_c C_p & C_c \\ C_p & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} I_m & D_c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

We can easily see from this realization that $\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} I_m & D_c \\ 0 & 0 \end{bmatrix}$ and the assumption $D_{11} \neq 0$ is satisfied. As in the proof of Proposition 2, we calculate $P(z) = T_{yr_1}(z) T_{ur_1}^{-1}(z)$ to get (17a). Since it has n_c uncontrollable, observable modes, this realization is not minimal. This shows that the controllable, observable part of (17a) is the state space realization of the original plant. Similarly, we can compute the state space model of the controller. □

It may be noted that the transfer matrices $P(z)$, $C(z)$ of (16) is thoroughly determined by the realization of the deterministic component of (8). However, since the obtained transfer matrices of the plant and controller have the higher dimension than the original ones, we need to apply some model reduction technique to obtained transfer matrices in order to recover lower dimensional models.

Example: To better understand the pole-zero cancellation issue discussed in Propositions 2 and 3, we consider a deterministic SISO system, where the plant and controller are given by

$$P(z) = \frac{n_p(z)}{d_p(z)}, \quad C(z) = \frac{n_c(z)}{d_c(z)} \tag{18}$$

Since the closed loop system is stable, the characteristic polynomial of the closed loop system

$$d_{cl}(z) = d_p(z) d_c(z) + n_p(z) n_c(z)$$

is stable. As in (Verhaegen, 1993), it follows from Fig. 1 and (18) that

$$\begin{bmatrix} u(t) \\ y(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} = \frac{1}{d_{cl}(z)} \begin{bmatrix} d_p(z)d_c(z) & d_p(z)n_c(z) \\ n_p(z)d_c(z) & n_p(z)n_c(z) \\ -n_p(z)n_c(z) & d_p(z)n_c(z) \\ -n_p(z)d_c(z) & d_p(z)d_c(z) \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} \tag{19}$$

It should be noted that all the transfer functions in the 2-inputs 4-outputs state space model of (19) have a common denominator $d_{cl}(z)$, which is canceled in each computation of relevant transfer functions.

Case 1: Consider the case where r_1 satisfies the PE condition, but r_2 does not. In this case, we can correctly identify transfer functions $T_{ur_1}(z), T_{yr_1}(z), T_{e_1r_1}(z), T_{e_2r_1}(z)$. We see from (19) that

$$T_{ur_1}(z) = \frac{d_pd_c(z)}{d_{cl}(z)}, \quad T_{yr_1}(z) = \frac{n_pd_c(z)}{d_{cl}(z)},$$

where $d_pd_c(z)$ implies that we can obtain only the estimate of the product of two polynomials $d_p(z)$ and $d_c(z)$ (Verhaegen, 1993). Thus the plant and controller are computed by

$$P(z) = \frac{T_{yr_1}(z)}{T_{ur_1}(z)} = \frac{n_pd_c(z)}{d_pd_c(z)} \tag{20a}$$

$$C(z) = \frac{T_{e_1r_1}(z)}{T_{e_2r_1}(z)} = \frac{n_cn_p(z)}{d_cn_p(z)} \tag{20b}$$

To get $P(z)$ from (20a), we first factorize the estimates of denominator and numerator into the products as $\hat{d}_p(z)\hat{d}_c(z)$ and $\hat{n}_p(z)\hat{d}_c(z)$. Then $\hat{d}_c(z)$ is canceled out by the model reduction technique to get a reduced order estimate of the plant transfer function. Since in this paper a model reduction procedure that can be applied to both stable and unstable systems, $\hat{d}_c(z)$ need not be be stable. Similarly, to get the controller $C(z)$ from (20b), $\hat{n}_p(z)$ should be cancelled.

Case 2: Consider the case where the exogenous input r_2 is a PE signal, but r_1 is not. In this case, we identify transfer functions $T_{ur_2}(z)$, $T_{yr_2}(z)$, $T_{e_1r_2}(z)$, $T_{e_2r_2}(z)$. It also follows from (19) that the plant and controller are obtained as

$$P(z) = \frac{T_{yr_2}(z)}{T_{ur_2}(z)} = \frac{n_pn_c(z)}{d_pn_c(z)} \tag{21a}$$

$$C(z) = \frac{T_{ur_2}(z)}{T_{e_2r_2}(z)} = \frac{n_cd_p(z)}{d_cd_p(z)} \tag{21b}$$

Case 3: Suppose that both the exogenous inputs r_1, r_2 satisfy PE condition as in the present setting (see A3). In this case, we can use either (20a) or (21a) for computing $P(z)$, and either (20b) or (21b) for computing $C(z)$.

