Approximation and Convergence in Stochastic Linear Complementarity Problems[1]

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Abstract. The stochastic linear complementarity problem is formulated as a problem of minimizing an expected residual defined by an NCP function. We generate observations by the quasi-Monte Carlo methods and show that every accumulation point of minimizers of discrete approximation problems is a minimum residual solution of the stochastic linear complementarity problem. We show that a sufficient condition for the existence of a solution to the residual minimization problem and its discrete approximations is that there is an observation ω^i such that the cost matrix $M(\omega^i)$ is an R_0 matrix. Furthermore, we show that a class of fixed cost problems are continuously differentiable and can be solved without using discrete approximation.

1 Introduction

The linear complementarity problem (LCP) is to find $x \in R^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad (Mx + q)^T x = 0,$$

where M is an $n \times n$ matrix and q is an n-dimensional vector.

We call $\phi : R^2 \to R$ an NCP function if ϕ satisfies

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \ b \geq 0, \ ab = 0.$$

Obviously, x is a solution of the LCP if and only if x is a solution of the nonlinear equations

$$\Phi(x) := \begin{pmatrix} \phi((Mx+q)_1, x_1) \\ \vdots \\ \phi((Mx+q)_n, x_n) \end{pmatrix} = 0.$$

Two commonly used NCP functions are the "min" function

$$\phi(a, b) = \min(a, b)$$

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and the Fischer-Burmeister (FB) function [4]

$$\phi(a,b) = a + b - \sqrt{a^2+b^2}.$$

All NCP functions including the "min" function and FB function are equivalent in the sense that the nonlinear equations $\Phi(x) = 0$ defined by the NCP functions have the same solution. Moreover, some NCP functions have the same growth rate. In particular, Tseng [11] showed

$$\frac{2}{\sqrt{2}+2}|\min(a,b)| \le |a+b-\sqrt{a^2+b^2}| \le (\sqrt{2}+2)|\min(a,b)|. \tag{1.1}$$

In the last decade, NCP functions have become an important tool for solving linear and nonlinear complementarity problems [1, 3, 5, 6, 7, 8, 10, 12].

In engineering and economic applications, elements of M and q often involve uncertain data, which are at best characterized by probability distributions. More specifically, the problem may be written as

$$M(\omega)x + q(\omega) \ge 0, \quad x \ge 0, \quad (M(\omega)x + q(\omega))^T x = 0, \tag{1.2}$$

where ω is a random vector with support $\Omega \subseteq R^m$. In general, however, there is no x that satisfies (1.2) for all $\omega \in \Omega$. So we may try to find a vector $x \in R^n_+ := \{x \in R^n \mid x \ge 0\}$ that minimizes an *expected residual* defined by an NCP function:

$$\min_{x \in R^n_+} E_\Omega[\|\Phi(x,\omega)\|^2], \tag{1.3}$$

where E_Ω stands for the expected value with respect to the distribution of ω, and

$$\Phi(x,\omega) := \begin{pmatrix} \phi((M(\omega)x + q(\omega))_1, x_1) \\ \vdots \\ \phi((M(\omega)x + q(\omega))_n, x_n) \end{pmatrix}.$$

We call problem (1.3) a *stochastic linear complementarity problem* (SLCP). Throughout, we assume that $M(\omega)$ and $q(\omega)$ are continuous functions of ω.

If Ω has only one realization, then (1.3) reduces to the standard LCP and a solution of (1.3) does not depend on the choice of NCP functions. However, the following example shows that we do not have such equivalence if Ω has more than one realization.

Example 1. Let $n = 1$, $m = 1$, $\Omega = \{\omega^1, \omega^2\} = \{0, 1\}$, $M(\omega) = \omega(1-\omega)$ and $q(\omega) = 1 - 2\omega$. Then we have $\|\cdot\| = \|\cdot\|_2$, $M(\omega^1) = M(\omega^2) = 0$, $q(\omega^1) = 1$, $q(\omega^2) = -1$ and

$$E_\Omega[\|\Phi(x,\omega)\|^2] = \frac{1}{2}\sum_{i=1}^{2}\|\Phi(x,\omega^i)\|_2^2.$$

The objective function of the SLCP (1.3) defined by the "min" function is

$$\frac{1}{2}[(\min(1,x))^2 + (\min(-1,x))^2] = \begin{cases} x^2 & x \le -1 \\ \frac{1}{2}(x^2+1) & -1 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

and the problem has a unique solution $x^* = 0$. However, problem (1.3) defined by the FB function has no solution as the objective function

$$\frac{1}{2}[(1 + x - \sqrt{1 + x^2})^2 + (-1 + x - \sqrt{1 + x^2})^2]$$

is monotonically decreasing on $[0, \infty)$.

In order to find a solution of a SLCP numerically, it is necessary to study the objective function of the SLCP defined by a NCP function. There are many NCP functions [1, 7, 8, 10, 12]. In this paper, we focus on the "min" function and the FB function. We use $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ to distinguish the function $\Phi(x, \omega)$ defined by the "min" function and the FB function, respectively. We use $\Phi(x, \omega)$ to represent both $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ when we discuss their common properties.

