A Smoothing Implicit Programming Approach for Solving a Class of Stochastic Generalized Semi-Infinite Programming Problems

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Abstract

This paper discusses a generalized semi-infinite programming problem under uncertainty. The expected value approach is applied to define a deterministic version of the problem. We propose a new reformulation by using the first order optimality conditions of the second stage optimization problem. We then present a smoothing implicit programming method to solve the problem with finite discrete distribution. Global convergence results are obtained under mild conditions.

Key words. Stochastic generalized semi-infinite programming problem, complementarity constraint, smoothing implicit programming method, global convergence.

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Short title. Stochastic Generalized Semi-Infinite Programms

1 Introduction

A generalized semi-infinite programming (GSIP) problem is a constrained optimization problem in which the constraints are given by a possibly infinite index set that depends upon the decision variable x:

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s. t.} & g(x,u) \leq 0, \ \forall\, u \in T(x), \end{array} \tag{1.1}$$

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where $T(x) = \{u \in \mathcal{R}^r \mid h(x, u) \leq 0\}$. Here, $f : \mathcal{R}^n \to \mathcal{R}$, $g : \mathcal{R}^n \times \mathcal{R}^r \to \mathcal{R}$, $h : \mathcal{R}^n \times \mathcal{R}^r \to \mathcal{R}^J$, $T : \mathcal{R}^n \to 2^{\mathcal{R}^r}$, and $2^{\mathcal{R}^r}$ is the set of all subsets in $\mathcal{R}^r$.

When the set-valued mapping T is constant, the GSIP problem reduces to a standard semi-infinite programming problem and will be abbreviated by SIP. Moreover, if T is a finite set, then SIP reduces to an ordinary nonlinear programming problem.

Recently, the GSIP problem becomes an active research topic in applied mathematics, as it arises in various fields of engineering such as the design problem, the problem of maneuverability of robots, and the reverse Chebyshev approximation problem, see, e.g., [6, 8, 12]. The first-order and second-order optimality conditions for the GSIP problem are studied in [9, 10, 14, 16]. Some numerical aspects of the GSIP problem are discussed in [17, 18].

Stochastic programming is another important branch of mathematical programming in which optimal decisions are sought under uncertainty. Modeling the uncertainty by random objects may lead to diverse stochastic programming problems. Various numerical methods for solving stochastic programming have been studied extensively, see [1, 2, 19].

In this paper, we consider the following stochastic version of the GSIP problem (1.1):

$$\begin{aligned} \min_x \quad & E_\omega[f(x, \omega)] \\ \text{s. t.} \quad & g(x, u, \omega) \leq 0, \\ & u \in T(x, \omega),\ \omega \in \Omega,\ a.s., \end{aligned} \tag{1.2}$$

where Ω is a sample space, $T(x, \omega) = \{u \in \mathcal{R}^r \mid h(x, u, \omega) \leq 0\}$ is a constraint index set correlated with a decision variable x and a random variable $\omega \in \Omega$, the abbreviation $a.s.$ means that the constraints hold almost surely, i.e., for all $\omega \in \Omega$ except for a set with zero probability. We assume that $f : \mathcal{R}^n \times \Omega \to \mathcal{R}$, $g : \mathcal{R}^n \times \mathcal{R}^r \times \Omega \to \mathcal{R}$, $h : \mathcal{R}^n \times \mathcal{R}^r \times \Omega \to \mathcal{R}^J$ are continuous, $T : \mathcal{R}^n \times \Omega \to 2^{\mathcal{R}^r}$ and Ω is a compact set in $\mathcal{R}^s$. We call problem (1.2) a stochastic generalized semi-infinite programming (SGSIP) problem. Obviously, if Ω is a singleton, then the problem (1.2) reduces to an ordinary GSIP problem. For each fixed $\omega \in \Omega$, the problem (1.2) is a GSIP problem, and can be reformulated as

$$\begin{aligned} \min_x \quad & E_\omega[f(x, \omega)] \\ \text{s. t.} \quad & v(x, \omega) \leq 0,\ \omega \in \Omega, a.s., \end{aligned} \tag{1.3}$$

where $v(x, \omega)$ is defined as

$$v(x, \omega) = \sup_u \{g(x, u, \omega) \mid u \in T(x, \omega)\}.$$

In this paper, we apply the expected value approach to the constraints of (1.3) and propose a deterministic version of SGSIP problem as follows:

$$\begin{aligned} \min_x \quad & E_\omega[f(x, \omega)] \\ \text{s. t.} \quad & E_\omega[v(x, \omega)] \leq 0. \end{aligned} \tag{1.4}$$

The expected value approach has been studied for stochastic variational inequality problems by Gürkan, Özge and Robinson [7]. The GSIP problem is a hard problem with an infinite constraint index set that may vary since it is correlated with decision variable x. Presence of an additional random variable makes the SGSIP problem even harder to solve than the GSIP problem. The aim of this paper is to propose a globally convergent algorithm for solving (1.4).