4.2 Realization of Stochastic Component

Now we turn to the realization of stochastic component. Let the subspace $\tilde{\mathcal{W}}$ be spanned by stochastic component w_s. Let $\tilde{\mathcal{W}}_t^+$ and $\tilde{\mathcal{W}}_t^-$ denote the linear spaces generated by the future and past of stochastic component $w_s(t)$, respectively. According to Section 3 of (Picci and Katayama, 1996*b*), the predictor space $\tilde{\mathcal{X}}_t^{+/-}$ for the stochastic component is defined by

$$\tilde{\mathcal{X}}_t^{+/-} := \hat{E}\{\tilde{\mathcal{W}}_t^+ \mid \tilde{\mathcal{W}}_t^-\}$$

Suppose that $\dim \tilde{\mathcal{X}}_t^{+/-} = \tilde{n}$, which is the sum of orders of the plant, controller and two noise filters. Let $x_s(t)$ be a basis in $\tilde{\mathcal{X}}_t^{+/-}$. Then, a particular realization of the stochastic system having the predictor space $\tilde{\mathcal{X}}_t^{+/-}$ as the state space is given by

$$x_s(t+1) = \bar{A}x_s(t) + [K_1 \;\; K_2]\begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix} \tag{22a}$$

$$\begin{bmatrix} u_s(t) \\ y_s(t) \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} x_s(t) + \begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix} \tag{22b}$$

where $[K_1 \; K_2]$ is the steady-state Kalman gain, and $\nu(t) := \begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix}$ is the innovation process of $w_s(t) = \begin{bmatrix} u_s(t) \\ y_s(t) \end{bmatrix}$ based on the past $\tilde{\mathcal{W}}_t^-$ (Picci and Katayama, 1996*b*). It can be also shown that $\nu(t)$ is the conditional innovation of $w(t)$ given observations of $r(\tau)$, $\tau \le t$, namely $\nu(t) = w(t) - \hat{E}\{w(t) \mid \mathcal{R}_{t+1}^- \vee \mathcal{W}_t^-\}$.

Substituting $\nu_1(t)$ of (22b) into (22a) yields

$$x_s(t+1) = (\bar{A} - K_1\bar{C}_1)x_s(t) + K_1 u_s(t) + K_2 \nu_2(t) \tag{23a}$$

$$y_s(t) = \bar{C}_2 x_s(t) + \nu_2(t) \tag{23b}$$

Proposition 4 *The realizations of $P(z)$ and $H_2(z)$ in (11) are given by*

$$P(z) = \left[\begin{array}{c|c} \bar{A} - K_1\bar{C}_1 & K_1 \\ \hline \bar{C}_2 & 0 \end{array}\right] \tag{24a}$$

$$H(z) = \left[\begin{array}{c|c} \bar{A} - K_1\bar{C}_1 & K_2 \\ \hline \bar{C}_2 & I_p \end{array}\right] \tag{24b}$$

Proof: It follows from (23) that

$$\begin{aligned} y_s(t) &= \bar{C}_2(zI - \bar{A} + K_1\bar{C}_1)^{-1}K_1 u_s(t) \\ &\quad + [I_p + \bar{C}_2(zI - \bar{A} + K_1\bar{C}_1)^{-1}K_2]\nu_2(t) \end{aligned}$$

Since $\nu_2(t)$ is a white noise, we get (24a) and (24a) by comparing with the above equation with the first equation of (11). □

Remark 2 *Though we get a similar expression for u_s by deleting ν_2 from (22), the resulting equation cannot be used for estimating $C(z)$ in (11). For, in this case, $C(z)$ so estimated is necessarily restricted to strictly proper matrix due to the fact that no noises are assumed at the output of the controller. In order to consider the controller that may not be strictly proper, the noise model should be treated more carefully allowing the possible correlations between ν_1 and ν_2 along the line of (Van der Klauw* et al.*, 1991).*

It should be noted that the realization of (24a) theoretically equals to that of (16a). As in the deterministic case, the identified models have higher dimensions than the original ones; hence we also need model reduction to get the reduced order models.

5 Closed Loop Subspace Identification Algorithm

In this section, we present a subspace identification method based on finite data. Suppose that finite input-output data $r_1(t)$, $r_2(t)$, $u(t)$, $y(t)$ for $t = 0, 1, \cdots, N + 2k - 2$ are given with $k > 0$ and N sufficiently large. We assume that they are samples from jointly stationary processes with means zero and finite covariance matrices. Recall that the exogenous input r and joint input-output w are defined as

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} \in \mathbb{R}^l, \quad w(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^l$$

As usual, we fix a present time $t = k$ and define the block Hankel matrices $R_{0|k-1}$, $R_{k|2k-1}$ of dimension $lk \times N$ formed by past and future reference inputs as

$$R_{0|k-1} = \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r(1) & r(2) & \cdots & r(N) \\ \vdots & \vdots & \ddots & \vdots \\ r(k-1) & r(k) & \cdots & r(N+k-2) \end{bmatrix} \in \mathbb{R}^{kl \times N}$$

and

$$R_{k|2k-1} = \begin{bmatrix} r(k) & r(k+1) & \cdots & r(k+N-1) \\ r(k+1) & r(k+2) & \cdots & r(k+N) \\ \vdots & \vdots & \ddots & \vdots \\ r(2k-1) & r(2k) & \cdots & r(N+2k-2) \end{bmatrix} \in \mathbb{R}^{kl \times N}$$

where the first and the second subscripts denote the indices of the upper-left and the lower-left element of the block Hankel matrix, respectively. In terms of $\{w(0), w(1), \cdots, w(N+2k-2)\}$, the block Hankel matrices $W_{0|k-1}$, $W_{k|2k-1} \in \mathbb{R}^{kl \times N}$ are defined similarly. The row subspaces $\mathcal{R}_{0|k-1}$, $\mathcal{W}_{0|k-1}$ generated by rows of the Hankel matrices $R_{0|k-1}$, $W_{0|k-1}$ are the past reference input and joint input-output subspaces, respectively. Future subspaces $\mathcal{R}_{k|2k-1}$, $\mathcal{W}_{k|2k-1}$ are defined similarly. Let us also define the stacked $2kl \times N$ block Hankel matrices

$$R_{0|2k-1} := \begin{bmatrix} R_{0|k-1} \\ R_{k|2k-1} \end{bmatrix}, \quad W_{0|2k-1} := \begin{bmatrix} W_{0|k-1} \\ W_{k|2k-1} \end{bmatrix}$$

and row spaces $\mathcal{R}_{0|2k-1}$, $\mathcal{W}_{0|2k-1}$.

