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GEOMETRIC AND SUBEXPONENTIAL ASYMPTOTICS OF MARKOV CHAINS OF M/G/1 TYPE

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Abstract

This paper considers the steady-state solution of Markov chains of M/G/1 type. We first derive the matrix product-form solution of the steady-state probability. This formula is considered as a natural generalization of the matrix-geometric solution of quasi birth-and-death processes to Markov chains of M/G/1 type. Based on this formula, we study the asymptotics of the tail distribution. For the light-tailed case, we show a sufficient condition for the geometric asymptotics of the tail distribution, employing the Markov key renewal theorem. Contrary to the previous works, some periodic characteristics of transitions in upper levels are taken into account explicitly and a new geometric asymptotic formula is established. Furthermore, for the heavy-tailed case, we show a subexponential asymptotics formula for the tail distribution under a mild condition.

1 Introduction

This paper considers the steady-state solution of a bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$, where X_n takes a nonnegative integer value and S_n takes a value in $\mathcal{M}_0 = \{1, \ldots, M_0\}$ when $X_n = 0$ and a value in $\mathcal{M} = \{1, \ldots, M\}$ when $X_n \geq 1$. We assume that the bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ is of M/G/1 type (Neuts 1989), and its transition probability matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{B}(0) & \boldsymbol{B}(1) & \boldsymbol{B}(2) & \boldsymbol{B}(3) & \cdots \\ \boldsymbol{C}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

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where $\boldsymbol{A}(k)$ $(k = 0, 1, \ldots)$ are $M \times M$ matrices, $\boldsymbol{B}(0)$ is an $M_0 \times M_0$ matrix, $\boldsymbol{B}(k)$ $(k = 1, 2, \ldots)$ are $M_0 \times M$ matrices, and $\boldsymbol{C}(0)$ is an $M \times M_0$ matrix. We define $\boldsymbol{\beta}_A$ and $\boldsymbol{\beta}_B$ as

$$\boldsymbol{\beta}_A = \sum_{k=1}^{\infty} k\boldsymbol{A}(k)\boldsymbol{e}, \qquad \boldsymbol{\beta}_B = \sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e},$$

where $\boldsymbol{e}$ denotes a column vector of ones with an appropriate dimension. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote

$$\boldsymbol{A} = \sum_{k=0}^{\infty} \boldsymbol{A}(k), \qquad \boldsymbol{B} = \sum_{k=1}^{\infty} \boldsymbol{B}(k),$$

respectively. Note that $\boldsymbol{A}$ is a stochastic matrix and $\boldsymbol{B}(0)\boldsymbol{e} + \boldsymbol{B}\boldsymbol{e} = \boldsymbol{e}$. Let $\boldsymbol{\pi}$ denote a $1 \times M$ vector which satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{A}, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1, \tag{2}$$

and we define ρ as

$$\rho = \boldsymbol{\pi}\boldsymbol{\beta}_A.$$

Throughout the paper, we assume the following.

Assumption 1

(a) The bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ is irreducible,

(b) $\boldsymbol{A}$ is irreducible,

(c) $\boldsymbol{\beta}_B$ is a finite vector, and

(d) $\rho < 1$.

Remark 1 *The irreducibility of $\boldsymbol{A}$ implies the uniqueness of $\boldsymbol{\pi}$ satisfying (2). Further, the irreducible bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ is positive recurrent if and only if Assumption 1 (c) and (d) hold (Neuts 1989).*

We define X and S as generic random variables for X_n and S_n in steady state, respectively. Let $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \ldots)$ denote the steady-state solution of the bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$, where $\boldsymbol{x}(0)$ denotes a $1 \times M_0$ vector whose jth element represents $\Pr(X = 0, S = j)$ and $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ denotes a $1 \times M$ vector whose jth element represents $\Pr(X = k, S = j)$. An algorithmic solution method to compute the $\boldsymbol{x}(k)$ is known (Neuts 1989), which is called the M/G/1 paradigm, where an $M \times M$ stochastic matrix $\boldsymbol{G}$ satisfying

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}(k)\boldsymbol{G}^k. \tag{3}$$

plays a central role. Note that $\boldsymbol{G}$ represents the transition probability matrix of the second variable S in the fundamental period, i.e., the (i, j)th element of $\boldsymbol{G}$ represents the probability that for $k \geq 2$, the first passage time from state $(k + 1, i)$ to any state in $\{(k, 1), \ldots, (k, M)\}$

ends with state (k, j), and $\boldsymbol{G}$ is given by a minimal nonnegative solution of (3) (Neuts 1989). Following the M/G/1 paradigm, we obtain the boundary vector $\boldsymbol{x}(0)$ by (Schellhaas 1990)

$$\boldsymbol{x}(0) = \left[1 + \frac{\boldsymbol{\kappa}}{1-\rho}\left\{\boldsymbol{\beta}_B + \left(\boldsymbol{B} - \sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k\right)[\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{\pi}]^{-1}\boldsymbol{\beta}_A\right\}\right]^{-1}\boldsymbol{\kappa},$$

where $\boldsymbol{\kappa}$ denotes the invariant probability row vector of stochastic matrix $\boldsymbol{K}$:

$$\boldsymbol{K} = \boldsymbol{B}(0) + \sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^{k-1}\left(\boldsymbol{I} - \sum_{m=1}^{\infty}\boldsymbol{A}(m)\boldsymbol{G}^{m-1}\right)^{-1}\boldsymbol{C}(0),$$

so that $\boldsymbol{\kappa}$ satisfies

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \qquad \boldsymbol{\kappa}\boldsymbol{e} = \boldsymbol{e}.$$

Further the $\boldsymbol{x}(k)$ is recursively determined by (Ramaswami 1988, Schellhaas 1990)

$$\boldsymbol{x}(k) = \left(\boldsymbol{x}(0)\boldsymbol{V}(k-1) + \sum_{j=1}^{k-1}\boldsymbol{x}(j)\boldsymbol{U}(k-j)\right)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \qquad k = 1, 2, \ldots, \tag{4}$$

where for $k = 0, 1, \ldots$,

$$\boldsymbol{U}(k) = \sum_{m=k+1}^{\infty}\boldsymbol{A}(m)\boldsymbol{G}^{m-k-1}, \qquad \boldsymbol{V}(k) = \sum_{m=k+1}^{\infty}\boldsymbol{B}(m)\boldsymbol{G}^{m-k-1}. \tag{5}$$

Note that the special case $\boldsymbol{C}(0) = \boldsymbol{A}(0)$ (and therefore $\mathcal{M}_0 = \mathcal{M}$) is encountered quite often in application. In this case, $\boldsymbol{K}$ is reduced to

$$\boldsymbol{K} = \sum_{k=0}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k$$

and therefore

$$\boldsymbol{x}(0) = \left[1 + \frac{\boldsymbol{\kappa}}{1-\rho}\left\{\boldsymbol{\beta}_B + (\boldsymbol{B}(0) + \boldsymbol{B} - \boldsymbol{I})[\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{\pi}]^{-1}\boldsymbol{\beta}_A\right\}\right]^{-1}\boldsymbol{\kappa},$$

Further, in the special case of $\boldsymbol{C}(0) = \boldsymbol{A}(0)$ and $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \ldots$, the results become much more simpler, yet it is highly important in application because the time-average queue length distribution in the stationary BMAP/G/1 queue is identical to the solution of a certain Markov chain of this special case (Takine 2000). Note that in this special case, $\boldsymbol{K} = \boldsymbol{G}$ and $\boldsymbol{x}(0)$ is given by

$$\boldsymbol{x}(0) = (1-\rho)\boldsymbol{g},$$

where $\boldsymbol{g}$ denotes an invariant probability vector of $\boldsymbol{G}$:

$$\boldsymbol{g} = \boldsymbol{g}\boldsymbol{G}, \qquad \boldsymbol{g}\boldsymbol{e} = \boldsymbol{e}.$$

In this paper, we first derive a matrix product-form solution for the $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ in terms of $\boldsymbol{x}(0)$ and $\boldsymbol{G}$. Note that this formula is a natural generalization of the matrix-geometric solution of quasi birth-and-death processes and is closely related to Wiener-Hopf factorization (Asmussen 1989). Contrary to Asmussen (1989), however, our derivation of the formula is quite straightforward. Because the formula is given in a closed form, it is suitable for studying the asymptotics of the tail distribution of the $\boldsymbol{x}(k)$.

Let $\overline{\boldsymbol{x}}(k)$ $(k = 0, 1, \dots)$ denote a $1 \times M$ vector whose jth element represents $\Pr(X > k, S = j)$ in steady state, i.e.,

$$\overline{\boldsymbol{x}}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{x}(m).$$

We then show the geometric asymptotics of the $\overline{\boldsymbol{x}}(k)$, assuming the $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ are light-tailed. More precisely, we show a sufficient condition under which there exists θ such that

$$\lim_{k\to\infty} \theta^{\zeta+k\tau}\overline{\boldsymbol{x}}(\zeta + k\tau) = \boldsymbol{c}_\zeta > \boldsymbol{0}, \qquad \zeta = 0, 1, \dots, \tau - 1, \tag{6}$$

where τ denote a span of a certain Markov renewal process associated with the original Markov chain of M/G/1 type. Further, when $\tau = 1$, we show a sufficient condition under which there exists θ such that

$$\lim_{k\to\infty} \theta^{k}\overline{\boldsymbol{x}}(k) = \boldsymbol{c} > \boldsymbol{0}. \tag{7}$$

The approach in this paper is based on the Markov key renewal theorem (e.g., see Çinlar (1975)), which was successfully applied to study the asymptotic tail distribution of risk processes (Asmussen 1999, Miyazawa 2001) and the virtual waiting time distribution in a single-server queue with Markovian arrival streams (Miyazawa 2001). Note that both Asmussen (1999) and Miyazawa (2001) consider continuous-time processes, so that they exclude the periodic nature of the processes. While the aperiodic assumption is natural for continuous process, it seems to be restrictive for discrete-time processes. As you will see, we show a simple periodic example for $\tau = 2$.

To the best of our knowledge, the geometric asymptotics of Markov chains of M/G/1 type has been studied only for the case $\tau = 1$. Among those, Abate, Choudhury and Whitt (1994) attacked the problem with the help of the Tauberian theorem (Feller 1971), and a necessary condition for the geometric asymptotics (7) for $\tau = 1$ was obtained. In Møller (2001), the geometric asymptotic tail (7) for $\tau = 1$ was considered by studying the regenerative cycles and its maxima. The author tried to show that there exists some unknown vector $\boldsymbol{c}$ that satisfies (7). However, the proof in Møller (2001) is incomplete. Namely, in the proof, some quantities are claimed to be positive, whereas they are nonnegative in general. Note also that the approach in Møller (2001) does not yield the explicit expression of the asymptotic constant $\boldsymbol{c}$. Contrary to these works, we take account of some periodic nature associated with the original process and identify the asymptotic constants $\boldsymbol{c}_\zeta$ in (6) when $\tau \geq 2$. See Falkenberg (1994), too, which studied a stronger form of the geometric asymptotics of the tail distribution.

Finally, we consider the subexponential asymptotics of $\overline{\boldsymbol{x}}(k)$, assuming at least one of the equilibrium distributions of the $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ is subexponential. That is, we show the asymptotic formula of a form:

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y}(k)} = \boldsymbol{c},$$

for some subexponential complementary distribution function $\overline{Y}(k)$ and some positive vector $\boldsymbol{c}$. The line of the proof is similar to that in Takine (2001), which shows the subexponential asymptotics of the tail distributions of virtual and actual waiting times in a single-server queue with Markovian arrival streams. The subexponential asymptotics in a more general structured Markov chain was studied in Asmussen and Møller (1999), where the Markov chain of M/G/1 type is included as a special case. However, a sufficient condition for the subexponentiality given in Asmussen and Møller (1999) is more restrictive than ours.

The rest of this paper is divided into three sections. In section 2, we derive the matrix product-form solution for the $\boldsymbol{x}(k)$. In section 3, we study the geometric asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$, applying the Markov key renewal theorem. Finally, in section 4, we show a subexponential asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$.

2 Matrix product-form solution for the steady-state probability

In this section, we derive the matrix product-form solution for the $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ in terms of $\boldsymbol{x}(0)$ and $\boldsymbol{G}$. Let $\boldsymbol{F}(k)$ $(k = 1, 2, \ldots)$ and $\boldsymbol{E}(k)$ $(k = 0, 1, \ldots)$ denote $M \times M$ and $M_0 \times M$ matrices, which are given by

$$\boldsymbol{F}(k) = \begin{cases} \boldsymbol{O}, & k = 0, \\ \boldsymbol{U}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, & k = 1, 2, \ldots, \end{cases}$$
$$\boldsymbol{E}(k) = \boldsymbol{V}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \qquad k = 0, 1, \ldots,$$

respectively, where $\boldsymbol{U}(k)$ and $\boldsymbol{V}(k)$ are given by (5). Further we define $\boldsymbol{F}^{(n)}(k)$ $(n = 1, 2, \ldots,$ $k = n-1, n, \ldots)$ recursively as

$$\boldsymbol{F}^{(1)}(k) = \boldsymbol{E}(k), \qquad k = 0, 1, \ldots,$$

and for $n = 2, 3, \ldots,$

$$\boldsymbol{F}^{(n)}(k) = \sum_{j=n-2}^{k-1} \boldsymbol{F}^{(n-1)}(j)\boldsymbol{F}(k-j), \qquad k = n-1, n, \ldots.$$

Theorem 1 *Under Assumption 1, $\boldsymbol{x}(k)$ is given in terms of $\boldsymbol{x}(0)$:*

$$\boldsymbol{x}(k) = \boldsymbol{x}(0) \sum_{n=1}^{k} \boldsymbol{F}^{(n)}(k-1), \qquad k = 1, 2, \ldots. \tag{8}$$

Remark 2 *Suppose $\boldsymbol{A}(k) = \boldsymbol{O}$ for all k $(k = 3, 4, \ldots)$. Then (8) is reduced to be*

$$\boldsymbol{x}(k) = \boldsymbol{x}(0) \sum_{n=1}^{k} \boldsymbol{E}(k-n)\boldsymbol{R}^{n-1} \qquad k = 1, 2, \ldots, \tag{9}$$

where $\boldsymbol{E}(k) = \left(\sum_{m=k+1}^{\infty} \boldsymbol{B}(m)\boldsymbol{G}^{m-k-1}\right)(\boldsymbol{I} - \boldsymbol{A}(1) - \boldsymbol{A}(2)\boldsymbol{G})^{-1}$ and $\boldsymbol{R} = \boldsymbol{A}(2)(\boldsymbol{I} - \boldsymbol{A}(1) - \boldsymbol{A}(2)\boldsymbol{G})^{-1}$. Further, if $\boldsymbol{B}(k) = \boldsymbol{O}$ for all k $(k = 2, 3, \ldots)$, (9) is reduced to be

$$\begin{aligned} \boldsymbol{x}(k) &= \boldsymbol{x}(0)\boldsymbol{B}(1)\left[\boldsymbol{I} - \boldsymbol{A}(1) - \boldsymbol{A}(2)\boldsymbol{G}\right]^{-1}\boldsymbol{R}^{k-1} \\ &= \boldsymbol{x}(1)\boldsymbol{R}^{k-1}, \qquad\qquad k = 1, 2, \ldots, \end{aligned}$$

which is well known as the matrix-geometric solution of quasi birth-and-death processes (Neuts 1981). Thus Theorem 1 is a natural generalization of the matrix-geometric solution of quasi birth-and-death processes to Markov chains of M/G/1 type. See Asmussen (1989) for the connection between the matrix product-form solution and the Wiener-Hopf factorization method.

