Hybrid Algorithms with Index Addition and Subtraction Strategies for Solving Mathematical Programs with Complementarity Constraints *

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Abstract. Recently, the authors have presented a hybrid approach for solving mathematical programs with complementarity constraints. This approach employs an active-set identification function and possesses a finite termination property under some appropriate conditions including the so-called asymptotically weak nondegeneracy. In this paper, we continue this work and propose two modified methods, one of which makes use of an index addition strategy and the other utilizes the converse strategy. Both the methods do not require the above-mentioned assumption and also have a finite termination property under very weak assumptions. Numerical experience shows that the proposed approaches are effective.

Key Words. mathematical program with complementarity constraints, MPCC-LICQ, (B-, S-, M-, C-) stationarity, asymptotically weak nondegeneracy, weak second-order necessary conditions, upper-level strict complementarity condition, identification function.

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1 Introduction

We consider the following mathematical program with complementarity constraints (MPCC):

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g(z) \leq 0,\ h(z) = 0 \\ & G(z) \geq 0,\ H(z) \geq 0 \\ & G(z)^T H(z) = 0, \end{array} \tag{P}$$

where $f : \Re^n \to \Re$, $g : \Re^n \to \Re^p$, $h : \Re^n \to \Re^q$, and $G, H : \Re^n \to \Re^m$ are all twice continuously differentiable functions. This problem plays an important role in many fields such as engineering design, economic equilibrium, multilevel game, and mathematical programming itself, see [19], and has attracted much attention in the recent literature, see the attached references.

A well-known difficulty in solving problem (P) is that the problem fails to satisfy a standard constraint qualification at any feasible point [4], so that the developed theory for conventional nonlinear programs is likely to fail for this problem. There have been proposed a number of approaches such as sequential quadratic programming approach [8, 14, 19], implicit programming approach [3, 19], penalty function approach [11, 12, 18, 19, 20, 24], ϵ-active set approach [10], and reformulation approach [7, 9, 15, 16, 23]. However, these methods require to solve an infinite sequence of nonlinear programs generally.

Recently, Lin and Fukushima [17] proposed a hybrid approach by applying an active set identification technique to some existing methods and showed, under the so-called asymptotically weak nondegeneracy assumption [9] and some other mild conditions, that a point with some kind of stationarity to problem (P) can be found by solving a finite sequence of nonlinear programs. This new approach is therefore interesting and attractive. In this paper, we continue to develop algorithms with a finite termination property. Two modified approaches, one of which uses an index addition strategy and the other utilizes an index subtraction strategy, will be proposed for solving MPCCs. We will show that the new methods do not require the asymptotically weak nondegeneracy assumption and also have a finite termination property under very weak assumptions. Our numerical experience shows that the proposed methods are quite effective.

Organization of the paper: In the next section, we recall some basic concepts for MPCCs and standard nonlinear programming problems. In Section 3, after reviewing the smoothing continuation method [9] and the hybrid approach [17], we describe our new algorithms and present a comprehensive theoretical analysis as well. Some comments and extensions are also given in this section. We report some numerical results in Section 4 and make concluding remarks in Section 5.

Notation: We denote by $\mathcal{F}$ the feasible region of problem (P) throughout the paper and, for a given function $F : \Re^n \to \Re^m$ and a vector $z \in \Re^n$, we let

$$\mathcal{I}_F(z) := \{i \mid \ F_i(z) = 0\}$$

stand for the active index set of F at z. In addition, for two vectors $u, v \in \Re^n$, $\min(u, v)$ is understood to be taken componentwise, i.e.,

$$\min(u, v) = \Big(\min\{u_1, v_1\}, \cdots, \min\{u_m, v_m\}\Big)^T.$$

2 Preliminaries

In this section, we state some basic concepts for MPCCs and standard nonlinear programming problems. We first consider the standard nonlinear programming problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & c_i(z) \leq 0, \quad i = 1, 2, \cdots, l, \\ & c_i(z) = 0, \quad i = l+1, \cdots, s, \end{array} \tag{2.1}$$

where $f : \Re^n \to \Re$ and $c : \Re^n \to \Re^s$ are twice continuously differentiable.

Definition 2.1 We say z to be a *stationary* point of problem (2.1) if it is feasible to (2.1) and there exists a Lagrange multiplier vector $\lambda \in \Re^s$ such that

$$\begin{array}{l} \nabla f(z) + \nabla c(z)\lambda = 0, \\ \lambda_i \geq 0, \ \lambda_i c_i(z) = 0, \quad\ \ i = 1, 2, \cdots, l. \end{array}$$

Definition 2.2 Let z be a stationary point of problem (2.1) and λ be a Lagrange multiplier vector corresponding to z. We say the *weak second-order necessary condition* (WSONC) holds at z if we have

$$d^T \Big(\nabla^2 f(z) + \sum_{i=1}^{s} \lambda_i \nabla^2 c_i(z)\Big) d \geq 0, \quad \forall d \in \mathcal{T}(z),$$

where $\mathcal{T}(z)$ is defined by

$$\mathcal{T}(z) := \Big\{ d \in \Re^n \ | \ \ d^T \nabla c_i(z) = 0, \ \ \forall \ i \in \mathcal{I}_c(z) \Big\}.$$

Next, we recall some concepts related to problem (P).

Definition 2.3 The MPCC-*linear independence constraint qualification* (MPCC-LICQ) is said to hold at $\bar{z} \in \mathcal{F}$ if

$$\Big\{ \nabla g_l(\bar{z}), \nabla h_r(\bar{z}), \nabla G_i(\bar{z}), \nabla H_j(\bar{z}) \ | \\ \ l \in \mathcal{I}_g(\bar{z}), \ r = 1, 2, \cdots, q, \ i \in \mathcal{I}_G(\bar{z}), \ j \in \mathcal{I}_H(\bar{z}) \Big\}$$

is linearly independent.

This condition is not particularly stringent [25] and has been assumed often in the literature on MPCCs [9, 11, 12, 15, 17, 23]. Note that this definition is different from the standard definition of LICQ in nonlinear programming theory that would require the gradient of the function $G(z)^T H(z)$ be linearly independent of the above vectors, which cannot happen in any case actually.

In the study of MPCCs, there are several kinds of stationarity defined for problem (P). Among them, the strongest stationarity is the following one.

Definition 2.4 We say $\bar{z} \in \mathcal{F}$ is a *Bouligand or B-stationary* point of problem (P) if it satisfies

$$d^T \nabla f(\bar{z}) \geq 0, \quad \forall d \in \mathcal{T}(\bar{z}, \mathcal{F}), \tag{2.2}$$

where

$$\mathcal{T}(\bar{z}, \mathcal{F}) := \{d \in \Re^n | \ \ t_k(z^k - \bar{z}) \to d, z^k \to \bar{z}, z^k \in \mathcal{F}, t_k \geq 0, k = 1, 2, \cdots\}$$

stands for the tangent cone of $\mathcal{F}$ at $\bar{z}$.

B-stationary points are obviously candidates for local minimizers of problem (P). However, it may be difficult to obtain such a point for MPCCs in general. Next, we introduce some other kinds of stationarity for problem (P).