5.1 Orthogonal Decomposition based on Finite Data

The first step of the subspace identification is the preliminary decomposition of w into the deterministic and stochastic components based on finite input-output data. This is performed by computing

$$\hat{W}^d_{t|t} = \hat{E}\left\{W_{t|t} \mid \mathcal{R}_{0|2k-1}\right\}, \quad \hat{W}^s_{t|t} = \hat{E}\left\{W_{t|t} \mid \mathcal{R}^{\perp}_{0|2k-1}\right\}$$

where we can use the LQ factorization for the orthogonal projection.

Let $W^d_{t|t}$ be defined as

$$W^d_{t|t} = [w_d(t) \; w_d(t+1) \; \cdots \; w_d(t+N-1)] \in \mathbb{R}^{l \times N}$$

It then follows from (15) that $W^d_{t|t}$ satisfies the matrix state space model

$$X^d_{t+1} = AX^d_t + [B_1 \;\; B_2]R_{t|t} \tag{25a}$$

$$W^d_{t|t} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} X^d_t + \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} R_{t|t} \tag{25b}$$

where $X^d_t \in \mathbb{R}^{n \times N}$ is constructed by the state vectors as

$$X^d_t = [x_d(t) \; x_d(t+1) \; \cdots \; x_d(t+N-1)] \in \mathbb{R}^{n \times N}$$

Lemma 2 *(Picci and Katayama, 1996*b*) Let the deterministic component $W^d_{t|t}$ be described by (25). Also, define*

$$\hat{E}\{W_{t|t} \mid \mathcal{R}_{0|2k-1}\} = \hat{E}\{W^d_{t|t} \mid \mathcal{R}_{0|2k-1}\} =: \hat{W}^d_{t|t}$$

Then $\hat{W}^d_{t|t}$ is described by the same state space model as (25), namely,

$$\hat{X}^d_{t+1} = A\hat{X}^d_t + [B_1 \;\; B_2]R_{t|t} \tag{26a}$$

$$\hat{W}^d_{t|t} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \hat{X}^d_t + \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} R_{t|t} \tag{26b}$$

with the initial state $\hat{X}^d_0 := \hat{E}\{X^d_0 \mid \mathcal{R}_{0|2k-1}\}$, which is different from X^d_0.

Proof: Follows from (9) by noting that $w_s(t) \perp \mathcal{R} \supset \mathcal{R}_{0|2k-1}$. □

Define $B := [B_1 \;\; B_2] \in \mathbb{R}^{n\times l}$, $C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \in \mathbb{R}^{l\times n}$ and $D := \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{l\times l}$. Let the extended observability matrix $\mathcal{O}_k$ be given by

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \in \mathbb{R}^{kl\times n}, \quad k > n$$

and the lower triangular Toeplitz matrix Ψ_k formed by the Markov parameters of the realization of (26) be defined as

$$\Psi_k = \begin{bmatrix} D & & & & \\ CB & D & & \mathbf{0} & \\ CAB & CB & D & & \\ \vdots & \vdots & \ddots & \ddots & \\ CA^{k-2}B & CA^{k-3}B & \cdots & CB & D \end{bmatrix} \in \mathbb{R}^{kl\times kl} \tag{27}$$

Thus from (26), we have the matrix input-output equation

$$\hat{W}^d_{k|2k-1} = \mathcal{O}_k\hat{X}^d_k + \Psi_k R_{k|2k-1} \tag{28}$$

where $\hat{W}^d_{k|2k-1}$ is the stacked matrix defined as

$$\hat{W}^d_{k|2k-1} = \begin{bmatrix} \hat{W}^d_{k|k} \\ \hat{W}^d_{k+1|k+1} \\ \vdots \\ \hat{W}^d_{2k-1|2k-1} \end{bmatrix} \in \mathbb{R}^{kl \times N}$$

5.2 Subspace identification of deterministic part

Now we consider the procedure of subspace identification of the plant and controller based on finite data. We first show that the finite-history of deterministic component is obtained by using the standard LQ factorization (Verhaegen, 1994)

$$\left[\begin{array}{c} R_{k|2k-1} \\ R_{0|k-1} \\ \hline W_{0|k-1} \\ W_{k|2k-1} \end{array}\right] = \left[\begin{array}{cc|cc} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ \hline L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{array}\right] \left[\begin{array}{c} Q_1^T \\ Q_2^T \\ \hline Q_3^T \\ Q_4^T \end{array}\right] \tag{29}$$

where $L_{11}, L_{22}, L_{33}, L_{44} \in \mathbb{R}^{lk \times lk}$ are the lower triangular matrices with $Q_i^T Q_j = I_{lk}\delta_{ij}$.