We consider the following stochastic linear complementarity problem:

$$\min_{x \geq 0} f(x) := \int_\Omega \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega, \tag{1.4}$$

where $\rho : \Omega \to R_+$ is a continuous probability density function satisfying

$$\int_\Omega \rho(\omega) d\omega = 1 \quad \text{and} \quad \int_\Omega \|\omega\|^2 \rho(\omega) d\omega < \infty. \tag{1.5}$$

Obviously, if $M(\omega) \equiv M$ and $q(\omega) \equiv q$, then (1.4) reduces to a standard linear complementarity problem. Throughout $\|\cdot\|$ denotes the Euclidean norm $\|\cdot\|_2$.

Note that, by the continuity of $\Phi_1(\cdot, \omega)$, the function

$$f_1(x) := \int_\Omega \|\Phi_1(x, \omega)\|^2 \rho(\omega) d\omega$$

is continuous. Moreover, by the continuous differentiability of $\|\Phi_2(\cdot, \omega)\|^2$ [6], the function

$$f_2(x) := \int_\Omega \|\Phi_2(x, \omega)\|^2 \rho(\omega) d\omega$$

is continuously differentiable.

In section 2, we show that a sufficient condition for the existence of minimizers of (1.4) and its discrete approximations is that there is an observation ω^i such that the cost matrix $M(\omega^i)$ is an R_0 matrix. Moreover, we prove that every accumulation point of minimizers of discrete approximation problems is a solution of the SLCP (1.4). Specially for the fixed cost SLCP with $M(\omega) \equiv M$, we show that M being an R_0 matrix is a necessary and sufficient condition for the boundedness of the solution sets of the SLCP and its discrete approximations with any $q(\omega)$. In section 3, we show that a class of fixed cost problems with the "min" function is smooth and it can be solved without using discrete approximation. In section 4, we illustrate the relation between stochastic linear complementarity problems and two stage stochastic programs with equilibrium constraints.

2 Existence of solutions

Let the level sets of functions f and f_i, $i = 1, 2$, be denoted

$$D(\gamma) := \{\, x \,|\, f(x) \le \gamma\}$$

and

$$D_i(\gamma) := \{\, x \,|\, f_i(x) \le \gamma\}, \quad i = 1, 2,$$

respectively. From (1.1) and the definitions of $f_1(x)$ and $f_2(x)$, we have

$$\frac{2}{3+2\sqrt{2}} f_1(x) \le f_2(x) \le (6+4\sqrt{2}) f_1(x).$$

This implies

$$D_2(\gamma) \subseteq D_1\Big(\frac{3+2\sqrt{2}}{2}\gamma\Big) \tag{2.1}$$

and

$$D_1(\gamma) \subseteq D_2((6+4\sqrt{2})\gamma).$$

Recall that M is called an R_0 matrix if

$$x \ge 0,\ Mx \ge 0,\ x^T M x = 0 \quad \Longrightarrow \quad x = 0.$$

Lemma 2.1 *If $M(\bar{\omega})$ is an R_0 matrix for some $\bar{\omega} \in \Omega$, then there is a closed sphere $B(\bar{\omega}, \delta) := \{\omega \,|\, \|\omega - \bar{\omega}\| \le \delta\}$ with $\delta > 0$ such that for every $\omega \in \bar{B} := B(\bar{\omega}, \delta) \cap \Omega$, $M(\omega)$ is an R_0 matrix.*

Proof: Assume that this lemma is not true. Then there is a sequence $\{\omega^k\} \subset \bar{B}$ such that

$$\lim_{k\to\infty} \omega^k = \bar{\omega}$$

and, for every $M(\omega^k)$, we can find $x^k \in R^n$ satisfying

$$x^k \ge 0, \quad x^k \ne 0, \quad M(\omega^k)x^k \ge 0, \quad (x^k)^T M(\omega^k) x^k = 0.$$

Put $v^k = \frac{x^k}{\|x^k\|}$. Then we have

$$v^k \ge 0, \quad \|v^k\| = 1, \quad M(\omega^k)v^k \ge 0, \quad (v^k)^T M(\omega^k) v^k = 0.$$

Letting $k \to \infty$, we obtain a vector $\bar{v} \in R^n$ satisfying

$$\bar{v} \ge 0, \quad \|\bar{v}\| = 1, \quad M(\bar{\omega})\bar{v} \ge 0, \quad \bar{v}^T M(\bar{\omega})\bar{v} = 0.$$

This contradicts the assumption that $M(\bar{\omega})$ is an R_0 matrix. ∎

Lemma 2.2 *Assume that there exists an $\bar{\omega} \in \Omega$ such that $\rho(\bar{\omega}) > 0$ and $M(\bar{\omega})$ is an R_0 matrix. Then, for any positive number γ, the level set $D(\gamma)$ is bounded.*