Definition 2.5 We say $\bar{z} \in \mathcal{F}$ is *weakly stationary* to problem (P) if there exist multiplier vectors $\bar{\lambda}_g \in \Re^p$, $\bar{\lambda}_h \in \Re^q$, and $\bar{\lambda}_G, \bar{\lambda}_H \in \Re^m$ such that

$$\nabla f(\bar{z}) + \nabla g(\bar{z})\bar{\lambda}_g + \nabla h(\bar{z})\bar{\lambda}_h - \nabla G(\bar{z})\bar{\lambda}_G - \nabla H(\bar{z})\bar{\lambda}_H = 0, \tag{2.3}$$

$$\bar{\lambda}_g \geq 0, \quad \bar{\lambda}_g^T g(\bar{z}) = 0, \tag{2.4}$$

$$\bar{\lambda}_{G,i} = 0, \quad i \notin \mathcal{I}_G(\bar{z}), \tag{2.5}$$

$$\bar{\lambda}_{H,i} = 0, \quad i \notin \mathcal{I}_H(\bar{z}). \tag{2.6}$$

Definition 2.6 Suppose that $\bar{z} \in \mathcal{F}$ is a weakly stationary point of (P) and the multiplier vectors $\bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_G, \bar{\lambda}_H$ satisfy conditions (2.3)–(2.6).

(1) $\bar{z}$ is called a *Clarke or C-stationary* point of (P) if

$$\bar{\lambda}_{G,i}\bar{\lambda}_{H,i} \geq 0, \quad \forall i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}); \tag{2.7}$$

(2) $\bar{z}$ is called a *Mordukhovich or M-stationary* point of (P) if

$$\text{either} \ \ \bar{\lambda}_{G,i} > 0, \ \bar{\lambda}_{H,i} > 0 \ \ \text{or} \ \ \bar{\lambda}_{G,i}\bar{\lambda}_{H,i} = 0, \quad \forall i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}); \tag{2.8}$$

(3) $\bar{z}$ is called a *Strong or S-stationary* point of (P) if

$$\bar{\lambda}_{G,i} \geq 0, \bar{\lambda}_{H,i} \geq 0, \quad \forall i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}). \tag{2.9}$$

Note that, if the MPCC-LICQ holds at $\bar{z}$, then $\bar{z}$ is B-stationary to (P) if and only if it is S-stationary to (P) [22]. In general, in order to get a desirable stationary point of (P), some conditions are assumed. Here, we introduce one of them.

Definition 2.7 A weakly stationary point $\bar{z} \in \mathcal{F}$ of problem (P) is said to satisfy the *upper-level strict complementarity* (ULSC) condition if there exist multiplier vectors $\bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_G$, and $\bar{\lambda}_H$ satisfying (2.3)–(2.6) with

$$\bar{\lambda}_{G,i}\bar{\lambda}_{H,i} \neq 0, \quad i \in \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}). \tag{2.10}$$

3 Modified Hybrid Algorithms for MPCC

In this section, we first briefly recall the smoothing continuation method considered in [9] and the hybrid method proposed in [17]. We then present our new algorithms with comprehensive theoretical analysis. In the following, for a point $\bar{z} \in \mathcal{F}$, we let $\alpha(\bar{z}), \beta(\bar{z})$ and $\gamma(\bar{z})$ be the index sets defined by

$$\begin{array}{rcl}
\alpha(\bar{z}) & := & \Big\{ i \mid \; G_i(\bar{z}) > 0, \; H_i(\bar{z}) = 0 \Big\} \; = \; \{1, 2, \cdots, m\} \setminus \mathcal{I}_G(\bar{z}), \\
\beta(\bar{z}) & := & \Big\{ i \mid \; G_i(\bar{z}) = 0, \; H_i(\bar{z}) = 0 \Big\} \; = \; \mathcal{I}_G(\bar{z}) \cap \mathcal{I}_H(\bar{z}), \\
\gamma(\bar{z}) & := & \Big\{ i \mid \; G_i(\bar{z}) = 0, \; H_i(\bar{z}) > 0 \Big\} \; = \; \{1, 2, \cdots, m\} \setminus \mathcal{I}_H(\bar{z}),
\end{array}$$

respectively. It is obvious that $\{\alpha(\bar{z}), \beta(\bar{z}), \gamma(\bar{z})\}$ is a partition of the set $\{1, 2, \cdots, m\}$.

3.1 Smoothing continuation method

The smoothing continuation method [9] uses the function $\Phi_\epsilon : \Re^n \to \Re^m$ defined by

$$\Phi_\epsilon(z) := \Big(\phi_\epsilon(G_1(z), H_1(z)), \cdots, \phi_\epsilon(G_m(z), H_m(z)) \Big)^T,$$

where $\epsilon \geq 0$ and $\phi_\epsilon : \Re^2 \to \Re$ is the perturbed Fischer-Burmeister function

$$\phi_\epsilon(a, b) := a + b - \sqrt{a^2 + b^2 + 2\epsilon^2},$$

and employs the problem

$$\begin{array}{ll}
\min & f(z) \\
\text{s.t.} & \Phi_\epsilon(z) = 0, \\
 & g(z) \leq 0, \; h(z) = 0
\end{array} \qquad (P_\epsilon)$$

as an approximate problem of (P). Note that $\Phi_\epsilon(z) = 0$ if and only if

$$G_i(z) \geq 0, \; H_i(z) \geq 0, \; G_i(z) H_i(z) = \epsilon^2, \quad i = 1, \cdots, m. \qquad (3.1)$$

Thus problem (P_0) is actually equivalent to the original problem (P). Moreover, the function Φ_ϵ is differentiable everywhere when $\epsilon > 0$. We assume that problem (P_ϵ) has a stationary point for each small scalar $\epsilon > 0$. The following convergence result is given in [9].

Theorem 3.1 *Let z^k be a stationary point of problem (P_{ϵ_k}) and the generated sequence $\{z^k\}$ converge to a point z^* as $\epsilon_k \to 0$. Suppose that the WSONC holds at each z^k, the MPCC-LICQ holds at z^*, and $\{z^k\}$ is asymptotically weakly nondegenerate. Then z^* is B-stationary to problem (P).*

Roughly speaking, the asymptotically weak nondegeneracy of $\{z^k\}$ means that, for each $i \in \mathcal{I}_G(z^*) \cap \mathcal{I}_H(z^*)$, $G_i(z^k)$ and $H_i(z^k)$ approach zero in the same order of magnitude. In particular, this condition is implied by the ULSC condition. See [9] or [17] for details.

Moreover, like in [11, 12, 15, 16, 23], we have the following well-known result, whose proof is omitted here.

Theorem 3.2 *Let z^k be stationary to problem (P_{ϵ_k}) and the sequence $\{z^k\}$ converge to z^* as $\epsilon_k \to 0$. Suppose that the MPCC-LICQ holds at z^*. Then*

(i) *z^* is C-stationary to problem (P);*

(ii) *if the WSONC holds at each z^k, z^* is M-stationary to problem (P);*

(iii) *if the WSONC holds at each z^k and the ULSC holds at z^*, z^* is B-stationary to problem (P).*

See [9] and [17] for more results about the smoothing continuation method.

3.2 Hybrid method with active set identification

In this subsection, we briefly recall the hybrid approach presented in [17]. Throughout Subsections 3.2, 3.3, and 3.4, we let $\{\epsilon_k\}$ be a positive sequence converging to zero and, for each k, let z^k stand for a stationary point of problem (P_{ϵ_k}). Also, we suppose that the sequence $\{z^k\}$ converges to z^*. It then follows that $z^* \in \mathcal{F}$.