Lemma 3 *(Picci and Katayama, 1996*a*; Chiuso and Picci, 2000) The deterministic component is obtained by*

$$\hat{W}^d_{0|2k-1} := \hat{E}\{W_{0|2k-1} \mid \mathcal{R}_{0|2k-1}\} = \begin{bmatrix} L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

Proof: This follows immediately by noting that the rows of $\begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$ in the LQ factorization of (29) form an orthonormal basis for the rowspace $\mathcal{R}_{0|2k-1}$. □

From this lemma and (28), we have

$$\hat{W}^d_{k|2k-1} = L_{41}Q_1^T + L_{42}Q_2^T = \Psi_k L_{11} Q_1^T + \mathcal{O}_k \hat{X}^d_k \tag{30}$$

In the algorithm below, we identify the state space model of the transfer matrix from r to w_d by MOESP like algorithm (Verhaegen, 1994).

Deterministic Subspace Identification Algorithm

Step 1: Since $Q_1^T Q_2 = 0$, it follows from (30) that

$$\mathcal{O}_k \hat{X}_k^d Q_2 = L_{42}$$

Compute the SVD of L_{42}, namely,

$$L_{42} = [\hat{U} \;\; \tilde{U}] \begin{bmatrix} \hat{S} & 0 \\ 0 & \tilde{S} \end{bmatrix} \begin{bmatrix} \hat{V}^T \\ \tilde{V}^T \end{bmatrix} \simeq \hat{U}\hat{S}\hat{V}^T$$

where $\hat{S}$ is obtained by neglecting singular values "nearly zero," so that the dimension of the state vector equals dim $\hat{S}$. It therefore follows that

$$\mathcal{O}_k \hat{X}_k^d Q_2 = L_{42} \simeq \hat{U}\hat{S}\hat{V}^T = \left(\hat{U}\hat{S}^{1/2}\right)\left(\hat{S}^{1/2}\hat{V}^T\right)$$

Thus the extended observability matrix is given by

$$\mathcal{O}_k = \hat{U}\hat{S}^{1/2}$$

Step 2: Compute the estimate of C and A as

$$C = \text{the first } l \text{ rows of } \mathcal{O}_k$$
$$A = \mathcal{O}_{k-1}^{\dagger}\overline{\mathcal{O}}_k$$

where $\overline{\mathcal{O}}_k$ the extended observability matrix with the first l rows deleted.

Step 3: Given the estimates of A and C, the estimates of B and D are obtained by solving the weighted optimization problem

$$\min_{B,D} \left\| \left(\tilde{U}\right)^T L_{41} - \left(\tilde{U}\right)^T \Psi_k(B,D) L_{11} \right\|^2$$

where L_{11} is given by (29) and $\tilde{U}$ satisfies $\tilde{U}^T \mathcal{O}_k = 0$, and where $D_{21} = 0$, $D_{22} = 0$ from the assumption that the transfer function of the plant is strictly proper. The notation $\Psi_k(B,D)$ emphasizes that the block Toeplitz matrix of (27) is linear with respect to B and D.

Thus, from Proposition 2, the estimates of the state space realizations of $P(z)$ and $C(z)$ are given by

$$P(z) = \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_1 D_{11}^{-1} \\ \hline C_2 & 0 \end{array}\right] \tag{31}$$

$$C(z) = \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_2 - B_1 D_{11}^{-1} D_{12} \\ \hline D_{11}^{-1} C_1 & D_{11}^{-1} D_{12} \end{array}\right] \tag{32}$$

Suppose that the estimate of the deterministic component $\hat{W}^d_{0|2k-1}$ is given by Lemma 3 in the last subsection. Then the estimates of stochastic component is computed by

$$\hat{W}^s_{0|2k-1} := W_{0|2k-1} - \hat{W}^d_{0|2k-1}$$

Define the future and past of stochastic components as $\hat{W}_p = \hat{W}^s_{0|k-1}$ and $\hat{W}_f = \hat{W}^s_{k|2k-1}$, respectively. We consider the LQ factorization

$$\frac{1}{\sqrt{N}}\begin{bmatrix}\hat{W}_p \\ \hat{W}_f\end{bmatrix} = \begin{bmatrix}R_{11} & 0 \\ R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}\bar{Q}_1^T \\ \bar{Q}_2^T\end{bmatrix} \tag{33}$$

where $\bar{Q}_1, \bar{Q}_2$ are orthogonal matrices. Then we define the covariance matrix of stochastic component by

$$\Sigma^s := \frac{1}{N}\begin{bmatrix}\hat{W}_p \\ \hat{W}_f\end{bmatrix}\begin{bmatrix}\hat{W}_p \\ \hat{W}_f\end{bmatrix}^T = \begin{bmatrix}\Sigma_{pp} & \Sigma_{pf} \\ \Sigma_{fp} & \Sigma_{ff}\end{bmatrix}$$

Thus from (33), we have

$$\Sigma_{pp} = R_{11}R_{11}^T, \quad \Sigma_{fp} = R_{21}R_{11}^T, \quad \Sigma_{ff} = R_{21}R_{21}^T + R_{22}R_{22}^T$$

We shall develop a stochastic subspace identification method based on (22), in which we define $\bar{C} := \begin{bmatrix}\bar{C}_1 \\ \bar{C}_2\end{bmatrix}$, and $K := [K_1 \ KB_2]$. The stochastic subspace identification method is summarized as follows (Katayama and Picci, 1999).

