A Hybrid Algorithm with Active Set Identification for Mathematical Programs with Complementarity Constraints*

Gui-Hua Lin[†] and Masao Fukushima[‡]

September 4, 2002

Abstract. We consider a mathematical program with complementarity constraints (MPCC). Our purpose is to develop methods that enable us to compute a solution or a point with some kind of stationarity to MPCC by solving a finite number of nonlinear programs. To this end, we first introduce an active set identification technique. Then, by applying this technique to a smoothing continuation method presented by Fukushima and Pang (1999), we propose a hybrid method for solving MPCC. Under reasonable assumptions, the hybrid algorithm is shown to possess a finite termination property. Numerical experience shows that the proposed approach is quite effective.

Key Words. mathematical program with complementarity constraints, MPCC-LICQ, stationarity, weak second-order necessary conditions, asymptotically weak nondegeneracy, upper-level strict complementarity condition, lower-level strict complementarity condition, identification function.

1 Introduction

Mathematical programs with equilibrium constraints (MPECs) have been receiving much attention in the optimization community. There are many practical problems in engineering and economics that can be modelled as MPECs. MPECs also find

*This work was supported in part by the Scientific Research Grant-in-Aid from the Ministry of Education, Science, Sports, and Culture of Japan.

[†]Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan./Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China. E-mail: ghlin@amp.i.kyoto-u.ac.jp.

[‡]Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan. E-mail: fuku@amp.i.kyoto-u.ac.jp.

applications in mathematical programming itself, for example, through reformulation of bilevel programming. For further applications, see the monograph [21] and the references therein.

In this paper, we consider the following mathematical program with complementarity constraints (MPCC), which constitutes an important subclass of MPECs:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g(z) \leq 0, \;\; h(z) = 0 \\ & G(z) \geq 0, \;\; H(z) \geq 0 \\ & G(z)^T H(z) = 0, \end{array} \qquad (P)$$

where $f : \Re^n \to \Re, g : \Re^n \to \Re^p, h : \Re^n \to \Re^q$, and $G, H : \Re^n \to \Re^m$ are all twice continuously differentiable functions. The major difficulty in solving problem (P) is that its constraints fail to satisfy a standard constraint qualification at any feasible point [4], so that standard methods are not guaranteed to solve this problem. There have been proposed several approaches such as sequential quadratic programming approach [9, 16, 21], implicit programming approach [3, 21], penalty function approach [13, 14, 20, 21, 22, 28], active-set approach [12], and reformulation approach [8, 10, 18, 19, 27]. In particular, Fukushima and Pang [10] considered a smoothing continuation method and showed, under the MPCC-linear independence constraint qualification and an additional condition called the asymptotically weak nondegeneracy, that an accumulation point of KKT points satisfying the second-order necessary conditions for the perturbed problems is a B-stationary point of the original problem. Subsequently, similar schemes were presented by Scholtes [27] and Lin and Fukushima [18,19]. However, these methods require to solve an infinite sequence of nonlinear programs.

The purpose of this paper is to develop methods that enable us to compute a solution or a point with some kind of stationarity to problem (P) by solving a finite number of nonlinear programs. To this end, we will introduce an active set identification technique and incorporate it with the smoothing continuation method [10]. We will show that, under reasonable assumptions, the hybrid algorithm possesses a finite termination property. Numerical experience shows that the proposed approach is quite effective.

The rest of this paper is organized as follows. In the next section, we recall some basic concepts for problem (P) and standard nonlinear programming problems. In Section 3, we first introduce the smoothing continuation method [10] and then apply an active set identification technique to the method and present our hybrid algorithm. Some notes and properties are also given. We report numerical results in Section 4 and make some concluding remarks in Section 5. Throughout the paper, we let $\mathcal{F}$ denote the feasible region of problem (P) and, for a function $F : \Re^n \to \Re^m$ and a given vector $z \in \Re^n$, let

$$\mathcal{I}_F(z) = \{i : \; F_i(z) = 0\}$$

stand for the active index set of F at z.

2 Preliminaries

First, we recall some basic concepts for the standard nonlinear programming problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & c_i(z) \leq 0, \quad i = 1, 2, \cdots, l, \\ & c_i(z) = 0, \quad i = l+1, \cdots, s, \end{array} \tag{2.1}$$

where $c : \Re^n \to \Re^s$ is twice continuously differentiable. We will use the standard definition of stationarity, i.e., z is a *stationary* point of problem (2.1) if z is feasible to problem (2.1) and there exists a Lagrange multiplier vector $\lambda \in \Re^s$ such that

$$\begin{array}{l} \nabla f(z) + \nabla c(z)\lambda = 0, \\ \lambda_i \geq 0, \;\; \lambda_i c_i(z) = 0, \qquad i = 1, 2, \cdots, l. \end{array}$$

For a point z satisfying the above stationarity conditions, we say the *weak second-order necessary condition* (WSONC) holds at z if, furthermore,

$$d^T \Big(\nabla^2 f(z) + \sum_{i=1}^{s} \lambda_i \nabla^2 c_i(z) \Big) d \geq 0, \quad \forall d \in \mathcal{T}(z),$$

where $\mathcal{T}(z)$ is defined by

$$\mathcal{T}(z) = \Big\{ d \in \Re^n : \; d^T \nabla c_i(z) = 0 \;\text{ for each } i \in \mathcal{I}_c(z) \Big\}.$$

For problem (P), we say the MPCC- *linear independence constraint qualification* (MPCC-LICQ) holds at a given point $\bar{z} \in \mathcal{F}$ if the set of vectors

$$\Big\{ \nabla g_l(\bar{z}), \nabla h_r(\bar{z}), \nabla G_i(\bar{z}), \nabla H_j(\bar{z}) : \\ l \in \mathcal{I}_g(\bar{z}),\; r = 1, 2, \cdots, q,\; i \in \mathcal{I}_G(\bar{z}),\; j \in \mathcal{I}_H(\bar{z}) \Big\}$$

is linearly independent. This condition is not particularly stringent for MPCC [29].

The usual LICQ for problem (P) would require, in addition, that the gradient of the function $G(z)^T H(z)$ be linearly independent of the above vectors, which, as we mentioned in the previous section, cannot happen in any case.

We say that $\bar{z} \in \mathcal{F}$ is a *Bouligand or B-stationary* point of problem (P) if it satisfies

$$d^T \nabla f(\bar{z}) \geq 0, \quad \forall d \in \mathcal{T}(\bar{z}, \mathcal{F}), \tag{2.2}$$

where $\mathcal{T}(\bar{z}, \mathcal{F})$ stands for the tangent cone of $\mathcal{F}$ at $\bar{z}$. A B-stationary point is obviously a candidate for a local minimizer of problem (P); however, it may be difficult to obtain such a point for MPECs in general. Next, we introduce some other kinds of stationarity for problem (P) [26]. A feasible point $\bar{z}$ is called *weakly stationary* to problem (P) if there exist multiplier vectors $\bar{\lambda}_g \in \Re^p, \bar{\lambda}_h \in \Re^q$, and $\bar{\lambda}_G, \bar{\lambda}_H \in \Re^m$ such that

$$\nabla f(\bar{z}) + \nabla g(\bar{z})^T \bar{\lambda}_g + \nabla h(\bar{z})^T \bar{\lambda}_h - \nabla G(\bar{z})^T \bar{\lambda}_G - \nabla H(\bar{z})^T \bar{\lambda}_H = 0, \tag{2.3}$$

$$\bar{\lambda}_g \geq 0, \quad \bar{\lambda}_g^T g(\bar{z}) = 0, \tag{2.4}$$

$$\bar{\lambda}_{G,i} = 0, \quad i \notin \mathcal{I}_G(\bar{z}), \tag{2.5}$$

$$\bar{\lambda}_{H,i} = 0, \quad i \notin \mathcal{I}_H(\bar{z}). \tag{2.6}$$

If the MPCC-LICQ holds at $\bar{z}$, then the definition (2.2) of B-stationarity can be restated in terms of Lagrange multipliers: A feasible point $\bar{z}$ is a B-stationary point of problem (P) if there exist multiplier vectors $\bar\lambda_g, \bar\lambda_h, \bar\lambda_G$, and $\bar\lambda_H$ such that (2.3)–(2.6) hold with

$$\bar\lambda_{G,i} \geq 0, \quad \bar\lambda_{H,i} \geq 0, \quad i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z). \tag{2.7}$$

Obviously any weakly stationary point $\bar z$ is B-stationary whenever $\bar z$ is *nondegenerate* in the sense that the *lower-level strict complementarity* condition

$$\mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z) = \emptyset \tag{2.8}$$

holds at $\bar z$.