The main issue in [17] is to identify the index sets $\alpha(z^*), \beta(z^*)$, and $\gamma(z^*)$ in finite steps. To this end, by means of an identification function $\rho : \Re^n \to [0, +\infty)$ satisfying

$$\lim_{k\to\infty} \rho(z^k) \ = \ 0 \tag{3.2}$$

and, for all k large enough,

$$\max_{i\in\beta(z^*)} \Big\{G_i(z^k), H_i(z^k)\Big\} \ \leq \ \rho(z^k), \tag{3.3}$$

$$\max_{i\in\alpha(z^*)\cup\gamma(z^*)} \Big\{\min\{G_i(z^k), H_i(z^k)\}\Big\} \ \leq \ \rho(z^k), \tag{3.4}$$

we define from z^k some index sets α^k, β^k, and γ^k that constitute a partition of $\{1, \cdots, m\}$, see (3.8)–(3.10). By the analysis in [17], conditions (3.2)–(3.4) ensure that

$$\alpha^k = \alpha(z^*),\ \beta^k = \beta(z^*),\ \gamma^k = \gamma(z^*) \tag{3.5}$$

hold for all k large enough. At each iteration, for the given index sets α^k, β^k and γ^k, we solve the nonlinear programming problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & G_i(z) \geq 0,\ H_i(z) = 0, \quad i \in \alpha^k, \\ & G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta^k, \\ & G_i(z) = 0,\ H_i(z) \geq 0, \quad i \in \gamma^k, \\ & g(z) \leq 0,\ h(z) = 0. \end{array} \tag{3.6}$$

Since problem (3.6) is no longer an MPCC, it is therefore easier to deal with. Denote by $\hat{z}^k$ a stationary point of problem (3.6). Note that $\hat{z}^k \in \mathcal{F}$ and $\beta^k \subseteq \beta(\hat{z}^k)$. If the Lagrange multipliers corresponding to the constraints

$$\begin{array}{ll} G_i(z) \geq 0,\ H_i(z) = 0, & i \in \alpha^k \cap \beta(\hat{z}^k), \\ G_i(z) = 0,\ H_i(z) = 0, & i \in \beta^k, \\ G_i(z) = 0,\ H_i(z) \geq 0, & i \in \gamma^k \cap \beta(\hat{z}^k) \end{array} \tag{3.7}$$

are all nonnegative, then $\hat{z}^k$ is a B-stationary point of the original problem (P) under the MPCC-LICQ assumption at the point [10]. Therefore, assuming that each subproblem (3.6) can be solved exactly, we may identify the sets $\alpha(z^*), \beta(z^*), \gamma(z^*)$ in finite steps with the help of a suitable function ρ and furthermore, we may solve the original problem in a finite number of iterations. Here, we state one of the hybrid algorithms proposed in [17].

Hybrid Algorithm — Algorithm H:

Step 0: Choose $\epsilon_0 > 0$ and set $k := 0$.

Step 1: Solve problem (P_{ϵ_k}) and denote by z^k one of its stationary points. Set

$$\alpha^k := \Big\{ i \mid \; G_i(z^k) > \rho(z^k), \;\; H_i(z^k) \le \rho(z^k) \Big\}, \tag{3.8}$$

$$\beta^k := \Big\{ i \mid \; G_i(z^k) \le \rho(z^k), \;\; H_i(z^k) \le \rho(z^k) \Big\}, \tag{3.9}$$

$$\gamma^k := \Big\{ 1, 2, \cdots, m \Big\} \setminus \Big(\alpha^k \cup \beta^k \Big) \tag{3.10}$$

and go to Step 2.

Step 2: If problem (3.6) is solvable, denote by $\hat{z}^k$ one of its stationary points and go to Step 3. Otherwise, go to Step 4.

Step 3: If the Lagrange multipliers corresponding to the constraints (3.7) in problem (3.6) are all nonnegative, then terminate. Else, go to Step 4.

Step 4: Choose an $\epsilon_{k+1} \in (0, \epsilon_k)$ and let $k := k + 1$. Go to Step 1.

The key issue is, of course, how to define an identification function ρ for Algorithm H. This is not a trivial task because it may depend on the unknown point z^* generally. The following theorem [17] provides two examples of identification functions.

Theorem 3.3 *Suppose that the sequence $\{z^k\}$ generated by Algorithm H is asymptotically weakly nondegenerate. Then, for two numbers $\tau > 0$ and $\sigma \in (0, 1)$, both*

$$\rho_1(z) := \tau \| \min(G(z), H(z)) \|^\sigma \tag{3.11}$$

and

$$\rho_2(z) := \tau \| \Phi_0(z) \|^\sigma \tag{3.12}$$

can serve as identification functions for Algorithm H.

The next theorem is the main result in [17].

Theorem 3.4 *Let ρ be a function defined by (3.11) or (3.12) with $\sigma \in (0, 1)$ and $\tau > 0$. Suppose that the sequence $\{z^k\}$ of stationary points of problems (P_{ϵ_k}) converges to z^* as $\epsilon_k \to 0$, the MPCC-LICQ holds at z^*, and the sequence $\{z^k\}$ is asymptotically weakly nondegenerate. Then, for all k sufficiently large, we have*

(i) *problem (P_{ϵ_k}) satisfies the standard LICQ at z^k;*

(ii) *the sets α^k, β^k, and γ^k defined by (3.8), (3.9), and (3.10) satisfy (3.5).*

This theorem indicates that, under some mild conditions, the index sets $\alpha(z^*), \beta(z^*)$, and $\gamma(z^*)$ may be identified correctly and hence subproblem (3.6) coincides with the following problem for some k:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & G_i(z) \geq 0,\ H_i(z) = 0, \quad i \in \alpha(z^*), \\ & G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta(z^*), \\ & G_i(z) = 0,\ H_i(z) \geq 0, \quad i \in \gamma(z^*), \\ & g(z) \leq 0,\ h(z) = 0. \end{array} \tag{3.13}$$

Then we may expect that Algorithm H terminates finitely by producing a B-stationary point of the original problem (P).

3.3 Modified hybrid method with index addition strategy

As indicated in the last subsection, the asymptotically weak nondegeneracy condition is a key assumption in [17]. This assumption is not excessively stringent because it is implied by the upper-level strict complementarity (ULSC) condition [17]. Nevertheless, it is certainly desirable to lessen the required assumptions. In this subsection, we introduce a modified approach with index addition strategy and in the next subsection, we describe another approach with the converse strategy. Both of these two methods do not require the assumption of asymptotically weak nondegeneracy.

Let ρ be a function defined by (3.11) or (3.12) with $\tau > 0$ and $\sigma \in (0, 1)$.

Hybrid Algorithm with Index Addition Strategy — Algorithm HIA:

Step 0: Choose $\theta_0 > 0$ and $\epsilon_0 > 0$. Set $k := 0$.

Step 1: Solve problem (P_{ϵ_k}) to obtain a stationary point z^k. Set

$$\alpha_0^k := \Big\{ i \mid \quad G_i(z^k) > \rho(z^k), \quad H_i(z^k) \leq \rho(z^k) \Big\}, \tag{3.14}$$

$$\beta_0^k := \Big\{ i \mid \quad G_i(z^k) \leq \rho(z^k), \quad H_i(z^k) \leq \rho(z^k) \Big\}, \tag{3.15}$$

$$\gamma_0^k := \Big\{ 1, 2, \cdots, m \Big\} \setminus \Big(\alpha_0^k \cup \beta_0^k \Big), \tag{3.16}$$

$$\delta_0^k := +\infty,$$

$$j := 0$$

and go to Step 2.

Step 2: If the problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & G_i(z) \geq 0,\ H_i(z) = 0, \quad i \in \alpha_j^k, \\ & G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta_j^k, \\ & G_i(z) = 0,\ H_i(z) \geq 0, \quad i \in \gamma_j^k, \\ & g(z) \leq 0,\ h(z) = 0 \end{array} \tag{3.17}$$

is solvable, let $\hat{z}_j^k$ denote one of its stationary points and go to Step 3. Otherwise, go to Step 4.