Proof: By the continuity of ρ and Lemma 2.1, there exist a closed sphere $B(\bar{\omega}, \delta)$ with $\delta > 0$ and a constant $\rho_0 > 0$ such that $M(\omega)$ is an R_0 matrix and $\rho(\omega) \geq \rho_0$ for all $\omega \in \bar{B} := B(\bar{\omega}, \delta) \cap \Omega$. Let us consider a sequence $\{x^k\} \subset R^n$. Then, by the continuity of $M(\cdot)$, $q(\cdot)$ and Φ, for each k, there exists an $\omega^k \in \bar{B}$ such that

$$\|\Phi(x^k, \omega^k)\| = \min_{\omega \in \bar{B}} \|\Phi(x^k, \omega)\|.$$

It then follows that

$$\begin{aligned} f(x^k) &\geq \int_{\bar{B}} \|\Phi(x^k, \omega)\|^2 \rho(\omega) d\omega \\ &\geq \|\Phi(x^k, \omega^k)\|^2 \rho_0 \int_{\bar{B}} d\omega \\ &\geq C\rho_0 \|\Phi(x^k, \omega^k)\|^2, \end{aligned}$$

where $C = \int_{\bar{B}} d\omega > 0$. To prove the lemma, it suffices show that $\|\Phi(x^k, \omega^k)\| \to +\infty$ whenever $\|x^k\| \to +\infty$.

Suppose $\|x^k\| \to +\infty$. It is not difficult to see that if $x_i^k \to -\infty$ or $(M(\omega^k)x^k + q(\omega^k))_i \to -\infty$ for some i, then we have $|\phi((M(\omega^k)x^k + q(\omega^k))_i, x_i^k)| \to +\infty$ and hence $\|\Phi(x^k, \omega^k)\| \to +\infty$. So we only need to consider the case where both $\{x_i^k\}$ and $\{(M(\omega^k)x^k + q(\omega^k))_i\}$ are bounded below for all i. Then, by dividing each element of these sequences by $\|x^k\|$ and passing to the limit, we obtain

$$(M(\hat{\omega})\hat{v})_i \geq 0, \quad \hat{v}_i \geq 0, \qquad i = 1, \ldots, n,$$

where $\hat{\omega}$ and $\hat{v}$ are accumulation points of $\{\omega^k\}$ and $\{\frac{x^k}{\|x^k\|}\}$, respectively. Note that $\hat{\omega} \in \bar{B}$ and $\|\hat{v}\| = 1$. Since $M(\hat{\omega})$ is an R_0 matrix and $\hat{v} \neq 0$, there must exist some i such that $(M(\hat{\omega})\hat{v})_i > 0$ and $\hat{v}_i > 0$. This implies $(M(\omega^k)x^k + q(\omega^k))_i \to +\infty$ and $x_i^k \to +\infty$, which in turn implies $|\phi((M(\omega^k)x^k + q(\omega^k))_i, x_i^k)| \to +\infty$. Hence we have $\|\Phi(x^k, \omega^k)\| \to +\infty$. This completes the proof. ∎

Now, we employ a quasi-Monte Carlo method for numerical integration [9]. In particular, we use a transformation function $\omega = u(\tilde{\omega})$ to go from an integral on Ω to the integral on the unit hypercube $[0, 1]^m$ and generate observations $\{\tilde{\omega}^i,\ i = 1, \ldots, N\}$ in the unit hypercube uniformly. The function $f(x)$ can then be written as

$$\begin{aligned} f(x) &= \int_{\Omega} \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega \\ &= \int_{[0,1]^m} \|\Phi(x, u(\tilde{\omega}))\|^2 \rho(u(\tilde{\omega})) u'(\tilde{\omega}) d\tilde{\omega} \\ &= \int_{[0,1]^m} \|\Phi(x, u(\tilde{\omega}))\|^2 \tilde{\rho}(\tilde{\omega}) d\tilde{\omega} \end{aligned}$$

where $\tilde{\rho}(\tilde{\omega}) = \rho(u(\tilde{\omega}))u'(\tilde{\omega})$.

To simplify the notation, without confusion, we suppose $\Omega = [0, 1]^m$ and let ω denote $\tilde{\omega}$ in the remainder of this section.

For each k, let

$$f^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi(x, \omega^i)\|^2 \rho(\omega^i)$$

where $\Omega_k := \{\omega^i,\ i = 1, \ldots, N_k\}$ is a set of observations such that $\Omega_k \subseteq \Omega_{k+1} \subset \Omega$ and $N_k \to \infty$ as $k \to \infty$.

By the continuity of $\Phi_1(\cdot, \omega)$, the function

$$f_1^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi_1(x, \omega^i)\|^2 \rho(\omega^i)$$

is continuous. Moreover, by the continuous differentiability of $\|\Phi_2(\cdot, \omega)\|^2$ [6], the function

$$f_2^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi_2(x, \omega^i)\|^2 \rho(\omega^i)$$

is continuously differentiable.