Other two kinds of stationarity concepts for problem (P) are Clarke or C-stationarity and Mordukhovich or M-stationarity [26], which are stronger than the weak stationarity but weaker than B-stationarity. We say a feasible point $\bar z$ is *C-stationary* to problem (P) if there exist multiplier vectors $\bar\lambda_g, \bar\lambda_h, \bar\lambda_G$, and $\bar\lambda_H$ such that (2.3)–(2.6) hold and

$$\bar\lambda_{G,i}\bar\lambda_{H,i} \geq 0, \quad i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z) \tag{2.9}$$

and we say $\bar z$ is *M-stationary* to problem (P) if, furthermore,

$$\text{either} \;\; \bar\lambda_{G,i} > 0, \; \bar\lambda_{H,i} > 0 \;\; \text{or} \;\; \bar\lambda_{G,i}\bar\lambda_{H,i} = 0 \tag{2.10}$$

for all $i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z)$.

In addition, a weakly stationary point $\bar z \in \mathcal{F}$ of problem (P) is said to satisfy the *upper-level strict complementarity* (ULSC) condition if there exist multiplier vectors $\bar\lambda_g, \bar\lambda_h, \bar\lambda_G$, and $\bar\lambda_H$ satisfying (2.3)–(2.6) with

$$\bar\lambda_{G,i}\bar\lambda_{H,i} \neq 0, \quad i \in \mathcal{I}_G(\bar z) \cap \mathcal{I}_H(\bar z). \tag{2.11}$$

The upper-level strict complementarity is much weaker than the lower-level strict complementarity, because the former vacuously holds when the latter holds. Also, it is obvious that any M-stationary point of problem (P) satisfying the upper-level strict complementarity condition is B-stationary to problem (P).

3 A Hybrid Algorithm for MPCCs

As mentioned earlier, the aim of this paper is to develop methods that enable us to compute a solution or some kind of stationary points of problem (P) by solving a finite sequence of nonlinear programs, unlike many existing methods that require to solve an infinite sequence of nonlinear programs. To this end, we apply an active set identification technique, which will be introduced in the second subsection, to a smoothing continuation method [10]. In the first subsection, we will describe the smoothing continuation method.

3.1 Smoothing continuation method for MPCC

Consider the smoothing continuation method [10] that uses the perturbed Fischer-Burmeister function

$$\phi_\epsilon(a, b) = a + b - \sqrt{a^2 + b^2 + 2\epsilon^2},$$

where $\epsilon \geq 0$. Define the function $\Phi_\epsilon : \Re^n \to \Re^m$ by

$$\Phi_\epsilon(z) = \Big(\phi_\epsilon(G_1(z), H_1(z)), \cdots, \phi_\epsilon(G_m(z), H_m(z))\Big)^T$$

and consider the nonlinear programming problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & \Phi_\epsilon(z) = 0, \\ & g(z) \leq 0,\ h(z) = 0. \end{array} \tag{P_ϵ}$$

Note that (P_0) is equivalent to problem (P) and Φ_ϵ is differentiable everywhere for any $\epsilon > 0$. We assume that problem (P_ϵ) has a solution (or a stationary point) z^ϵ for each small scalar $\epsilon > 0$. We may expect to find a solution or a point with some kind of stationarity to problem (P) by tracing the trajectory $\{z^\epsilon\}$ as $\epsilon \to 0$. Suppose that $\{\epsilon_k\}$ is a positive sequence converging to zero. The following convergence result is given in [10].

Theorem 3.1 *Let z^k be a stationary point of problem (P_{ϵ_k}) and the sequence $\{z^k\}$ converge to $\bar{z}$ as $\epsilon_k \to 0$. Suppose that the WSONC holds at each z^k, the MPCC-LICQ holds at $\bar{z}$, and $\{z^k\}$ is asymptotically weakly nondegenerate. Then $\bar{z}$ is B-stationary to problem (P).*

We now recall the asymptotically weak nondegeneracy [10], which is assumed in the above theorem and will often be employed in the subsequent analysis. Suppose the sequence $\{z^k\}$ converges to $\bar{z}$ as $\epsilon_k \to 0$. Then $\bar{z} \in \mathcal{F}$. It can be shown [10] that, for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$,

$$\nabla \Phi_{\epsilon_k,i}(z^k) = \frac{H_i(z^k)}{G_i(z^k) + H_i(z^k)} \nabla G_i(z^k) + \frac{G_i(z^k)}{G_i(z^k) + H_i(z^k)} \nabla H_i(z^k). \tag{3.1}$$

Therefore, every accumulation point r of $\left\{\nabla \Phi_{\epsilon_k,i}(z^k)\right\}$ can be represented as

$$r = \xi_i(r) \nabla G_i(\bar{z}) + \eta_i(r) \nabla H_i(\bar{z}) \tag{3.2}$$

for some $(\xi_i(r), \eta_i(r))$ with $(1 - \xi_i(r))^2 + (1 - \eta_i(r))^2 \leq 1$. We say that $\{z^k\}$ is *asymptotically weakly nondegenerate* if, for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$, neither $\xi_i(r)$ nor $\eta_i(r)$ vanishes for every accumulation point r of $\left\{\nabla \Phi_{\epsilon_k,i}(z^k)\right\}$.

Roughly speaking, the asymptotically weak nondegeneracy of $\{z^k\}$ means that, for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$, $G_i(z^k)$ and $H_i(z^k)$ approach zero in the same order of magnitude. This property is obviously weaker than the nondegeneracy, because it vacuously holds when $\bar{z}$ is nondegenerate. The next result shows that it is even weaker than the ULSC condition, which is often employed in the literature on MPCCs, see [13, 14, 18, 19, 27].

Lemma 3.2 *Let z^k be stationary to problem (P_{ϵ_k}) for each k and the sequence $\{z^k\}$ converge to $\bar{z}$ as $\epsilon_k \to 0$. Suppose that the MPCC-LICQ holds at $\bar{z}$. If the ULSC condition holds at $\bar{z}$, then $\{z^k\}$ is asymptotically weakly nondegenerate.*

A proof of this lemma is given in the appendix. By Theorem 3.1 and Lemma 3.2, we have the following result immediately.

Theorem 3.3 *Let z^k be a stationary point of problem (P_{ϵ_k}) and the sequence $\{z^k\}$ converge to $\bar{z}$ as $\epsilon_k \to 0$. Suppose that the WSONC holds at each z^k and both* *the MPCC-LICQ and the ULSC conditions hold at $\bar{z}$. Then $\bar{z}$ is B-stationary to problem (P).*

Recently, Lin and Fukushima [18, 19] have established some new convergence results for their relaxation methods for MPCCs. We can prove that the smoothing continuation method possesses similar convergence properties. Here, we state one of such results.

Theorem 3.4 *Let z^k be a stationary point of problem (P_{ϵ_k}) and, for each k, $(\lambda_g^k, \lambda_h^k, \lambda_\Phi^k)$ be a multiplier vector corresponding to z^k. Suppose that the sequence $\{z^k\}$ converges to $\bar{z}$ as $\epsilon_k \to 0$ and, for each k, $\nabla_z^2 L_{\epsilon_k}(z^k, \lambda_g^k, \lambda_h^k, \lambda_\Phi^k)$ is bounded below with constant $\alpha_k \geq 0$ on the corresponding tangent space $\mathcal{T}_{\varepsilon_k}(z^k)$, which means*

$$d^T \nabla_z^2 L_{\epsilon_k}(z^k, \lambda_g^k, \lambda_h^k, \lambda_\Phi^k) d \geq -\alpha_k ||d||^2, \quad \forall d \in \mathcal{T}_{\epsilon_k}(z^k), \tag{3.3}$$

where

$$\begin{aligned} L_\epsilon(z, \lambda_g, \lambda_h, \lambda_\Phi) &= f(z) + \lambda_g^T g(z) + \lambda_h^T h(z) + \lambda_\Phi^T \Phi_\epsilon(z), \\ \mathcal{T}_\epsilon(z) &= \Big\{ d \in R^n : \ \ d^T \nabla \Phi_{\epsilon,i}(z) = 0, \ i = 1, 2, \cdots, m; \\ &\qquad\qquad\qquad d^T \nabla g_l(z) = 0, \quad l \in \mathcal{I}_g(z); \\ &\qquad\qquad\qquad d^T \nabla h_r(z) = 0, \quad r = 1, 2, \cdots, q \Big\}. \end{aligned}$$

If the sequence $\{\alpha_k\}$ is bounded, $\{z^k\}$ is asymptotically weakly nondegenerate, and the MPEC-LICQ holds at $\bar{z}$, then $\bar{z}$ is a B-stationary point of problem (P).