Step 3: If, in (3.17), the Lagrange multipliers corresponding to the constraints

$$\begin{aligned} &G_i(z) \geq 0,\ H_i(z) = 0, \quad i \in \alpha_j^k \cap \beta(\hat{z}_j^k), \\ &G_i(z) = 0,\ H_i(z) = 0, \quad i \in \beta_j^k, \\ &G_i(z) = 0,\ H_i(z) \geq 0, \quad i \in \gamma_j^k \cap \beta(\hat{z}_j^k) \end{aligned} \tag{3.18}$$

are all nonnegative, then terminate. Else, if there is an $\hat{\imath} \in \alpha_j^k \cup \gamma_j^k$ such that

$$\min_{i \in \alpha_j^k \cup \gamma_j^k} \left\{ \max \left\{ G_i(z^k), H_i(z^k) \right\} \right\} = \max \left\{ G_{\hat{\imath}}(z^k), H_{\hat{\imath}}(z^k) \right\} < \theta_k, \tag{3.19}$$

then set

$$\delta_{j+1}^k := \min \left\{ \delta_j^k,\ \frac{1}{2} \min \left\{ G_i(\hat{z}_j^k) + H_i(\hat{z}_j^k) \mid \quad i \in \alpha(\hat{z}_j^k) \cup \gamma(\hat{z}_j^k) \right\} \right\} \tag{3.20}$$

and

$$\begin{aligned} \alpha_{j+1}^k &:= \alpha_j^k \setminus \{\hat{\imath}\}, && (3.21) \\ \beta_{j+1}^k &:= \beta_j^k \cup \{\hat{\imath}\}, && (3.22) \\ \gamma_{j+1}^k &:= \gamma_j^k \setminus \{\hat{\imath}\}, && (3.23) \\ j &:= j+1 \end{aligned}$$

and go to Step 2. Otherwise, let $j_k := j$ and $\delta_k := \delta_j^k$ and go to Step 4.

Step 4: Choose an $\epsilon_{k+1} \in (0, \epsilon_k)$ and set $\theta_{k+1} := \min \left\{ \theta_k,\ \delta_k \right\}$. Let $k := k+1$ and go to Step 1.

The next theorem describes the relations between the sets $\{\alpha_j^k, \beta_j^k, \gamma_j^k\},\ j = 0, \cdots, j_k$, and $\{\alpha(z^*), \beta(z^*), \gamma(z^*)\}$.

Theorem 3.5 *Suppose that the sequence $\{z^k\}$ generated by Algorithm HIA converges to z^* as $\epsilon_k \to 0$. Then there is an integer $k_0 \geq 0$ such that, for any $k \geq k_0$,*
(i) *$\alpha_0^k \supseteq \alpha(z^*),\ \beta_0^k \subseteq \beta(z^*),\ \gamma_0^k \supseteq \gamma(z^*)$;*
(ii) *if $\beta(z^*) = \emptyset$, namely, z^* is nondegenerate, then we have*

$$\alpha_0^k = \alpha(z^*), \tag{3.24}$$

$$\beta_0^k = \emptyset, \tag{3.25}$$

$$\gamma_0^k = \gamma(z^*); \tag{3.26}$$

(iii) *if $\beta(z^*) \neq \emptyset$ and, for some j, β_j^k is a proper subset of $\beta(z^*)$, then any index $\hat{\imath} \in \alpha_j^k \cup \gamma_j^k$ satisfying (3.19) belongs to $\beta(z^*)$ and hence we have*

$$\alpha_0^k \supseteq \alpha_1^k \supseteq \cdots \supseteq \alpha_j^k \supseteq \alpha_{j+1}^k \supseteq \alpha(z^*), \tag{3.27}$$

$$\beta_0^k \subseteq \beta_1^k \subseteq \cdots \subseteq \beta_j^k \subseteq \beta_{j+1}^k \subseteq \beta(z^*), \tag{3.28}$$

$$\gamma_0^k \supseteq \gamma_1^k \supseteq \cdots \supseteq \gamma_j^k \supseteq \gamma_{j+1}^k \supseteq \gamma(z^*). \tag{3.29}$$

Proof: First of all, we note that

$$\begin{aligned}
&\alpha_0^k \supseteq \alpha_1^k \supseteq \cdots \supseteq \alpha_j^k \supseteq \alpha_{j+1}^k,\\
&\beta_0^k \subseteq \beta_1^k \subseteq \cdots \subseteq \beta_j^k \subseteq \beta_{j+1}^k,\\
&\gamma_0^k \supseteq \gamma_1^k \supseteq \cdots \supseteq \gamma_j^k \supseteq \gamma_{j+1}^k
\end{aligned}$$

by (3.21)–(3.23). Next we show the existence of the integer k_0. We only consider the case where $\rho = \rho_2$. We may deal with the other case similarly.

Since $\{z^k\}$ converges to $z^* \in \mathcal{F}$, we have from the continuity of the function ρ that

$$\lim_{k\to\infty} \rho(z^k) = 0 \tag{3.30}$$

and hence, for each $i \in \alpha(z^*) \cup \gamma(z^*)$,

$$\begin{aligned}
\max\Big\{G_i(z^k), H_i(z^k)\Big\} &> \rho(z^k) \quad \Big(:= \tau\|\Phi_0(z^k)\|^\sigma\Big) && (3.31)\\
&\geq 2\ \|\Phi_0(z^k)\|\\
&\geq 2\phi_0(G_i(z^k), H_i(z^k))\\
&= 2\Big(G_i(z^k) + H_i(z^k) - \sqrt{(G_i(z^k))^2 + (H_i(z^k))^2}\Big)\\
&= \frac{4G_i(z^k)H_i(z^k)}{G_i(z^k) + H_i(z^k) + \sqrt{(G_i(z^k))^2 + (H_i(z^k))^2}}\\
&\geq \min\Big\{G_i(z^k), H_i(z^k)\Big\} && (3.32)
\end{aligned}$$

when k is large enough, where the second inequality follows from the fact that $\{\|\Phi_0(z^k)\|\}$ is convergent to 0 and the constant σ lies in the interval $(0, 1)$, and the last inequality follows from (3.1) and the fact that

$$\begin{aligned}
G_i(z^k) + H_i(z^k) + \sqrt{(G_i(z^k))^2 + (H_i(z^k))^2} &\leq 2(G_i(z^k) + H_i(z^k))\\
&\leq 4\max\Big\{G_i(z^k), H_i(z^k)\Big\}.
\end{aligned}$$

Thus, we have from (3.31)–(3.32) and the continuity of the functions G and H that, for any k sufficiently large,

$$G_i(z^k) > \rho(z^k), \quad H_i(z^k) \leq \rho(z^k), \qquad i \in \alpha(z^*), \tag{3.33}$$

$$G_i(z^k) \leq \rho(z^k), \quad H_i(z^k) > \rho(z^k), \qquad i \in \gamma(z^*). \tag{3.34}$$

Note that (3.14)–(3.16) imply

$$\gamma_0^k \supseteq \Big\{\ i \mid \quad G_i(z^k) \leq \rho(z^k), \quad H_i(z^k) > \rho(z^k)\ \Big\} \tag{3.35}$$

for each k. Moreover, it is obvious from (3.30) that $\beta_0^k \subseteq \beta(z^*)$ when k is sufficiently large. It then follows from (3.14)–(3.16) and (3.33)–(3.35) that there exists an integer $k_1 \geq 0$ such that, for any $k \geq k_1$,

$$\alpha_0^k \supseteq \alpha(z^*),\ \ \beta_0^k \subseteq \beta(z^*),\ \ \gamma_0^k \supseteq \gamma(z^*). \tag{3.36}$$

If z^* is nondegenerate, then (3.24)–(3.26) follow from (3.36) and the fact that both $\{\alpha_0^k, \beta_0^k, \gamma_0^k\}$ and $\{\alpha(z^*), \beta(z^*), \gamma(z^*)\}$ are partitions of the set $\{1, 2, \cdots, m\}$. Therefore, (i) and (ii) hold for all $k \geq k_1$.