Theorem 2.1 *Suppose that $M(\omega)$ and $q(\omega)$ are linear functions of ω. Then for any fixed $x \in R^n_+$*

$$f(x) = \lim_{k \to \infty} f^{(k)}(x).$$

Proof: By the assumption of this theorem, for any fixed $x \in R^n_+$, we have

$$\begin{aligned} \|\Phi_1(x, \omega)\| &= \|\min(M(\omega)x + q(\omega), x)\| \\ &\le \|M(\omega)x + q(\omega)\| + \|x\| \\ &\le \|M(\omega)\|\|x\| + \|q(\omega)\| + \|x\| \\ &\le c_0(\|x\| + 1)(\|\omega\| + 1), \end{aligned}$$

where c_0 is a positive constant and the last inequality follows from the fact that $M(\omega)$ and $q(\omega)$ are linear functions of ω. Hence the function $\|\Phi_1(x, \cdot)\|^2 \rho(\cdot)$ is continuous, nonnegative and bounded due to the condition (1.5).

Since Φ_2 and Φ_1 have the same growth rate (1.1), $\|\Phi_2(x, \cdot)\|^2 \rho(\cdot)$ is also continuous, nonnegative and bounded. Therefore, we can claim that $\|\Phi(x, \cdot)\|^2 \rho(\cdot)$ is integrable, that is, $0 \le f(x) < \infty$.

By the continuity of $\|\Phi(x, \cdot)\|^2 \rho(\cdot)$ and Theorem 2.13 in [9], we obtain

$$|f^{(k)}(x) - f(x)| \le 4 \sup_{\substack{\mu, \nu \in \Omega \\ \|\mu - \nu\| \le D_k(\Omega_k)^{1/m}}} \Big| \|\Phi(x, \mu)\|^2 \rho(\mu) - \|\Phi(x, \nu)\|^2 \rho(\nu) \Big|$$

with $\Omega_k = \{\omega^1, \ldots, \omega^{N_k}\}$ and

$$D_k(\Omega_k) := \sup_{B \in \mathcal{B}} \Big| \frac{1}{N_k} \sum_{i=1}^{N_k} \chi_B(\omega^i) - \lambda_m(B) \Big|,$$

where λ_m is the m-dimensional Lebesgue measure, $\mathcal{B}$ is the family of all closed subintervals of $[0, 1]^m$, and χ_B is the characteristic function of the set B, i.e.,

$\chi_B(\omega) = 1$ if $\omega \in B$, and $\chi_B(\omega) = 0$ otherwise. According to a classical result in the theory of uniform distribution of sequences [9, p. 17], we have

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{i=1}^{N_k} \chi_B(\omega^i) = \lambda_m(B).$$

This implies

$$\lim_{k\to\infty} D_k(\Omega_k) = 0.$$

From the continuity of $\|\Phi(x,\cdot)\|^2\rho(\cdot)$, we find that $f(x) = \lim_{k\to\infty} f^{(k)}(x)$. ∎

Let us denote the set of minimizers of $f^{(k)}$ on R^n_+ by

$$S_k := \operatorname{argmin}\{f^{(k)}(x) \,|\, x \in R^n_+\}.$$

Theorem 2.2 *Suppose that $M(\omega)$ and $q(\omega)$ are linear functions of ω. Assume that there is an $\bar\omega \in \Omega$ such that $\rho(\bar\omega) > 0$ and $M(\bar\omega)$ is an R_0 matrix. Then for all k, S_k is nonempty and bounded. Let $x^{(k)} \in S_k$ for each k. Then every accumulation point of the sequence $\{x^{(k)}\}$ is contained in the set* $\operatorname{argmin}\{f(x) \,|\, x \in R^n_+\}$.

Proof: We can show that S_k is nonempty and bounded in a similar manner to Lemma 2.2.

Let $\bar x$ be an accumulation point of $\{x^{(k)}\}$. For simplicity, we assume that $\{x^{(k)}\}$ itself converges to $\bar x$. Let $\gamma > f(\bar x)$. Then from the continuity of f, we have

$$f(x^{(k)}) \le \gamma \quad \text{for all large} \quad k,$$

that is, $x^{(k)} \in D(\gamma)$ for all large k.

Now we show

$$|f^{(k)}(x^{(k)}) - f^{(k)}(\bar x)| \to 0 \quad \text{as} \quad k \to \infty. \tag{2.2}$$

It is known that for any fixed ω, $\Phi(\cdot,\omega)$ is globally Lipschitzian, that is,

$$\|\Phi(x,\omega) - \Phi(y,\omega)\| \le L(\omega)\|x - y\| \quad \text{for all } x, y \in R^n,$$

where $L(\omega)$ is a positive constant depending on ω. Moreover, since $M(\omega)$ and $q(\omega)$ are linear functions of ω, we can show that

$$L(\omega) \le c_1(\|\omega\| + 1)$$

for some positive constant c_1.