Actually, the condition that problem (P_{ϵ_k}) satisfies the WSONC at z^k for each k means that $\nabla_z^2 L_{\epsilon_k}(z^k, \lambda_g^k, \lambda_h^k, \lambda_\Phi^k)$ is bounded below with constant 0 for each k. In consequence, Theorem 3.1 is a corollary of Theorem 3.4. Note that, for the matrix $\nabla_z^2 L_{\epsilon_k}(z^k, \lambda_g^k, \lambda_h^k, \lambda_\Phi^k)$, there must exist a number α_k such that (3.3) holds. For example, any nonnegative scalar α_k such that $(-\alpha_k)$ is less than the smallest eigenvalue of $\nabla_z^2 L_{\epsilon_k}(z^k, \lambda_g^k, \lambda_h^k, \lambda_\Phi^k)$ must satisfy (3.3). However, the WSONC means that the matrix should have some kind of semi-definiteness on the tangent space $\mathcal{T}_{\epsilon_k}(z^k)$. Therefore, the assumptions in Theorem 3.4 are significantly weaker than the conditions of Theorem 3.1. Since the proof of Theorem 3.4 is similar to that of Theorem 3.1, it is omitted here, see [10].

3.2 A hybrid algorithm for MPCCs

We first introduce an active set identification technique for MPCCs. Active set identification plays an important role in optimization theory [1, 2, 6, 7, 31]. Accurate identification of active constraints is important from both theoretical and practical points of view. For problem (P), by means of active set identification, the combinatorial constraints

$$G(z) \geq 0, \quad H(z) \geq 0, \quad G(z)^T H(z) = 0 \tag{3.4}$$

may be replaced by some equality or inequality constraints that are easier to deal with.

For a point $\bar{z} \in \mathcal{F}$, let $\alpha(\bar{z}), \beta(\bar{z})$ and $\gamma(\bar{z})$ be the index sets defined by

$$\begin{aligned} \alpha(\bar{z}) &= \{i : G_i(\bar{z}) > 0, \ H_i(\bar{z}) = 0\}, \\ \beta(\bar{z}) &= \{i : G_i(\bar{z}) = 0, \ H_i(\bar{z}) = 0\}, \\ \gamma(\bar{z}) &= \{i : G_i(\bar{z}) = 0, \ H_i(\bar{z}) > 0\}, \end{aligned}$$

respectively. Obviously, $\alpha(\bar{z}) \cup \beta(\bar{z}) \cup \gamma(\bar{z}) = \{1, \ldots, m\}$ and these index sets are mutually disjoint. Note that

$$\mathcal{I}_G(\bar{z}) = \beta(\bar{z}) \cup \gamma(\bar{z}), \quad \mathcal{I}_H(\bar{z}) = \alpha(\bar{z}) \cup \beta(\bar{z}),$$

and

$$\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}) = \beta(\bar{z}).$$

Thus, by (2.8), $\bar{z}$ is nondegenerate if and only if $\beta(\bar{z}) = \emptyset$. If $\beta(\bar{z}) \neq \emptyset$, we call $\bar{z}$ *degenerate*. Let a sequence $\{z^k\}$ be generated so that it converges to $\bar{z} \in \mathcal{F}$. If $\bar{z}$ is nondegenerate, it is generally not difficult to identify the correct index sets finitely. However, when $\bar{z}$ is degenerate, it is not necessarily easy to identify the active index sets. In the following, we will particularly be interested in the case where $\{z^k\}$ is convergent to a degenerate point $\bar{z} \in \mathcal{F}$.

Let $\{\epsilon_k\}$ be a positive sequence converging to zero and let z^k stand for a solution or a stationary point of problem (P_{ϵ_k}) for each k. Suppose that $\{z^k\}$ converges to some $\bar{z}$ throughout this section. Note that

$$G_i(z^k) > 0,\ H_i(z^k) > 0,\ G_i(z^k)H_i(z^k) = \epsilon_k^2 \tag{3.5}$$

for each k and each i. We try to estimate the index sets $\alpha(\bar{z})$, $\beta(\bar{z})$ and $\gamma(\bar{z})$ by some index sets α^k, β^k and γ^k, respectively, which are obtained from z^k and satisfy $\alpha^k \cup \beta^k \cup \gamma^k = \{1, \ldots, m\}$ with $\alpha^k, \beta^k, \gamma^k$ being mutually disjoint. Given the index sets α^k, β^k and γ^k, we then solve the nonlinear programming problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & G_i(z) \geq 0,\ H_i(z) = 0, \quad i \in \alpha^k, \\ & G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta^k, \\ & G_i(z) = 0,\ H_i(z) \geq 0, \quad i \in \gamma^k, \\ & g(z) \leq 0,\ h(z) = 0. \end{array} \tag{3.6}$$

This problem is no longer an MPCC and hence easier to deal with. Denote by $\hat{z}^k$ a stationary point of problem (3.6). Obviously, we always have $\beta^k \subseteq \beta(\hat{z}^k)$, and $\beta(\hat{z}^k)$ may contain some $i \in \alpha^k$ with $G_i(\hat{z}^k) = 0$ or some $i \in \gamma^k$ with $H_i(\hat{z}^k) = 0$. If the Lagrange multipliers corresponding to the constraints

$$G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta(\hat{z}^k)$$

are all nonnegative, then $\hat{z}^k$ is a B-stationary point of problem (P) under the MPCC-LICQ assumption at the point. Therefore, assuming that (3.6) can be solved exactly, we may terminate the method in finite steps, unlike the method in [10], which needs to solve an infinite sequence of nonlinear programs.

The key to success is to define the index sets α^k, β^k and γ^k such that

$$\alpha^k = \alpha(\bar{z}),\ \beta^k = \beta(\bar{z}),\ \gamma^k = \gamma(\bar{z}) \tag{3.7}$$

for all k large enough. To this end, we may use an identification function $\rho : \Re^n \to [0, +\infty)$ satisfying

$$\lim_{k\to\infty} \rho(z^k) \ = \ 0 \tag{3.8}$$

and, for all k large enough,

$$\max_{i\in\beta(\bar{z})} \Big\{G_i(z^k), H_i(z^k)\Big\} \;\leq\; \rho(z^k), \tag{3.9}$$

$$\max_{i\in\alpha(\bar{z})\cup\gamma(\bar{z})} \Big\{ \min\{G_i(z^k), H_i(z^k)\}\Big\} \;\leq\; \rho(z^k), \tag{3.10}$$

and consider the following hybrid algorithm that combines the smoothing continuation method with an active set identification technique.

Hybrid Algorithm

Step 0: Choose $\epsilon_0 > 0$ and set $k = 0$.

Step 1: Solve problem (P_{ϵ_k}) and denote by z^k one of its stationary points. Set

$$\alpha^k \;=\; \Big\{\, i: \;\; G_i(z^k) > \rho(z^k), \;\; H_i(z^k) \leq \rho(z^k) \,\Big\}, \tag{3.11}$$

$$\beta^k \;=\; \Big\{\, i: \;\; G_i(z^k) \leq \rho(z^k), \;\; H_i(z^k) \leq \rho(z^k) \,\Big\}, \tag{3.12}$$

$$\gamma^k \;=\; \Big\{\, i: \;\; G_i(z^k) \leq \rho(z^k), \;\; H_i(z^k) > \rho(z^k) \,\Big\}. \tag{3.13}$$

If $\alpha^k \cup \beta^k \cup \gamma^k = \{1, \ldots, m\}$, go to Step 2. Otherwise, go to Step 4.

Step 2: If problem (3.6) is solvable, denote by $\hat{z}^k$ one of its stationary points and go to Step 3. Otherwise, go to Step 4.