The rest of this paper is organized as follows. In Section 2, we reformulate problem (1.4) as a mathematical programming problem with complementarity constraints. In Section 3, we establish some properties of certain parametric smoothing approximations for the reformulated problem. In Section 4, we present global convergence analysis of a smoothing implicit programming algorithm for solving the problem with finite discrete distribution. Some final remarks are given in Section 5.

Here are a few words about the notation. We let $||\cdot||$ denote the Euclidean norm, I and O denote the identity and zero matrix with a suitable dimension, respectively, and U denote the closed unit ball in an Euclidean space with a suitable dimension. Unless stated otherwise, throughout this paper, all vectors (vector functions) are column vectors (vector functions). For a differentiable vector-valued function $F : \mathcal{R}^n \to \mathcal{R}^m$, we denote the transposed Jacobian of F at x by $\nabla F(x)$. For $u \in \mathcal{R}^n$ and $v \in \mathcal{R}^m$, (u, v) denotes the column vector $[u^T, v^T]^T$ in $\mathcal{R}^{n+m}$. Let $\mathcal{R}^\ell_+ = \{z \in \mathcal{R}^\ell \ \mid z \geq 0\}$.

2 A New Reformulation

In this section, we present a new reformulation of problem (1.4). Our main idea is to regard (1.4) as a two-stage optimization problem and use the first order optimality condition of the second stage optimization problem to deal with the constraints of (1.4).

Assumption A. For any $x \in \mathcal{R}^n$ and $\omega \in \Omega$, $g(x, \cdot, \omega)$ is twice continuously differentiable and pseudo-concave, $h(x, \cdot, \omega)$ is twice continuously differentiable and $y^T h(x, \cdot, \omega)$ is quasi-convex for any $y \in \mathcal{R}^J_+$.

For any $(x, \omega) \in \mathcal{R}^n \times \Omega$, we define a parametric programming problem

$$Q(x,\omega): \qquad \begin{array}{ll} \max\limits_{u} & g(x,u,\omega) \\ \text{s. t.} & u \in T(x,\omega). \end{array}$$

The first-order optimality conditions for problem $Q(x, \omega)$ are given by

$$\begin{aligned} &\nabla_u g(x,u,\omega) - \nabla_u h(x,u,\omega) y = 0, \\ &y^T h(x,u,\omega) = 0, \\ &h(x,u,\omega) \leq 0, \\ &y \geq 0. \end{aligned}$$

Definition 2.1 *We say that the linear independence constraint qualification (LICQ) is satisfied at $\bar{u}$ for problem $Q(x, \omega)$, if the vectors*

$$\nabla_u h_j(x,\bar{u},\omega), \quad j \in \mathcal{I}_h(x,\bar{u},\omega)$$

are linearly independent, where $\mathcal{I}_h(x, \bar{u}, \omega)$ is the index set of active constraints

$$\mathcal{I}_h(x,\bar{u},\omega) = \{j \mid h_j(x,\bar{u},\omega) = 0\}.$$

We say that the Mangasarian-Fromovitz constraint qualification (MFCQ) [14] is satisfied at $\bar{u}$ for problem $Q(x,\omega)$, if there exists a vector $\gamma_0 \in \mathcal{R}^r$ such that

$$\nabla_u h_j(x,\bar{u},\omega)^T \gamma_0 < 0, \;\; j \in \mathcal{I}_h(x,\bar{u},\omega).$$

Assumption B. For any $x \in \mathcal{R}^n$ and $\omega \in \Omega$, problem $Q(x,\omega)$ has a unique solution, which we denote $u(x,\omega)$. Moreover, the MFCQ is satisfied at $u(x,\omega)$ for problem $Q(x,\omega)$.