Stochastic Subspace Identification Algorithm

Step 1: Compute the SVD of the normalized covariance matrix

$$\Sigma_{ff}^{-1/2}\Sigma_{fp}\Sigma_{pp}^{-T/2} = USV^T \simeq \hat{U}\hat{S}\hat{V}^T$$

where as in the deterministic case, $\hat{S}$ is obtained by neglecting very small singular values of S. Define

$$\Sigma_{fp} = \Sigma_{ff}^{1/2}\hat{U}\hat{S}\hat{V}^T\Sigma_{pp}^{T/2}$$

Step 2: Define the extended observability and controllability matrices as

$$\mathcal{O}^s_k = \Sigma_{ff}^{1/2}\hat{U}\hat{S}^{1/2}, \quad \mathcal{C}^s_k = \hat{S}^{1/2}\hat{V}^T\Sigma_{pp}^{T/2}$$

Step 3: The estimate of the state vector is given by

$$\hat{X}_k^s = \mathcal{C}_k^s \Sigma_{pp}^{-1} \hat{W}_{0|k-1}^s = \hat{S}^{1/2} \hat{V}^T R_{11}^{-1} \hat{W}_{0|k-1}^s$$

where $\hat{X}_k^s = [x_s(k)\ x_s(k+1)\ \cdots\ x_s(N+k-1)]$. Thus the estimates of $(\bar{A}, \bar{C})$ are obtained by solving the overdetermined equation

$$\begin{bmatrix} \hat{X}_{k+1}^s \\ W_{k|k}^s \end{bmatrix} = \begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix} \hat{X}_k^s + \begin{bmatrix} \delta_k \\ \varepsilon_k \end{bmatrix}$$

by the least-squares method, where $\hat{X}_{k+1}^s$ is the shifted version of the state vector, and $\delta_k \in \mathbb{R}^{n\times N}$, $\varepsilon_k \in \mathbb{R}^{p\times N}$ are residuals.

Step 4: Compute the covariance matrices of the residuals as

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \delta_k \\ \varepsilon_k \end{bmatrix} \begin{bmatrix} \delta_k \\ \varepsilon_k \end{bmatrix}^T$$

Step 5: Solve the algebraic Riccati equation

$$Y = \bar{A}Y\bar{A}^T - (\bar{A}Y\bar{C}^T + S)(\bar{C}Y\bar{C}^T + R)^{-1}(\bar{C}Y\bar{A}^T + S^T) + Q$$

to obtain the stabilizing solution and define the Kalman gain as

$$K = (\bar{A}Y\bar{C}^T + S)(\bar{C}Y\bar{C}^T + R)^{-1} := [K_1\ K_2]$$

Then the state space models of plant and noise filter are given by

$$P(z) = \left[\begin{array}{c|c} \bar{A} - K_1\bar{C}_1 & K_1 \\ \hline \bar{C}_2 & 0 \end{array}\right], \quad H(z) = \left[\begin{array}{c|c} \bar{A} - K_1\bar{C}_1 & K_2 \\ \hline \bar{C}_2 & I_p \end{array}\right] \tag{34}$$

6 Model Reduction

It should be noted that the state vector x_d for the deterministic component of joint process contains the state vectors for the plant and the controller, and that the state vector x_s for the stochastic component contains that of the noise filter as well. Thus all the identified transfer matrices have higher dimensions than the true ones. Hence, it is necessary to recover reduced order models from (31), (32) and (34) by deleting uncontrollable and/or unobservable modes. Since the open loop plant is possibly unstable, we need the model reduction technique which can be applied to both stable and unstable transfer matrices (Zhou *et al.*, 1999; Varga, 2001).

In this section, for simplicity, we rewrite a stabilizable, detectable realization as $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. It is also assumed that A has no eigenvalues on the unit circle; this is a main restriction which cannot be avoided as in the spectral factorization problem for discrete-time systems . It may be noted that D matrix is irrelevant for model reduction.

According to (Zhou *et al.*, 1999; Varga, 2001), we define

$$P = \frac{1}{2\pi}\int_0^{2\pi}(e^{j\theta}I - A)^{-1}BB^T(e^{-j\theta} - A^T)^{-1}d\theta \tag{35a}$$

$$Q = \frac{1}{2\pi}\int_0^{2\pi}(e^{-j\theta}I - A^T)^{-1}C^TC(e^{j\theta}I - A)^{-1}d\theta \tag{35b}$$

In terms of P, Q above, the generalized Hankel singular values of $G(z)$ are defined by

$$\sigma_i = \sqrt{\lambda_i(PQ)}, \quad i = 1, \cdots, n$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

We see that if A is stable, P and Q are the controllability and observability Gramians of $G(z)$, satisfying the well-known Lyapunov equations.