Step 3: For $\hat{z}^k$, if the Lagrange multipliers corresponding to the constraints

$$G_i(z) = 0, \; H_i(z) = 0, \quad i \in \beta(\hat{z}^k) \tag{3.14}$$

are all nonnegative, then terminate. Else, go to Step 4.

Step 4: Choose an $\epsilon_{k+1} \in (0, \epsilon_k)$ and let $k := k + 1$. Go to Step 1.

Next, we make some remarks on the identification function ρ and the algorithm.

First of all, we have that, if $\beta(\bar{z}) \neq \emptyset$,

$$\max_{i\in\alpha(\bar{z})\cup\gamma(\bar{z})} \Big\{ \min\{G_i(z^k), H_i(z^k)\}\Big\} < \min_{i\in\beta(\bar{z})} \Big\{G_i(z^k), H_i(z^k)\Big\} \tag{3.15}$$

holds for all k large enough as long as $\alpha(\bar{z}) \cup \gamma(\bar{z}) \neq \emptyset$. In fact, we have from (3.5) that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\epsilon_k^2}{\min\{G_i(z^k), H_i(z^k)\}} &= \lim_{k\to\infty} \max\{G_i(z^k), H_i(z^k)\} \\
&= 0, \qquad\qquad\qquad i \in \beta(\bar{z})
\end{aligned}$$

and

$$\begin{aligned}
\lim_{k\to\infty} \frac{\epsilon_k^2}{\min\{G_j(z^k), H_j(z^k)\}} &= \lim_{k\to\infty} \max\{G_j(z^k), H_j(z^k)\} \\
&= G_j(\bar{z}) + H_j(\bar{z}) > 0, \qquad j \in \alpha(\bar{z}) \cup \gamma(\bar{z}).
\end{aligned}$$

In consequence,

$$\lim_{k\to\infty} \frac{\min\{G_j(z^k), H_j(z^k)\}}{\min\{G_i(z^k), H_i(z^k)\}} = 0 \tag{3.16}$$

holds for each $i \in \beta(\bar{z})$ and each $j \in \alpha(\bar{z}) \cup \gamma(\bar{z})$ and hence we have (3.15). This inequality means that condition (3.9) implies condition (3.10) for all k sufficiently large if $\beta(\bar{z}) \neq \emptyset$ and $\alpha(\bar{z}) \cup \gamma(\bar{z}) \neq \emptyset$.

Moreover, it is obvious that $\{\alpha^k, \beta^k, \gamma^k\}$ defined by (3.11)–(3.13) is mutually disjoint for each k. On the other hand, conditions (3.9) and (3.10) ensure that, when k is sufficiently large,

$$\min\{G_i(z^k), H_i(z^k)\} \leq \rho(z^k), \quad \forall i.$$

This means $\alpha^k \cup \beta^k \cup \gamma^k = \{1, \ldots, m\}$ and so $\{\alpha^k, \beta^k, \gamma^k\}$ defined in Step 1 is a partition of $\{1, \ldots, m\}$ for all k sufficiently large.

Furthermore, we have from (3.8) that, when k is sufficiently large,

$$\alpha^k \supseteq \alpha(\bar{z}),\ \ \beta^k \subseteq \beta(\bar{z}),\ \ \gamma^k \supseteq \gamma(\bar{z}).$$

In addition, it follows from (3.9) and (3.12) that $\beta(\bar{z}) \subseteq \beta^k$ for all k sufficiently large. Note that both $\{\alpha^k, \beta^k, \gamma^k\}$ and $\{\alpha(\bar{z}), \beta(\bar{z}), \gamma(\bar{z})\}$ are partitions of $\{1, \ldots, m\}$. Therefore, (3.7) holds when k is large enough.

The above facts indicate that the hybrid algorithm may possess a finite termination property. The key question is, of course, how to define the identification function ρ for the hybrid algorithm. This is not a trivial task because the function may depend on the unknown point $\bar{z}$ generally. Next, we consider the case where $\{z^k\}$ is asymptotically weakly nondegenerate and show the following results.

Theorem 3.5 *Suppose the sequence $\{z^k\}$ generated by the hybrid algorithm is asymptotically weakly nondegenerate. Let*

$$\rho_1(z) = \tau || \min(G(z), H(z))||^\sigma, \tag{3.17}$$

where $\tau > 0$ and $\sigma \in (0, 1)$ are constants and

$$\min(G(z), H(z)) = \Big(\min\{G_1(z), H_1(z)\}, \cdots, \min\{G_m(z), H_m(z)\}\Big)^T. \tag{3.18}$$

Then (3.8)–(3.10) hold with $\rho = \rho_1$, i.e., the function ρ_1 can serve as an identification function for the hybrid algorithm.

Proof: Note that the complementarity constraints (3.4) are equivalent to $\rho_1(z) = 0$. By the fact that $\{z^k\}$ is convergent to $\bar{z} \in \mathcal{F}$ and the continuity of ρ_1, we have

$$\lim_{k\to\infty} \rho_1(z^k) = \rho_1(\bar{z}) = 0.$$

Therefore, condition (3.8) holds with $\rho = \rho_1$. On the other hand, for any $i \in \alpha(\bar{z}) \cup \gamma(\bar{z})$, we have

$$\min\Big\{G_i(z^k), H_i(z^k)\Big\} \leq || \min(G(z^k), H(z^k))|| \leq \rho_1(z^k)$$

for all k sufficiently large, where the first inequality follows from (3.18) and the second inequality follows from the facts that $\{|| \min(G(z^k), H(z^k))||\}$ is convergent to 0 and the constant $\sigma \in (0, 1)$. This means that (3.10) with $\rho = \rho_1$ holds when k is sufficiently large.

We next prove (3.9) with $\rho = \rho_1$ holds when k is sufficiently large. We may assume $\beta(\bar{z}) \neq \emptyset$ because (3.9) holds vacuously if $\beta(\bar{z})$ is empty. Let $i \in \beta(\bar{z})$. Since the set of accumulation points of the sequence $\Big\{\nabla\Phi_{\epsilon_k,i}(z^k)\Big\}$ is compact, by the asymptotically weak nondegeneracy of $\{z^k\}$, the set of the coefficient pairs in (3.2) is a compact subset of

$$\Re^2_{++} = \Big\{(\xi, \eta)^T \in \Re^2 : \ \xi > 0, \eta > 0\Big\}.$$

Then, by (3.1) and (3.2), there exist positive constants $a_i < b_i$ such that

$$a_i \le \frac{G_i(z^k)}{H_i(z^k)} \le b_i, \qquad \forall k.$$

Let

$$a = \min_{i\in\beta(\bar z)} a_i, \quad b = \max_{i\in\beta(\bar z)} b_i.$$

It follows that

$$0 < a \le \frac{G_i(z^k)}{H_i(z^k)} \le b \tag{3.19}$$

for each k and each $i \in \beta(\bar z)$. We then have from (3.19) that

$$G_i(z^k) \le bH_i(z^k), \quad H_i(z^k) \le a^{-1}G_i(z^k)$$

and so

$$\max\{G_i(z^k), H_i(z^k)\} \le (a^{-1}+b)\min\{G_i(z^k), H_i(z^k)\} \tag{3.20}$$

for each k and each $i \in \beta(\bar z)$. By (3.20) and (3.18), we have

$$\begin{aligned}\max\Big\{G_i(z^k), H_i(z^k)\Big\} &\le (a^{-1}+b)||\min(G(z^k), H(z^k))|| \\ &\le \rho_1(z^k)\end{aligned}$$

for all k sufficiently large, where the last inequality follows from the same facts as above. This completes the proof of (3.9) and so the function ρ_1 given by (3.17) can serve as an identification function. □

Theorem 3.6 *Suppose that the sequence* $\{z^k\}$ *generated by the hybrid algorithm is asymptotically weakly nondegenerate. Then the function*

$$\rho_2(z) = \tau||\Phi_0(z)||^\sigma, \qquad \tau > 0,\ \sigma \in (0,1) \tag{3.21}$$

is an identification function for the algorithm.