Under Assumptions A and B, we show that problem (1.4) is equivalent to the following problem.

$$\begin{array}{lll} \tilde{P}: & \min\limits_{x} & E_\omega[f(x,\omega)] \\ & \text{s. t.} & E_\omega[g(x,u(x,\omega),\omega)] \leq 0, \end{array}$$

where $u(x,\omega)$, together with a vector $y(x,\omega) \in R^J$, satisfies the following first-order optimality conditions for problem $Q(x,\omega)$:

$$\begin{array}{l} \nabla_u g(x,u(x,\omega),\omega) - \nabla_u h(x,u(x,\omega),\omega) y(x,\omega) = 0, \\ \min(y(x,\omega), -h(x,u(x,\omega),\omega)) = 0. \end{array}$$

Lemma 2.1 *Suppose that Assumptions A and B hold. Then, $\tilde{x}$ is a feasible solution of problem (1.4) if and only if $\tilde{x}$ is a feasible solution of problem $\tilde{P}$.*

Proof. Let $\tilde{x}$ be a feasible solution of problem (1.4), that is, $E_\omega[g(\tilde{x},u(\tilde{x},\omega),\omega)] \leq 0$, where $u(\tilde{x},\omega)$ is the unique solution of $Q(\tilde{x},\omega)$. By Assumption B, for every $\omega \in \Omega$, there exists a vector y such that $(u(\tilde{x},\omega), y)$ satisfies the first-order optimality conditions of $Q(\tilde{x},\omega)$ at $u(\tilde{x},\omega)$, which implies that $\tilde{x}$ is a feasible solution of problem $\tilde{P}$.

Conversely, let $\tilde{x}$ be a feasible solution of problem $\tilde{P}$, that is, there exists a pair $(u(\tilde{x},\omega), y)$ such that $(\tilde{x}, u(\tilde{x},\omega), y)$ satisfies the constraints of $\tilde{P}$. From Assumption A, the first-order optimality conditions imply

$$g(\tilde{x},u(\tilde{x},\omega),\omega) = v(\tilde{x},\omega).$$

Hence, $E_\omega[v(\tilde{x},\omega)] \leq 0$, that is, $\tilde{x}$ is a feasible solution of (1.4). The proof is complete. ∎

From Lemma 2.1, we readily obtain the following theorems. The proof is omitted.

Theorem 2.1 *Suppose that Assumptions A and B hold. Then $\tilde{x}$ is a global optimal solution of problem (1.4) if and only if $\tilde{x}$ is a global optimal solution of problem $\tilde{P}$.*

Theorem 2.2 *Suppose that Assumptions A and B hold. Then $\tilde{x}$ is a local optimal solution of problem (1.4) if and only if $\tilde{x}$ is a local optimal solution of problem $\tilde{P}$.*

3 Smoothing Approximation for $\tilde{P}$

In this section, we study a smoothing approach for solving problem $\tilde{P}$.

Let $\varepsilon \in \mathcal{R}_+$ be a smoothing parameter. Define a function $\phi_\varepsilon : \mathcal{R}^2 \to \mathcal{R}$ by

$$\phi_\varepsilon(s,t) = \frac{1}{2}\big(s + t - \sqrt{(s-t)^2 + 4\varepsilon^2}\big)$$

which is called the CHKS (Chen-Harker-Kanzow-Smale) smoothing function for the function $\min(s,t)$.

Proposition 3.1 *[11] For any $\varepsilon \in \mathcal{R}_+$, we have*

1. $|\phi_\varepsilon(s,t) - \min(s,t)| \le \varepsilon$,

2. $\phi_\varepsilon(s,t) = 0 \iff s \ge 0,\, t \ge 0,\, st = \varepsilon^2$,

3. *$\phi_\varepsilon(s,t)$ is a C^∞ function of (s,t) for a fixed $\varepsilon > 0$.*

Let us define the function $\Psi : \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^r \times \mathcal{R}^J \times \Omega \to \mathcal{R}^{r+J}$ by

$$\Psi(\varepsilon, x, u, y, \omega) = \begin{pmatrix} \nabla_u g(x,u,\omega) - \nabla_u h(x,u,\omega) y \\ \phi_\varepsilon(y_1, -h_1(x,u,\omega)) \\ \vdots \\ \phi_\varepsilon(y_J, -h_J(x,u,\omega)) \end{pmatrix}.$$

Then, a parametric smooth approximation to problem $\tilde{P}$ can be formulated as

$$\begin{array}{lll} \tilde{P}(\varepsilon,\delta): & \min\limits_x & E_\omega[f(x,\omega)] \\ & \text{s. t.} & E_\omega[g(x, u(\varepsilon,x,\omega), \omega)] \le \delta, \end{array}$$

where ε, $\delta > 0$ are parameters, and $u(\varepsilon,x,\omega)$, together with a vector $y(\varepsilon,x,\omega) \in R^J$, satisfies

$$\Psi(\varepsilon, x, u(\varepsilon,x,\omega), y(\varepsilon,x,\omega), \omega) = 0.$$

We denote the feasible regions of $\tilde{P}(\varepsilon,\delta)$ and $\tilde{P}$ by $\mathcal{F}(\varepsilon,\delta)$ and $\tilde{\mathcal{F}}$, respectively. It is clear that if $(\varepsilon,\delta) = 0$ then $\tilde{P}(\varepsilon,\delta)$ coincides with $\tilde{P}$, and hence $\mathcal{F}(0,0)$ is identical to $\tilde{\mathcal{F}}$. In the next section, we will present an algorithm for solving problem $\tilde{P}$ by solving a sequence of problems $\tilde{P}(\varepsilon,\delta)$. In the rest of this section, we concentrate on establishing some properties of $\tilde{P}(\varepsilon,\delta)$. To this end, we state two lemmas at first. Their proofs are omitted since they can be found in some text books on matrix analysis.