In a way similar to the proof of Theorem 2.1, for any $x \in D(\gamma)$, we obtain

$$\|\Phi(x,\omega)\| \le c_0(\|x\| + 1)(\|\omega\| + 1)$$

for some constant $c_0 > 0$. Furthermore, since $D(\gamma)$ is closed and bounded by Lemma 2.2, we may define

$$c_2 := \max\{\|x\| \mid x \in D(\gamma)\}.$$

Therefore, for any $x, y \in D(\gamma)$, we obtain

$$\begin{array}{rcl} \left| \|\Phi(x,\omega)\|^2 - \|\Phi(y,\omega)\|^2 \right| & = & (\|\Phi(x,\omega)\| + \|\Phi(y,\omega)\|) \, |\|\Phi(x,\omega)\| - \|\Phi(y,\omega)\|| \\ & \le & c_0 c_1 (2 + \|x\| + \|y\|)(\|\omega\| + 1)^2 \|x - y\| \\ & \le & C(\|\omega\| + 1)^2 \|x - y\|, \end{array}$$

where $C := 2c_0c_1(1 + c_2)$.

By the assumption (1.5) on the density function ρ, we obtain

$$\begin{array}{rcl} |f^{(k)}(x^{(k)}) - f^{(k)}(\bar{x})| & \le & \displaystyle\frac{1}{N_k} \sum_{i=1}^{N_k} \left| \|\Phi(x^{(k)}, \omega^i)\|^2 - \|\Phi(\bar{x}, \omega^i)\|^2 \right| \rho(\omega^i) \\ & \le & \displaystyle\frac{1}{N_k} \sum_{i=1}^{N_k} C(\|\omega^i\| + 1)^2 \rho(\omega^i) \|x^{(k)} - \bar{x}\| \\ & \le & K\|x^{(k)} - \bar{x}\|, \end{array}$$

where K is a constant satisfying

$$K \ge \frac{C}{N_k} \sum_{i=1}^{N_k} (\|\omega^i\| + 1)^2 \rho(\omega^i) \quad \text{for all large } k.$$

Hence (2.2) holds. Now by Theorem 2.1 and (2.2), we find

$$\begin{array}{rcl} |f^{(k)}(x^{(k)}) - f(\bar{x})| & \le & |f^{(k)}(x^{(k)}) - f^{(k)}(\bar{x})| + |f^{(k)}(\bar{x}) - f(\bar{x})| \\ & \to & 0 \qquad \text{as} \quad k \to \infty. \end{array}$$

However, by definition, we have

$$f^{(k)}(x^{(k)}) \le f^{(k)}(x) \quad \text{for all } x \in R^n_+.$$

Therefore, combining the above results, we obtain

$$f(\bar{x}) = \lim_{k\to\infty} f^{(k)}(x^{(k)}) \le \lim_{k\to\infty} f^{(k)}(x) = f(x) \quad \text{for all } x \in R^n_+.$$

This completes the proof. ∎

The following lemma shows that the converse of Lemma 2.2 is true, when $M(\omega) \equiv M$ and $q(\omega)$ is a linear function of ω.

Lemma 2.3 *Suppose that $M(\omega) \equiv M$. If M is not an R_0 matrix, and $q(\omega)$ is a linear function of ω, then there is a $\gamma > 0$ such that the level set $D(\gamma)$ is unbounded.*

Proof: Since M is not an R_0 matrix, there is an $x \neq 0$ such that

$$x \ge 0, \;\; Mx \ge 0, \;\; x^T Mx = 0,$$

which in particular implies that either $x_i = 0$ or $(Mx)_i = 0$ holds for each i. Hence we have

$$\min(x_i, (Mx + q(\omega))_i) = \begin{cases} 0 & x_i = 0,\ (Mx + q(\omega))_i \geq 0 \\ (Mx + q(\omega))_i & x_i = 0,\ (Mx + q(\omega))_i \leq 0 \\ x_i & (Mx)_i = 0,\ q_i(\omega) \geq x_i \\ q_i(\omega) & (Mx)_i = 0,\ q_i(\omega) \leq x_i. \end{cases}$$

Note that $|(Mx + q(\omega)_i| \leq |q_i(\omega)|$ if $(Mx + q(\omega))_i \leq 0$. This implies that

$$\frac{1}{\sqrt{2}+2}|\Phi_2(x,\omega)_i| \leq |\Phi_1(x,\omega)_i| = |\min(x_i, (Mx + q(\omega))_i)| \leq |q_i(\omega)|.$$

Thus we find

$$f(x) \leq (\sqrt{2}+2)^2 \int_\Omega \|q(\omega)\|^2 \rho(\omega) d\omega =: \gamma.$$

From the assumption that $q(\omega)$ is a linear function of ω and the assumption (1.5) on $\rho(\omega)$, we have $\gamma < \infty$.