Proof: Noting that, for each i and each k,

$$\begin{aligned}\frac{2}{2+\sqrt{2}}||\min(G(z^k), H(z^k))|| &\le ||\Phi_0(z^k)|| \\ &\le (2+\sqrt{2})||\min(G(z^k), H(z^k))||\end{aligned}$$

(see [30]), we have

$$\tau\Big(\frac{2}{2+\sqrt{2}}\Big)^{\sigma}||\min(G(z^k),H(z^k))||^{\sigma} \;\leq\; \rho_2(z^k) \\ \leq\; \tau(2+\sqrt{2})^{\sigma}||\min(G(z^k),H(z^k))||^{\sigma}. \tag{3.22}$$

Let

$$\hat{\rho}_1(z) = \tau\Big(\frac{2}{2+\sqrt{2}}\Big)^{\sigma}||\min(G(z),H(z))||^{\sigma}$$

and

$$\bar{\rho}_1(z) = \tau(2+\sqrt{2})^{\sigma}||\min(G(z),H(z))||^{\sigma}.$$

By Theorem 3.5, both $\hat{\rho}_1$ and $\bar{\rho}_1$ are identification functions for the hybrid algorithm. As a result, condition (3.8) holds with $\bar{\rho}_1$. This, together with the second inequality in (3.22), implies that condition (3.8) holds with $\rho = \rho_2$. On the other hand, the first inequality in (3.22), together with the fact that conditions (3.9) and (3.10) with $\rho = \hat{\rho}_1$, respectively hold for all k sufficiently large, means that (3.9) and (3.10) hold with $\rho = \rho_2$ when k is sufficiently large. In consequence, the function ρ_2 satisfies conditions (3.8)–(3.10). This completes the proof. □

The next lemma indicates that problem (P_ϵ) satisfies the standard LICQ under some appropriate assumptions, unlike problem (P), which fails to satisfy any constraint qualification at any feasible point. A proof of the lemma may be found in [10].

Lemma 3.7 *If the MPCC-LICQ holds at $\bar{z}$, then there exist a neighborhood $U(\bar{z})$ of $\bar{z}$ and a positive constant $\bar{\epsilon}$ such that, for any $\epsilon \in (0,\bar{\epsilon})$, problem ($P_\epsilon$) satisfies the standard LICQ at any feasible point in $U(\bar{z})$.*

Summarizing the above arguments, we have the following concluding results.

Theorem 3.8 *Let $\sigma \in (0,1), \tau > 0$, and ρ be the function ρ_1 defined by (3.17) or ρ_2 defined by (3.21). Suppose that the sequence $\{z^k\}$ of stationary points of problems (P_{ϵ_k}) converges to $\bar{z}$ as $\epsilon_k \to 0$, the MPCC-LICQ holds at $\bar{z}$, and the sequence $\{z^k\}$ is asymptotically weakly nondegenerate. Then, for all k sufficiently large, we have*

i) *problem* (P_{ϵ_k}) *satisfies the standard LICQ at* z^k*;*

ii) *the sets* α^k, β^k*, and* γ^k *defined by (3.11)–(3.13) satisfy (3.7).*

This theorem indicates that, by means of the technique introduced above, we can identify the active sets $\alpha(\bar{z}), \beta(\bar{z})$, and $\gamma(\bar{z})$ in a finite number of iterations under some mild conditions. As a result, if the problem

$$\begin{array}{lll} \min & f(z) & \\ \text{s.t.} & G_i(z) \geq 0,\ H_i(z) = 0, & i \in \alpha(\bar{z}), \\ & G_i(z) = 0,\ H_i(z) = 0, & i \in \beta(\bar{z}), \\ & G_i(z) = 0,\ H_i(z) \geq 0, & i \in \gamma(\bar{z}), \\ & g(z) \leq 0,\ h(z) = 0 & \end{array} \tag{3.23}$$

can be solved exactly, we may expect that the hybrid algorithm terminates in finite steps. In particular, if $\bar{z}$ is a stationary point obtained by solving (3.23), then, under the assumptions of one of Theorems 3.1, 3.3, and 3.4, the algorithm terminates in a finite number of iterations by producing a B-stationary point of problem (P). Note that it is possible that the algorithm terminates by producing a B-stationary point of (P) before we solve problem (3.23).

Remark 3.1. The stopping criterion in Step 3 is used to check the B-stationarity of the point $\hat{z}^k$. We may replace it by some other conditions for checking M-stationarity or C-stationarity, see (2.9) and (2.10). Moreover, in order to ensure

$$\alpha^k \cup \beta^k \cup \gamma^k = \{1, 2, \ldots, m\} \tag{3.24}$$

in Step 1 for each k, we may define

$$\beta^k = \{1, \ldots, m\} \setminus (\alpha^k \cup \gamma^k) \tag{3.25}$$

instead of (3.12) in the hybrid algorithm.

Remark 3.2. Since problem (P_ϵ) is equivalent to

$$\begin{array}{lll} \min & f(z) & \\ \text{s.t.} & g(z) \leq 0,\ h(z) = 0 & \\ & G(z) + H(z) \geq 0 & (P'_\epsilon) \\ & G_i(z)H_i(z) = \epsilon^2,\ i = 1, \cdots, m, & \end{array}$$

we may use problem (P'_{ϵ_k}) instead of (P_{ϵ_k}) in Step 1 of the hybrid algorithm for each k. Note that, for any $\epsilon > 0$, the constraints

$$G_i(z) + H_i(z) \geq 0, \quad i = 1, 2, \cdots, m$$

are always inactive and so problem (P'_ϵ) seems simpler than (P_ϵ).

4 Computational Results

We have tested the hybrid algorithm on various small scale problems of MPCCs. These problems were generated by the MATLAB generator QPECgen of Jiang and Ralph [15]. In our testing, we used the MATLAB 6.0 built-in solver function *fmincon* to solve problems (P_{ϵ_k}) and (3.6) at each iteration. The computational results indicate that the proposed method finds an optimal solution of MPCC in a small number of iterations. We report the details below.

The QPECgen generator is a MATLAB program which that uses a set of parameters, see Table 1 or [15] for detail. Once these parameters have been specified, the program can randomly generate a quadratic program with linear complementarity constraints

$$\begin{array}{ll} \min & \frac{1}{2}(x^T, y^T)P\begin{pmatrix} x \\ y \end{pmatrix} + c^T x + d^T y \\ \text{s.t.} & A\begin{pmatrix} x \\ y \end{pmatrix} + a \leq 0, \\ & y \geq 0,\ Nx + My + q \geq 0, \\ & y^T(Nx + My + q) = 0, \end{array}$$

Table 1: Problems generated by QPECgen

Parameters in QPECgen	Input Data		
	Problem 4.1	Problem 4.2	Problem 4.3
`qpec_type`	300	300	300
`(n, m)`	(8, 5)	(10, 10)	(6, 12)
`(l, p)`	(4, 5)	(5, 10)	(4, 12)
`cond_P`	100	100	100
`scale_P`	100	100	100
`concex_f`	1	1	1
`symm_M`	1	1	1
`mono_M`	1	1	1
`cond_M`	200	200	200
`scale_M`	200	200	200
`second_deg`	2	5	6
`first_deg`	2	3	2
`mix_deg`	2	3	4
`tol_deg`	1.0e-6	1.0e-6	1.0e-6
`implicit`	1	0	1
`rand_seed`	0	0	0
`output`	3	3	3

where P, A, N, M are constant matrices and c, d, a, q are constant vectors with appropriate dimensions. QPECgen also outputs an approximate solution of a generated problem.

We set the QPECgen parameters as in Table 1 to generate test problems 4.1, 4.2, and 4.3. In particular, `(n, m)` denotes the dimension of the variable (x, y) and `second_deg` stands for the cardinality of the lower-level degenerate index set at a solution.

Some data obtained by the QPECgen generator are summarized in Table 2, in which $(\bar{x}, \bar{y})$ denotes the solution given by QPECgen and $\beta(\bar{x}, \bar{y})$ stands for the lower-level degenerate index set at $(\bar{x}, \bar{y})$, i.e.,

$$\beta(\bar{x}, \bar{y}) = \{i : \ (N\bar{x} + M\bar{y} + q)_i = 0 = \bar{y}_i\}.$$

Table 2: Some data obtained by QPECgen for Problems 4.1–4.3

Problem 4.1	$\bar{x}$	(0.0723, 0.4345, 0.2387, -0.4003, -0.2870, -0.5857, -0.4421, 0.2286)
	$\bar{y}$	(0, 0, 0, 0, 0)
	$\beta(\bar{x}, \bar{y})$	{1, 2}
Problem 4.2	$\bar{x}$	(0.6369, 0.6371, 0.1739, -0.7158, -0.8703, -0.7478, -0.4383, 0.1886, 0.0741, 0.1494)
	$\bar{y}$	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
	$\beta(\bar{x}, \bar{y})$	{1, 2, 3, 4, 5}
Problem 4.3	$\bar{x}$	(-0.1018, 0.0166, -0.3092, 0.1948, -0.4273, 0.0296)
	$\bar{y}$	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.8546, 0.3146)
	$\beta(\bar{x}, \bar{y})$	{1, 2, 3, 4, 5, 6}

Recall that the purpose of this paper is to identify the set $\beta(\bar{x}, \bar{y})$.