Lemma 3.1 *Let*

$$T = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where $A \in R^{p\times p}$, $B \in R^{p\times q}$, $C \in R^{q\times q}, p \le q$. Then the following two statements are true:
(1) If A is negative semidefinite, C is positive definite and the row rank of B is p, then T is nonsingular.
(2) If A is negative definite and C is positive definite, then T is nonsingular.

Lemma 3.2 *Let*

$$T = \begin{pmatrix} A & BC & D \\ B^T & I - C & O \\ D^T & O & O \end{pmatrix},$$

where $A \in \mathcal{R}^{p\times p}$*,* $B \in \mathcal{R}^{p\times q}$*,* $C \in \mathcal{R}^{q\times q}$*,* $D \in \mathcal{R}^{p\times s}$*. If* A *and* $C^T - C^TC$ *are negative definite and positive semidefinite, respectively, and the column rank of* (B, D) *is* $q + s$*, then* T *is nonsingular.*

Using Lemmas 3.1 and 3.2, we can investigate the nonsingularity of (generalized) Jacobian of Φ with respect to the variable (u, y), which plays an important role in the rest of this section and the convergence analysis of the algorithm presented in Section 4. We first recall the conception of generalized Jacobian in Clarke's sense for locally Lipschitz functions [4].

Definition 3.1 *Suppose* $F : \mathcal{R}^p \to \mathcal{R}^q$ *is a locally Lipschitz function. The generalized Jacobian of* F *at* x*, denoted by* $\partial F(x)$*, is the convex hull of all* $q \times p$ *matrices* W *obtained as the limits of all sequences* $\{\nabla F(x^k)^T\}$ *such that* $x^k \to x$*,* $x^k \in D_F$*, where* D_F *is the set of all points at which* F *is differentiable.*

Proposition 3.2 *The function* Ψ *is locally Lipschitz and regular.*

Proof. It is similar to [5, Lemma 1]. ∎

Proposition 3.3 *Let* $\Psi(\bar{\varepsilon}, \bar{x}, \bar{u}, \bar{y}, \bar{\omega}) = 0$*. Suppose that*

$$\bar{A}(\bar{\omega}) = \nabla^2_{uu} g(\bar{x}, \bar{u}, \bar{\omega}) - \sum_{j=1}^{J} \bar{y}_j \nabla^2_{uu} h_j(\bar{x}, \bar{u}, \bar{\omega})$$

is negative definite. In addition, suppose that the LICQ is satisfied at $\bar{u}$ *for problem* $Q(\bar{x}, \bar{\omega})$ *if* $\bar{\varepsilon} = 0$*. Then all matrices in* $\partial_{(u,y)}\Psi(\bar{\varepsilon}, \bar{x}, \bar{u}, \bar{y}, \bar{\omega})$ *are nonsingular.*

Proof. We only show the conclusion in the case where $\bar{\varepsilon} = 0$. The conclusion in the case where $\bar{\varepsilon} > 0$ can be shown similarly by using Lemma 3.1. We assume without loss of generality that

$$\begin{cases} -h_j(\bar{x}, \bar{u}, \bar{\omega}) > \bar{y}_j, & j = 1, \cdots, J_1, \\ -h_j(\bar{x}, \bar{u}, \bar{\omega}) = \bar{y}_j, & j = J_1 + 1, \cdots, J_2, \\ -h_j(\bar{x}, \bar{u}, \bar{\omega}) < \bar{y}_j, & j = J_2 + 1, \cdots, J, \end{cases}$$

and write

$$\begin{cases} \bar{B}_1 = [\nabla_u h_1(\bar{x}, \bar{u}, \bar{\omega}), \cdots, \nabla_u h_{J_1}(\bar{x}, \bar{u}, \bar{\omega})], \\ \bar{B}_2 = [\nabla_u h_{J_1+1}(\bar{x}, \bar{u}, \bar{\omega}), \cdots, \nabla_u h_{J_2}(\bar{x}, \bar{u}, \bar{\omega})], \\ \bar{B}_3 = [\nabla_u h_{J_2+1}(\bar{x}, \bar{u}, \bar{\omega}), \cdots, \nabla_u h_J(\bar{x}, \bar{u}, \bar{\omega})], \\ \bar{C}_2 = \mathrm{diag}[\bar{c}_1, \cdots, \bar{c}_{J_2-J_1}],\ 0 \le \bar{c}_j \le 1,\ j = 1, \cdots, J_2 - J_1. \end{cases}$$