Suppose $\beta(z^*) \neq \emptyset$. If $\beta(z^*) = \{1, 2, \cdots, m\}$, then we have $\beta_{j+1}^k \subseteq \beta(z^*)$ for any k. Next we suppose that $\beta(z^*)$ is a proper subset of $\{1, 2, \cdots, m\}$. We will show that, when k is large enough, if there is an $\hat{i} \in \alpha_j^k \cup \gamma_j^k$ satisfying (3.19) and β_j^k is a proper subset of $\beta(z^*)$, then $\hat{i} \in \beta(z^*)$. Note that

$$\lim_{k\to\infty} \max\Big\{G_i(z^k), H_i(z^k)\Big\} = G_i(z^*) + H_i(z^*) > 0, \qquad i \notin \beta(z^*), \tag{3.37}$$

$$\lim_{k\to\infty} \max\Big\{G_i(z^k), H_i(z^k)\Big\} = 0, \qquad i \in \beta(z^*). \tag{3.38}$$

We have from (3.37) and (3.38) that there exists an integer $k_0 \geq k_1$ such that, for any $k \geq k_0$,

$$\max_{i\in\beta(z^*)} \Big\{ \max\Big\{G_i(z^k), H_i(z^k)\Big\}\Big\} < \min_{i\notin\beta(z^*)} \Big\{ \max\Big\{G_i(z^k), H_i(z^k)\Big\}\Big\}. \tag{3.39}$$

This inequality means that the index $\hat{i}$ satisfying (3.19) must be in $\beta(z^*)$ as long as β_j^k is a proper subset of $\beta(z^*)$, namely,

$$\Big(\alpha_j^k \cup \gamma_j^k\Big) \cap \beta(z^*) = \beta(z^*) \setminus \beta_j^k \neq \emptyset.$$

By (3.22), we have $\beta_{j+1}^k \subseteq \beta(z^*)$ and therefore, (3.27)–(3.29) hold for all $k \geq k_0$. Note that, since $k_0 \geq k_1$, (i) and (ii) also hold for all $k \geq k_0$. This completes the proof. □

We proceed to analyzing convergence properties of Algorithm HIA in detail. First, we make the following assumption:

A1: Even if the identical subproblem appears in Step 2 at infinitely many iterations and this problem may have an infinite number of solutions, we always obtain the same solution, or at most finitely many different solutions.

This assumption seems reasonable in practice, since an iterative method applied to solve a subproblem will generate an identical sequence as long as the same starting point is chosen.

In Algorithm HIA, for each k, the parameter θ_k is expectantly used as a positive lower bound of

$$\min_{i\notin\beta(z^*)} \max\Big\{G_i(z^*), H_i(z^*)\Big\} = \min_{i\notin\beta(z^*)} \Big(G_i(z^*) + H_i(z^*)\Big) > 0. \tag{3.40}$$

In fact, a key technique for obtaining a finite termination property of Algorithm HIA is to choose the index sets α_j^k, β_j^k and γ_j^k so that

$$\alpha_{j'_k}^k = \alpha(z^*),\ \beta_{j'_k}^k = \beta(z^*),\ \gamma_{j'_k}^k = \gamma(z^*) \tag{3.41}$$

hold for some index $j'_k \in \{0, 1, \cdots, j_k\}$ when k is large enough. In order to ensure this, the number θ_k needs to be small enough to exclude all the indices in $\alpha(z^*) \cup \gamma(z^*)$ from $\beta_{j'_k}^k$ for all k sufficiently large. Another requirement is that all the indices in $\beta(z^*)$ remain in $\beta_{j'_k}^k$ when k is sufficiently large.

Since the index set $\{1, 2, \cdots, m\}$ has a finite number of partitions, there are a finite number of subproblems (3.17). By assumption A1, the set

$$S := \left\{ \hat{z}_j^k \mid \quad 0 \le j \le j_k, \ k = 0, 1, \cdots \right\} \tag{3.42}$$

is a finite set. Recall that $\hat{z}_j^k \in \mathcal{F}$ for any k and j. We consider the following two cases.

Case I: $\bigcup_{\hat{z}_j^k \in S} \left(\alpha(\hat{z}_j^k) \cup \gamma(\hat{z}_j^k) \right) \neq \emptyset$. In this case, we have

$$\min_{\hat{z}_j^k \in S} \left\{ G_i(\hat{z}_j^k) + H_i(\hat{z}_j^k) \mid \quad i \in \alpha(\hat{z}_j^k) \cup \gamma(\hat{z}_j^k) \right\} > 0,$$

since S is a finite set. It then follows from (3.20) and the way of updating δ_k in Step 3 that the parameter δ_k stays at a positive constant when k is sufficiently large. So, by the updating rule of θ_k, there exists an integer $\bar{k} > 0$ such that

$$\theta_k = \theta_{\bar{k}} > 0, \quad k \ge \bar{k}. \tag{3.43}$$

Since

$$\lim_{k \to \infty} \max \left\{ G_i(z^k), H_i(z^k) \right\} = 0, \quad \forall i \in \beta(z^*),$$

we have

$$\max \left\{ G_i(z^k), H_i(z^k) \right\} < \theta_{\bar{k}}, \quad \forall i \in \beta(z^*) \tag{3.44}$$

for all sufficiently large k. Moreover, by the definition of $\beta(z^*)$, we have

$$\max_{i \in \beta(z^*)} \left\{ \max \left\{ G_i(z^k), H_i(z^k) \right\} \right\} < \min_{i \notin \beta(z^*)} \left\{ \max \left\{ G_i(z^k), H_i(z^k) \right\} \right\} \tag{3.45}$$

for any k large enough. Taking into account (3.19), we deduce from (3.44) and (3.45) that the indices in $\beta(z^*)$ is inevitably included by some β_j^k and, in Step 3, these indices must be chosen earlier than the indices in $\alpha(z^*) \cup \gamma(z^*)$. As a result, there is some $j \in \{0, 1, \cdots, j_k\}$ such that $\beta(z^*) \subseteq \beta_j^k$ whenever k is sufficiently large. This together with Theorem 3.5 means that, when k is large sufficiently, there must be some index $j'_k \in \{0, 1, \cdots, j_k\}$ satisfying (3.41), and then problem (3.17) with $j = j'_k$ is actually equivalent to problem (3.13). As long as a solution of problem (3.17) with $j = j'_k$ yields a B-stationary point of problem (P), Algorithm HIA may terminate in a finite number of iterations. Of course, it may happen that the algorithm stops by getting another B-stationary point of (P) before we identify the correct index sets.

Furthermore, we make the following assumption, which is most likely to hold when problem (P) has finitely many B-stationary points:

A2: The limit point z^* of $\{z^k\}$, which is a sequence of stationary points of (P_{ϵ_k}), is a B-stationary point of problem (P) and it belongs to the set S given by (3.42).

Then, it follows from (3.20) and the updating rule of θ_k that the number $\theta_{\bar{k}}$ in (3.43) is actually a positive lower bound of (3.40). Thus, our requirements are fulfilled: That is, in Step 3,

(a) all the indices in $\alpha(z^*) \cup \gamma(z^*)$ are excluded from $\beta_{j_k}^k$ eventually;

(b) all the indices in $\beta(z^*)$ remain in $\beta^k_{j_k}$ eventually.
Therefore, we are able to identify the correct index sets $\alpha(z^*), \beta(z^*)$, and $\gamma(z^*)$ in a finite number of iterations and furthermore, we may terminate the algorithm finitely by getting z^*. Note that Algorithm HIA may stop prematurely by producing another B-stationary point of problem (P).