Suppose that A is unstable. Let a right-coprime factorization over the ring of stable FDLTI transfer matrices be given by

$$(zI - A)^{-1}B = N(z)M^{-1}(z)$$

where $M(z)$ is an inner matrix with $M^T(e^{-j\theta})M(e^{j\theta}) = I_m$. Let X be a stabilizing nonnegative definite solution of the following Riccati equation

$$X = A^T(X - XB[I + B^TXB]^{-1}B^TX)A \tag{36}$$

and let $F = -(I + B^TXB)^{-1}B^TXA$. Then it can be shown (Zhou *et al.*, 1999) that the factors are given by

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline C & 0 \\ F & I_m \end{array}\right] W$$

where $WW^T = (I + B^TXB)^{-1}$, and where the controllability Gramian satisfies

$$P = (A+BF)^TP(A+BF) + BWW^TB^T \tag{37}$$

Also, consider a left-coprime factorization

$$C(zI - A)^{-1} = \tilde{M}^{-1}(z)\tilde{N}(z)$$

where $\tilde{M}(z)$ is an inner matrix with $\tilde{M}(e^{j\theta})\tilde{M}^T(e^{-j\theta}) = I_p$. Let $\tilde{X}$ be a stablizing solution of the Riccati equation

$$\tilde{X} = A(\tilde{X} - \tilde{X}C^T[I + C\tilde{X}C^T]^{-1}C\tilde{X})A \tag{38}$$

and let $\tilde{F} = -A\tilde{X}C^T(I + C\tilde{X}C^T)^{-1}$. Then it can be shown that the factors are given by

$$[\tilde{N}(z)\ \ \tilde{M}(z)] = V \left[\begin{array}{c|cc} A + \tilde{F}C & I & \tilde{F} \\ \hline C & 0 & I \end{array}\right]$$

where $V^TV = (I + CYC^T)^{-1/2}$, and the observability Gramian satisfies

$$Q = (A + \tilde{F}C)Q(A + \tilde{F}C)^T + C^TV^TVC \tag{39}$$

Model Reduction Algorithm

Suppose that the Gramians P and Q are computed by (37) and (39), respectively.

Step 1: Perform the Cholesky factorizations

$$P = SS^T, \quad Q = R^TR$$

to get S and R.

Step 2: Compute the following SVD of RS, namely,

$$RS = [U_1\ \ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

where $\Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$, and $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \cdots, \sigma_n)$.

Step 3: Let $T_1 \in \mathbb{R}^{r\times n}$ and $T_2 \in \mathbb{R}^{n\times r}$ be given by

$$T_1 = \Sigma_1^{-1/2}U_1^TR, \quad T_2 = SV_1\Sigma_1^{-1/2}$$

Then, the reduced model with $T_1AT_2 \in \mathbb{R}^{r\times r}$ is given by

$$G_r(z) = \left[\begin{array}{c|c} T_1AT_2 & T_1B \\ \hline CT_2 & 0 \end{array}\right]$$

The above algorithm is available from SLICOT website.

7 Simulation Results

Some simulation results are included to show the applicability of the present technique. In the first example, we concentrate on the identification of deterministic parts, namely the plant and controller. The second example is devoted to the identification of the plant and noise filter.

7.1 Example 1

The model in this example is partially borrowed from (Verhaegen, 1993) and is also used in (Van Overschee and Moor, 1997). The plant is a discrete time laboratory model given by

$$P(z) = \frac{10^{-3}\left(0.98z^4 + 12.990z^3 + 18.589z^2 + 3.2987z - 0.02\right)}{z^5 - 4.3986z^4 + 8.0852z^3 - 7.8233z^2 + 3.9954z - 0.8588}$$

Also, a stabilizing controller is chosen as

$$C(z) = \frac{0.63z^4 - 2.083z^3 + 2.8222z^2 - 1.865z + 0.4978}{z^4 - 2.65z^3 + 3.11z^2 - 1.75z + 0.39}$$

It may be noted that $P(z)$ has poles at $z = 1, 0.9674 \pm 0.1493j, 0.7319 \pm 0.6005j$; thus the plant is marginally stable, but the controller $C(z)$ is stable and minimal phase. The configuration of the plant and controller is the same as the one depicted in Fig. 1, where the noise filter is $H(z) = 1$. The r_1 and r_2 are Gaussian white noises with variances $\sigma_{r_1}^2 = 0.5$ and $\sigma_{r_2}^2 = 1$, respectively, and ξ is a Gaussian white noise with variance $\sigma_\xi^2 = 1/9$.

We take the number of data points $N = 5000$ and the block rows $k = 15$, and generated 30 data set, each with the same inputs r_1 and r_2 but with a different noise sequence ξ. Moreover, it is assumed that the orders of the plant and controller are known.

Fig. 2 shows the estimated eigenvalues of the matrix $A_1 - B_{11}D_{11}^{-1}C_{11} \in \mathbb{R}^{9\times 9}$ in Proposition 2, where $*$ denotes the estimated eigenvalues, and $+$ and $\times$ the true poles of the plant and controller, respectively. We see from Fig. 2 that the poles of the 9th-order model are identified very well. FBR procedure is then applied to the 9th-order models for deriving 5th-order models, whose poles are depicted in Fig. 3, where $*$ denotes the estimated poles over 30 experiments, and $+$ the true poles. Bode plot of estimated plant is depicted in Fig. 4, where the dashed line shows the gain of the true plant and the solid line the average over 30 experiments. From these figures, we see that the proposed algorithm performs very well in the identification of the plant. Since, in this case, there are no unstable pole-zero cancellations, the model reduction by FBR is carried out without any problem.