Then, by direct calculation, it is not difficult to obtain

$$\partial_{(u,y)}\Psi(0,\bar{x},\bar{u},\bar{y},\bar{\omega}) =$$
$$\left\{ \begin{pmatrix} \bar{A} & -\bar{B}_1 & -\bar{B}_2 & -\bar{B}_3 \\ O & I & O & O \\ -\bar{C}_2\bar{B}_2^T & O & I-\bar{C}_2 & O \\ -\bar{B}_3^T & O & O & O \end{pmatrix} \;\middle|\; \begin{array}{l} 0 \le \bar{c}_j \le 1, \\ j = 1,\cdots,J_2 - J_1 \end{array} \right\}.$$

It is easy to see that the matrix

$$\begin{pmatrix} \bar{A} & -\bar{B}_1 & -\bar{B}_2 & -\bar{B}_3 \\ O & I & O & O \\ -\bar{C}_2\bar{B}_2^T & O & I-\bar{C}_2 & O \\ -\bar{B}_3^T & O & O & O \end{pmatrix}$$

is also nonsingular as the matrix

$$\begin{pmatrix} \bar{A} & \bar{B}_2 & \bar{B}_3 \\ \bar{C}_2\bar{B}_2^T & I-\bar{C}_2 & O \\ \bar{B}_3^T & O & O \end{pmatrix}.$$

Hence, by the given conditions and Lemma 3.2, all matrices in $\partial_{(u,y)}\Psi(0,\bar{x},\bar{u},\bar{y},\bar{\omega})$ are nonsingular. The proof is complete. ∎

We now focus our discussion on problem $\tilde{P}$ where Ω is a finite discrete set. Specifically, let $\Omega = \{\omega_1,\omega_2,\cdots,\omega_L\}$. For every ω_l, $l = 1,2,\cdots,L$, we denote

$$f^l(\cdot) = f(\cdot,\omega_l), \quad g^l(\cdot,\cdot) = g(\cdot,\cdot,\omega_l), \quad h^l(\cdot,\cdot) = h(\cdot,\cdot,\omega_l)$$

and the jth-component of $h^l(\cdot,\cdot)$ by $h^l_j(\cdot,\cdot)$.

For the sake of clarity, throughout the rest of this paper, we let u_l and y_l denote the variables u and y, respectively, which correspond to the index l in the constraints of $\tilde{P}$. Then, problem $\tilde{P}$ can be rewritten as

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s. t.} & G(x,\mathbf{u}) \le 0, \end{array} \tag{3.5}$$

where $f(x) = \sum_{l=1}^L p_l f^l(x)$, $G(x,\mathbf{u}) = \sum_{l=1}^L p_l g^l(x,u_l)$, $p_l \ge 0$, $\sum_{l=1}^L p_l = 1$, and

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_L \end{pmatrix} \in \mathcal{R}^{rL}.$$

Here $u_l \in \mathcal{R}^r$, together with a vector $y_l \in \mathcal{R}^J$, satisfies

$$\begin{cases} \nabla_{u_l} g^l(x,u_l) - \nabla_{u_l} h^l(x,u_l) y_l = 0, \\ \min(y_l, -h^l(x,u_l)) = 0, \end{cases}$$

which constitutes the first-order optimality conditions for the problem

$$Q_l(x): \qquad \begin{array}{ll} \max\limits_{u} & g^l(x,u) \\ \text{s. t.} & h^l(x,u) \leq 0. \end{array}$$

On the other hand, problem $\tilde{P}(\varepsilon,\delta)$ can be rewritten as

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s. t.} & G(x,\mathbf{u}(\varepsilon,x) \leq \delta, \end{array} \tag{3.6}$$

where $G(x,\mathbf{u}(\varepsilon,x) = \sum_{l=1}^{L} p_l g^l(x,u_l(\varepsilon,x))$ and

$$\mathbf{u}(\varepsilon,x) = \begin{pmatrix} u_1(\varepsilon,x) \\ \vdots \\ u_L(\varepsilon,x) \end{pmatrix}.$$