Proof. Note first that (4) is equivalent to

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \sum_{j=1}^{k-1} \boldsymbol{x}(j)\boldsymbol{F}(k-j), \qquad k = 1, 2, \ldots. \tag{10}$$

For $k = 1$, (8) holds because (10) implies $\boldsymbol{x}(1) = \boldsymbol{x}(0)\boldsymbol{E}(0) = \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(0)$. Suppose (8) holds for $j = 1, \ldots, k-1$ ($k \geq 2$), i.e.,

$$\boldsymbol{x}(j) = \boldsymbol{x}(0) \sum_{n=1}^{j} \boldsymbol{F}^{(n)}(j-1), \qquad j = 1, \ldots, k-1. \tag{11}$$

Substituting (11) into (10), we have

$$\begin{aligned}
\boldsymbol{x}(k) &= \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \sum_{j=1}^{k-1} \boldsymbol{x}(0) \sum_{n=1}^{j} \boldsymbol{F}^{(n)}(j-1)\boldsymbol{F}(k-j) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \sum_{j=n}^{k-1} \boldsymbol{F}^{(n)}(j-1)\boldsymbol{F}(k-j) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \sum_{j=n-1}^{k-2} \boldsymbol{F}^{(n)}(j)\boldsymbol{F}(k-1-j) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \boldsymbol{F}^{(n+1)}(k-1) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=2}^{k} \boldsymbol{F}^{(n)}(k-1),
\end{aligned}$$

which shows (8) holds for $j = k$, too. □

In the following two sections, we consider the geometric and subexponential asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$, respectively. For this purpose, we rewrite Theorem 1 in a more convenient form.

Corollary 1 *Under Assumption 1, $\boldsymbol{x}(k)$ ($k = 1, 2, \ldots$) is given by*

$$\boldsymbol{x}(k) = \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k-1), \qquad k = 1, 2, \ldots. \tag{12}$$

where the $\boldsymbol{F}^{(n)}(k)$ is recursively determined by

$$\boldsymbol{F}^{(1)}(k) = \boldsymbol{E}(k), \qquad k = 0, 1, \ldots,$$

and for $n = 2, 3, \ldots$,

$$\boldsymbol{F}^{(n)}(k) = \sum_{j=0}^{k} \boldsymbol{F}^{(n-1)}(j)\boldsymbol{F}(k-j), \qquad k = 0, 1, \ldots. \tag{13}$$

Proof. By definition, $\boldsymbol{F}(0) = \boldsymbol{O}$, so that we have from (13)

$$\boldsymbol{F}^{(n)}(j) = \boldsymbol{O}, \qquad n = 2, 3, \dots, \ j = 0, \dots, n-2.$$

Thus Corollary 1 is equivalent to Theorem 1. □

Corollary 1 will serve as the basis for the asymptotic analysis in this paper. Note that, with Corollary 1, we have for $k = 0, 1, \dots$

$$\begin{aligned}
\overline{\boldsymbol{x}}(k) &= \sum_{m=k+1}^{\infty} \boldsymbol{x}(m) \\
&= \sum_{m=k+1}^{\infty} \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(m-1) \\
&= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \sum_{m=k}^{\infty} \boldsymbol{F}^{(n)}(m) \\
&= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k),
\end{aligned}$$

where

$$\overline{\boldsymbol{F}^{(n)}}(k) = \sum_{m=k}^{\infty} \boldsymbol{F}^{(n)}(m), \qquad n = 1, 2, \dots.$$

Remark 3 *Note that $\sum_{n=1}^{k} \boldsymbol{F}^{(n)}(k-1)$ is considered as a matrix rate function, i.e., the (i,j)th element of $\sum_{n=1}^{k} \boldsymbol{F}^{(n)}(k-1)$ represents the mean number of visits to state (k,j) during the recurrence time of a set of states $\{(0,1), \dots, (0,M_0)\}$, given that the recurrence time starts from state $(0,i)$. Therefore under Assumption 1,*

$$\sum_{n=1}^{k} \boldsymbol{F}^{(n)}(k-1) = \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k-1) > \boldsymbol{O}, \tag{14}$$

for all $k = 1, 2, \dots$, because the Markov chain $\{(X_n, S_n);\ n = 0, 1, \dots\}$ is assumed to be irreducible.

3 Geometric asymptotics

In this section, we derive a sufficient condition under which the $\overline{\boldsymbol{x}}(k)$ has a geometric tail.

3.1 Preliminary

Let $\boldsymbol{F}$ and $\boldsymbol{E}$ denote

$$\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{F}(k), \qquad \boldsymbol{E} = \sum_{k=0}^{\infty} \boldsymbol{E}(k),$$

respectively. Note here that $\boldsymbol{E}$ is bounded because $\sum_{k=0}^{\infty} \boldsymbol{E}(k) = \sum_{k=0}^{\infty} \boldsymbol{V}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}$ and $\sum_{k=0}^{\infty} \boldsymbol{V}(k)\boldsymbol{e} = \boldsymbol{\beta}_B$ is a finite vector by Assumption 1 (c).

Lemma 1 *$\boldsymbol{F}$ and $\boldsymbol{E}$ are given by*

$$\boldsymbol{F} = \boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{A} + (\boldsymbol{e} - \boldsymbol{\beta}_A)\boldsymbol{g})(\boldsymbol{I} - \boldsymbol{G} + \boldsymbol{eg})^{-1}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \tag{15}$$

$$\boldsymbol{E} = \left(\boldsymbol{B} - \sum_{k=1}^{\infty} \boldsymbol{B}(k)\boldsymbol{G}^k + \boldsymbol{\beta}_B\boldsymbol{g}\right)(\boldsymbol{I} - \boldsymbol{G} + \boldsymbol{eg})^{-1}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \tag{16}$$

respectively.

The proof of Lemma 1 is given in Appendix A.1.

Lemma 2 *Under Assumption 1, $\boldsymbol{F}$ is a nonnegative matrix whose spectral radius is less than one.*

Proof. The nonnegativity of $\boldsymbol{F}$ is clear from the definition. Note here that

$$\begin{aligned}\sum_{k=1}^{\infty} \boldsymbol{x}(k) &= \sum_{k=1}^{\infty} \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k-1) \\ &= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \boldsymbol{F}^{(n)}(k).\end{aligned}$$

By definition,

$$\sum_{k=0}^{\infty} \boldsymbol{F}^{(1)}(k) = \boldsymbol{E},$$

and for $n = 2, 3, \ldots,$

$$\begin{aligned}\sum_{k=0}^{\infty} \boldsymbol{F}^{(n)}(k) &= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \boldsymbol{F}^{(n-1)}(m)\boldsymbol{F}(k-m) \\ &= \sum_{m=0}^{\infty} \boldsymbol{F}^{(n-1)}(m) \sum_{k=m}^{\infty} \boldsymbol{F}(k-m) \\ &= \boldsymbol{E}\boldsymbol{F}^{n-1}, \qquad n = 2, 3, \ldots,\end{aligned}$$

by induction. Thus

$$\sum_{k=1}^{\infty} \boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{E} \sum_{n=0}^{\infty} \boldsymbol{F}^n. \tag{17}$$

Because the bivariate Markov chain is irreducible and positive recurrent, the infinite sum on the right hand side converges to $(\boldsymbol{I} - \boldsymbol{F})^{-1}$, and the convergent is equivalent to the statement of the lemma. □

As stated in Remark 3, $\sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k)$ is considered as a matrix rate function. Therefore we first consider its dual that leads to a Markov renewal equation with a defective semi-Markov kernel. Next we transform it to a Markov renewal equation with a proper semi-Markov kernel, by means of change of measure. Finally, applying the Markov key renewal theorem to it, we obtain the desired result.

3.2 Markov renewal equation

In this subsection, we derive a Markov renewal equation that is closely related to $\sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k)$. We define $\widetilde{\boldsymbol{A}}(k)$ and $\widetilde{\boldsymbol{B}}(k)$ ($k = 0, 1, \dots$) as nonnegative $M \times M$ and $M \times M_0$ matrices, which are given by

$$\widetilde{\boldsymbol{A}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{A}(k)^T \Delta_{\boldsymbol{\pi}}, \qquad \widetilde{\boldsymbol{B}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{B}(k)^T, \qquad k = 0, 1, \dots,$$

respectively, where $\Delta_{\boldsymbol{x}}$ represents an $M \times M$ diagonal matrix whose (j, j)th element is equal to the jth element of vector $\boldsymbol{x}$, and T represents the transpose operator. Let $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$ denote $M \times M$ and $M \times M_0$ matrices given by

$$\widetilde{\boldsymbol{A}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{A}}(k), \qquad \widetilde{\boldsymbol{B}} = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{B}}(k),$$

respectively. Note that $\widetilde{\boldsymbol{A}}$ is a stochastic matrix satisfying $\boldsymbol{\pi}\widetilde{\boldsymbol{A}} = \boldsymbol{\pi}$. We define $\tilde{\boldsymbol{\alpha}}^A$ and $\tilde{\boldsymbol{\alpha}}^B$ as

$$\tilde{\boldsymbol{\alpha}}^A = \boldsymbol{\pi} \sum_{k=1}^{\infty} k\widetilde{\boldsymbol{A}}(k), \qquad \tilde{\boldsymbol{\alpha}}^B = \boldsymbol{\pi} \sum_{k=1}^{\infty} k\widetilde{\boldsymbol{B}}(k),$$

respectively. We then have

$$\tilde{\boldsymbol{\alpha}}^A \boldsymbol{e} = \rho < 1.$$

We define $\widetilde{\boldsymbol{G}}$ as

$$\widetilde{\boldsymbol{G}} = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{G}^T \Delta_{\boldsymbol{\pi}}.$$

It then follows from (3) that $\widetilde{\boldsymbol{G}}$ satisfies

$$\widetilde{\boldsymbol{G}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{G}}^k \widetilde{\boldsymbol{A}}(k).$$

The maximum eigenvalue of $\widetilde{\boldsymbol{G}}$ is equal to one because $\tilde{\boldsymbol{\alpha}}^A \boldsymbol{e} = \rho < 1$ (Neuts 1981) and its associated left and right eigenvectors are given by $\boldsymbol{\pi}$ and $\tilde{\boldsymbol{g}}^T = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{g}^T$, respectively.

$$\boldsymbol{\pi}\widetilde{\boldsymbol{G}} = \boldsymbol{\pi}, \qquad \widetilde{\boldsymbol{G}}\tilde{\boldsymbol{g}}^T = \tilde{\boldsymbol{g}}^T.$$

It is easy to see that $\boldsymbol{\pi}\tilde{\boldsymbol{g}}^T = 1$.

We then define $\widetilde{\boldsymbol{F}}(k)$ and $\widetilde{\boldsymbol{E}}(k)$ ($k = 0, 1, \dots$) as

$$\widetilde{\boldsymbol{F}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{F}(k)^T \Delta_{\boldsymbol{\pi}}, \qquad \widetilde{\boldsymbol{E}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{E}(k)^T,$$

respectively. Note that $\widetilde{\boldsymbol{F}}(k)$ and $\widetilde{\boldsymbol{E}}(k)$ are represented by

$$\widetilde{\boldsymbol{F}}(k) = \begin{cases} \boldsymbol{O}, & k = 0, \\ (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{U}}(k), & k = 1, 2, \dots, \end{cases}$$
$$\widetilde{\boldsymbol{E}}(k) = (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{V}}(k), \quad k = 0, 1, \dots,$$

respectively, where for $k = 0, 1, \dots$

$$\widetilde{\boldsymbol{U}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{U}(k)^T \Delta_{\boldsymbol{\pi}} = \sum_{m=k+1}^{\infty} \widetilde{\boldsymbol{G}}^{m-k-1} \widetilde{\boldsymbol{A}}(m),$$
$$\widetilde{\boldsymbol{V}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{V}(k)^T = \sum_{m=k+1}^{\infty} \widetilde{\boldsymbol{G}}^{m-k-1} \widetilde{\boldsymbol{B}}(m).$$

We define $\widetilde{\boldsymbol{F}}$ and $\widetilde{\boldsymbol{E}}$ as

$$\widetilde{\boldsymbol{F}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{F}}(k), \qquad \widetilde{\boldsymbol{E}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{E}}(k),$$

respectively. Note that $\widetilde{\boldsymbol{F}} = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{F}^T \Delta_{\boldsymbol{\pi}}$ and $\widetilde{\boldsymbol{E}} = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{E}^T$. Thus from Lemma 1, we obtain

$$\widetilde{\boldsymbol{F}} = \boldsymbol{I} - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{G}} + \tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{A}} + \tilde{\boldsymbol{g}}^T(\boldsymbol{\pi} - \tilde{\boldsymbol{\alpha}}^A)), \tag{18}$$

$$\widetilde{\boldsymbol{E}} = (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{G}} + \tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1} \left(\widetilde{\boldsymbol{B}} - \sum_{k=1}^{\infty} \widetilde{\boldsymbol{G}}^k \widetilde{\boldsymbol{B}}(k) + \tilde{\boldsymbol{g}}^T \tilde{\boldsymbol{\alpha}}^B \right). \tag{19}$$

Lemma 3 *Under Assumption 1, $\widetilde{\boldsymbol{F}}$ is a strictly substochastic matrix.*

Proof. It is clear that $\widetilde{\boldsymbol{F}}$ is a nonnegative matrix because $\widetilde{\boldsymbol{F}}(k)$ is nonnegative. From (18), we have

$$\begin{aligned} \widetilde{\boldsymbol{F}}\boldsymbol{e} &= \boldsymbol{e} - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{G}} + \tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\tilde{\boldsymbol{g}}^T(1-\rho) \\ &= \boldsymbol{e} - (1-\rho)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}\tilde{\boldsymbol{g}}^T \\ &< \boldsymbol{e}, \end{aligned}$$

where the second equality comes from $(\boldsymbol{I} - \widetilde{\boldsymbol{G}} + \tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\tilde{\boldsymbol{g}}^T = \tilde{\boldsymbol{g}}^T$, and the inequality comes from $\tilde{\boldsymbol{g}}^T > \boldsymbol{0}$, $(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}\boldsymbol{e} > \boldsymbol{e}$ and $\rho < 1$. □

We define $\overline{\overline{\widetilde{\boldsymbol{F}}}}(k)$ and $\overline{\overline{\widetilde{\boldsymbol{E}}}}(k)$ as

$$\overline{\overline{\widetilde{\boldsymbol{F}}}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}(m), \qquad \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{E}}(m), \qquad k = 0, 1, \ldots,$$

respectively. Let $\widetilde{\boldsymbol{F}}^{(n)}(k)$ ($n = 1, 2, \ldots$, $k = 0, 1, \ldots$) denote

$$\widetilde{\boldsymbol{F}}^{(n)}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{F}^{(n)}(k)^T, \qquad n = 1, 2, \ldots, \; k = 0, 1, \ldots.$$

We then define $\widetilde{\boldsymbol{Q}}(k)$ ($k = 0, 1, \ldots$) as

$$\widetilde{\boldsymbol{Q}}(k) = \sum_{n=1}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(k), \qquad k = 0, 1, \ldots,$$

where

$$\overline{\widetilde{\boldsymbol{F}}^{(n)}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m).$$