Since the argument above holds for λx with any $\lambda > 0$, that is, $f(\lambda x) \leq \gamma$, we complete the proof. ∎

Theorem 2.3 *Suppose $M(\omega) \equiv M$ and $q(\omega)$ is a linear function of ω, i.e., $q(\omega) = \bar{q} + T\omega$, where $\bar{q} \in R^n$ and $T \in R^{n\times m}$.*

1. *If M is an R_0 matrix, then for any $\bar{q}$ and T, the sets S_k are nonempty and bounded. Let $x^{(k)} \in S_k$ for each k. Then any accumulation point of $\{x^{(k)}\}$ is a solution of the SLCP (1.3).*

2. *If M is not an R_0 matrix, then there are $\bar{q}$ and T such that the sets S_k are unbounded for all k.*

Proof: Part 1 follows from Theorem 2.2 directly. To prove Part 2, let $x \neq 0$ satisfy

$$x \geq 0, \quad Mx \geq 0, \quad x^T Mx = 0$$

and choose $\bar{q}$ and T such that

$$x_i = 0 \implies q_i(\omega) \geq 0 \quad \text{for all } \omega \in \Omega,$$
$$x_i > 0 \implies q_i(\omega) \equiv 0 \quad \text{for all } \omega \in \Omega.$$

Then we have

$$\|\Phi_2(\lambda x, \omega)\| \leq (\sqrt{2}+2)\|\Phi_1(\lambda x,\omega)\| = (\sqrt{2}+2)\|\min(\lambda x, M(\lambda x) + q(\omega))\| = 0$$

for all $\omega \in \Omega$ and $\lambda > 0$. Hence $f^{(k)}(\lambda x) = 0$, that is, $\lambda x \in S_k$ for all $\lambda > 0$. Since $x \neq 0$, S_k is unbounded. ∎

Remark It is well known that for the LCP, the matrix M being a P matrix is a necessary and sufficient condition for the existence and uniqueness of the solution of the LCP with all q. Theorem 2.3 states that the matrix M being an R_0 matrix is a necessary and sufficient condition for the boundedness of the solution set of the fixed cost SLCP with any linear function $q(\omega)$. Furthermore, the condition that M and q are linear functions can be replaced by the condition that there is a constant $c > 0$ such that $\|M(\omega)\| + \|q(\omega)\| \leq c(\|\omega\| + 1)$.

3 Fixed cost problems

In this section, we consider a class of fixed cost SLCP, where

$$M(\omega) \equiv M \quad \text{and} \quad q(\omega) \equiv \bar{q} + T\omega,$$

where $M \in R^{n\times n}$, $\bar{q} \in R^n$ and $T \in R^{n\times m}$ are given constants. Moreover we assume that matrix T has at least one nonzero element in each row.

Lemma 3.1 *Suppose that $\rho : R \to [0, \infty)$ is piecewise continuous. Then for any $c \neq 0$, the function $\psi : R^2 \to R$ defined by*

$$\psi(a, b) := \int_{-\infty}^{\infty} (\min(a, b + c\omega))^2 \rho(\omega) d\omega$$

is continuously differentiable.

Proof: It is easy to verify

$$\begin{aligned} \psi(a, b) &= \int_{-\infty}^{\infty} (a - \max(0, a - b - c\omega))^2 \rho(\omega) d\omega \\ &= a^2 - 2a \int_{-\infty}^{\infty} \max(0, a - b - c\omega) \rho(\omega) d\omega \\ &\quad + \int_{-\infty}^{\infty} (\max(0, a - b - c\omega))^2 \rho(\omega) d\omega. \end{aligned} \tag{3.1}$$

Obviously the first term is continuously differentiable. If $c > 0$, then the second term is a class of smoothing function for the function $\max(0, \cdot)$. By the results in [2], it is continuously differentiable. If $c < 0$, we set $\tilde{\omega} = -\omega$ and $\tilde{\rho}(\tilde{\omega}) := \rho(\omega)$. Then $\tilde{\rho} : R \to [0, \infty)$ is piecewise continuous. Hence we can use the results in [2] to claim that

$$\begin{aligned} &\int_{-\infty}^{\infty} \max(0, a - b - c\omega) \rho(\omega) d\omega \\ &= \int_{-\infty}^{\infty} \max(0, a - b - |c|\tilde{\omega}) \tilde{\rho}(\tilde{\omega}) d\tilde{\omega} \end{aligned}$$

is also continuously differentiable with respect to a and b.

The continuously differentiability of the third term in (3.1) follows from the fact that $(\max(0, a - b - c\omega))^2 \rho(\omega)$ is continuously differentiable with respect to a and b. ∎

We illustrate Lemma 3.1 by the following uniform density function

$$\rho(\omega) = \begin{cases} 1 & \omega \in [0,\, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $c > 0$. We can deal with the case $c < 0$ in a similar manner. Calculating the integrals, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \max(0, a-b-c\omega)\rho(\omega)d\omega \\ &= \begin{cases} 0 & a-b \le 0 \\ \frac{1}{2c}(a-b)^2 & 0 \le a-b \le c \\ a-b-\frac{c}{2} & a-b \ge c \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty} (\max(0, a-b-c\omega))^2 \rho(\omega)d\omega \\ &= \begin{cases} 0 & a-b \le 0 \\ \frac{1}{3c}(a-b)^3 & 0 \le a-b \le c \\ \frac{1}{3c}[(a-b)^3 - (a-b-c)^3] & a-b \ge c. \end{cases} \end{aligned}$$

Summarizing these terms gives the function

$$\psi(a,b) = \begin{cases} a^2 & a-b \le 0 \\ a^2 - \frac{a}{c}(a-b)^2 + \frac{1}{3c}(a-b)^3 & 0 \le a-b \le c \\ a^2 - 2a(a-b) + ac + \frac{1}{3c}[(a-b)^3 - (a-b-c)^3] & a-b \ge c, \end{cases}$$

which is continuously differentiable.