In our tests, we set $\epsilon_0 = 10^{-2}$ and updated this parameter by $\epsilon_{k+1} = 10^{-2}\epsilon_k$. We use the point (x^k, y^k) as the starting point for the next step in both the hybrid algorithm and the smoothing continuation method. We employed

$$\rho_1(x, y) = || \min(Nx + My + q,\ y)||^{\frac{1}{2}}$$

and

$$\rho_2(x, y) = ||\Phi_0(x, y)||^{\frac{1}{2}},$$

where

$$\Phi_0(x, y) = \Big(\phi_0((Nx + My + q)_1, y_1), \cdots, \phi_0((Nx + My + q)_m, y_m)\Big)^T,$$

to be the identification function ρ in Step 1 of the hybrid algorithm, respectively, and we obtained almost the same numerical results for the examples. This is not surprising because $\{\rho_1(x^k, y^k)\}$ and $\{\rho_2(x^k, y^k)\}$ tend to 0 in the same order as $(x^k, y^k) \to (\bar{x}, \bar{y})$.

The computational results with the identification function ρ_1 for test problems 4.1–4.3 are reported in Tables 3–5. In the tables, we only list the values of the variable x, which are sufficient to show the performance of the tested algorithms.

Table 3: Computational results for Problem 4.1

<table>
<tr><td rowspan="4">Hybrid Algorithm</td><td colspan="2">Initial point</td><td>(0.5, $\cdots$, 0.5)</td></tr>
<tr><td rowspan="3">$\epsilon_0 = 10^{-2}$</td><td>$\hat{x}^0$</td><td>(0.0723,0.4345,0.2386,-0.4003, -0.2870,-0.5857,-0.4421, 0.2287)</td></tr>
<tr><td>Ite</td><td>24+8</td></tr>
<tr><td>β^0</td><td>{1, 2}</td></tr>
<tr><td rowspan="5">Smoothing Continuation Method</td><td colspan="2">Initial point</td><td>(3, 0, 0)</td></tr>
<tr><td rowspan="2">$\epsilon_0 = 10^{-2}$</td><td>x^0</td><td>(0.0730,0.4335,0.2367,-0.4022, -0.2890,-0.5841,-0.4399, 0.2314)</td></tr>
<tr><td>Ite</td><td>24</td></tr>
<tr><td rowspan="2">$\epsilon_1 = 10^{-4}$</td><td>x^1</td><td>(0.0724,0.4345,0.2386,-0.4003, -0.2870,-0.5857,-0.4421, 0.2286)</td></tr>
<tr><td>Ite</td><td>19</td></tr>
</table>

We denote by $(\hat{x}^k, \hat{y}^k)$ and (x^k, y^k) the points obtained by the hybrid algorithm and the smoothing continuation method at iteration k, respectively, and denote by Ite the number of iterations spent by the solver $fmincon$. Especially, for the hybrid algorithm, $Ite = n_1 + n_2$ means that n_1 and n_2 are the numbers of the iterations spent by $fmincon$ to solve problems (P_{ϵ_k}) and (3.6), respectively. In addition, β^k denotes the lower-level degenerate index set estimated at the point $(\hat{x}^k, \hat{y}^k)$.

The results shown in the tables reveal that it was not difficult to identify the active sets by the hybrid algorithm, at least for the test problems used in our numerical experiments. In fact, we got the correct active sets in only one step for Problem 4.1 and in two steps for Problems 4.2 and 4.3. Moreover, since the computed points satisfy the B-stationarity conditions, the hybrid algorithm terminated.

5 Conclusions

By applying an active set identification technique to a smoothing continuation method, we have presented a hybrid method for MPCC which enables us to compute a solution or a point with some kind of stationarity to problem (P) by solving a finite number of nonlinear programs. The proposed approach is different from the

Table 4: Computational results for Problem 4.2

Hybrid Algorithm	Initial point		(0.5, $\cdots$, 0.5)
	$\epsilon_0 = 10^{-2}$	$\hat{x}^0$	(0.7361,0.4854,0.3201,-0.6325,-0.6232, -0.6439,-0.1454,-0.0451,-0.2336,0.2999)
		Ite	86+10
		β^0	$\{2, 4, 7\}$
	$\epsilon_1 = 10^{-4}$	$\hat{x}^1$	(0.6368,0.6371,0.1739,-0.7158,-0.8704, -0.7478,-0.4383,0.1886,0.0741,0.1494)
		Ite	28+8
		β^1	$\{1, 2, 3, 4, 5\}$
Smoothing Continuation Method	Initial point		(0.5, $\cdots$, 0.5)
	$\epsilon_0 = 10^{-2}$	x^0	(0.6053,0.7516,0.1275,-0.6123,-0.6650, -0.8600,-0.0628,0.0174,-0.0832,0.2838)
		Ite	86
	$\epsilon_1 = 10^{-4}$	x^1	(0.6358,0.6367,0.1739,-0.7155,-0.8697, -0.7497,-0.4375,0.1888,0.0754,0.1497)
		Ite	28
	$\epsilon_2 = 10^{-6}$	x^2	(0.6366,0.6368,0.1736,-0.7157,-0.8698, -0.7490,-0.4379,0.1885,0.0746,0.1495)
		Ite	29
	$\epsilon_3 = 10^{-8}$	x^3	(0.6370,0.6366,0.1738,-0.7158,-0.8703, -0.7486,-0.4382,0.1886,0.0744,0.1499)
		Ite	42

existing methods [10, 18, 19, 27] that require to solve an infinite sequence of nonlinear programs. However, since those methods share similar convergence properties, problem (P_ϵ) in Step 1 may be replaced by some other subproblems used in [18] or [19] or [27].

The proposed method is based on an identification function satisfying (3.8)–(3.10), by means of which we may obtain (3.7) in a finite number of iterations. Under the assumption of the asymptotically weakly nondegeneracy of the sequence $\{z^k\}$ generated in Step 1, we have shown two identification functions in Theorems 3.5 and 3.6. We remark that, as shown in Lemma 3.2, the asymptotically weak nondegeneracy is implied by the upper-level strict complementarity condition and

Table 5: Computational results for Problem 4.3

Hybrid Algorithm	Initial point		(0.5, $\cdots$, 0.5)
	$\epsilon_0 = 10^{-2}$	$\hat{x}^0$	(-0.1018,0.0167,-0.3091,0.1947,-0.4275,0.0296)
		Ite	17+2
		β^0	$\emptyset$
	$\epsilon_1 = 10^{-4}$	$\hat{x}^1$	(-0.1018,0.0167,-0.3091,0.1947,-0.4275,0.0296)[a]
		Ite	20+2
		β^1	{1, 2, 3, 4, 5, 6}
Smoothing Continuation Method	Initial point		(0.5, $\cdots$, 0.5)
	$\epsilon_0 = 10^{-2}$	x^0	(-0.0915,0.0196,-0.3204,0.2065,-0.4307,0.0359)
		Ite	17
	$\epsilon_1 = 10^{-4}$	x^1	(-0.0991,0.0163,-0.3107,0.1972,-0.4279,0.0324)
		Ite	20
	$\epsilon_2 = 10^{-6}$	x^2	(-0.1030,0.0150,-0.3080,0.1958,-0.4340,0.0289)
		Ite	28
	$\epsilon_3 = 10^{-8}$	x^3	(-0.1002,0.0159,-0.3091,0.1947,-0.4276,0.0294)
		Ite	39

[a] Actually, $(\hat{x}^1, \hat{y}^1)$ equals $(\hat{x}^0, \hat{y}^0)$ approximately, not exactly.

hence, it is not a very stringent assumption. It is still an open question to find a new identification function under weaker conditions.