Lemma 4 *Under Assumption 1, $\widetilde{\boldsymbol{Q}}(k)$ satisfies*

$$\widetilde{\boldsymbol{Q}}(k) = \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \overline{\overline{\widetilde{\boldsymbol{F}}}}(k+1)(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}} + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j)\widetilde{\boldsymbol{Q}}(k-j), \qquad k = 0, 1, \ldots. \tag{20}$$

Remark 4 *(20) is considered as a Markov renewal equation for $\widetilde{\boldsymbol{Q}}(k)$ with defective semi-Markov kernel $\widetilde{\boldsymbol{F}}(j)$ ($j = 0, 1, \ldots$) (Çinlar, E. 1975).*

Proof. For $k = 0, 1, \ldots$, we have

$$\begin{aligned}
\widetilde{\boldsymbol{Q}}(k) &= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{n=2}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(k) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \sum_{j=0}^{m} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{F}}^{(n-1)}(m-j) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \left(\sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{F}}^{(n-1)}(m-j) + \sum_{j=k+1}^{m} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{F}}^{(n-1)}(m-j) \right) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{m=k}^{\infty} \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j) + \sum_{m=k+1}^{\infty} \sum_{j=k+1}^{m} \widetilde{\boldsymbol{F}}(j) \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{Q}}(k-j) + \sum_{j=k+1}^{\infty} \widetilde{\boldsymbol{F}}(j) \sum_{m=j}^{\infty} \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j) \\
&= \overline{\overline{\widetilde{\boldsymbol{E}}}}(k) + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{Q}}(k-j) + \overline{\overline{\widetilde{\boldsymbol{F}}}}(k+1)(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1} \widetilde{\boldsymbol{E}},
\end{aligned}$$

which completes the proof. □

We define $\boldsymbol{A}^*(z)$ and $\boldsymbol{B}^*(z)$ as

$$\boldsymbol{A}^*(z) = \sum_{k=0}^{\infty} \boldsymbol{A}(k) z^k, \qquad \boldsymbol{B}^*(z) = \sum_{k=1}^{\infty} \boldsymbol{B}(k) z^k, \tag{21}$$

respectively. Let r_A and r_B denote the convergence radiuses of $\boldsymbol{A}^*(z)$ and $\boldsymbol{B}^*(z)$ in (21), respectively. Let $\delta(z)$ denote the Perron-Frobenius eigenvalue of $\boldsymbol{A}^*(z)$, and $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ denote the left and right eigenvectors associated with $\delta(z)$, respectively, with normalizing condition $\boldsymbol{u}(z)\boldsymbol{v}(z) = \boldsymbol{u}(z)\boldsymbol{e} = 1$.

Assumption 2

(a) The convergence radiuses r_A of $\boldsymbol{A}^*(z)$ and r_B of $\boldsymbol{B}^*(z)$ are greater than one, and

(b) there exists θ such that $1 < \theta < \min(r_A, r_B)$ and $\theta = \delta(\theta)$.

Remark 5 *Suppose $r_A > 1$. Then there exists at most one θ such that $1 < \theta < r_A$ and $\theta = \delta(\theta)$, because $\delta(z)$ is a convex function of z and $\frac{d}{dz}\delta(z)|_{z=1} = \rho < 1$ (Neuts 1989).*

We define $\widetilde{\boldsymbol{A}}^*(z)$ and $\widetilde{\boldsymbol{B}}^*(z)$ as

$$\widetilde{\boldsymbol{A}}^*(z) = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{A}}(k) z^k, \qquad \widetilde{\boldsymbol{B}}^*(z) = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{B}}(k) z^k,$$

respectively. Note that $\widetilde{\boldsymbol{A}}^*(z) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{A}^*(z)^T \Delta_{\boldsymbol{\pi}}$ and $\widetilde{\boldsymbol{B}}^*(z) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{B}^*(z)^T$. Note also that $\widetilde{\boldsymbol{A}}^*(z)$ has the same eigenvalues as $\boldsymbol{A}^*(z)$ (Neuts 1981). Let $\widetilde{\boldsymbol{u}}(z)^T$ and $\widetilde{\boldsymbol{v}}(z)^T$ denote

$$\widetilde{\boldsymbol{u}}(z)^T = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{u}(z)^T, \qquad \widetilde{\boldsymbol{v}}(z)^T = \boldsymbol{v}(z)^T \Delta_{\boldsymbol{\pi}}.$$

We then have

$$\widetilde{\boldsymbol{A}}^*(z)\widetilde{\boldsymbol{u}}(z)^T = \delta(z)\widetilde{\boldsymbol{u}}(z)^T, \qquad \widetilde{\boldsymbol{v}}(z)^T\widetilde{\boldsymbol{A}}^*(z) = \delta(z)\widetilde{\boldsymbol{v}}(z)^T,$$

and $\widetilde{\boldsymbol{v}}(z)^T\widetilde{\boldsymbol{u}}(z)^T = 1$.

We define $\widehat{\boldsymbol{Q}}(k)$ $(k = 0, 1, \ldots)$ as

$$\widehat{\boldsymbol{Q}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{Q}}(k).$$

Let $\widehat{\overline{\overline{\boldsymbol{F}}}}(k)$, $\widehat{\overline{\overline{\boldsymbol{E}}}}(k)$, $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{E}}$ denote

$$\widehat{\overline{\overline{\boldsymbol{F}}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\overline{\overline{\boldsymbol{F}}}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad \widehat{\overline{\overline{\boldsymbol{E}}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\overline{\overline{\boldsymbol{E}}}}(k),$$

$$\widehat{\boldsymbol{F}} = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{F}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad \widehat{\boldsymbol{E}} = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{E}},$$

respectively.

Lemma 5 *Under Assumptions 1 and 2, $\theta^k\widehat{\boldsymbol{Q}}(k)$ $(k = 0, 1, \ldots)$ satisfies*

$$\theta^k\widehat{\boldsymbol{Q}}(k) = \theta^k\widehat{\overline{\overline{\boldsymbol{E}}}}(k) + \theta^k\widehat{\overline{\overline{\boldsymbol{F}}}}(k+1)(\boldsymbol{I} - \widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} + \sum_{j=0}^{k}\theta^j\widehat{\boldsymbol{F}}(j)\cdot\theta^{k-j}\widehat{\boldsymbol{Q}}(k-j), \tag{22}$$

which is considered as a Markov renewal equation with a proper kernel $\theta^k\widehat{\boldsymbol{F}}(k)$ because $\theta^k\widehat{\boldsymbol{F}}(k) \geq \boldsymbol{O}$ and

$$\sum_{k=0}^{\infty}\theta^k\widehat{\boldsymbol{F}}(k)\boldsymbol{e} = \boldsymbol{e}. \tag{23}$$

Proof. From Lemma 4, we have

$$\widehat{\boldsymbol{Q}}(k) = \widehat{\overline{\overline{\boldsymbol{E}}}}(k) + \widehat{\overline{\overline{\boldsymbol{F}}}}(k+1)(\boldsymbol{I} - \widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} + \sum_{j=0}^{k}\widehat{\boldsymbol{F}}(j)\widehat{\boldsymbol{Q}}(k-j). \tag{24}$$

Multiplying both sides of (24) by θ^k, we obtain (22). $\theta^k\widehat{\boldsymbol{F}}(k) \geq \boldsymbol{O}$ is clear from the definition. To show (23), we need the following lemma.

Lemma 6 *Let $\widetilde{\boldsymbol{F}}^*(z)$ and $\widetilde{\boldsymbol{E}}^*(z)$ denote*

$$\widetilde{\boldsymbol{F}}^*(z) = \sum_{k=0}^{\infty}\widetilde{\boldsymbol{F}}(k)z^k, \qquad \widetilde{\boldsymbol{E}}^*(z) = \sum_{k=0}^{\infty}\widetilde{\boldsymbol{E}}(k)z^k.$$

Under Assumptions 1 and 2, for $1 < z < r_A$, $\widetilde{\boldsymbol{F}}^*(z)$ is given by

$$\widetilde{\boldsymbol{F}}^*(z) = \boldsymbol{I} + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1}(\widetilde{\boldsymbol{A}}^*(z)/z - \boldsymbol{I}). \tag{25}$$

Also, for $1 < z < r_B$, $\widetilde{\boldsymbol{E}}^*(z)$ is given by

$$\widetilde{\boldsymbol{E}}^*(z) = (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1}\frac{1}{z}\left(\widetilde{\boldsymbol{B}}^*(z) - \sum_{k=1}^{\infty}\widetilde{\boldsymbol{G}}^k\widetilde{\boldsymbol{B}}(k)\right). \tag{26}$$

The proof of Lemma 6 is given in Appendix A.2.

We define $\widehat{\boldsymbol{F}}^*(z)$ as

$$\widehat{\boldsymbol{F}}^*(z) = \sum_{k=0}^{\infty} \widehat{\boldsymbol{F}}(k) z^k,$$

where $\widehat{\boldsymbol{F}}(k)$ $(k = 0, 1, \ldots)$ is given by

$$\widehat{\boldsymbol{F}}(k) = \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \widetilde{\boldsymbol{F}}(k) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad k = 0, 1, \ldots.$$

Note that $\widehat{\boldsymbol{F}}^*(z) = \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \widetilde{\boldsymbol{F}}^*(z) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}$. It then follows from Lemma 6 that

$$\begin{aligned}
\sum_{k=0}^{\infty} \theta^k \widehat{\boldsymbol{F}}(k) \boldsymbol{e} &= \widehat{\boldsymbol{F}}^*(\theta) \boldsymbol{e} \\
&= \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \widetilde{\boldsymbol{F}}^*(\theta) \widetilde{\boldsymbol{u}}(\theta) \\
&= \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \left(\widetilde{\boldsymbol{u}}(\theta)^T + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)^{-1} \left(\widetilde{\boldsymbol{A}}^*(\theta) \widetilde{\boldsymbol{u}}(\theta)^T - \theta \widetilde{\boldsymbol{u}}(\theta)^T \right) / \theta \right) \\
&= \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \widetilde{\boldsymbol{u}}(\theta)^T \\
&= \boldsymbol{e},
\end{aligned}$$

which completes the proof. □

Remark 6 *Let $\widehat{\boldsymbol{\nu}}(\theta)$ denote*

$$\widehat{\boldsymbol{\nu}}(\theta) = \widetilde{\boldsymbol{v}}(\theta)^T (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0)) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

Note that from Lemma 6

$$\widehat{\boldsymbol{\nu}}(\theta) \widehat{\boldsymbol{F}}^*(\theta) = \widehat{\boldsymbol{\nu}}(\theta).$$

3.3 Results

We shall show the geometric asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$ using the Markov key renewal theorem. In Appendix B, we summarize the results for discrete-time Markov renewal processes which will be used. As shown in Appendix B, we have to identify the structure of $\widehat{\boldsymbol{F}}^*(\theta)$ to use the Markov key renewal theorem.

Lemma 7 *Under Assumptions 1 and 2, the M-state Markov chain with transition probability matrix $\widehat{\boldsymbol{F}}^*(\theta) = \sum_{k=0}^{\infty} \theta^k \widehat{\boldsymbol{F}}(k)$ has a single irreducible absorbing class $\mathcal{M}_1 \subset \mathcal{M}$. That is, $\widehat{\boldsymbol{F}}^*(\theta)$ and $\theta^k \widehat{\boldsymbol{F}}(k)$ take forms:*

$$\widehat{\boldsymbol{F}}^*(\theta) = \begin{array}{c c} & \begin{array}{cc} \mathcal{M}_1 & \mathcal{M}_2 \end{array} \\ \begin{array}{c} \mathcal{M}_1 \\ \mathcal{M}_2 \end{array} & \begin{pmatrix} \widehat{\boldsymbol{F}}_1^*(\theta) & \boldsymbol{O} \\ \widehat{\boldsymbol{F}}_{2,1}(\theta) & \widehat{\boldsymbol{F}}_2(\theta) \end{pmatrix} \end{array}, \qquad \theta^k \widehat{\boldsymbol{F}}(k) = \begin{array}{c c} & \begin{array}{cc} \mathcal{M}_1 & \mathcal{M}_2 \end{array} \\ \begin{array}{c} \mathcal{M}_1 \\ \mathcal{M}_2 \end{array} & \begin{pmatrix} \theta^k \widehat{\boldsymbol{F}}_1(k) & \boldsymbol{O} \\ \theta^k \widehat{\boldsymbol{F}}_{2,1}(k) & \theta^k \widehat{\boldsymbol{F}}_2(k) \end{pmatrix} \end{array}, \qquad (27)$$

where $\widehat{\boldsymbol{F}}_1^*(\theta)$ is an irreducible stochastic matrix and states in $\mathcal{M}_2 = \mathcal{M} - \mathcal{M}_1$, if any, are transient.

Proof. In order to identify the structure of $\widehat{\boldsymbol{F}}^*(\theta)$, we first consider the structure of $\boldsymbol{F}$. For this purpose, we consider a bivariate Markov chain $\{(Y_n, S_n);\ n = 0, 1, \ldots\}$ whose transition probability matrix $\boldsymbol{T}_A$ is given by

$$\boldsymbol{T}_A = \begin{pmatrix} \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{28}$$

Lemma 8 *Under Assumption 1 (b) and 2, there exists* n *such that* $P(Y_n > k \mid Y_0 = 0, S_0 = i) > 0$ *for all* $k = 0, 1, \ldots$ *and all* $i \in \mathcal{M}$.

The proof of Lemma 8 is given in Appendix A.3.

Let $F_{i,j}$ denote the (i, j)th element of $\boldsymbol{F}$. For $i, j \in \mathcal{M}$, we denote $i \to j$ if there exists $l_m \in \mathcal{M}$ $(m = 1, \ldots, n)$ such that

$$F_{i,l_1} F_{l_1,l_2} \cdot \cdots \cdots F_{l_{n-1},l_n} F_{l_n,j} > 0$$

for some n. Note that $i \to j$ if and only if there exists n such that

$$\Pr(X_m > k\ (m = 1, \ldots, n), S_n = j \mid X_0 = k \geq 1, S_0 = i) > 0.$$

Owing to Lemma 8, there exists $i, j \in \mathcal{M}$ such that for some n

$$\Pr(k < X_m < k + M\ (m = 1, \ldots, n-1), X_n \geq k + M, S_n = j \mid X_0 = k \geq 1, S_0 = i) > 0.$$

Further, because $\boldsymbol{A}$ is irreducible, all states $l \in \mathcal{M}$ $(l \neq j)$ are reachable from state j by $\boldsymbol{A}$ within $M - 1$ steps, and $X_{n+M-1} > k$ owing to the skip free to the left property. Therefore $i \to l$ for all $l \in \mathcal{M}$ (including state i itself). As a result, there exists state $i \in \mathcal{M}$ such that $i \to j$ for all $j \in \mathcal{M}$.