Case II: $\alpha(\hat{z}^k_j) \cup \gamma(\hat{z}^k_j) = \emptyset,\ \forall \hat{z}^k_j \in S.$ In this case, we have from the updating rules of δ_k and θ_k that δ_k remains to be $+\infty$ and so

$$\theta_k \equiv \theta_0, \qquad \forall k.$$

Since the strategy in Algorithm HIA is to add some indices to β^k_j one after another, by Theorem 3.5, we have the same conclusion as in Case I: When k is sufficiently large, there exists some index $j'_k \in \{0, 1, \cdots, j_k\}$ satisfying (3.41). Furthermore, suppose that Assumption A2 holds. In the present case, this means

$$\beta(z^*) = \{1, 2, \cdots, m\}.$$

Recall that $\theta_k \equiv \theta_0$ for all k. As a result, all indices should be chosen to be in $\beta^k_{j_k}$ eventually when k becomes large enough, i.e., we also can identify the index sets $\alpha(z^*), \beta(z^*)$, and $\gamma(z^*)$ in a finite number of steps.

The following result [9] indicates that problem (P_ϵ) satisfies the standard LICQ under appropriate conditions.

Theorem 3.6 *If the MPCC-LICQ holds at z^*, then there exist a neighborhood $U(z^*)$ of z^* and a positive constant $\bar{\epsilon}$ such that, for any $\epsilon \in (0, \bar{\epsilon})$, problem (P_ϵ) satisfies the standard LICQ at any feasible point in $U(z^*)$.*

The preceding analysis together with Theorem 3.6 yields the following concluding result.

Theorem 3.7 *Suppose that the sequence $\{z^k\}$ generated by Algorithm HIA converges to z^* as $\epsilon_k \to 0$ and the MPCC-LICQ holds at z^*. Then, under Assumption A1, we have that, for any sufficiently large k,*

(i) *problem (P_{ϵ_k}) satisfies the standard LICQ at z^k;*

(ii) *there exists some index $j'_k \in \{0, 1, \cdots, j_k\}$ such that $\alpha^k_{j'_k}, \beta^k_{j'_k}$, and $\gamma^k_{j'_k}$ satisfy condition (3.41).*

If furthermore, z^* belongs to the set S given by (3.42), then

$$\alpha^k_{j_k} = \alpha(z^*),\ \beta^k_{j_k} = \beta(z^*),\ \gamma^k_{j_k} = \gamma(z^*) \tag{3.46}$$

hold for all k sufficiently large.

In consequence, without the assumption of asymptotically weak nondegeneracy, we have attained the same target as in [17], which is to identify the index sets $\alpha(z^*), \beta(z^*)$, and $\gamma(z^*)$ finitely. Thus, as we discussed above, Algorithm HIA terminates in a finite number of iterations by producing a B-stationary point of problem (P).

On the other hand, if the sequence $\{z^k\}$ is asymptotically weakly nondegenerate, the sets $\alpha^k_0, \beta^k_0, \gamma^k_0$ given in Step 1 of Algorithm HIA are the same as the sets $\alpha^k, \beta^k, \gamma^k$ given in Step 1 of Algorithm H when k is large enough. Theorem 3.4(ii) immediately yields the following result.

Theorem 3.8 *Suppose that the sequence $\{z^k\}$ generated by Algorithm HIA converges to z^* as $\epsilon_k \to 0$, the MPCC-LICQ holds at z^*, and the sequence $\{z^k\}$ is asymptotically weakly nondegenerate. Then the sets α_0^k, β_0^k, and γ_0^k satisfy (3.41) with $j'_k = 0$ for all sufficiently large k.*

The main strategy in Algorithm HIA is to add those indices to β_j^k which are chosen from $\alpha_j^k \cup \gamma_j^k$. Since $\alpha_j^k \cup \gamma_j^k$ contains $\beta(z^*)$ for any k large enough, condition (3.41) holds for some $j'_k \in \{0, 1, \cdots, j_k\}$ when k is large enough. In order to ensure this, the inclusions in (3.36) are necessary. In the above discussion, we suppose that $\tau > 0$ and $\sigma \in (0, 1)$, just as in [17]. Actually, from the proof of Theorem 3.5, (3.36) remains true for the case where $\tau \geq 2$ and $\sigma = 1$ and furthermore, so do Theorems 3.5 and 3.7. Moreover, since the functions G and H play a symmetric role in problem (P), we may exchange the definitions of α_0^k and γ_0^k in Step 1 of Algorithm HIA, namely, let

$$
\begin{aligned}
\gamma_0^k &:= \Big\{ i \mid \ \ G_i(z^k) \leq \rho(z^k), \ \ H_i(z^k) > \rho(z^k) \Big\}, \\
\alpha_0^k &:= \Big\{ 1, 2, \cdots, m \Big\} \setminus \Big(\beta_0^k \cup \gamma_0^k \Big)
\end{aligned}
$$

instead of (3.16) and (3.14), respectively.

3.4 Modified hybrid method with index subtraction strategy

In this subsection, we consider another hybrid algorithm that adopts an index subtraction strategy. One advantage of this algorithm is that the function ρ employed in the last subsection can be replaced by a sequence of positive numbers.

Hybrid Algorithm with Index Subtraction Strategy — Algorithm HIS:

Step 0: Choose $\eta > 0$, $\theta_0 > 0$, $\xi_0 > 0$, and $\epsilon_0 > 0$. Set $k := 0$.

Step 1: Solve problem (P_{ϵ_k}) and let z^k denote one of its stationary points. Set

$$
\begin{aligned}
\alpha_0^k &:= \Big\{ i \mid \ \ G_i(z^k) > \eta, \ \ H_i(z^k) \leq \xi_k \Big\}, && (3.47) \\
\gamma_0^k &:= \Big\{ i \mid \ \ G_i(z^k) \leq \xi_k, \ \ H_i(z^k) > \eta \Big\} \setminus \alpha_0^k, && (3.48) \\
\beta_0^k &:= \Big\{ 1, 2, \cdots, m \Big\} \setminus \Big(\alpha_0^k \cup \gamma_0^k \Big), && (3.49) \\
\delta_0^k &:= +\infty,
\end{aligned}
$$

and $j := 0$. Go to Step 2.

Step 2: If problem (3.17) is solvable, let $\hat{z}_j^k$ denote one of its stationary points and go to Step 3. Otherwise, go to Step 4.

Step 3: If, in (3.17), the Lagrange multipliers corresponding to the constraints (3.18) are all nonnegative, then terminate. Else, if there is an $\hat{i} \in \beta_j^k$ such that

$$
\max_{i \in \beta_j^k} \Big\{ \max \Big\{ G_i(z^k), H_i(z^k) \Big\} \Big\} = \max \Big\{ G_{\hat{i}}(z^k), H_{\hat{i}}(z^k) \Big\} > \theta_k, \tag{3.50}
$$

then set

$$
\delta_{j+1}^k := \min \Big\{ \delta_j^k, \ \frac{1}{2} \min \Big\{ G_i(\hat{z}_j^k) + H_i(\hat{z}_j^k) \ \mid \ \ i \in \alpha(\hat{z}_j^k) \cup \gamma(\hat{z}_j^k) \Big\} \Big\} \tag{3.51}
$$

and

$$
\begin{aligned}
\alpha_{j+1}^k &:= \begin{cases} \alpha_j^k \cup \{\hat{i}\}, & \text{if } G_{\hat{i}}(z^k) \geq H_{\hat{i}}(z^k) \\ \alpha_j^k, & \text{otherwise,} \end{cases} \\
\beta_{j+1}^k &:= \beta_j^k \setminus \{\hat{i}\}, \\
\gamma_{j+1}^k &:= \begin{cases} \gamma_j^k, & \text{if } G_{\hat{i}}(z^k) \geq H_{\hat{i}}(z^k) \\ \gamma_j^k \cup \{\hat{i}\}, & \text{otherwise,} \end{cases} \\
j &:= j+1
\end{aligned}
$$

and go to Step 2. Otherwise, let $j_k := j$ and $\delta_k := \delta_j^k$ and go to Step 4.