On the other hand, under the assumption of the asymptotically weakly nondegeneracy of $\{z^k\}$, we may define $\{\alpha^k, \beta^k, \gamma^k\}$ in a different manner in Step 1 of the hybrid algorithm. For example, we may put

$$\alpha^k = \Big\{ i : \ G_i(z^k) > \sqrt{\epsilon_k}, \ \ H_i(z^k) \leq \sqrt{\epsilon_k} \Big\}, \tag{5.1}$$

$$\beta^k = \Big\{ i : \ G_i(z^k) \leq \sqrt{\epsilon_k}, \ \ H_i(z^k) \leq \sqrt{\epsilon_k} \Big\}, \tag{5.2}$$

$$\gamma^k = \Big\{ i : \ G_i(z^k) \leq \sqrt{\epsilon_k}, \ \ H_i(z^k) > \sqrt{\epsilon_k} \Big\} \tag{5.3}$$

instead of (3.11)–(3.13). In fact, the proof of Theorem 3.5 indicates that there exist two positive numbers a and b such that (3.19) holds for each $i \in \beta(\bar{z})$ and each k. Therefore, by the equality in (3.5), we can deduce that

$$G_i(z^k) = O(\epsilon_k), \quad H_i(z^k) = O(\epsilon_k), \qquad i \in \beta(\bar{z}) \tag{5.4}$$

and

$$\min\{G_i(z^k), H_i(z^k)\} = O(\epsilon_k^2), \qquad i \in \alpha(\bar{z}) \cup \gamma(\bar{z}). \tag{5.5}$$

Since $\{\sqrt{\epsilon_k}\}$ approaches zero slower than both $\{\epsilon_k\}$ and $\{\epsilon_k^2\}$, we have from (5.4) and (5.5) that

$$\min\{G_i(z^k), H_i(z^k)\} \le \sqrt{\epsilon_k} \tag{5.6}$$

for each i and all k is sufficiently large. This means (3.24) holds when k is sufficiently large. Noticing that $\{\alpha^k, \beta^k, \gamma^k\}$ defined by (5.1)–(5.3) is mutually disjoint for each k, we have that $\{\alpha^k, \beta^k, \gamma^k\}$ given by (5.1)–(5.3) is a partition of $\{1, \ldots, m\}$ for all k sufficiently large. On the other hand, it is obvious that, when k is sufficiently large,

$$\alpha^k \supseteq \alpha(\bar{z}), \;\; \beta^k \subseteq \beta(\bar{z}), \;\; \gamma^k \supseteq \gamma(\bar{z}).$$

Moreover, it follows from (5.2) and (5.4) that $\beta(\bar{z}) \subseteq \beta^k$ for all k sufficiently large. Note that both $\{\alpha^k, \beta^k, \gamma^k\}$ and $\{\alpha(\bar{z}), \beta(\bar{z}), \gamma(\bar{z})\}$ are partitions of $\{1, \ldots, m\}$. Therefore, (3.7) holds when k is large enough. So, we may employ (5.1)–(5.3) instead of (3.11)–(3.13) in Step 1 of the hybrid algorithm. From the computational point of view, (5.1)–(5.3) are simpler than (3.11)–(3.13).

6 Appendix: Proof of Lemma 3.2

The lemma trivially holds when $\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}) = \emptyset$. So we assume $\mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}) \neq \emptyset$. First, we show that $\bar{z}$ is a weakly stationary point of problem (P). By the continuity of the functions involved and the assumptions that $\{z^k\}$ converges to $\bar{z}$ and the MPCC-LICQ holds at $\bar{z}$, we have that, for all k large enough,

$$\mathcal{I}_G(z^k) \subseteq \mathcal{I}_G(\bar{z}), \;\; \mathcal{I}_H(z^k) \subseteq \mathcal{I}_H(\bar{z}), \;\; \mathcal{I}_g(z^k) \subseteq \mathcal{I}_g(\bar{z}) \tag{6.1}$$

and the set

$$\begin{aligned} &\Big\{\nabla g_l(z^k), \nabla h_r(z^k) : \;\; l \in \mathcal{I}_g(\bar{z}), \; r = 1, \cdots, q\Big\} \\ \cup \;\; &\Big\{\nabla G_i(z^k), \nabla H_i(z^k) : \;\; i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})\Big\} \end{aligned}$$

$$\cup \quad \left\{\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)}\nabla H_i(z^k) : \quad i \in \mathcal{I}_G(\bar z) \setminus \mathcal{I}_H(\bar z)\right\}$$
$$\cup \quad \left\{\nabla H_i(z^k) + \frac{H_i(z^k)}{G_i(z^k)}\nabla G_i(z^k) : \quad i \in \mathcal{I}_H(\bar z) \setminus \mathcal{I}_G(\bar z)\right\} \tag{6.2}$$

is linearly independent. It follows from the stationarity of z^k that there exist Lagrange multiplier vectors $\lambda_g^k \in R^p, \lambda_h^k \in R^q$, and $\lambda_\Phi^k \in R^m$ such that

$$\nabla f(z^k) + \nabla g(z^k)^T\lambda_g^k + \nabla h(z^k)^T\lambda_h^k - \nabla\Phi_{\epsilon_k}(z^k)^T\lambda_\Phi^k = 0, \tag{6.3}$$
$$\lambda_g^k \geq 0, \quad g(z^k)^T\lambda_g^k = 0. \tag{6.4}$$

Note that (3.1) holds for each i and k. Moreover, by (6.1) and (6.4), we have

$$\lambda_{g,l}^k = 0, \quad l \notin \mathcal{I}_g(\bar z).$$

Therefore, the equation (6.3) becomes

$$\begin{aligned}
0 \;=\;& \nabla f(z^k) + \sum_{l\in\mathcal{I}_g(\bar z)} \lambda_{g,l}^k \nabla g_l(z^k) + \nabla h(z^k)^T\lambda_h^k \\
&- \sum_{i\in\mathcal{I}_G(\bar z)\cap\mathcal{I}_H(\bar z)} \left(\frac{\lambda_{\Phi,i}^k H_i(z^k)}{G_i(z^k)+H_i(z^k)}\nabla G_i(z^k) + \frac{\lambda_{\Phi,i}^k G_i(z^k)}{G_i(z^k)+H_i(z^k)}\nabla H_i(z^k)\right) \\
&- \sum_{i\in\mathcal{I}_G(\bar z)\setminus\mathcal{I}_H(\bar z)} \frac{\lambda_{\Phi,i}^k H_i(z^k)}{G_i(z^k)+H_i(z^k)}\left(\nabla G_i(z^k) + \frac{G_i(z^k)}{H_i(z^k)}\nabla H_i(z^k)\right) \\
&- \sum_{i\in\mathcal{I}_H(\bar z)\setminus\mathcal{I}_G(\bar z)} \frac{\lambda_{\Phi,i}^k G_i(z^k)}{G_i(z^k)+H_i(z^k)}\left(\nabla H_i(z^k) + \frac{H_i(z^k)}{G_i(z^k)}\nabla G_i(z^k)\right).
\end{aligned} \tag{6.5}$$

By the linear independence of the set (6.2), we have that all the multiplier sequences

$$\left\{\lambda_{g,l}^k : \; l \in \mathcal{I}_g(\bar z)\right\}, \quad \left\{\lambda_{h,r}^k : \; r = 1,2,\cdots,q\right\},$$
$$\left\{\frac{\lambda_{\Phi,i}^k H_i(z^k)}{G_i(z^k)+H_i(z^k)} : \; i \in \mathcal{I}_G(\bar z)\right\}, \quad \left\{\frac{\lambda_{\Phi,j}^k G_j(z^k)}{G_j(z^k)+H_j(z^k)} : \; j \in \mathcal{I}_H(\bar z)\right\}$$

are convergent. Define $\bar\lambda_g, \bar\lambda_h, \bar\lambda_G$, and $\bar\lambda_H$ as follows:

$$\bar\lambda_{g,l} \;=\; \begin{cases} \lim_{k\to\infty} \lambda_{g,l}^k & , \quad l \in \mathcal{I}_g(\bar z) \\ 0 & , \quad l \notin \mathcal{I}_g(\bar z) \end{cases}; \tag{6.6}$$
$$\bar\lambda_{h,r} \;=\; \lim_{k\to\infty} \lambda_{h,r}^k, \quad r = 1,2,\cdots,q; \tag{6.7}$$
$$\bar\lambda_{G,i} \;=\; \begin{cases} \lim_{k\to\infty} \frac{\lambda_{\Phi,i}^k H_i(z^k)}{G_i(z^k)+H_i(z^k)} & , \quad i \in \mathcal{I}_G(\bar z) \\ 0 & , \quad i \notin \mathcal{I}_G(\bar z) \end{cases}; \tag{6.8}$$
$$\bar\lambda_{H,j} \;=\; \begin{cases} \lim_{k\to\infty} \frac{\lambda_{\Phi,j}^k G_j(z^k)}{G_j(z^k)+H_j(z^k)} & , \quad j \in \mathcal{I}_H(\bar z) \\ 0 & , \quad j \notin \mathcal{I}_H(\bar z) \end{cases}. \tag{6.9}$$

Letting $k \to \infty$ in (6.5) and taking into account (6.4) and (6.6)–(6.9), we have that (2.3)–(2.6) hold, i.e., $\bar{z}$ is a weakly stationary point of problem (P).