Let $\mathcal{M}_1 \subset \mathcal{M}$ denote the set of states i such that $i \to l$ for all $l \in \mathcal{M}$. We then classify the state space $\mathcal{M}$ into $\mathcal{M}_1$ and its complement $\mathcal{M}_2 = \mathcal{M} - \mathcal{M}_1$. The above discussion then implies that $\boldsymbol{F}$ is rewritten to be

$$\boldsymbol{F} = \begin{array}{c} \\ \mathcal{M}_1 \\ \mathcal{M}_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathcal{M}_1 & \mathcal{M}_2 \end{array} \\ \begin{pmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_{1,2} \\ \boldsymbol{O} & \boldsymbol{F}_2 \end{pmatrix} \end{array}$$

where $\mathcal{M}_1 \neq \phi$, $\boldsymbol{F}_1$ is irreducible and every state $j \in \mathcal{M}_2$, if any, is reachable from any state $i \in \mathcal{M}_1$. Lemma 7 then follows immediately, because the structure of $\widehat{\boldsymbol{F}}^*(\theta)$ is identical to that of $\boldsymbol{F}^T$. $\square$

Let τ denote the span of states in $\mathcal{M}_1$ in the semi-Markov process with the kernel $\theta^k \widehat{\boldsymbol{F}}(k)$. Further let $\tau_{i,j}$ $(i, j \in \mathcal{M})$ denote the first jump point of the distribution of the first passage time from state i to state j. Applying (74) in Proposition 1, we have the following theorem.

Theorem 2 *Suppose Assumptions 1 and 2 holds. If $\tau \geq 2$, then for $\zeta = 0, 1, \ldots, \tau - 1$,*

$$\lim_{k\to\infty}\left[\theta^{\zeta+k\tau}\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(\zeta+k\tau)\right]_{i,j} = \frac{\tau}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\sum_{l=1}^{M}\left(\sum_{k=0}^{\infty}\theta^{\zeta+k\tau-\tau_{j,l}}\sum_{m=\zeta+k\tau-\tau_{j,l}}^{\infty}[\boldsymbol{E}(m)]_{i,l}\right.$$
$$\left.+\sum_{k=0}^{\infty}\theta^{\zeta+k\tau-\tau_{j,l}}\left[\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\sum_{m=\zeta+k\tau-\tau_{j,l}+1}^{\infty}\boldsymbol{F}(m)\right]_{i,l}\right)$$
$$\cdot\,[(\boldsymbol{I}-\boldsymbol{U}(0))(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta)]_l[\boldsymbol{u}(\theta)]_j, \qquad (29)$$

and

$$\lim_{k\to\infty}\theta^{\zeta+k\tau}[\overline{\boldsymbol{x}}(\zeta+k\tau)]_j = \frac{\tau}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\sum_{l=1}^{M}\left(\sum_{k=0}^{\infty}\theta^{\zeta+k\tau-\tau_{j,l}}\sum_{m=\zeta+k\tau-\tau_{j,l}}^{\infty}[\boldsymbol{x}(0)\boldsymbol{E}(m)]_l\right.$$
$$\left.+\sum_{k=0}^{\infty}\theta^{\zeta+k\tau-\tau_{j,l}}\left[\boldsymbol{\pi}^*\sum_{m=\zeta+k\tau-\tau_{j,l}+1}^{\infty}\boldsymbol{F}(m)\right]_l\right)$$
$$\cdot\,[(\boldsymbol{I}-\boldsymbol{U}(0))(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta)]_l[\boldsymbol{u}(\theta)]_j, \qquad (30)$$

where $\boldsymbol{F}(m) = \boldsymbol{E}(m) = \boldsymbol{O}$ for $m < 0$ and $\boldsymbol{\pi}^*$ is given by

$$\begin{aligned}\boldsymbol{\pi}^* &= \boldsymbol{x}(0)\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\\ &= (\boldsymbol{x}(0)\{\boldsymbol{B}+\boldsymbol{\beta}_B\boldsymbol{g}\}-\boldsymbol{x}(1)\boldsymbol{A}(0))\left(\boldsymbol{I}-\boldsymbol{A}+(\boldsymbol{e}-\boldsymbol{\beta}_A)\boldsymbol{g}\right)^{-1}.\end{aligned} \qquad (31)$$

Further the right hand sides of (29) and (30) are positive constants.

Remark 7 *The periodicity of the semi-Markov kernel $\theta^k\widehat{\boldsymbol{F}}(k)$ is closely related to the periodic structure of the $\boldsymbol{A}(k)$. We shall show such an example. Suppose $\boldsymbol{B}(k) > \boldsymbol{O}$ for all k ($k = 0, 1, \ldots$), $\boldsymbol{\beta}_B$ is a finite vector and*

$$\boldsymbol{A}(0) = \begin{pmatrix} 0 & 1-q \\ 1-q & 0 \end{pmatrix}, \qquad \boldsymbol{A}(2) = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix},$$

for some q ($0 < q < 1/2$). Then Assumption 1 holds. Further we assume that Assumption 2 holds, i.e., $r_B > [1+\sqrt{1-4q(1-q)}]/2q$. It is easy to see that

$$\boldsymbol{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{U}(0) = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \qquad \boldsymbol{U}(1) = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix},$$

and $\boldsymbol{U}(k) = \boldsymbol{O}$ for all k ($k = 2, 3, \ldots$). Thus we have

$$\boldsymbol{F}(1) = \begin{pmatrix} 0 & q/(1-q) \\ q/(1-q) & 0 \end{pmatrix},$$

and $\boldsymbol{F}(k) = \boldsymbol{O}$ for all k ($k = 2, 3, \ldots$). Therefore the semi-Markov process with the kernel $\theta^k\widehat{\boldsymbol{F}}(k)$ has span 2, i.e., $\tau = 2$, because

$$\theta\widehat{\boldsymbol{F}}(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\theta^k\widehat{\boldsymbol{F}}(k) = \boldsymbol{O}$ for all k ($k = 2, 3, \ldots$).

Proof. We shall apply the Markov key renewal theorem to (22). Recall that the semi-Markov kernel $\theta^k \boldsymbol{F}(k)$ has a single irreducible absorbing class. Thus in order to apply (74) of Proposition 1 in Appendix B, we need to show the direct Riemann integrability (Çinlar 1975, Rolski et al. 1998) of $\theta^k \overline{\overline{\boldsymbol{E}}}(k)$ and $\theta^k \overline{\overline{\boldsymbol{F}}}(k+1)$. Note first that $\overline{\overline{\boldsymbol{E}}}(k)$ and $\overline{\overline{\boldsymbol{F}}}(k)$ are nonincreasing functions. Thus from Lemma 6.1.4 in Rolski et al. (1998), it is sufficient to show the finiteness of $\sum_{k=0}^{\infty} \theta^k \overline{\overline{\boldsymbol{E}}}(k)$ and $\sum_{k=0}^{\infty} \theta^k \overline{\overline{\boldsymbol{F}}}(k+1)$.

Lemma 9

$$\sum_{k=0}^{\infty} \theta^k \overline{\overline{\boldsymbol{E}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{\theta}{\theta-1} \widetilde{\boldsymbol{E}}^*(\theta) - \frac{1}{\theta-1} \widetilde{\boldsymbol{E}} \right),$$

$$\sum_{k=0}^{\infty} \theta^k \overline{\overline{\boldsymbol{F}}}(k+1) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{1}{\theta-1} \widetilde{\boldsymbol{F}}^*(\theta) - \frac{1}{\theta-1} \widetilde{\boldsymbol{F}} \right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

The proof of Lemma 9 is given in Appendix A.4.

Thus applying (74) in Proposition 1 to (22), we have for $\zeta = 0, 1, \ldots, \tau-1$,

$$\begin{aligned}\lim_{k\to\infty} \theta^{\zeta+k\tau} \left[\widehat{\boldsymbol{Q}}(\zeta+k\tau)\right]_{i,j} &= \frac{\tau}{\widehat{\boldsymbol{\nu}}(\theta) \sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k)\boldsymbol{e}} \sum_{l=1}^{M} [\widehat{\boldsymbol{\nu}}(\theta)]_l \sum_{k=0}^{\infty} \Big[\theta^{\zeta+k\tau-\tau_{i,l}} \widehat{\overline{\boldsymbol{E}}}(\zeta+k\tau-\tau_{i,l}) \\ &\qquad + \theta^{\zeta+k\tau-\tau_{i,l}} \widehat{\overline{\boldsymbol{F}}}(\zeta+k\tau-\tau_{i,l}+1) \left(\boldsymbol{I}-\widehat{\boldsymbol{F}}\right)^{-1} \widehat{\boldsymbol{E}}\Big]_{l,j}. \end{aligned} \tag{32}$$

Further, to obtain the final result, we need the following lemma.

Lemma 10

$$\widehat{\boldsymbol{\nu}}(\theta) \sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k)\boldsymbol{e} = \frac{d}{dz}\delta(z)|_{z=\theta} - 1 > 0. \tag{33}$$

The proof of Lemma 10 is given in Appendix A.5.

On the other hand, by definition, we have

$$\begin{aligned}\left[\sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k)\right]_{i,j} &= \left[\Delta_{\boldsymbol{\pi}} \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \widehat{\boldsymbol{Q}}(k)\right]_{j,i} \\ &= \left[\widehat{\boldsymbol{Q}}(k)\right]_{j,i} [\boldsymbol{u}(\theta)]_j \end{aligned} \tag{34}$$

It then follows from (32), Lemma 10 and (34) that

$$\begin{aligned}\lim_{k\to\infty} \left[\sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(\zeta+k\tau)\right]_{i,j} &= \frac{\tau}{\frac{d}{dz}\delta(z)|_{z=\theta} - 1} \sum_{l=1}^{M} [\widehat{\boldsymbol{\nu}}(\theta)]_l \sum_{k=0}^{\infty} \Big[\theta^{\zeta+k\tau-\tau_{j,l}} \widehat{\overline{\boldsymbol{E}}}(\zeta+k\tau-\tau_{j,l}) \\ &\qquad + \theta^{\zeta+k\tau-\tau_{j,l}} \widehat{\overline{\boldsymbol{F}}}(\zeta+k\tau-\tau_{j,l}+1) \left(\boldsymbol{I}-\widehat{\boldsymbol{F}}\right)^{-1} \widehat{\boldsymbol{E}}\Big]_{l,i} [\boldsymbol{u}(\theta)]_j. \end{aligned}$$

Thus using the definitions of $\widehat{\boldsymbol{\nu}}(\theta)$, $\widehat{\overline{\boldsymbol{E}}}(k)$, $\widehat{\overline{\boldsymbol{F}}}(k)$, $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{E}}$, we obtain (29). (30) follows from (12) and (29).

Note here that (4) for $k = 1$ yields

$$\boldsymbol{x}(1) = \boldsymbol{x}(0) \sum_{k=1}^{\infty} \boldsymbol{B}(k)\boldsymbol{G}^{k-1} + \boldsymbol{x}(1) \sum_{k=1}^{\infty} \boldsymbol{A}(k)\boldsymbol{G}^{k-1},$$

from which it follows that

$$\begin{aligned}\boldsymbol{x}(0)\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k &= \boldsymbol{x}(0)\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^{k-1}\cdot\boldsymbol{G} \\ &= \boldsymbol{x}(1)\left(\boldsymbol{G}-\sum_{k=1}^{\infty}\boldsymbol{A}(k)\boldsymbol{G}^k\right) \\ &= \boldsymbol{x}(1)\boldsymbol{A}(0). \end{aligned} \tag{35}$$

(31) is obtained by Lemma 1 and (35).

The rest is to show the positivity of the right hand sides of (29) and (30). Because $\boldsymbol{x}(0) > \mathbf{0}$, it is sufficient to show the positivity of the right hand side of (29), which is equivalent to the positivity of the right hand side of (32). In what follows, we shall show the latter.

Owing to Lemma 7, $\widehat{\boldsymbol{\nu}}(\theta)$ is rewritten to be

$$\widehat{\boldsymbol{\nu}}(\theta) = (\widehat{\boldsymbol{\nu}}_1(\theta), \mathbf{0}), \tag{36}$$

where $\widehat{\boldsymbol{\nu}}_1(\theta)$ is a positive vector satisfying $\widehat{\boldsymbol{\nu}}_1(\theta)\widehat{\boldsymbol{F}}_1^*(\theta) = \widehat{\boldsymbol{\nu}}_1(\theta)$.

On the other hand, from (14) we have

$$\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\boldsymbol{F}^{(n)}(k-1) > \boldsymbol{O}.$$

Therefore noting $(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} = \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1}\Delta_{\boldsymbol{\pi}}^{-1}[\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}]^T$, we obtain

$$(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} > \boldsymbol{O}. \tag{37}$$

As a result, it suffices to show

$$\sum_{l\in\mathcal{M}_1}\left[\sum_{k=0}^{\infty}\overline{\widehat{\boldsymbol{F}}}(\zeta+k\tau-\tau_{i,l}+1)\boldsymbol{e}\right]_l > 0. \tag{38}$$

Note here that there exists $l^* \in \mathcal{M}_1$ such that $\tau_{i,l^*} = \zeta + k\tau$ for some $k = k^*$. Further $\left[\overline{\widehat{\boldsymbol{F}}}(1)\boldsymbol{e}\right]_{l^*} > 0$, so that (38) holds. □

We now assume the aperiodicity (i.e., $\tau = 1$) to simplify Theorem 2.

Assumption 3 *In the semi-Markov process with the kernel $\theta^k\widehat{\boldsymbol{F}}(k)$, the recurrence time of every state j ($j \in \mathcal{M}_1$) in the irreducible absorbing class $\mathcal{M}_1$ is aperiodic, i.e., $\tau = 1$.*

Sufficient conditions for Assumption 3 is either (a) all diagonal elements of $\boldsymbol{A}(2)$ are positive, or (b) $\sum_{k=2}^{\infty}\boldsymbol{A}(k)\boldsymbol{e} > \mathbf{0}$ and $\boldsymbol{A}(1)$ is irreducible. To see those, note first that the aperiodicity in Assumption 3 holds if all the diagonal elements of $\widehat{\boldsymbol{F}}(1)$ are positive, and this is equivalent to the condition that all the diagonal elements of $\boldsymbol{F}(1)$ are positive. By definition,

$$\begin{aligned}\boldsymbol{F}(1) &= \boldsymbol{U}(1)(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\ &= \left(\boldsymbol{A}(2)+\sum_{m=3}^{\infty}\boldsymbol{A}(m)\boldsymbol{G}^{m-2}\right)\left(\boldsymbol{I}+\sum_{n=1}^{\infty}\left(\boldsymbol{A}(1)+\sum_{m=2}^{\infty}\boldsymbol{A}(m)\boldsymbol{G}^{m-1}\right)^n\right).\end{aligned}$$

Thus all the diagonal elements of $\boldsymbol{F}(1)$ are positive if either all diagonal elements of $\boldsymbol{A}(2)$ are positive, or $[\boldsymbol{A}(2)+\sum_{m=3}^{\infty}\boldsymbol{A}(m)\boldsymbol{G}^{m-2}]\boldsymbol{e} = \sum_{k=2}^{\infty}\boldsymbol{A}(k)\boldsymbol{e} > \mathbf{0}$ and $\boldsymbol{A}(1)$ is irreducible.

Theorem 3 *Under Assumptions 1, 2 and 3,*

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{\left(\boldsymbol{B}^*(\theta)-\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k\right)\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta), \tag{39}$$

and

$$\lim_{k\to\infty}\theta^k\overline{\boldsymbol{x}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{(\boldsymbol{x}(0)\boldsymbol{B}^*(\theta)-\boldsymbol{x}(1)\boldsymbol{A}(0))\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta). \tag{40}$$

Further the right hand sides of (39) and (40) are a positive matrix and positive vector, respectively.