Here, $u_l(\varepsilon,x)$, together with $y_l(\varepsilon,x)$, satisfies the system

$$\Phi_l(\varepsilon,x,u_l(\varepsilon,x),y_l(\varepsilon,x)) := \begin{pmatrix} \nabla_{u_l} g^l(x,u_l(\varepsilon,x)) - \nabla_{u_l} h^l(x,u_l(\varepsilon,x))y_l(\varepsilon,x) \\ \phi_\varepsilon((y_l(\varepsilon,x))_1, -h_1^l(x,u_l(\varepsilon,x))) \\ \vdots \\ \phi_\varepsilon((y_l(\varepsilon,x))_J, -h_J^l(x,u_l(\varepsilon,x))) \end{pmatrix} = 0 \tag{3.7}$$

for $l = 1,2,\cdots,L$. Moreover, we set

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_L \end{pmatrix} \in \mathcal{R}^{JL}$$

and define a nonlinear operator $\Phi : \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^{(r+J)L} \to \mathcal{R}^{(r+J)L}$ by

$$\Phi(\varepsilon,x,\mathbf{u},\mathbf{y}) = \begin{pmatrix} \Phi_1(\varepsilon,x,u_1,y_1) \\ \vdots \\ \Phi_L(\varepsilon,x,u_L,y_L) \end{pmatrix}.$$

Proposition 3.4 *Let $\bar{\varepsilon} \in \mathcal{R}_+$ and $\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) = 0$. Suppose that*

$$\bar{A}_l = \nabla^2_{u_l u_l} g^l(\bar{x},\bar{u}_l) - \sum_{j=1}^{J} (\bar{y}_l)_j \nabla^2_{u_l u_l} h_j^l(\bar{x},\bar{u}_l)$$

is negative definite for each $l = 1,2,\cdots,L$, and the LICQ is satisfied at $\bar{u}_l$ for problem $Q_l(\bar{x})$. Then there exist a neighborhood $(\bar{\varepsilon}-\varepsilon',\bar{\varepsilon}+\varepsilon') \times N(\bar{x})$ of $(\bar{\varepsilon},\bar{x})$ and a continuous function $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot))$: $\{(\bar{\varepsilon}-\varepsilon',\bar{\varepsilon}+\varepsilon') \cap \mathcal{R}_+\} \times N(\bar{x}) \to \mathcal{R}^{(r+J)L}$ such that for each $(\varepsilon,x) \in \{(\bar{\varepsilon}-\varepsilon',\bar{\varepsilon}+\varepsilon') \cap \mathcal{R}_+\} \times N(\bar{x})$,

$$\Phi(\varepsilon,x,\mathbf{u}(\varepsilon,x,\mathbf{y}(\varepsilon,x) = 0. \tag{3.8}$$

Proof. According to the corollary of [4, Theorem 7.1.1], it suffices to check that the projection $\Pi_{(\mathbf{u},\mathbf{y})}\partial\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ of $\partial\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ on the space of the variable $(\mathbf{u},\mathbf{y})$ is comprised of nonsingular matrices. We only show the conclusion in the case where $\bar{\varepsilon}=0$. The conclusion in the case where $\bar{\varepsilon}>0$ can be shown similarly. By [4, Proposition 2.6.2 (e)] and the definition of the projection operator, we have

$$\begin{aligned}\Pi_{(\mathbf{u},\mathbf{y})}\partial\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) &\subseteq \Pi_{(\mathbf{u},\mathbf{y})}\begin{pmatrix}\partial\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\\ \vdots\\ \partial\Phi_s(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\end{pmatrix}\\ &\subseteq \begin{pmatrix}\Pi_{(\mathbf{u},\mathbf{y})}[\partial\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})]\\ \vdots\\ \Pi_{(\mathbf{u},\mathbf{y})}[\partial\Phi_s(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})]\end{pmatrix},\end{aligned}\tag{3.9}$$

where $s=(r+J)L$. Recall that Φ is regular by Proposition 3.2. It then follows from [4, Proposition 2.3.15] that

$$\partial\Phi_i(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\subseteq\partial_{(\varepsilon,x)}\Phi_i(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\times\partial_{(\mathbf{u},\mathbf{y})}\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}),\quad i=1,\cdots,s,$$

and hence

$$\begin{pmatrix}\Pi_{(\mathbf{u},\mathbf{y})}[\partial\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})]\\ \vdots\\ \Pi_{(\mathbf{u},\mathbf{y})}[\partial\Phi_s(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})]\end{pmatrix}\subseteq\begin{pmatrix}\partial_{(\mathbf{u},\mathbf{y})}\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\\ \vdots\\ \partial_{(\mathbf{u},\mathbf{y})}\Phi_s(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\end{pmatrix}.\tag{3.10}$$