Furthermore, the identification results of the controller are depicted in Figs. 5-8. It should be noted that in order to obtain the controller with the same order as the true one, we perform the unstable pole-zero cancellation at $z = 1$. We computed 4th-order models from the 9th-order models by FBR. Fig. 5 shows the pole estimates of the controller, where $\times$ denotes the true poles and $*$ the estimated ones, and Fig. 6 shows Bode plot of the true gain (dashed line) and the average gain over 30 experiments (solid line). We see from Figs. 5 and 6 that the model reduction technique failed to cancel the marginally stable pole and zero of higher order models of $C(z)$ located at $z = 1$. In order to avoid the unstable pole-zero cancellation, we computed 5th-order models with poles and zeros near $z = 1$ intact. Figs. 7 and 8 show the poles and Bode plot of the 5th-order model so obtained; we see from these figures that the controller is estimated very well via 5th-order model.

7.2 *Example 2*

Consider the following model used in (Codrons *et al.*, 2000):

$$
\begin{aligned}
P(z) &= \frac{0.1208z + 0.1812}{z^4 - 1.992z^3 + 2.203z^2 - 1.841z + 0.8941} \\
C(z) &= \frac{0.5517z^4 - 1.765z^3 + 2.113z^2 - 1.296z + 0.4457}{z^3(z-1)} \\
H(z) &= \frac{z^4}{z^4 - 1.992z^3 + 2.203z^2 - 1.841z + 0.8941}
\end{aligned}
$$

We see that all the poles of the plant are located near the unit circle (see Fig. 9), and a zero is at $z = -1.5$. Thus $P(z)$ is stable, but non-minimal phase. The controller has poles at $z = 1$ and $z = 0$, and has zeros at $z = 1.2629 \pm 0.2j$. Thus $C(z)$ is marginally stable, and non-minimal phase. The denominator of $H(z)$ is the same as that of the plant. We assume that the inputs r_1, r_2 are Gaussian white noises with mean zero and variances $\sigma_{r_1}^2 = 1$, $\sigma_{r_2}^2 = 1$, and that ξ is a Gaussian white noise with mean zero and variance σ_ξ^2. For simulation studies, we take $N = 3000$, and $k = 15$, and generated 30 data set, where in each case different Gaussian random numbers with specific variances are generated for (r_1, r_2, ξ).

First we show the identification results of deterministic part for $\sigma_\xi^2 = (0.3)^2$. The estimated poles of $P(z)$ by (16a) and the corresponding Bode plot are displayed in Figs. 9 and 10, respectively. We see that the identification result is quite good; though not included here if we decrease the noise variance to $\sigma_\xi^2 = (0.05)^2$, then the better estimates are obtained due to the relative increase of the power of deterministic component. Next, the noise filter $H(z)$ is estimated by using (24b). The Bode plot of the estimates of noise filter is depicted in Fig. 11, where $\sigma_\xi^2 = (0.3)^2$. But if we decrease the noise variance to $\sigma_\xi^2 = (0.05)^2$, the performance of the estimates is degraded as shown in Fig. 12, although the performance of the deterministic part improves. This may be inferred from the fact that the relative power of stochastic component increases as the variance σ_ξ^2 increases.

8 Conclusions

In this paper, we have developed a subspace method of identifying the plant, the controller and the noise filter operating in closed loop based on the realization results for stochastic systems with exogenous inputs. In particular, it can be shown that the identification of closed loop system is divided into two subproblems by the preliminary orthogonal decomposition of the joint input-output process: the identification of the deterministic part, or the plant and controller, and the stochastic part, or the noise filter. Thus the plant and the controller can be obtained from the realization of deterministic component of the joint process, and the noise filter can be retrieved from the realization of stochastic component. In each case, a model reduction procedure should be used for deriving lower order models from obtained higher order models by canceling poles and zeros situated very closely. Numerical results show the applicability of the present technique, although there exists inherent difficulty associated with possible pole-zero cancellations.

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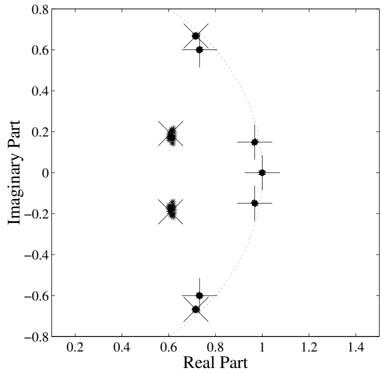

Fig. 2. Estimated eigenvalues of $A_1 - B_{11}D_{11}^{-1}C_{11}$ in (16), where $+$ and $\times$ denote the eigenvalues of true plant and controller, respectively.

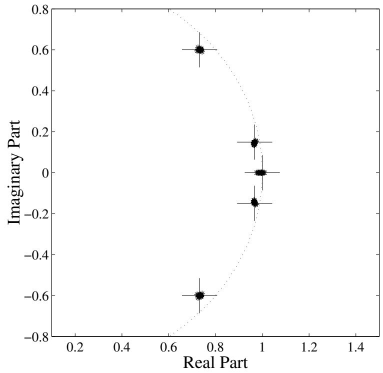

Fig. 3. Estimated poles ($*$) of 5th-order model of the plant $P(z)$, where $+$ denotes the true poles.