Remark 8 *When $\boldsymbol{C}(0)=\boldsymbol{A}(0)$, (39) and (40) are reduced to*

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{(\boldsymbol{B}(0)+\boldsymbol{B}^*(\theta)-\boldsymbol{K})\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta),$$

$$\lim_{k\to\infty}\theta^k\overline{\boldsymbol{x}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{\boldsymbol{x}(0)(\boldsymbol{B}(0)+\boldsymbol{B}^*(\theta)-\boldsymbol{I})\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta),$$

because $\boldsymbol{x}(0)=\boldsymbol{x}(0)\boldsymbol{B}(0)+\boldsymbol{x}(1)\boldsymbol{A}(0)$ and $\boldsymbol{x}(0)=\boldsymbol{x}(0)\boldsymbol{K}$. Further when $\boldsymbol{C}(0)=\boldsymbol{A}(0)$ and $\boldsymbol{B}(k)=\boldsymbol{A}(k)$ for all $k=0,1,\ldots$, they are reduced to

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{(\theta\boldsymbol{I}-\boldsymbol{G})\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta),$$

$$\lim_{k\to\infty}\theta^k\overline{\boldsymbol{x}}(k)=\frac{(1-\rho)\boldsymbol{g}\boldsymbol{v}(\theta)}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\boldsymbol{u}(\theta),$$

because $\boldsymbol{A}^*(\theta)\boldsymbol{v}(\theta)=\delta(\theta)\boldsymbol{v}(\theta)=\theta\boldsymbol{v}(\theta)$ and $\boldsymbol{x}(0)=(1-\rho)\boldsymbol{g}$.

Proof. Applying (80) in Proposition 2 to (22) and using Lemma 10, we have

$$\lim_{k\to\infty}\theta^k\widehat{\boldsymbol{Q}}(k)=\frac{\boldsymbol{e}\widehat{\boldsymbol{\nu}}(\theta)}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\left(\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{E}}}}(k)+\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{F}}}}(k+1)(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}}\right). \tag{41}$$

On the other hand, by definition, we have

$$\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)=\left(\Delta_{\boldsymbol{\pi}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}\widehat{\boldsymbol{Q}}(k)\right)^T. \tag{42}$$

Thus we obtain

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\left(\sum_{k=0}^{\infty}\theta^k\sum_{m=k}^{\infty}\boldsymbol{E}(m)+\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\sum_{k=0}^{\infty}\theta^k\sum_{m=k+1}^{\infty}\boldsymbol{F}(m)\right)$$
$$\cdot(\boldsymbol{I}-\boldsymbol{U}(0))(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta)\boldsymbol{u}(\theta). \tag{43}$$

After lengthy but straightforward calculations, we can obtain

$$\sum_{k=0}^{\infty}\theta^k\sum_{m=k}^{\infty}\boldsymbol{E}(m)(\boldsymbol{I}-\boldsymbol{U}(0))(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta)$$
$$=\frac{1}{\theta-1}\left[\left(\boldsymbol{B}^*(\theta)-\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k\right)\right.$$
$$\left.-\left(\boldsymbol{B}-\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k+\boldsymbol{\beta}_B\boldsymbol{g}\right)(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1}(\boldsymbol{I}-\boldsymbol{G}/\theta)\right]\boldsymbol{v}(\theta), \tag{44}$$

$$\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\sum_{k=0}^{\infty}\theta^k\sum_{m=k+1}^{\infty}\boldsymbol{F}(m)(\boldsymbol{I}-\boldsymbol{U}(0))(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta)$$
$$=\frac{1}{\theta-1}\left(\boldsymbol{B}-\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k+\boldsymbol{\beta}_B\boldsymbol{g}\right)(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1}(\boldsymbol{I}-\boldsymbol{G}/\theta)\boldsymbol{v}(\theta). \tag{45}$$

Thus (39) follows from (43), (44) and (45).

Further

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{x}}(k)=\boldsymbol{x}(0)\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)$$
$$=\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\cdot\frac{\left(\boldsymbol{x}(0)\boldsymbol{B}^*(\theta)-\boldsymbol{x}(0)\sum_{k=1}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k\right)\boldsymbol{v}(\theta)}{\theta-1}\cdot\boldsymbol{u}(\theta). \tag{46}$$

(40) immediately follows from (35) and (46).

The remaining is to show the positivities of (39) and (40). Because the former implies the latter, we shall show that the right hand side of (39) is a positive matrix. It follows from (42) that

$$\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k)>\boldsymbol{O}\ \Leftrightarrow\ \lim_{k\to\infty}\theta^k\widehat{\boldsymbol{Q}}(k)>\boldsymbol{O},$$

and from (41),

$$\widehat{\boldsymbol{\nu}}(\theta)\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{F}}}}(k+1)(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}}>\boldsymbol{O}\ \Rightarrow\ \lim_{k\to\infty}\theta^k\widehat{\boldsymbol{Q}}(k)>\boldsymbol{O}.$$

Thus it suffices to show

$$\widehat{\boldsymbol{\nu}}(\theta)\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{F}}}}(k+1)(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}}>\boldsymbol{O}. \tag{47}$$

From (27), (36) and the irreducibility of $\widehat{\boldsymbol{F}}_1^*(\theta)$, $\widehat{\boldsymbol{\nu}}(\theta)\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{F}}}}(k+1)$ takes a form

$$\widehat{\boldsymbol{\nu}}(\theta)\sum_{k=0}^{\infty}\theta^k\overline{\overline{\widehat{\boldsymbol{F}}}}(k+1)=(\boldsymbol{x},\boldsymbol{0}), \tag{48}$$

with some positive vector $\boldsymbol{x}$. Thus (47) follows from (37) and (48), which completes the proof. □

Remark 9 *In Abate, Choudhury and Whitt (1994), a necessary condition for the geometric asymptotics is considered, and the same expression as the right hand side of (40) was obtained* *under the assumption that* $\lim_{k\to\infty}\theta^k \boldsymbol{x}(k)$ *exists, as well as Assumptions 1 and 2, whereas the positivity was posed as an assumption. On the other hand, we consider a sufficient condition for the geometric asymptotics. Note that, following our approach, the existence of* $\lim_{k\to\infty}\theta^k \boldsymbol{x}(k)$ *is shown to be equivalent to Assumption 3. Further, owing to Lemma 8, Assumptions 1 and 2 are a sufficient condition for the positivity of the right hand side of (40).*

4 Subexponential asymptotics

In this section, we consider the subexponential asymptotics of the steady-state solution $\boldsymbol{x}(k)$.

4.1 Preliminaries

For any nonnegative discrete random variable Y with finite mean y, let $Y(k)$ $(k = 0, 1, \dots)$ denote the probability distribution function of Y, i.e., $Y(k) = \Pr(Y \le k)$. Further let $\overline{Y}(k)$ denote the complementary distribution function of $Y(k)$, i.e.,

$$\overline{Y}(k) = 1 - Y(k), \qquad k = 0, 1, \dots.$$

We then define Y_e as the equilibrium random variable associated with Y. Note that the equilibrium distribution function $Y_e(k) = \Pr(Y_e \le k)$ is given by

$$Y_e(k) = \sum_{m=0}^{k} y^{-1}\overline{Y}(m), \qquad k = 0, 1, \dots.$$

Let $\overline{Y_e}(k)$ denote the complementary distribution function of $Y_e(k)$:

$$\overline{Y_e}(k) = 1 - Y_e(k), \qquad k = 0, 1, \dots.$$

A probability distribution function $Y(k)$ and the corresponding discrete random variable Y are called heavy-tailed if $\overline{Y}(k) > 0$ for all $k = 0, 1, \dots$ and

$$\lim_{k\to\infty} \frac{\overline{Y}(k+m)}{\overline{Y}(k)} = 1, \qquad \text{for all } m = 1, 2, \dots. \tag{49}$$

Let $Y^{(n)}(k)$ denote the n-fold convolution of $Y(k)$ with itself, i.e.,

$$Y^{(n)}(k) = \sum_{l=0}^{k} \Pr(Y = l) Y^{(n-1)}(k - l), \qquad n = 2, 3, \dots,$$

with $Y^{(1)}(k) = Y(k)$, and let $\overline{Y^{(n)}}(k)$ denote

$$\overline{Y^{(n)}}(k) = 1 - Y^{(n)}(k).$$

Then the heavy-tailed $Y(k)$ (and the corresponding random variable Y) is called subexponential if

$$\lim_{k\to\infty} \frac{\overline{Y^{(2)}}(k)}{\overline{Y}(k)} = 2.$$

Readers are referred to Embrechts, Klüppelberg and Mikosch (1997) Jelenković and Lazar (1998), Sigman (1999) for details. In what follows, we provide some lemmas that we need in the sequel.

Lemma 11 (Corollary 3.3 in Sigman (1999)) *If the equilibrium distribution function $Y_e(k)$ of $Y(k)$ is heavy-tailed,*

$$\lim_{k\to\infty} \frac{\overline{Y}(k)}{\overline{Y_e}(k)} = 0.$$

Lemma 12 (Theorem 2.7 in Embrechts, Klüppelberg and Mikosch (1997)) *Suppose $Y(k)$ is subexponential and $Z(k)$ is any distribution function such that*

$$\lim_{k\to\infty} \frac{\overline{Z}(k)}{\overline{Y}(k)} = c < \infty.$$

If $c > 0$, $Z(k)$ is subexponential.

Lemma 13 (Lemma 10 in Jelenković and Lazar (1998)) *Let Y denote a subexponential random variable and let Z_i's ($i = 1, \ldots, n$) be independent random variables. Suppose*

$$\lim_{k\to\infty} \frac{\Pr(Z_i > k)}{\Pr(Y > k)} = c_i \in [0, \infty), \qquad i = 1, \ldots, n.$$

(a). The following equality holds.

$$\lim_{k\to\infty} \frac{\Pr(Z_1 + \cdots + Z_n > k)}{\Pr(Y > k)} = \sum_{i=1}^{n} c_i. \tag{50}$$

(b) Let c^* denote $\max(c_1, \ldots, c_n)$. Then for each $\epsilon > 0$, there exists some constant $N^* = N(\epsilon, c^*) < \infty$ such that

$$\frac{\Pr(Z_1 + \cdots + Z_n > k)}{\Pr(Y > k)} \le N^*(1+\epsilon)^n, \qquad \textit{for all } k = 0, 1, \ldots,$$

where N^* is independent of n.

Remark 10 *All the results in Lemmas 11 to 13 were originally stated for general random variables, and the statements here are adaptations to discrete random variables.*

Let $F_{i,j}$ and $E_{i,j}$ denote the (i,j)th elements of $\boldsymbol{F}$ and $\boldsymbol{E}$, respectively. Also, let $F_{i,j}(k)$ and $E_{i,j}(k)$ denote the (i,j)th elements of $\boldsymbol{F}(k)$ and $\boldsymbol{E}(k)$, respectively.

Lemma 14

(a). Let $\boldsymbol{\pi}^*$ denote a $1 \times M$ vector whose jth element represents $\Pr(X \ge 1, S = j)$ in steady state. Then $\boldsymbol{\pi}^*$ and $\boldsymbol{x}(0)$ are related by

$$\boldsymbol{\pi}^* = \boldsymbol{x}(0)\boldsymbol{E}(\boldsymbol{I} - \boldsymbol{F})^{-1}. \tag{51}$$

(b). If $F_{i,j} > 0$ for some $i, j \in \mathcal{M}$, $F_{i,j}(k)/F_{i,j}$ is a mass function for a proper nonnegative random variable. Similarly, if $E_{i,j} > 0$ for some $i \in \mathcal{M}_0$ and some $j \in \mathcal{M}$, $E_{i,j}(k)/E_{i,j}$ is a mass function for a proper nonnegative random variable.

(c) $\boldsymbol{\pi}$ and $\boldsymbol{F}$ are related by

$$\boldsymbol{\pi}(\boldsymbol{I}-\boldsymbol{F}) = (1-\rho)\boldsymbol{g}\left(\boldsymbol{I}-\boldsymbol{U}(0)\right)^{-1}.$$

Proof. Part (a) immediately follows from (17), because

$$\boldsymbol{\pi}^* = \sum_{k=1}^{\infty} \boldsymbol{x}(k).$$

Part (b) for $F_{i,j}(k)/F_{i,j}$ comes from the fact that $F_{i,j}(k)$ is nonnegative and by definition

$$\sum_{k=0}^{\infty} \frac{F_{i,j}(k)}{F_{i,j}} = 1,$$

if $F_{i,j} > 0$. Part (b) for $E_{i,j}(k)/E_{i,j}$ can be shown in the same way. Finally, we consider part (c). Pre-multiplying both sides of (15) by $\boldsymbol{\pi}$, and noting $\boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{\pi}$ and $\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1} = \boldsymbol{g}$, we have part (c). □

4.2 Results

Let $\overline{\boldsymbol{A}}(k)$ and $\overline{\boldsymbol{B}}(k)$ $(k = 0, 1, \dots)$ denote

$$\overline{\boldsymbol{A}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{A}(l), \qquad \overline{\boldsymbol{B}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{B}(l).$$

Assumption 4 *There exists a random variable Y with finite mean y such that*

(a) the equilibrium random variable Y_e of Y is subexponential,

(b) for $\overline{Y}(k) = \Pr(Y > k)$, the limits

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}(k)} = \boldsymbol{C}^A/y, \qquad \lim_{k\to\infty} \frac{\overline{\boldsymbol{B}}(k)}{\overline{Y}(k)} = \boldsymbol{C}^B/y,$$

exist, where $\boldsymbol{C}^A$ and $\boldsymbol{C}^B$ are finite nonnegative matrices, and

(c) $\boldsymbol{C}^A \neq \boldsymbol{O}$ or $\boldsymbol{C}^B \neq \boldsymbol{O}$.

Note that the subexponentiality of Y_e implies $\Pr(Y > k) > 0$ for all $k = 0, 1, \dots$, because $\Pr(Y_e > k) = \sum_{m=k+1}^{\infty} \overline{Y}(m)/E[Y] > 0$ for all $k = 0, 1, \dots$.

Remark 11 *Suppose Assumption 4 holds and the (i,j)th element $C_{i,j}^A$ of $\boldsymbol{C}^A$ is positive for some $i, j \in \mathcal{M}$. Let $A_{i,j}(k)$ and $A_{i,j}$ denote the (i,j)th elements of $\boldsymbol{A}(k)$ and $\boldsymbol{A}$, respectively. Then $a_{i.j}(k) = A_{i.j}(k)/A_{i.j}$ is a probability mass function of a proper random variable $J^A(i,j)$ representing the amount of a jump given the state transition from state i $(i > 0)$ to state j. It can be shown that*

$$\lim_{k\to\infty} \frac{\Pr(J_e^A(i,j) > k)}{\Pr(Y_e > k)} = \frac{y C_{i,j}^A}{\sum_{k=1}^{\infty} k a_{i,j}(k)}.$$

Thus Lemma 12 implies that the equilibrium distribution of the amount $J^A(i,j)$ of the conditional jump is subexponential. The same discussion is applicable to $\boldsymbol{B}(k)$, too. Note that the subexponentiality of the equilibrium random variable $J_e^A(i,j)$ implies neither the subexponentiality nor the heavy-tailed property of the original random variable $J^A(i,j)$ (Sigman 1999). Thus, even when Y is light-tailed, $\boldsymbol{x}(k)$ can be subexponential. Note that, in Asmussen and Møller (1999), the subexponentiality of Y is assumed, even though they study a more general structured Markov chain including the M/G/1 type as a special case.