Theorem 3.1 *Let $\Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_m, \beta_m]$ with $\alpha_i < \beta_i$, $j = 1, \ldots, m$. Suppose that $\omega_j, j = 1, \ldots, m$ are independent and the density function ρ is continuous on Ω. Then the function*

$$f_1(x) = \int_\Omega \|\min(x, Mx + \bar{q} + T\omega)\|^2 \rho(\omega)d\omega$$

is continuously differentiable.

Proof: For each j, let ρ_j denote the density function for ω_j. Due to the structure of the problem, f_1 can be written as

$$f_1(x) = \sum_{i=1}^{n} \int_{\alpha_m}^{\beta_m} \cdots \int_{\alpha_1}^{\beta_1} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_1(\omega_1) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_m.$$

Recall that, for each i, there exists at least one j such that $T_{ij} \neq 0$. From Lemma 3.1, it then follows that

$$\eta_{ij}(x, \omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m) := \int_{\alpha_j}^{\beta_j} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_j(\omega_j) d\omega_j$$

is continuously differentiable. Hence

$$\begin{aligned} f_{1,i}(x) &:= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_m}^{\beta_m} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_1(\omega_1) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_m \\ &= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_{j-1}}^{\beta_{j-1}} \int_{\alpha_{j+1}}^{\beta_{j+1}} \cdots \int_{\alpha_m}^{\beta_m} \eta_{ij}(x, \omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m) \\ &\quad \rho_1(\omega_1) \cdots \rho_{j-1}(\omega_{j-1}) \rho_{j+1}(\omega_{j+1}) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_{j-1} d\omega_{j+1} \cdots d\omega_m \end{aligned}$$

is a continuously differentiable function of x. Since each row of T has at least one nonzero element, we can claim that the function $f_1 = \sum_{i=1}^n f_{1,i}$ is continuously differentiable. ∎

This theorem suggests that it is possible to solve some special SLCP without using discrete approximation. For example, we consider the following case: For each i, let the ith row of matrix T has just one positive element t_i, and the density function ρ is defined by

$$\rho(\omega) = \begin{cases} 1 & \omega \in [0, 1]^m \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

In this case, we can write f_1 explicitly as

$$f_1(x) := \sum_{i=1}^n f_{1,i}(x),$$

where

$$f_{1,i}(x) := \begin{cases} x_i^2 & x_i - y_i \leq 0 \\ x_i^2 - \frac{x_i}{t_i}(x_i - y_i)^2 + \frac{1}{3t_i}(x_i - y_i)^3 & 0 \leq x_i - y_i \leq t_i \\ x_i^2 - 2x_i(x_i - y_i) + x_i t_i + \frac{1}{3t_i}[(x_i - y_i)^3 & \\ \qquad -(x_i - y_i - t_i)^3] & x_i - y_i \geq t_i \end{cases}$$

and $y_i = (Mx + \bar{q})_i$. Moreover, it is notable, in this case, the SLCP (1.3) defined by the "min" function is continuously differentiable. Therefore, we even do not need to use smoothing approximations.

In other cases, for example, ω_j are distributed normally or exponentially. In practice, we may always restrict ourselves to their 99% confidence interval $[\alpha_j, \beta_j]$. Hence Theorem 3.1 is particularly important for the purpose of applications. Moreover, we can use a transformation function to go from an integral on R^m to the integral of $[0, 1]^m$.

Now let us turn our attention to the FB function. The SLCP (1.3) defined by the FB function is continuously differentiable. In the case where the ith row of matrix T has only one positive element t_i for each $i = 1, \ldots, n$, and the density function ρ is defined by (3.2), the function f_2 can be written as

$$\begin{aligned} f_2(x) &= \sum_{i=1}^{n} (x_i + y_i)^2 + \frac{t_i}{2} + \frac{1}{2t_i}\left(y_i\sqrt{x_i^2 + y_i^2} - (y_i + t_i)\sqrt{x_i^2 + (y_i + t_i)^2} \right) \\ &\quad + \frac{x_i^2}{2t_i} \log\left(\frac{\sqrt{x_i^2 + y_i^2} + y_i}{\sqrt{x_i^2 + (y_i + t_i)^2} + y_i + t_i} \right). \end{aligned}$$

4 Some remarks

Problem (1.3) may be regarded as a constrained nonlinear least squares problem. Moreover, as shown below, it may also be viewed as a two stage stochastic program:

$$\min_{x \geq 0} E_\Omega[\|r(x, \omega)\|^2] \tag{4.1}$$

where $r(x, \omega)$ is the unique solution of the equilibrium problem

$$\begin{aligned} x - h(x, r) \geq 0, \quad M(\omega)x + q(\omega) - g(x, r) \geq 0, \\ (x - h(x, r))^T (M(\omega)x + q(\omega) - g(x, r)) = 0. \end{aligned} \tag{4.2}$$

Here $h(x, r) = (h_1(x_1, r_1), \ldots, h_n(x_n, r_n))$ and $g(x, r) = (g_1(x_1, r_1), \ldots, g_n(x_n, r_n))$ are dependent on the NCP function.