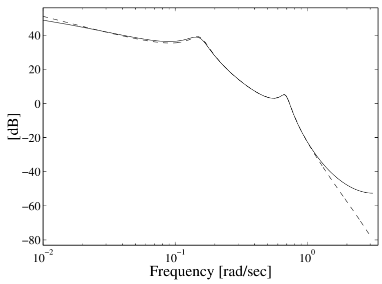

Fig. 4. Bode plot of true (dashed line) and average (solid line) transfer functions of the plant $P(z)$ over 30 experiments.

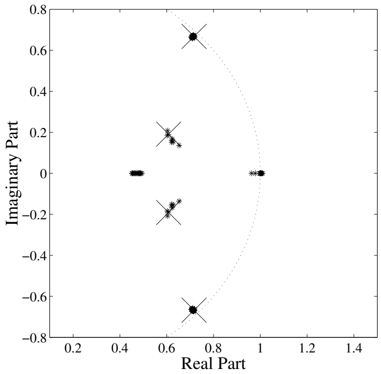

Fig. 5. Estimated poles ($*$) of 4th-order model of the controller $C(z)$, where $\times$ denotes the true poles.

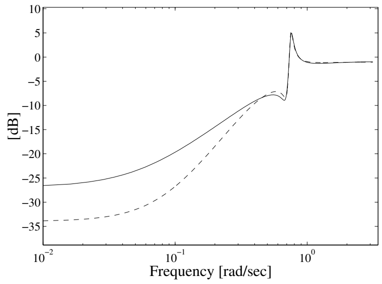

Fig. 6. Bode plot of true (dashed line) and average (solid line) transfer functions of 4th-order model of the controller $C(z)$ over 30 experiments.

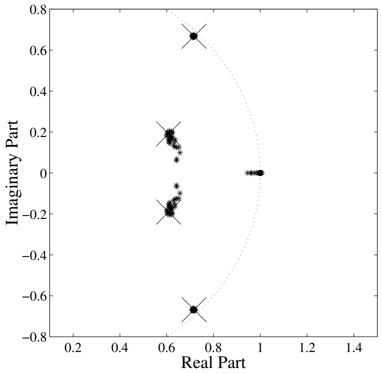

Fig. 7. Estimated poles ($*$) of 5th-order model of the controller $C(z)$, where $\times$ denotes the true poles.

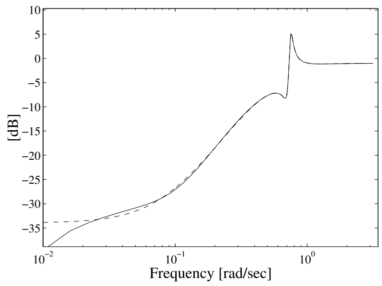

Fig. 8. Bode plot of true (dashed line) and average (solid line) transfer functions of 5th-order model of the controller $C(z)$ over 30 experiments.

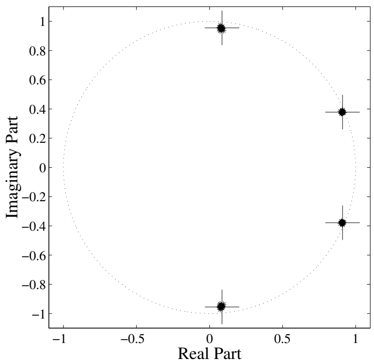

Fig. 9. True $(+)$ and estimated $(*)$ poles of the plant $P(z)$ by (16a), where $\sigma_\xi^2 = (0.3)^2$.

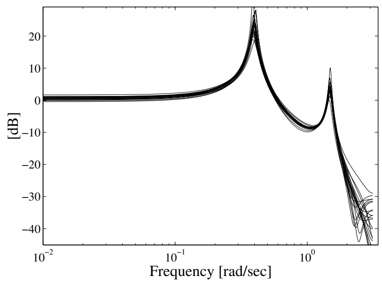

Fig. 10. Bode plot of estimated transfer functions of the plant $P(z)$ over 30 experiments, where $\sigma_\xi^2 = (0.3)^2$.

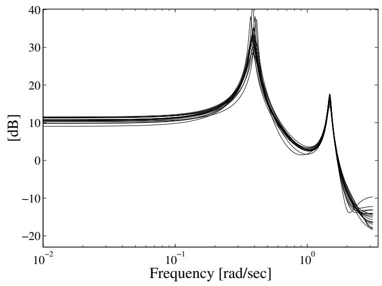

Fig. 11. Bode plot of estimated transfer functions of the noise filter $H(z)$ by (24b), where $\sigma_\xi^2 = (0.3)^2$.

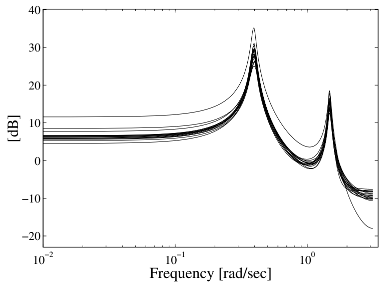

Fig. 12. Bode plot of estimated transfer functions of the noise filter $H(z)$ by (24b), where $\sigma_{\xi}^{2} = (0.05)^{2}$.