Let $\overline{Y_e}(k)$ denote the complementary equilibrium distribution function associated with Y:

$$\overline{Y_e}(k) = y^{-1} \sum_{m=0}^{\infty} \overline{Y}(k+m+1).$$

Theorem 4 *Under Assumptions 1 and 4, $\overline{\boldsymbol{x}}(k)$ satisfies*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \frac{\boldsymbol{x}(0)\boldsymbol{C}^B\boldsymbol{e} + \boldsymbol{\pi}^*\boldsymbol{C}^A\boldsymbol{e}}{1-\rho} \cdot \boldsymbol{\pi}.$$

Remark 12 *Owing to Lemma 12, Theorem 4 implies that the $\boldsymbol{x}(k)$ is subexponential because $Y_e(k)$ is subexponential.*

Remark 13 *When $\boldsymbol{C}(0) = \boldsymbol{A}(0)$ and $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \ldots$, we have $\boldsymbol{C}^B = \boldsymbol{C}^A$ and $\boldsymbol{\pi}^* = \boldsymbol{\pi} - \boldsymbol{x}(0)$, so that*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \frac{\boldsymbol{\pi}\boldsymbol{C}^A\boldsymbol{e}}{1-\rho} \cdot \boldsymbol{\pi}.$$

Proof. Let $\overline{\boldsymbol{F}}(k)$ and $\overline{\boldsymbol{E}}(k)$ denote

$$\overline{\boldsymbol{F}}(k) = \sum_{l=k}^{\infty} \boldsymbol{F}(l), \qquad \overline{\boldsymbol{E}}(k) = \sum_{l=k}^{\infty} \boldsymbol{E}(l),$$

respectively. Recall that $\overline{\boldsymbol{E}}(k) = \overline{\boldsymbol{F}^{(1)}}(k)$.

Lemma 15 *Under Assumptions 1 and 4,*

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{\overline{Y}_e(k)} = \boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \qquad \lim_{k\to\infty} \frac{\overline{\boldsymbol{E}}(k)}{\overline{Y}_e(k)} = \boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}.$$

The proof of Lemma 15 is given in Appendix A.6.

Lemma 16 *Under Assumptions 1 and 4,*

$$\begin{aligned}\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)} &= \boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1} \\ &\quad + \sum_{m=1}^{n-1} \boldsymbol{E}\boldsymbol{F}^{m-1}\boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1-m}. \end{aligned} \tag{52}$$

Further for each $\epsilon > 0$ and $n = 1, 2, \ldots$, there exists some constant $N^* < \infty$ such that

$$\frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)} \le N^*(1+\epsilon)^n \boldsymbol{E}\boldsymbol{F}^{n-1}, \qquad \text{for all } k \ge 0, \tag{53}$$

where N^* is independent of n.

Proof. We define random variables $H^F_{i,j}$ ($i,j \in \mathcal{M}$) and $H^E_{i,j}$ ($i \in \mathcal{M}_0,\ j \in \mathcal{M}$) whose mass functions $H^F_{i,j}(k)$ and $H^E_{i,j}(k)$ are given by

$$H^F_{i,j}(k) = \begin{cases} \dfrac{F_{i,j}(k)}{F_{i,j}}, & \text{if } F_{i,j} > 0, \\ 1, & \text{otherwise,} \end{cases} \qquad H^E_{i,j}(k) = \begin{cases} \dfrac{E_{i,j}(k)}{E_{i,j}}, & \text{if } E_{i,j} > 0, \\ 1, & \text{otherwise,} \end{cases}$$

respectively. Recall that $H^F_{i,j}(k)$ and $H^E_{i,j}(k)$ are mass functions for proper random variables (see Lemma 14 (b)). We define $\overline{H^F_{i,j}}(k)$ and $\overline{H^E_{i,j}}(k)$ as

$$\overline{H^F_{i,j}}(k) = \sum_{m=k+1}^{\infty} H^F_{i,j}(m), \qquad \overline{H^E_{i,j}}(k) = \sum_{m=k+1}^{\infty} H^E_{i,j}(m).$$

It then follows from Lemma 15 that

$$\lim_{k\to\infty} \frac{\overline{H^F_{i,j}}(k)}{\overline{Y_e}(k)} = D^F_{i,j} \quad (i,j \in \mathcal{M}), \qquad \lim_{k\to\infty} \frac{\overline{H^E_{i,j}}(k)}{\overline{Y_e}(k)} = D^E_{i,j} \quad (i \in \mathcal{M}_0, j \in \mathcal{M}),$$

where

$$D^F_{i,j} = \begin{cases} \displaystyle\sum_{l=1}^{M}\sum_{m=1}^{M} \frac{[\boldsymbol{C}^A]_{i,l}[\boldsymbol{g}]_m[(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{m,j}}{F_{i,j}}, & \text{if } F_{i,j} \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$D^E_{i,j} = \begin{cases} \displaystyle\sum_{l=1}^{M}\sum_{m=1}^{M} \frac{[\boldsymbol{C}^B]_{i,l}[\boldsymbol{g}]_m[(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{m,j}}{E_{i,j}}, & \text{if } E_{i,j} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\begin{aligned} \frac{\left[\overline{\boldsymbol{F}^{(n)}}(k)\right]_{i,j}}{\overline{Y_e}(k)} &= \sum_{l_1\in\mathcal{M}} \cdots \sum_{l_{n-1}\in\mathcal{M}} E_{i,l_1}F_{l_1,l_2}\cdots F_{l_{n-2},l_{n-1}}F_{l_{n-1},j} \\ &\qquad \cdot \frac{\Pr(H^E_{i,l_1} + H^F_{l_1,l_2} + \cdots + H^F_{l_{n-2},l_{n-1}} + H^F_{l_{n-1},j} > k)}{\overline{Y_e}(k)}. \end{aligned} \tag{54}$$

Applying Lemma 13 (a) to (54), we have

$$\begin{aligned} \lim_{k\to\infty} \frac{\left[\overline{\boldsymbol{F}^{(n)}}(k)\right]_{i,j}}{\overline{Y_e}(k)} &= \sum_{l_1\in\mathcal{M}} \cdots \sum_{l_{n-1}\in\mathcal{M}} E_{i,l_1}F_{l_1,l_2}\cdots F_{l_{n-2},l_{n-1}}F_{l_{n-1},j} \\ &\qquad \cdot \left(D^E_{i,l_1} + D^F_{l_1,l_2} + \cdots + D^F_{l_{n-2},l_{n-1}} + D^F_{l_{n-1},j}\right). \end{aligned}$$

Therefore, noting for all $i,j \in \mathcal{M}$

$$F_{i,j}D^F_{i,j} = [\boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{i,j},$$

and for all $i \in \mathcal{M}_0$ and all $j \in \mathcal{M}$

$$E_{i,j}D^E_{i,j} = [\boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{i,j},$$

we obtain (52).

On the other hand, the second part of Lemma 16 is obtained by applying Lemma 13 (b) to (54), where N^* depends only on ϵ and $H^* = \max(D^F_{i,j}\ (i,j \in \mathcal{M}), D^E_{i,j}\ (i \in \mathcal{M}_0, j \in \mathcal{M}))$, and independent of n. □

Because the spectral radius of nonnegative matrix $\boldsymbol{F}$ is less than one (see Lemma 2), we can choose ϵ in such a way that the spectral radius of $(1+\epsilon)\boldsymbol{F}$ is less than one. It then follows from (53) and bounded convergence theorem that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \boldsymbol{x}(0) \sum_{n=1}^{\infty} \lim_{k\to\infty} \frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)}.$$

Then applying (52) to the above equation, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} &= \boldsymbol{x}(0)\boldsymbol{C}^B\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1} \\
&\qquad + \boldsymbol{x}(0)\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \boldsymbol{E}\boldsymbol{F}^{m-1}\boldsymbol{C}^A\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1-m} \\
&= \boldsymbol{x}(0)\Big[\boldsymbol{C}^B\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1} \\
&\qquad + \sum_{m=1}^{\infty} \boldsymbol{E}\boldsymbol{F}^{m-1}\boldsymbol{C}^A\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\sum_{n=m+1}^{\infty}\boldsymbol{F}^{n-1-m}\Big] \\
&= \boldsymbol{x}(0)\Big[\boldsymbol{C}^B\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1} \\
&\qquad + \boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\boldsymbol{C}^A\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1}\Big] \\
&= \boldsymbol{x}(0)\Big[\boldsymbol{C}^B + \boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1}\boldsymbol{C}^A\Big]\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1} \\
&= \Big[\boldsymbol{x}(0)\boldsymbol{C}^B + \boldsymbol{\pi}^*\boldsymbol{C}^A\Big]\boldsymbol{eg}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1}.
\end{aligned}$$

Finally, applying Lemma 14 (a) and (c) to the above equation, we obtain

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \Big[\boldsymbol{x}(0)\boldsymbol{C}^B + \boldsymbol{\pi}^*\boldsymbol{C}^A\Big]\boldsymbol{e}\frac{\boldsymbol{\pi}}{1-\rho},$$

which completes the proof. □

A Proofs of Lemmas

A.1 Proof of Lemma 1

By definition, we have

$$\begin{aligned}
\boldsymbol{F} &= \sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-1}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{m=2}^{\infty} \boldsymbol{A}(m)\sum_{k=0}^{m-2}\boldsymbol{G}^k(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \boldsymbol{A}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} - \sum_{m=1}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-1} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \sum_{m=1}^{\infty} \boldsymbol{A}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} + \boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{U}(0))^{-1}
\end{aligned}
$$

Note here that

$$
\begin{aligned}
\sum_{m=1}^{\infty} \boldsymbol{A}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k (\boldsymbol{I} - \boldsymbol{G} + \boldsymbol{eg}) &= \sum_{m=1}^{\infty} \boldsymbol{A}(m)(\boldsymbol{I} - \boldsymbol{G}^m) + \sum_{m=1}^{\infty} m\boldsymbol{A}(m)\boldsymbol{eg} \\
&= \boldsymbol{A} - \boldsymbol{G} + \boldsymbol{\beta}_A \boldsymbol{g}.
\end{aligned}
$$

Thus

$$
\boldsymbol{F} = (\boldsymbol{A} - \boldsymbol{G} + \boldsymbol{\beta}_A \boldsymbol{g})(\boldsymbol{I} - \boldsymbol{G} + \boldsymbol{eg})^{-1} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} + \boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{U}(0))^{-1},
$$

from which (15) follows.

Similarly,

$$
\begin{aligned}
\boldsymbol{E} &= \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \boldsymbol{B}(m) \boldsymbol{G}^{m-k-1} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \sum_{m=1}^{\infty} \boldsymbol{B}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k (\boldsymbol{I} - \boldsymbol{U}(0))^{-1},
\end{aligned}
$$

and

$$
\sum_{m=1}^{\infty} \boldsymbol{B}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k (\boldsymbol{I} - \boldsymbol{G} + \boldsymbol{eg}) = \boldsymbol{B} - \sum_{m=1}^{\infty} \boldsymbol{B}(m) \boldsymbol{G}^m + \boldsymbol{\beta}_A \boldsymbol{g},
$$

from which (16) immediately follows. □

A.2 Proof of Lemma 6

Because $\widetilde{\boldsymbol{F}}(0) = \boldsymbol{O}$, we have

$$
\begin{aligned}
\widetilde{\boldsymbol{F}}^*(z) &= \sum_{k=1}^{\infty} (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{U}}(k) z^k \\
&= \sum_{k=0}^{\infty} (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{U}}(k) z^k - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{U}}(0) \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \widetilde{\boldsymbol{G}}^{m-k-1} \widetilde{\boldsymbol{A}}(m) z^k + \boldsymbol{I} - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{m=0}^{\infty} \widetilde{\boldsymbol{G}}^m \sum_{k=m+1}^{\infty} \widetilde{\boldsymbol{A}}(k) z^{k-m-1} + \boldsymbol{I} - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}. \qquad (55)
\end{aligned}
$$

Note here that

$$
\begin{aligned}
\sum_{m=0}^{\infty} \widetilde{\boldsymbol{G}}^m \sum_{k=m+1}^{\infty} \widetilde{\boldsymbol{A}}(k) z^{k-m-1} &= \sum_{k=1}^{\infty} \left(\sum_{m=0}^{k-1} \widetilde{\boldsymbol{G}}^m \right) \widetilde{\boldsymbol{A}}(k) z^{k-m-1} \\
&= \sum_{k=1}^{\infty} \left(\sum_{m=0}^{k-1} (\widetilde{\boldsymbol{G}}/z)^m \right) \widetilde{\boldsymbol{A}}(k) z^{k-1}.
\end{aligned}
$$

Because $z > 1$ and the maximum eigenvalue of $\widetilde{\boldsymbol{G}}$ is equal to one, $(\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1}$ exists. Thus

$$\begin{aligned}
&\sum_{m=0}^{\infty} \widetilde{\boldsymbol{G}}^m \sum_{k=m+1}^{\infty} \widetilde{\boldsymbol{A}}(k) z^{k-m-1} \qquad (56)\\
&\quad = \sum_{k=1}^{\infty} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \left(\boldsymbol{I} - (\widetilde{\boldsymbol{G}}/z)^k\right) \widetilde{\boldsymbol{A}}(k) z^{k-1} \\
&\quad = (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \left(\sum_{k=1}^{\infty} \widetilde{\boldsymbol{A}}(k) z^{k-1} - \sum_{k=1}^{\infty} (\widetilde{\boldsymbol{G}}/z)^k \widetilde{\boldsymbol{A}}(k)/z \right) \\
&\quad = (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \left((\widetilde{\boldsymbol{A}}^*(z) - \widetilde{\boldsymbol{A}}(0))/z - (\widetilde{\boldsymbol{G}} - \widetilde{\boldsymbol{A}}(0))/z \right) \\
&\quad = (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} (\widetilde{\boldsymbol{A}}^*(z) - \widetilde{\boldsymbol{G}})/z \\
&\quad = (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \widetilde{\boldsymbol{A}}^*(z)/z + \boldsymbol{I} - (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1}. \qquad (57)
\end{aligned}$$

Substituting (57) into (55), we obtain

$$\begin{aligned}
\widetilde{\boldsymbol{F}}^*(z) &= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \widetilde{\boldsymbol{A}}^*(z)/z - (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} + \boldsymbol{I} \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} \left(\widetilde{\boldsymbol{A}}^*(z)/z - \boldsymbol{I} \right) + \boldsymbol{I},
\end{aligned}$$

from which (25) follows.