Step 4: Choose $\epsilon_{k+1} \in (0, \epsilon_k)$, $\xi_{k+1} \in (0, \xi_k]$, and set $\theta_{k+1} := \min\Big\{\theta_k,\ \delta_k\Big\}$. Let $k := k+1$ and go to Step 1.

Note that, since $G_i(z^k) \to 0$ for each $i \in \beta(z^*) \cup \gamma(z^*)$, the index set α_0^k determined in Step 1 will eventually consist of indices in $\alpha(z^*)$ only. Similarly, γ_0^k will eventually consist of indices in $\gamma(z^*)$ only. Therefore, we must have

$$
\alpha_0^k \subseteq \alpha(z^*),\ \beta_0^k \supseteq \beta(z^*),\ \gamma_0^k \subseteq \gamma(z^*) \tag{3.52}
$$

for all k sufficiently large. This is the key requirement for Algorithm HIS, which subtracts some indices from β_j^k so that condition (3.41) holds for some $j_k' \in \{0, 1, \cdots, j_k\}$ when k is large enough.

In Algorithm HIS, the strict positivity of η is essential, and the sequence $\{\xi_k\}$ plays only a subsidiary role: The sets α_0^k and γ_0^k should be chosen not too large so that the key condition (3.52) holds as early as possible. To this end, we may choose $\{\xi_k\}$ to be a null sequence. On the other hand, from the computational viewpoint, it is desirable that the set β_0^k is as small as possible, i.e., the sets α_0^k and γ_0^k are as large as possible. In consequence, it would be important to choose the constant $\eta > 0$ and the sequence $\{\xi_k\}$ appropriately.

Another practical choice is simply to remove $\{\xi_k\}$ from Algorithm HIS. For example, we may define

$$
\begin{aligned}
\alpha_0^k &:= \Big\{\, i \mid \quad G_i(z^k) > \eta \Big\}, \\
\gamma_0^k &:= \Big\{\, i \mid \quad H_i(z^k) > \eta \Big\} \setminus \alpha_0^k, \\
\beta_0^k &:= \Big\{\, 1, 2, \cdots, m \Big\} \setminus \Big(\alpha_0^k \cup \gamma_0^k\Big)
\end{aligned}
$$

instead of (3.47)–(3.49), or preferably, let

$$
\begin{aligned}
\alpha_0^k &:= \Big\{\, i \mid \quad G_i(z^k) > \eta,\ H_i(z^k) \leq \eta \Big\}, \\
\gamma_0^k &:= \Big\{\, i \mid \quad G_i(z^k) \leq \eta,\ H_i(z^k) > \eta \Big\}, \\
\beta_0^k &:= \Big\{\, 1, 2, \cdots, m \Big\} \setminus \Big(\alpha_0^k \cup \gamma_0^k\Big),
\end{aligned}
$$

which is equivalent to letting $\xi_k \equiv \eta$ $(\forall k)$ in (3.47)–(3.49).

Moreover, the parameter θ_k is also expected to be a positive lower bound of (3.40) so that the indices outside $\beta(z^*)$ can be removed from $\beta_{j_k}^k$ eventually in Step 3. As analyzed in the last subsection, Assumptions A1 and A2 guarantee that the parameter θ_k satisfies this requirement, i.e., θ_k can serve as a positive lower bound of (3.40) when k is large enough.

In a similar way to the last subsection, a comprehensive and detailed analysis can be given for Algorithm HIS. In particular, under Assumption A1, θ_k stays at a positive constant when k is sufficiently large in both Cases I and II considered in the last subsection. It then follows from (3.50) that all the indices in $\beta(z^*)$ cannot be selected in Step 3 for all k large enough, i.e., they remain in $\beta^k_{j_k}$ eventually. Thus, we have the following result.

Theorem 3.9 *Suppose that the sequence $\{z^k\}$ generated by Algorithm HIS converges to z^* as $\epsilon_k \to 0$ and Assumption A1 holds. Then, for any sufficiently large k, we have*

$$\begin{aligned}
&\alpha_0^k \subseteq \alpha_1^k \subseteq \cdots \subseteq \alpha_{j_k}^k \subseteq \alpha(z^*),\\
&\beta_0^k \supseteq \beta_1^k \supseteq \cdots \supseteq \beta_{j_k}^k \supseteq \beta(z^*),\\
&\gamma_0^k \subseteq \gamma_1^k \subseteq \cdots \subseteq \gamma_{j_k}^k \subseteq \gamma(z^*).
\end{aligned}$$

If furthermore, Assumption A2 holds, then we have (3.46) for all k sufficiently large.

This theorem indicates that, under similar assumptions to the previous subsection, Algorithm HIS also has a finite termination property.

3.5 Some remarks

In the last two subsections, we have presented two hybrid algorithms by using index addition and subtraction strategies, respectively. Here, we make some comments and suggest possible extensions of these algorithms.

Remark 3.1 In the preceding discussion, we assume that the sequence $\{z^k\}$ converges to z^*. Actually, the sequence $\{z^k\}$ may have multiple limit points in general. In this case, it is easy to see that, as long as one of the limit points satisfies the assumptions made for z^*, we may obtain similar conclusions. Especially, Assumption A2 can be restated as follows :

A2 $'$: The set S given by (3.42) contains an accumulation point of the sequence $\{z^k\}$ that is a B-stationary point of problem (P).

This suggests that our assumptions are really not very restrictive.

Remark 3.2 The stopping criterion in Step 3 of Algorithms HIA and HIS is used to check the B-stationarity of the point $\hat{z}^k$. We may replace it by some other conditions for checking M-stationarity or C-stationarity, see (2.8) and (2.7), and then the algorithms may terminate finitely by producing an M- or C-stationary point of problem (P) under some weaker conditions, see Theorem 3.2.

Remark 3.3 Comparing with Algorithm H, Algorithms HIA and HIS may need to solve more subproblems (3.17). On the other hand, Algorithm H may have to solve more subproblems (P_ϵ) than the other two, see the numerical results in the next section. From both theoretical and computational points of view, problem (3.17) is easier to deal with than problem (P_ϵ). For example, under the condition that the functions G, H, h are all affine and each g_i $(1 \le i \le p)$ is convex, problem (3.17) has a convex feasible region but problem (P_ϵ) does in general not.

Remark 3.4 The proposed algorithms may be modified by using another subproblem instead of problem (P_ϵ) in Step 1. For example, since problem (P_ϵ) is equivalent to

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g(z) \le 0,\ h(z) = 0 \\ & G(z) + H(z) \ge 0 \\ & G_i(z)H_i(z) = \epsilon^2,\ i = 1, \cdots, m, \end{array} \tag{3.53}$$

we may use problem (3.53) instead of (P_ϵ) in Step 1 of Algorithms HIA and HIS. Note that, for any $\epsilon > 0$, the constraints

$$G_i(z) + H_i(z) \ge 0, \quad i = 1, 2, \cdots, m$$

are always inactive and so problem (3.53) seems simpler than (P_ϵ). It is easily seen that all the conclusions obtained in this section remain true. On the other hand, it is possible to apply the proposed approach to many existing methods including the reformulation methods [23, 15, 16] and the penalty methods [11, 12].

4 Computational Results

In this section, we report some numerical results obtained by applying the proposed approaches to some examples of MPCCs. Comprehensive numerical results shown in [17] have revealed that Algorithm H is quite comparable to some existing methods such as the smoothing continuation method [9], the penalty function method [11], and the regularization method [23]. Here, we only compare Algorithms HIA and HIS with Algorithm H. We state the details below.