Next we suppose that the ULSC holds at $\bar{z}$. It then follows from (2.11) and (6.8)–(6.9) that, for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$,

$$\bar{\lambda}_{G,i} = \lim_{k\to\infty} \frac{\lambda^k_{\Phi,i} H_i(z^k)}{G_i(z^k) + H_i(z^k)} \neq 0, \tag{6.10}$$

$$\bar{\lambda}_{H,i} = \lim_{k\to\infty} \frac{\lambda^k_{\Phi,i} G_i(z^k)}{G_i(z^k) + H_i(z^k)} \neq 0. \tag{6.11}$$

Since both $G_i(z^k)$ and $H_i(z^k)$ are positive for each i and each k, it follows from (6.10) and (6.11) that

$$0 < |\bar{\lambda}_{G,i}| \leq \liminf_{k\to\infty} \ |\lambda^k_{\Phi,i}|$$

and

$$\lim_{k\to\infty} \lambda^k_{\Phi,i} = \bar{\lambda}_{G,i} + \bar{\lambda}_{H,i}$$

for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$. Therefore, $\left\{\lambda^k_{\Phi,i}\right\}$ is convergent to a nonzero number for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$. We then have from (6.10) and (6.11) that, for each $i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z})$,

$$\lim_{k\to\infty} \frac{H_i(z^k)}{G_i(z^k) + H_i(z^k)} = \frac{\bar{\lambda}_{G,i}}{\bar{\lambda}_{G,i} + \bar{\lambda}_{H,i}} \neq 0,$$

$$\lim_{k\to\infty} \frac{G_i(z^k)}{G_i(z^k) + H_i(z^k)} = \frac{\bar{\lambda}_{H,i}}{\bar{\lambda}_{G,i} + \bar{\lambda}_{H,i}} \neq 0.$$

This, together with (3.1), implies that the sequence $\{z^k\}$ is asymptotically weakly nondegenerate. This completes the proof of Lemma 3.2. □

References

[1] J.V. Burke, *On the identification of active constraints II: The nonconvex case*, SIAM Journal on Numerical Analysis, 27 (1990), 1081–1102.

[2] J.V. Burke and J.J. Moré, *On the identification of active constraints*, SIAM Journal on Numerical Analysis, 25 (1988), 1197–1211.

[3] X. Chen and M. Fukushima, *A smoothing method for a mathematical program with P-matrix linear complementarity constraints*, Technical Report 2001-008, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2001.

[4] Y. Chen and M. Florian, *The nonlinear bilevel programming problem: Formulations, regularity and optimality conditions*, Optimization, 32 (1995), 193–209.

[5] R.W. Cottle, J.S. Pang, and R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, NY, 1992.

[6] F. Facchinei, A. Fischer, and C. Kanzow, *On the accurate identification of active constraints*, SIAM Journal on Optimization, 9 (1998), 14–32.

[7] F. Facchinei, A. Fischer, and C. Kanzow, *On the identification of zero variables in an interior-point framework*, SIAM Journal on Optimization, 10 (2000), 1058–1078.

[8] F. Facchinei, H. Jiang, and L. Qi, *A smoothing method for mathematical programs with equilibrium constraints*, Mathematical Programming, 85 (1999), 107–134.

[9] M. Fukushima, Z.Q. Luo, and J.S. Pang, *A globally convergent sequential quadratic programming algorithm for mathematical programs with linear complementarity constraints*, Computational Optimization and Applications, 10 (1998), 5–34.

[10] M. Fukushima and J.S. Pang, *Convergence of a smoothing continuation method for mathematical problems with complementarity constraints*, Illposed Variational Problems and Regularization Techniques, Lecture Notes in Economics and Mathematical Systems, Vol. 477, M. Théra and R. Tichatschke (eds.), Springer-Verlag, Berlin/Heidelberg, 1999, 105–116.

[11] M. Fukushima and J.S. Pang, *Some feasibility issues in mathematical programs with equilibrium constraints*, SIAM Journal on Optimization, 8 (1998), 673–681.

[12] M. Fukushima and P. Tseng, *An implementable active-set algorithm for computing a B-stationary point of the mathematical program with linear complementarity constraints*, SIAM Journal on Optimization, 12 (2002), 724-739.

[13] X. Hu and D. Ralph, *Convergence of a penalty method for mathematical programming with equilibrium constraints*, manuscript, Department of Mathematics and Statistics, University of Melbourne, Australia, 2001.

[14] X.X. Huang, X.Q. Yang, and D.L. Zhu, *A sequential smooth penalization approach to mathematical programs with complementarity constraints*, manuscript, Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong, 2001.

[15] H. Jiang and D. Ralph, *QPECgen, a MATLAB generator for mathematical programs with quadratic objectives and affine variational inequality constraints*, Computational Optimization and Applications, 13 (1999), 25–59.

[16] H. Jiang and D. Ralph, *Smooth SQP methods for mathematical programs with nonlinear complementarity constraints*, SIAM Journal on Optimization, 10 (2000), 779–808.

[17] C. Kanzow and H. Jiang, *A continuation method for (strongly) monotone variational inequalities*, Mathematical Programming, 81 (1998), 103–125.

[18] G.H. Lin and M. Fukushima, *A modified relaxation scheme for mathematical programs with complementarity constraints*, Technical Report 2002-02, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2002.

[19] G.H. Lin and M. Fukushima, *A new relaxation method for mathematical programs with complementarity constraints*, Journal of Optimization Theory and Applications, to appear.

[20] G.H. Lin and M. Fukushima, *Some exact penalty results for nonlinear programs and their applications to mathematical programs with equilibrium constraints*, Journal of Optimization Theory and Applications, to appear.

[21] Z.Q. Luo, J.S. Pang, and D. Ralph, Mathematical Programs with Equilibrium Constraints, Cambridge University Press, Cambridge, United Kingdom, 1996.

[22] Z.Q. Luo, J.S. Pang, D. Ralph, and S.Q. Wu, *Exact penalization and stationary conditions of mathematical programs with equilibrium constraints*, Mathematical Programming, 75 (1996), 19–76.

[23] J.V. Outrata, *On optimization problems with variational inequality constraints*, SIAM Journal on Optimization, 4 (1994), 340–357.

[24] J.V. Outrata and J. Zowe, *A numerical approach to optimization problems with variational inequality constraints*, Mathematical Programming, 68 (1995), 105–130.

[25] J.S. Pang and M. Fukushima, *Complementarity constraint qualifications and simplified B-stationarity conditions for mathematical programs with equilibrium constraints*, Computational Optimization and Applications, 13 (1999), 111–136.

[26] H. Scheel and S. Scholtes, *Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity*, Mathematics of Operations Research, 25 (2000), 1–22.

[27] S. Scholtes, *Convergence properties of a regularization scheme for mathematical programs with complementarity constraints*, SIAM Journal on Optimization, 11 (2001), 918–936.

[28] S. Scholtes and M. Stöhr, *Exact penalization of mathematical programs with complementarity constraints*, SIAM Journal on Control and Optimization, 37 (1999), 617–652.

[29] S. Scholtes and M. Stöhr, *How stringent is the linear independence assumption for mathematical programs with complementarity constraints*, Mathematics of Operations Research, 26 (2001), 851–863.

[30] P. Tseng, *Growth behavior of a class of merit functions for the nonlinear complementarity problem*, Journal of Optimization Theory and Applications, 89 (1996), 17–37.

[31] N. Yamashita, H. Dan, and M. Fukushima, *On the identification of degenerate indices in the nonlinear complementarity problem with the proximal point algorithm*, Technical Report 2001-003, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2001.