On the other hand, from the very special structure of the function ϕ_ε, we have

$$\begin{pmatrix}\partial_{(\mathbf{u},\mathbf{y})}\Phi_1(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\\ \vdots\\ \partial_{(\mathbf{u},\mathbf{y})}\Phi_s(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\end{pmatrix}=\partial_{(\mathbf{u},\mathbf{y})}\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}),$$

see [4]. The above formula, together with (3.9) and (3.10), implies

$$\Pi_{(\mathbf{u},\mathbf{y})}\partial\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\subseteq\partial_{(\mathbf{u},\mathbf{y})}\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}).$$

Hence, we obtain, by Proposition 3.3, that $\Pi_{(\mathbf{u},\mathbf{y})}\partial\Phi(\bar{\varepsilon},\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ is comprised of nonsingular matrices. The proof is complete. ∎

Let $\mathcal{S}$ denote the set of all points $(x,\mathbf{u},\mathbf{y})$ satisfying $\Phi(0,x,\mathbf{u},\mathbf{y})=0$ and $G(x,\mathbf{u})\le 0$, that is,

$$\mathcal{S}:=\left\{(x,\mathbf{u},\mathbf{y})\in\mathcal{R}^{n+(r+J)L}\mid\Phi(0,x,\mathbf{u},\mathbf{y})=0,\ G(x,\mathbf{u})\le 0\right\}.\tag{3.11}$$

Proposition 3.5 *Let $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\in\mathcal{S}$. Suppose that for every $l=1,2,\cdots,L$, $\bar{A}_l$ is negative definite, and the LICQ is satisfied at $\bar{u}_l$ for problem $Q_l(\bar{x})$. Then, there exist two positive numbers $\bar{\varepsilon}$ and $\bar{\tau}$, a neighborhood $N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ of $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$, and a continuous function $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot)):[0,\bar{\varepsilon})\times\Pi_xN(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\to\mathcal{R}^{(r+J)L}$, such that for any $(\varepsilon,x,\mathbf{u},\mathbf{y})\in(0,\bar{\varepsilon})\times(N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})\cap\mathcal{S})$,*

$$\Phi(\varepsilon,x,\mathbf{u}(\varepsilon,x),\mathbf{y}(\varepsilon,x))=0$$

and

$$||\mathbf{u}(\varepsilon,x)-\mathbf{u}|| \le 2\sqrt{LJ}\bar{\tau}\varepsilon, \quad ||\mathbf{y}(\varepsilon,x)-\mathbf{y}|| \le 2\sqrt{LJ}\bar{\tau}\varepsilon. \tag{3.12}$$

Proof. Firstly, by Proposition 3.4, there exist a positive number $\hat{\varepsilon}$, a neighborhood $N(\bar{x})$ of $\bar{x}$ and a continuous function $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot)) : [0,\hat{\varepsilon}) \times N(\bar{x}) \to R^{(r+J)L}$, such that for any $(\varepsilon,x) \in (0,\bar{\varepsilon}) \times N(\bar{x})$,

$$\Phi(\varepsilon,x,\mathbf{u}(\varepsilon,x),\mathbf{y}(\varepsilon,x)) = 0. \tag{3.13}$$

Secondly, it is not difficult to see that $\Phi(\varepsilon,x,\mathbf{u},\mathbf{y})$ is smooth and $\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\varepsilon,x,\mathbf{u},\mathbf{y})$ is nonsingular for any $\varepsilon>0$ and $(x,\mathbf{u},\mathbf{y})$ close enough to $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$. We now show that there exist a neighborhood $N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ with $\Pi_x N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \subset N(\bar{x})$ and a positive number $\bar{\varepsilon} \in (0,\hat{\varepsilon})$ such that (3.12) holds for any $(\varepsilon,x,\mathbf{u},\mathbf{y}) \in (0,\bar{\varepsilon}) \times (N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \cap \mathcal{S})$. To this end, we show that there exist a positive number $\bar{\varepsilon}$, a neighborhood $N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ of $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ and a positive constant $\bar{\tau}$ such that for any $(\varepsilon,x,\mathbf{u},\mathbf{y}) \in (0,\bar{\varepsilon}) \times (N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \cap \mathcal{S})$,

$$||\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon},x,\tilde{\mathbf{u}},\tilde{\mathbf{y}})^{-1}|| \le \bar{\tau}\ , \tag{3.14}$$