Similarly, we have

$$\begin{aligned}
\widetilde{\boldsymbol{E}}^*(z) &= \sum_{k=0}^{\infty} (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \widetilde{\boldsymbol{V}}(k) z^k \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \widetilde{\boldsymbol{G}}^{m-k-1} \widetilde{\boldsymbol{B}}(m) z^k \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{m=0}^{\infty} \widetilde{\boldsymbol{G}}^m \sum_{k=m+1}^{\infty} \widetilde{\boldsymbol{B}}(k) z^{k-m-1} \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{k=1}^{\infty} \left(\sum_{m=0}^{k-1} (\widetilde{\boldsymbol{G}}/z)^m \right) \widetilde{\boldsymbol{B}}(k) z^{k-1} \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} \sum_{k=1}^{\infty} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/z)^{-1} (\boldsymbol{I} - (\widetilde{\boldsymbol{G}}/z)^k) \widetilde{\boldsymbol{B}}(k) z^{k-1},
\end{aligned}$$

from which (26) follows. □

A.3 Proof of Lemma 8

Note first that the statement of the lemma depends only on the location of positive elements of $\boldsymbol{A}(k)$ $(k = 0, 1, \ldots)$, and not on their magnitude. Thus instead of the $\boldsymbol{A}(k)$, we consider

$$\boldsymbol{P}(k, z) = \frac{z^k}{\delta(z)} \Delta_{\boldsymbol{v}(z)}^{-1} \boldsymbol{A}(k) \Delta_{\boldsymbol{v}(z)}, \qquad k = 0, 1, \ldots,$$

where $0 < z < r_A$. Note here that

$$\sum_{k=0}^{\infty} \boldsymbol{P}(k, z) = \frac{1}{\delta(z)} \Delta_{\boldsymbol{v}(z)}^{-1} \boldsymbol{A}^*(z) \Delta_{\boldsymbol{v}(z)},$$

so that

$$\sum_{k=0}^{\infty} \boldsymbol{P}(k,z)\boldsymbol{e} = \boldsymbol{e}, \qquad \boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)} \sum_{k=0}^{\infty} \boldsymbol{P}(k,z) = \boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)}$$

Therefore $\sum_{k=0}^{\infty} \boldsymbol{P}(k,z)$ is a stochastic matrix with the invariant probability vector $\boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)}$, because $\boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)}\boldsymbol{e} = 1$. We then consider a Markov chain $\{(Z_n, S_n);\ n = 0, 1, \ldots\}$ whose transition probability matrix is given by

$$\begin{pmatrix} \boldsymbol{P}(0,z) & \boldsymbol{P}(1,z) & \boldsymbol{P}(2,z) & \boldsymbol{P}(3,z) & \cdots \\ \boldsymbol{P}(0,z) & \boldsymbol{P}(1,z) & \boldsymbol{P}(2,z) & \boldsymbol{P}(3,z) & \cdots \\ \boldsymbol{O} & \boldsymbol{P}(0,z) & \boldsymbol{P}(1,z) & \boldsymbol{P}(2,z) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{P}(0,z) & \boldsymbol{P}(1,z) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Because $\boldsymbol{A}$ is irreducible, $\sum_{k=0}^{\infty} \boldsymbol{P}(k,z)$ is also irreducible. The mean drift of this Markov chain is given by

$$\begin{aligned} \boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)} \sum_{k=0}^{\infty} k\boldsymbol{P}(k,z)\boldsymbol{e} &= \frac{z}{\delta(z)}\boldsymbol{u}(z) \sum_{k=0}^{\infty} kz^{k-1}\boldsymbol{A}(k)\boldsymbol{v}(z) \\ &= \frac{z}{\delta(z)}\boldsymbol{u}(z)\frac{d}{dz}\boldsymbol{A}^*(z)\boldsymbol{v}(z). \end{aligned}$$

Because (Neuts 1989)

$$\boldsymbol{u}(z)\frac{d}{dz}\boldsymbol{A}^*(z)\boldsymbol{v}(z) = \frac{d}{dz}\delta(z),$$

we have

$$\boldsymbol{u}(z)\Delta_{\boldsymbol{v}(z)} \sum_{k=0}^{\infty} k\boldsymbol{P}(k,z)\boldsymbol{e} = \frac{z}{\delta(z)}\frac{d}{dz}\delta(z).$$

Thus for $z = \theta$, we have

$$\boldsymbol{u}(\theta)\Delta_{\boldsymbol{v}(\theta)} \sum_{k=0}^{\infty} k\boldsymbol{P}(k,\theta)\boldsymbol{e} = \frac{d}{dz}\delta(z)|_{z=\theta} > 1,$$

where the inequality follows from $\theta = \delta(\theta)$, the convexity of $\delta(z)$ and $\frac{d}{dz}\delta(z)|_{z=1} = \rho < 1$. Because the mean drift of the $\boldsymbol{P}(k,\theta)$ is greater than one and $\sum_{k=0}^{\infty} \boldsymbol{P}(k,\theta)$ is irreducible, there exists n such that $P(Z_n > k, \mid Z_0 = 0, S_0 = i) > 0$ for all $k = 0, 1, \ldots$ and all $i \in \mathcal{M}$, from which Lemma 8 follows. □

A.4 Proof of Lemma 9

Note that

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k \overline{\overline{\boldsymbol{E}}}(k) &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{k=0}^{\infty} \theta^k \overline{\boldsymbol{E}}(k) \\ &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{k=0}^{\infty} \theta^k \sum_{m=k}^{\infty} \widetilde{\boldsymbol{E}}(m) \\ &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \theta^k \widetilde{\boldsymbol{E}}(m) \end{aligned}$$

$$
\begin{aligned}
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{m=0}^{\infty} \frac{\theta^{m+1}-1}{\theta-1} \widetilde{\boldsymbol{E}}(m) \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{\theta}{\theta-1} \widetilde{\boldsymbol{E}}^*(\theta) - \frac{1}{\theta-1} \widetilde{\boldsymbol{E}} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{\infty} \theta^k \overline{\widehat{\boldsymbol{F}}}(k+1) &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{k=0}^{\infty} \theta^k \overline{\widetilde{\boldsymbol{F}}}(k+1) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \frac{1}{\theta} \sum_{k=1}^{\infty} \theta^k \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}(m) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \frac{1}{\theta} \sum_{m=1}^{\infty} \sum_{k=1}^{m} \theta^k \widetilde{\boldsymbol{F}}(m) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{m=1}^{\infty} \frac{\theta^m - 1}{\theta - 1} \widetilde{\boldsymbol{F}}(m) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{1}{\theta-1} \widetilde{\boldsymbol{F}}^*(\theta) - \frac{1}{\theta-1} \widetilde{\boldsymbol{F}} \right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T},
\end{aligned}
$$

which completes the proof. □

A.5 Proof of Lemma 10

Recall that

$$
\sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \sum_{k=1}^{\infty} k\theta^k \widetilde{\boldsymbol{F}}(k) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.
$$

It follows from Lemma 6 that

$$
\begin{aligned}
\sum_{k=1}^{\infty} k\theta^k \widetilde{\boldsymbol{F}}(k) &= \theta \frac{d}{dz} \widetilde{\boldsymbol{F}}^*(z)|_{z=\theta} \\
&= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)^{-1} (\widetilde{\boldsymbol{G}}/\theta) (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)^{-1} (\widetilde{\boldsymbol{A}}^*(\theta)/\theta - \boldsymbol{I}) \\
&\quad + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)^{-1} \left(\frac{d}{dz} \widetilde{\boldsymbol{A}}^*(z)|_{z=\theta} - \widetilde{\boldsymbol{A}}^*(\theta)/\theta \right).
\end{aligned}
$$

Using

$$
(\widetilde{\boldsymbol{A}}^*(\theta)/\theta - \boldsymbol{I}) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \boldsymbol{e} = (\widetilde{\boldsymbol{A}}^*(\theta)/\theta - \boldsymbol{I}) \widetilde{\boldsymbol{u}}(\theta)^T = \boldsymbol{0},
$$

we have

$$
\begin{aligned}
\widehat{\boldsymbol{\nu}}(\theta) \sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k) \boldsymbol{e} &= \widehat{\boldsymbol{\nu}}(\theta) \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{G}}/\theta)^{-1} \\
&\qquad \cdot \left(\frac{d}{dz} \widetilde{\boldsymbol{A}}^*(z)|_{z=\theta} - \widetilde{\boldsymbol{A}}^*(\theta)/\theta \right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \boldsymbol{e} \\
&= \frac{\widetilde{\boldsymbol{v}}(\theta)^T}{c(\widehat{\boldsymbol{\nu}}(\theta))} \left(\frac{d}{dz} \widetilde{\boldsymbol{A}}^*(z)|_{z=\theta} - 1 \right) \widetilde{\boldsymbol{u}}(\theta)^T. \qquad (58)
\end{aligned}
$$

Note here that $\widetilde{\boldsymbol{v}}(\theta)^T \widetilde{\boldsymbol{u}}(\theta)^T = 1$ and therefore (Neuts 1989)

$$
\widetilde{\boldsymbol{v}}(\theta)^T \frac{d}{dz} \widetilde{\boldsymbol{A}}^*(z)|_{z=\theta} \widetilde{\boldsymbol{u}}(\theta)^T = \frac{d}{dz} \delta(z)|_{z=\theta}. \qquad (59)
$$

Substituting (59) into (58) yields the equality in (33). The inequality in (33) comes from the convexity of $\delta(z)$ and $d\delta(z)/dz|_{z=1} = \rho$. □

A.6 Proof of Lemma 15

By definition, we have

$$\begin{aligned}
\overline{\boldsymbol{F}}(k) &= \sum_{l=k}^{\infty} \boldsymbol{U}(l)(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \boldsymbol{U}(k+l)(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \sum_{m=k+l+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-l-1}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \boldsymbol{A}(k+l+1+m)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{m=0}^{\infty} \overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}.
\end{aligned}$$

Similarly we have

$$\overline{\boldsymbol{E}}(k) = \sum_{m=0}^{\infty} \overline{\boldsymbol{B}}(m+k)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}.$$

In what follows, we shall show

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} = \boldsymbol{C}^A \boldsymbol{e}\boldsymbol{g},$$

which is equivalent to the first equation of the lemma. We note that for any integer $n > 0$

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} &= \sum_{m=0}^{n-1} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} \\
&\le \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}_e(k)} \sum_{m=0}^{n-1} \boldsymbol{G}^m + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} \\
&= \frac{\overline{Y}(k)}{\overline{Y}_e(k)} \cdot \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}(k)} \sum_{m=0}^{n-1} \boldsymbol{G}^m + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)}.
\end{aligned}$$

Therefore, we obtain for any $n > 0$

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} &\le 0 \cdot \frac{\boldsymbol{C}^A}{y} \sum_{m=0}^{n-1} \boldsymbol{G}^m + \limsup_{k\to\infty} \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} \\
&= \limsup_{k\to\infty} \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)},
\end{aligned} \tag{60}$$

where we use Lemma 11.

Note here that $\boldsymbol{G}$ is a stochastic matrix with an invariant probability vector $\boldsymbol{g}$, so that

$$\lim_{m\to\infty} \boldsymbol{G}^m = \boldsymbol{e}\boldsymbol{g}.$$

Thus for any $\epsilon > 0$, we can choose an integer n^* ($n^* \ge 1$) such that

$$\boldsymbol{G}^m \le (1+\epsilon)\boldsymbol{e}\boldsymbol{g}, \qquad \text{for all } m \ge n^*. \tag{61}$$

It then follows from (61) that

$$\begin{aligned}\sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y}_e(k)} &\le (1+\epsilon) \sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{eg}}{\overline{Y}_e(k)} \\ &\le (1+\epsilon) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)\boldsymbol{eg}}{\overline{Y}_e(k)}.\end{aligned}$$

To proceed further, we need the following lemma.

Lemma 17 *Under Assumptions 1 and 4, we have*

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} = \boldsymbol{C}^A, \qquad \lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{B}}(k+m+1)}{\overline{Y}_e(k)} = \boldsymbol{C}^B.$$

Proof. Assumption 4 (b) implies that for any $\epsilon > 0$, there exists $k^* > 0$ enough large so that

$$\frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}(k+m+1)} \le (1+\epsilon)\frac{\boldsymbol{C}^A}{y} \qquad \text{for all } k \ge k^*, m \ge 0,$$

from which it follows that for all $k \ge k^*$,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}(k+m+1)} \cdot \frac{\overline{Y}(k+m+1)}{\overline{Y}_e(k)} \\ &\le (1+\epsilon)\frac{\boldsymbol{C}^A}{y} \sum_{m=0}^{\infty} \frac{\overline{Y}(k+m+1)}{\overline{Y}_e(k)} \\ &= (1+\epsilon)\boldsymbol{C}^A.\end{aligned}$$

Since ϵ is arbitrary, we have

$$\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} \le \boldsymbol{C}^A. \tag{62}$$

On the other hand, the lower bound is obtained as follows. We first note that

$$\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} \ge \sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)}, \qquad \text{for all } j = 0, 1, \ldots.$$

Thus it suffices to show that

$$\liminf_{k\to\infty} \sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} \ge \boldsymbol{C}^A, \qquad \text{for some } j \ge 0. \tag{63}$$

Note that

$$\begin{aligned}\sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}_e(k)} &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1+j)}{\overline{Y}_e(k)} \\ &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1+j)}{\overline{Y}(k+m+1+j)} \cdot \frac{\overline{Y}(k+m+1+j)}{\overline{Y}_e(k)}.\end{aligned} \tag{64}$$

It follows from Assumption 4 (b) that for any $\epsilon > 0$, there exists $j^* > 0$ such that

$$\frac{\overline{\boldsymbol{A}}(k+j^*)}{\overline{Y}(k+j^*)} \ge (1-\epsilon)\frac{\boldsymbol{C}^A}{y}, \qquad \text{for all } k \ge 0. \tag{65}$$

Thus using (64) and (65), we obtain that

$$\begin{aligned}\sum_{m=j^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} &\ge (1-\epsilon)\boldsymbol{C}^A \sum_{m=0}^{\infty} \frac{\overline{Y}(k+m+1+j^*)}{y\overline{Y_e}(k)} \\ &= (1-\epsilon)\boldsymbol{C}^A \frac{\overline{Y_e}(k+j^*)}{\overline{Y_e}(k)}. \end{aligned} \tag{66}$$

Since ϵ is arbitrary, (63) immediately follows from (66) and the heavy-tailed property (49), which completes the proof for the first equation of Lemma 17. The second equation can be proved exactly in the same way, so that it is omitted. □

Thus using (60) and Lemma 17, we have

$$\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \le \limsup_{k\to\infty} \sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \le \boldsymbol{C}^A \boldsymbol{eg}.$$

On the other hand, the lower bound is obtained as follows. We first note that

$$\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \ge \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)}, \qquad \text{for all } n = 0, 1, \ldots,$$

and for any $\epsilon_1 > 0$, there exists an integer m^* large enough so that

$$\boldsymbol{G}^m \ge (1-\epsilon_1)\boldsymbol{eg}, \qquad \text{for all } \ m \ge m^*+1.$$

Therefore it suffices to show that

$$\liminf_{k\to\infty} \sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \ge \boldsymbol{C}^A \boldsymbol{eg}, \tag{67}$$

Note here that

$$\begin{aligned}\sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} &\ge (1-\epsilon_1) \sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{eg}}{\overline{Y_e}(k)} \\ &= (1-\epsilon_1) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{eg}}{\overline{Y_e}(k)} \\ &= (1-\epsilon_1) \sum_{m=0}^{\infty} \frac{\overline{Y_e}(k+m^*)}{\overline{Y_e}(k)} \cdot \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{eg}}{\overline{Y_e}(k+m^*)}. \end{aligned}$$

Further for any $\epsilon_2 > 0$, there exists k^* large enough so that

$$\frac{\overline{Y_e}(k+m^*)}{\overline{Y_e}(k)} \ge 1-\epsilon_2, \qquad \text{for all } k \ge k^*,$$

because of the heavy-tailed property (49). Thus we have for all $k > k^*$

$$\sum_{m=m^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \ge (1-\epsilon_1)(1-\epsilon_2) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{eg}}{\overline{Y_e}(k+m^*)}. \tag{68}$$