Our test problems were generated by the program QPECgen of Jiang and Ralph [13]. The QPECgen generator is a MATLAB program that uses a set of parameters and, once these parameters are specified, the program randomly generates a quadratic program with linear complementarity constraints

$$\begin{array}{ll} \min & \frac{1}{2}(x^T, y^T)P\begin{pmatrix} x \\ y \end{pmatrix} + c^T x + d^T y \\ \text{s.t.} & A\begin{pmatrix} x \\ y \end{pmatrix} + a \le 0, \\ & y \ge 0,\ Nx + My + q \ge 0, \\ & y^T(Nx + My + q) = 0, \end{array}$$

where P, A, N, M are constant matrices and c, d, a, q are constant vectors with appropriate dimensions. This program may also output an approximate solution of a generated problem, see the original reference [13] for detail.

We state the QPECgen parameters in Table 1 to generate Problems 4.1 and 4.2 and summarize some data output by the QPECgen generator in Table 2, in which $(\bar{x}, \bar{y})$ denotes the

Table 1: Parameters for Test Problems in QPECgen

Problem	qpec_type	(n, m)	(l, p)	cond_P	scale_P	convex_f	symm_M	mono_M	cond_M
# 4.1	300	(10, 10)	(5, 10)	100	100	1	1	1	
# 4.2	300	(8, 14)	(4, 14)	100	100	1	1	1	200
Problem	scale_M	second_deg	first_deg	mix_deg	tol_deg	implicit	rand_seed	output	
# 4.1	200	4	3	3	1.0e-6	0	0	3	
# 4.2	200	4	2	1	1.0e-6	0	0	3	

Table 2: Some Data Obtained by QPECgen for Test Problems

Problem	$\bar{x}$	$\bar{y}$	$\beta(\bar{x},\bar{y})$
# 4.1	(0.6369, 0.6371, 0.1739, -0.7158, -0.8703, -0.7478, -0.4383, 0.1886, 0.0741, 0.1494)	(0, 0, $\cdots$, 0)	{1, 2, 3, 4}
# 4.2	(-0.7330, -0.2090, -0.4140, 0.0168, -0.7084, -0.1104, 0.0030, -0.4658)	(0,$\cdots$,0,0.4681,0.4739,0.2088)	{1, 2, 3, 4}

solution given by QPECgen and $\beta(\bar{x},\bar{y})$ stands for the lower-level degenerate index set at $(\bar{x},\bar{y})$, i.e.,

$$\beta(\bar{x},\bar{y}) := \{i \mid \ (N\bar{x}+M\bar{y}+q)_i = 0 = \bar{y}_i\}.$$

In our experiments, we employed the MATLAB 6.5 built-in solver function *fmincon* to solve the subproblems at each iteration, and for all algorithms tested, we set $\epsilon_0 = 10^{-2}$ and updated this parameter by $\epsilon_{k+1} = 10^{-1}\epsilon_k$. The point (x^k, y^k) obtained by solving (P_{ϵ_k}) is used as the starting point for the subproblems at Step 2 of the algorithms and the next iteration as well. Moreover, we employ

$$\rho(x,y) = \|\min(Nx+My+q,y)\|^{\frac{4}{5}}$$

as the identification function in Step 1 of both Algorithm H and Algorithm HIA and, in Algorithm HIS, we use the sequence $\{\xi_k\}$ given by

$$\xi_k = \rho(x^k, y^k), \qquad k = 0, 1, 2, \cdots,$$

which is a reasonable choice to compare with the other two methods. See the tables for the setting of the other parameters involved.

The computational results for Problems 4.1 and 4.2 are reported in Tables 3 and 4, respectively. In the tables, `distance` denotes the distance between the obtained point and the solution $(\bar{x},\bar{y})$ measured by the infinity norm. In the `# iterations` column, a sum $\nu_1+\nu_2$ means that ν_1 is the number of iterations spent by *fmincon* for solving problem (P_ϵ) and ν_2 denotes the number of iterations spent by solving subproblem (3.17), whereas a single number ν stands for the number of iterations spent by solving subproblem (3.17) (as there is no need to solve problem (P_ϵ) in these cases).

The results shown in the tables reveal that it was not difficult to identify the active sets by the hybrid algorithm, at least for the test problems used in our numerical experiments. It is interesting to observe that Algorithm HIA as well as Algorithm H was able to obtain a solution before the active sets were completely identified. As mentioned in the previous section, Algorithms HIA and HIS need to solve more problems of the form (3.17) than Algorithm H, whereas the latter has to solve more problems of the form (P_ϵ). Our experiments show that problem (3.17) can be solved in fewer iterations than problem (P_ϵ) generally.

Table 3: Computational Results for Problem 4.1 [a]

	ϵ_k	degenerate set	distance	# iterations
Algorithm H	10^{-2}	$\beta^k = \emptyset$	0.0007	25+10
	10^{-3}	$\beta^k = \emptyset$	0.0010	29+10
	10^{-4}	$\beta^k = \{4\}$	0.0005	23+11
Algorithm HIA	10^{-2}	$\beta_0^k = \emptyset$	0.0007	25+10
		$\beta_1^k = \{4\}$	0.0007	10
		$\beta_2^k = \{2, 4\}$	0.0005	11
		$\beta_3^k = \{1, 2, 4\}$	0.0004	10
		$\beta_4^k = \{1, 2, 3, 4\}$	0.0004	10
Algorithm HIS	10^{-2}	$\beta_0^k = \{1, 2, 3, 4, 7, 9, 10\}$	1.5944	25+6
		$\beta_1^k = \{1, 2, 3, 4, 9, 10\}$	1.2922	5
		$\beta_2^k = \{1, 2, 3, 4, 10\}$	0.4131	9
		$\beta_3^k = \{1, 2, 3, 4\}$	0.0004	10

[a] We used $(1, 1, \cdots, 1)$ as the initial point for all methods and we set $\theta_0 = 0.2$ in Algorithms HIA and HIS. In addition, the parameter η in Algorithm HIS is set to be 0.5.

Table 4: Computational Results for Problem 4.2 [a]

	ϵ_k	degenerate set	distance	# iterations
Algorithm H	10^{-2}	$\beta^k = \emptyset$	0.0003	27+6
	10^{-3}	$\beta^k = \{1, 2, 3\}$	0.0004	19+5
	10^{-4}	$\beta^k = \{1, 2, 3, 4\}$	0.0003	35+7
Algorithm HIA	10^{-2}	$\beta_0^k = \emptyset$	0.0003	27+6
		$\beta_1^k = \{2\}$	0.0003	6
		$\beta_2^k = \{1, 2\}$	0.0009	3
		$\beta_3^k = \{1, 2, 3\}$	0.0009	3
		$\beta_4^k = \{1, 2, 3, 4\}$	0.0009	3
Algorithm HIS	10^{-2}	$\beta_0^k = \{1, 2, 3, 4, 11\}$	0.8554	27+6
		$\beta_1^k = \{1, 2, 3, 4\}$	0.0009	3

[a] We employed $(1, 1, \cdots, 1)$ as the initial point for all methods. We set $\theta_0 = 0.2$ in Algorithms HIA and HIS and, $\eta = 0.2$ in Algorithm HIS.

5 Conclusions

Active set identification plays an important role in optimization theory [1, 2, 5, 6, 17, 26]. Accurate identification of active constraints is important from both theoretical and practical points of view. In this paper, we have proposed some new approaches for solving MPCCs, which can be regarded as modifications of the hybrid approach suggested in [17]. Theoretical analysis indicates that, without the key assumption in [17] of the asymptotically weak nondegeneracy, the new algorithms may attain the same goal as in [17]. That is, the active sets at a point with some kind of stationarity to an MPCC may be identified in a finite number of iterations.

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