Proposition 4.1 *For any $a, b, c \in R$ such that $b \geq 0$, we have*

1. $\min(a, b) = c \quad \Longleftrightarrow \quad \min(a - c, b - c) = 0.$
2. $a + b - \sqrt{a^2 + b^2} = c \quad \Longleftrightarrow \quad a - s + b - t - \sqrt{(a - s)^2 + (b - t)^2} = 0,$

where

$$s = \begin{cases} 0 & a = 0 \\ c & a \neq 0, \end{cases} \qquad t = \begin{cases} 0 & a = 0 \ \text{ or } a = c \\ \dfrac{(c - 2a)c}{2(c - a)} & a \neq 0 \ \text{ and } \ a \neq c. \end{cases}$$

Proof: Part 1 is very easy to verify. We only show part 2. Clearly the equivalence holds for $a = 0$. Note that $c = 0$ in this case. In the following proof, we assume $a \neq 0$ and $a \neq c$.

Let $a + b - \sqrt{a^2 + b^2} = c$. Then we have

$$a - c = \sqrt{a^2 + b^2} - b > 0$$

and

$$2ab - 2(a + b)c + c^2 = 0. \tag{4.3}$$

From (4.3) we get

$$b - \frac{c-2a}{2(c-a)}c = \frac{2b(c-a)-(c-2a)c}{2(c-a)} = \frac{2ab-2(a+b)c+c^2}{2(a-c)} = 0.$$

Hence it is easy to see that $a-s+b-t-\sqrt{(a-s)^2+(b-t)^2}=0$ holds with $s=c$ and $t=\frac{c-2a}{2(c-a)}c$.

Conversely, suppose that $a-s+b-t-\sqrt{(a-s)^2+(b-t)^2}=0$. Then we have $a-s\geq 0$, $b-t\geq 0$ and

$$(a-s)(b-t)=0. \tag{4.4}$$

From (4.4) and the definitions of s and t, we have $a>c$ and

$$0 = 2(a-c)\Big(b-\frac{c-2a}{2(c-a)}c\Big) = 2(a-c)+c^2-2ac.$$

This shows (4.3), which in turn implies $(a+b-c)^2=a^2+b^2$. On the other hand, since $a>c$ implies $b=\frac{c-2a}{2(c-a)}c$, we have

$$a+b-c = \frac{a^2+(a-c)^2}{2(a-c)} \geq 0.$$

Hence we obtain $a+b-\sqrt{a^2+b^2}=c$. ∎

By Proposition 4.1, the functions $h(x,r)$ and $g(x,r)$ in (4.1)–(4.2) are

$$h(x,r)=r, \qquad g(x,r)=r$$

for the "min" function, and

$$h_i(x_i,r_i)=\begin{cases} r_i & y_i\neq 0 \\ 0 & \text{otherwise,}\end{cases} \qquad g_i(x_i,r_i)=\begin{cases} \dfrac{r_i-2y_i}{2(r_i-y_i)}r_i & y_i\neq 0 \\ 0 & \text{otherwise.}\end{cases}$$

for the BF function, where $y_i=(M(\omega)x+q(\omega))_i$.

Moreover, by using a positive definite diagonal matrix $D(\omega)$, we may consider the following problem, which is equivalent to (1.2):

$$D(\omega)^{-1}(M(\omega)x+q(\omega))\geq 0, \quad x\geq 0, \quad (D(\omega)^{-1}(M(\omega)x+q(\omega)))^Tx=0.$$

Then the equilibrium problem (4.2) in the second stage of problem (4.1) becomes

$$x-h(x,r)\geq 0, \quad D(\omega)^{-1}(M(\omega)x+q(\omega))-g(x,r)\geq 0$$
$$(x-h(x,r))^T(D(\omega)^{-1}(M(\omega)x+q)-g(x,r))=0,$$

which is equivalent to

$$x-h(x,r)\geq 0, \quad M(\omega)x+q(\omega)-D(\omega)g(x,r)\geq 0$$
$$(x-h(x,r))^T(M(\omega)x+q-D(\omega)g(x,r))=0.$$

From the viewpoint of smoothness, problem (1.4) defined by the BF function may be considered more desirable than that defined by the "min" function. However, by the simplicity of $h(x, r)$ and $g(x, r)$ in the second stage of problem (4.1) and the boundedness of level sets (cf. Example 1), one may prefer using the "min" function. Furthermore, for the fixed cost problems treated in section 3, the "min" function may be an appropriate NCP function for SLCP. We are aware that the "min" function and the BF function have some limitations in dealing with the monotone linear complementarity problem. NCP functions for monotone LCP have been studied in [1, 3, 7, 10, 12]. It may be of interest to study monotone SLCP with these NCP functions.

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