Because ϵ_1 and ϵ_2 are arbitrary, (67) follows from Lemma 17 and (68), which completes the proof for the first equation of Lemma 15. The proof of the second equation can be shown exactly in the same way, so that it is omitted. □

B Key renewal theorem for discrete-time Markov renewal processes

The materials in this appendix are based on Çinlar (1975). Thus for detailed discussions, see Çinlar (1975). Consider a renewal equation

$$q(k) = w(k) + \sum_{m=0}^{k} f(m)q(k-m),$$

where $f(m)$ denotes a possibly defective probability mass function for a nonnegative random variable. To avoid trivialities, we assume that $\sum_{m=1}^{\infty} f(m) > 0$. Let $f^{(0)}(0) = 1$, $f^{(0)}(k) = 0$ for $k = 1, 2, \ldots$, and

$$f^{(n+1)}(k) = \sum_{m=0}^{k} f(m)f^{(n)}(k-m), \qquad n = 0, 1, \ldots.$$

Then from (9.2.3) in Çinlar (1975)

$$q(k) = \sum_{m=0}^{k} z(m)w(k-m),$$

where

$$z(k) = \sum_{n=0}^{\infty} f^{(n)}(k).$$

When $f(k)$ is a defective probability mass function, we have from (9.2.4) in Çinlar (1975)

$$\lim_{k\to\infty} \sum_{m=0}^{k} z(m)w(k-m) = \frac{w(\infty)}{1-f(\infty)},$$

and if $\sum_{k=0}^{\infty} w(k) < \infty$, we have

$$\lim_{k\to\infty} \sum_{m=0}^{k} z(m)w(k-m) = 0. \tag{69}$$

We now assume $f(k)$ is a proper probability mass function. If $w(k)$ is directly Riemann integrable and $z(k)$ is periodic with span τ, it follows from (9.2.10) in Çinlar (1975) that

$$\lim_{k\to\infty} \sum_{m=0}^{k} z(m\tau)w(\zeta+(k-m)\tau) = \frac{\tau}{\sum_{m=1}^{\infty} mf(m)} \sum_{m=0}^{\infty} w(\zeta+m\tau), \qquad \zeta = 0, 1, \ldots, \tau-1. \tag{70}$$

Note that if $\tau = 1$ (i.e., $z(k)$ is aperiodic), (70) is reduced to be

$$\lim_{k\to\infty} \sum_{m=0}^{k} z(m)w(k-m) = \frac{1}{\sum_{m=1}^{\infty} mf(m)} \sum_{m=0}^{\infty} w(m). \tag{71}$$

We now consider a Markov renewal equation

$$\boldsymbol{Q}(k) = \boldsymbol{W}(k) + \sum_{m=0}^{k} \boldsymbol{F}(m)\boldsymbol{Q}(k-m),$$

where $\boldsymbol{Q}(k)$ and $\boldsymbol{W}(k)$ are nonnegative $M \times M_0$ matrices, $\boldsymbol{F}(m)$ is an $M \times M$ nonnegative matrix and $\sum_{m=0}^{\infty} \boldsymbol{F}(m)$ is stochastic. To avoid trivialities, we assume that $\sum_{m=1}^{\infty} \boldsymbol{F}(m) \neq \boldsymbol{O}$. Let $\boldsymbol{F}^{(0)}(0) = \boldsymbol{I}$ and $\boldsymbol{F}^{(0)}(k) = \boldsymbol{O}$ for all $k = 1, 2, \ldots$, and

$$\boldsymbol{F}^{(n+1)}(k) = \sum_{m=0}^{k} \boldsymbol{F}(m)\boldsymbol{F}^{(n)}(k-m), \qquad n = 1, 2, \ldots.$$

Then from (10.3.9) in Çinlar (1975)

$$\boldsymbol{Q}(k) = \sum_{m=0}^{k} \boldsymbol{Z}(m)\boldsymbol{W}(m-k),$$

where

$$\boldsymbol{Z}(k) = \sum_{n=0}^{\infty} \boldsymbol{F}^{(n)}(k).$$

Consider the semi-Markov process whose kernel is given by $\boldsymbol{F}(k)$ $(k = 0, 1, \ldots)$. Let $\boldsymbol{H}(k)$ denote an $M \times M$ matrix whose (i, j)th element $H(i, j, k)$ represents: for $i \neq j$, the probability that the first passage time from state i to state j is equal to k, and for $i = j$, the probability that the recurrence time of state i is equal to k. We define $H^{<n>}(j, j, k)$ as

$$H^{<0>}(j,j,k) = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \ldots, \end{cases} \qquad H^{<n+1>}(j,j,k) = \sum_{m=0}^{k} H(j,j,m)H^{<n>}(j,j,k-m).$$

Then, from (3.2.8) and (3.2.9) in Çinlar (1975), we have

$$[\boldsymbol{Z}(k)]_{i,j} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} H^{<n>}(j,j,k), & i = j, \\ \displaystyle\sum_{m=0}^{k} H(i,j,m)[\boldsymbol{Z}(k-m)]_{j,j}, & i \neq j. \end{cases}$$

Without loss of generality, we assume that the stochastic matrix $\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{F}(k)$ has N irreducible absorbing classes $\mathcal{M}_n$ $(n = 1, \ldots, N)$, i.e., $\boldsymbol{F}(k)$ $(k = 0, 1, \ldots)$ and $\boldsymbol{F}$ take forms:

$$\boldsymbol{F}(k) = \begin{pmatrix} \boldsymbol{F}_1(k) & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{F}_2(k) & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{F}_N(k) & \boldsymbol{O} \\ \boldsymbol{S}_1(k) & \boldsymbol{S}_2(k) & \cdots & \boldsymbol{S}_n(k) & \boldsymbol{S}(k) \end{pmatrix},$$

and

$$\boldsymbol{F} = \begin{pmatrix} \boldsymbol{F}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{F}_2 & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{F}_N & \boldsymbol{O} \\ \boldsymbol{S}_1 & \boldsymbol{S}_2 & \cdots & \boldsymbol{S}_n & \boldsymbol{S}(k) \end{pmatrix},$$

respectively. Note that $\boldsymbol{F}_n = \sum_{k=0}^{\infty} \boldsymbol{F}_n(k)$'s $(n = 1, \ldots, N)$ are irreducible matrices. Let $\mathcal{M}_{N+1}$ denote the set of transient states.

Let $\boldsymbol{\eta}$ denote a $1 \times M$ vector whose jth element $\eta(j)$ represents the inverse of the mean recurrence time of state j in the semi-Markov processes defined by $\boldsymbol{F}_n(k)$ for $j \in \mathcal{M}_n$, and $\eta(j) = 0$ if $j \in \mathcal{M}_{N+1}$, i.e., state j is transient. We define $\boldsymbol{\nu}_n$ $(n = 1, \ldots, N)$ as a positive row vector that satisfies

$$\boldsymbol{\nu}_n \boldsymbol{F}_n = \boldsymbol{\nu}_n.$$

Note that $\boldsymbol{\nu}_n > \mathbf{0}$ is unique up to a multiplicative factor. We then have for $j \in \mathcal{M}_n$

$$\eta(j) = \frac{\nu_n(j)}{\boldsymbol{\nu}_n \sum_{m=1}^{\infty} m \boldsymbol{F}_n(m) \boldsymbol{e}}, \qquad j \in \mathcal{M}_n, \tag{72}$$

where $\nu_n(j)$ denotes the element of $\boldsymbol{\nu}_n$ corresponding to state $j \in \mathcal{M}_n$.

Note that if $\boldsymbol{F}$ has only one irreducible absorbing class (i.e., $N = 1$), it has an invariant row vector $\boldsymbol{\nu}$ that is unique up to a multiplicative factor:

$$\boldsymbol{\nu} \boldsymbol{F} = \boldsymbol{\nu}.$$

Further if $N = 1$, $\boldsymbol{\eta}$ is given by

$$\boldsymbol{\eta} = \frac{\boldsymbol{\nu}}{\boldsymbol{\nu} \sum_{m=1}^{\infty} m \boldsymbol{F}(m) \boldsymbol{e}}. \tag{73}$$

Let $Q_{i,j}(x)$, $W_{i,j}(x)$ and $Z_{i,j}(x)$ denote the (i,j)th elements of $\boldsymbol{Q}(x)$, $\boldsymbol{W}(x)$ and $\boldsymbol{Z}(x)$, respectively.

Proposition 1 *Assume that $\boldsymbol{W}(k)$ is directly Riemann integrable and the recurrence time $Z_{j,j}(k)$ of every irreducible absorbing state j is periodic with the same span τ. Let $\tau_{i,j}$ denote the first jump point of the distribution of the first passage time from state i to state j. For simplicity in description, we assume $W_{i,j}(n) = 0$ for $n < 0$. Then*

$$\lim_{k \to \infty} Q_{i,j}(\zeta + k\tau) = \tau \sum_{l=1}^{M} \eta(l) \sum_{m=0}^{\infty} W_{l,j}(\zeta + m\tau - \tau_{i,l}), \qquad \zeta = 0, 1, \ldots, \tau - 1,$$

where $\eta(l)$ is given by (72), and if $\boldsymbol{F}$ has a single irreducible absorbing class (i.e., $N = 1$),

$$\lim_{k \to \infty} Q_{i,j}(\zeta + k\tau) = \frac{\tau \sum_{l=1}^{M} \nu(l) \sum_{m=0}^{\infty} W_{l,j}(\zeta + m\tau - \tau_{i,l}),}{\boldsymbol{\nu} \sum_{m=1}^{\infty} m \boldsymbol{F}(m) \boldsymbol{e}}, \qquad \zeta = 0, 1, \ldots, \tau - 1. \tag{74}$$

Proof. Because

$$\lim_{k \to \infty} \boldsymbol{Q}(\zeta + k\tau) = \lim_{k \to \infty} \sum_{m=0}^{\zeta + k\tau} \boldsymbol{Z}(m) \boldsymbol{W}(\zeta + k\tau - m),$$

we have

$$\lim_{k \to \infty} Q_{i,j}(\zeta + k\tau) = \sum_{l=1}^{M} \lim_{k \to \infty} \sum_{m=0}^{\zeta + k\tau} Z_{i,l}(m) W_{l,j}(\zeta + k\tau - m). \tag{75}$$

Thus for $i \neq l$, we have

$$\begin{aligned}
&\sum_{m=0}^{\zeta+k\tau} Z_{i,l}(m) W_{l,j}(\zeta+k\tau-m) \\
&\qquad = \sum_{m=0}^{\zeta+k\tau} \sum_{n=0}^{m} H(i,l,m-n) Z_{l,l}(n) W_{l,j}(\zeta+k\tau-m) \\
&\qquad = \sum_{n=0}^{\zeta+k\tau} Z_{l,l}(n) \sum_{m=n}^{\zeta+k\tau} H(i,l,m-n) W_{l,j}(\zeta+k\tau-m) \\
&\qquad = \sum_{n=0}^{k} Z_{l,l}(n\tau) \sum_{m=n\tau}^{\zeta+k\tau} H(i,l,m-n\tau) W_{l,j}(\zeta+k\tau-m) \\
&\qquad = \sum_{n=0}^{k} Z_{l,l}(n\tau) \sum_{m=0}^{\zeta+(k-n)\tau} H(i,l,\zeta+(k-n)\tau-m) W_{l,j}(m) \qquad (76)
\end{aligned}$$

We now assume that state l is in $\mathcal{M}_n$. Because $\boldsymbol{W}(m)$ is directly Riemann integrable, by (9.2.16) (d) in Çinlar (1975), the convolution of $H(i,l,k)$ and $W_{l,j}(m)$ is also directly Riemann integrable. Thus applying (70) to (76), we obtain

$$\lim_{k\to\infty} \sum_{m=0}^{\zeta+k\tau} Z_{i,l}(m) W_{l,j}(\zeta+k\tau-m) = \tau\eta(l) \sum_{m=0}^{\infty} \sum_{n=0}^{\zeta+m\tau} H(i,l,n) W_{l,j}(\zeta+m\tau-n).$$

Let $\tau_{i,l}$ denote

$$\tau_{i,l} = \min\{n;\ H(i,l,n) > 0\}.$$

Note here that $\tau_{i,i} = 0$. We then have $H(i,l,n) = 0$ for $n \neq \tau_{i,l} + m\tau$ $(m = 0, 1, \ldots)$. Thus

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{m=0}^{\zeta+k\tau} Z_{i,l}(m) W_{l,j}(\zeta+k\tau-m) \qquad (77) \\
&\qquad = \tau\eta(l) \sum_{m=0}^{\infty} \sum_{n=\tau_{i,l}}^{\zeta+m\tau} H(i,l,n) W_{l,j}(\zeta+m\tau-n) \\
&\qquad = \tau\eta(l) \sum_{m=0}^{\infty} \sum_{n=0}^{m-1\{\tau_{i,l}-\zeta\}} H(i,l,\tau_{i,l}+n\tau) W_{l,j}(\zeta+(m-n)\tau-\tau_{i,l}) \\
&\qquad = \tau\eta(l) \sum_{m=0}^{\infty} W_{l,j}(\zeta+m\tau-\tau_{i,l}), \qquad (78)
\end{aligned}$$

where $W_{l,j}(n) = 0$ for $n < 0$. Note that in the above derivation, we use $\sum_{m=0}^{\infty} H(i,l,\tau_{i,l}+m\tau) = 1$ which comes from the irreducibility of state l in $\mathcal{M}_n$.

Similarly, for $i = l$, we have

$$\lim_{k\to\infty} \sum_{m=0}^{\zeta+k\tau} Z_{l,l}(m) W_{l,j}(\zeta+k\tau-m) = \tau\eta(l) \sum_{m=0}^{\infty} W_{l,j}(\zeta+m\tau). \qquad (79)$$

Because $\tau_{l,l} = 0$, (78) holds for all $i \in \mathcal{M}_n$.

On the other hand, if state l is transient, it follows from (71) and (76) that

$$\lim_{k\to\infty} \sum_{m=0}^{k} Z_{i,l}(m) W_{l,j}(k-m) = 0.$$

Note here that $\eta(l) = 0$ for all transient state l, so that the expressions in (78) hold for all states $l \in \mathcal{M}$. Proposition 1 now follows from (73), (75), (78). $\Box$

Proposition 2 *Assume that $\boldsymbol{W}(k)$ is directly Riemann integrable and the recurrence time $Z_{j,j}(k)$ of every irreducible absorbing state j is aperiodic. Then*

$$\lim_{k\to\infty} \boldsymbol{Q}(k) = \boldsymbol{e}\boldsymbol{\eta} \sum_{m=0}^{\infty} \boldsymbol{W}(m),$$

and if $\boldsymbol{F}$ has a unique invariant probability vector $\boldsymbol{\nu}$ (i.e., $N = 1$),

$$\lim_{k\to\infty} \boldsymbol{Q}(k) = \frac{\boldsymbol{e}\boldsymbol{\nu} \sum_{m=0}^{\infty} \boldsymbol{W}(m)}{\boldsymbol{\nu} \sum_{m=1}^{\infty} m\boldsymbol{F}(m)\boldsymbol{e}}. \tag{80}$$

The proof is omitted because it is very similar to that of Proposition 1.

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