where $0<\tilde{\varepsilon}<\varepsilon$, $\mathbf{u}<\tilde{\mathbf{u}}<\mathbf{u}(\varepsilon,x)$, which means that the every component of $\tilde{\mathbf{u}}$ is in the open segment connecting the corresponding component of $\mathbf{u}$ and $\mathbf{u}(\varepsilon,x)$, and $\mathbf{y}<\tilde{\mathbf{y}}<\mathbf{y}(\varepsilon,x)$. Here, in different rows of $\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon},x,\tilde{\mathbf{u}},\tilde{\mathbf{y}})$, the values of $\tilde{\varepsilon}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$ may not be the same, but for the sake of simplicity, they are still written as $\tilde{\varepsilon}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$. Suppose on the contrary that (3.14) does not hold, then there exist a sequence $\{\varepsilon_k\}$ with $\varepsilon_k \downarrow 0$ and $\{(x^k,\mathbf{u}^k,\mathbf{y}^k)\}$ with $(x^k,\mathbf{u}^k,\mathbf{y}^k) \to (\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$, such that

$$||\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon}_k,x^k,\tilde{\mathbf{u}}^k,\tilde{\mathbf{y}}^k)^{-1}|| \to \infty \tag{3.15}$$

for some $0<\tilde{\varepsilon}_k<\varepsilon_k$, $\mathbf{u}^k<\tilde{\mathbf{u}}^k<\mathbf{u}(\varepsilon_k,x^k)$ and $\mathbf{y}^k<\tilde{\mathbf{y}}^k<\mathbf{y}(\varepsilon_k,x^k)$. Since $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot))$ is a continuous function and $(x^k,\mathbf{u}^k,\mathbf{y}^k) \to (\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ and $\varepsilon_k \downarrow 0$, it follows that $\tilde{\varepsilon}_k \to 0$ and $(\tilde{\mathbf{u}}^k,\tilde{\mathbf{y}}^k) \to (\bar{\mathbf{u}},\bar{\mathbf{y}})$. On the other hand, by direct computation, we obtain

$$\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon}_k,x^k,\tilde{\mathbf{u}}^k,\tilde{\mathbf{y}}^k)^T = \begin{pmatrix} \tilde{A}_1^k & -\tilde{B}_1^k & & & \\ (\tilde{C}_1^k+I)\tilde{B}_1^{kT} & \tilde{C}_1^k-I & & & \\ & & \ddots & & \\ & & & \tilde{A}_L^k & -\tilde{B}_L^k \\ & & & (\tilde{C}_L^k+I)\tilde{B}_L^{kT} & \tilde{C}_L^k-I \end{pmatrix},$$

where $\tilde{A}_l^k$, $\tilde{B}_l^k$, $\tilde{C}_l^k$, $l=1,2,\cdots,L$, are matrices given by

$$\begin{cases} \tilde{A}_l^k = \nabla^2_{u_lu_l}g_l(x^k,\tilde{u}_l^k) - \sum_{j=1}^J(\tilde{y}_l^k)_j\nabla^2_{u_lu_l}h_j^l(x^k,\tilde{u}_l^k), \\ \tilde{B}_l^k = [\nabla_{u_l}h_1^l(x^k,\tilde{u}_l^k),\cdots,\nabla_{u_l}h_J^l(x^k,\tilde{u}_l^k)], \\ \tilde{C}_l^k = \mathrm{diag}[\tilde{c}_1^k(l),\cdots,\tilde{c}_J^k(l)],\ \tilde{c}_j^k(l) = \frac{(\tilde{y}_l^k)_j+h_j^l(x^k,\tilde{u}_l^k)}{\sqrt{((\tilde{y}_l^k)_j+h_j^l(x^k,\tilde{u}_l^k))^2+4\tilde{\varepsilon}_k^2}}, \\ l=1,2,\cdots,L. \end{cases}$$

Since $\tilde{\varepsilon}_k \to 0$ and $(x^k,\tilde{\mathbf{u}}^k,\tilde{\mathbf{y}}^k) \to (\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$, it follows that

$$\lim_{k\to\infty}\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon}_k,x^k,\tilde{\mathbf{u}}^k,\tilde{\mathbf{y}}^k)^T \in \partial_{(\mathbf{u},\mathbf{y})}\Phi(0,\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}).$$

By Proposition 3.3, all matrices in $\partial_{(\mathbf{u},\mathbf{y})}\Phi(0,\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$ are nonsingular. Hence, there exists a positive constant $\bar{\tau}$ such that

$$||\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon}_k, x^k, \tilde{\mathbf{u}}^k, \tilde{\mathbf{y}}^k)^{-1}|| \leq \bar{\tau},$$

for k large enough, which contradicts (3.15). Therefore, (3.14) holds.

We assume, without loss of generality, that $(0,\bar{\varepsilon})\times\Pi_x N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \subset (0,\hat{\varepsilon})\times N(\bar{x})$. Take any $(\varepsilon, x, \mathbf{u}, \mathbf{y}) \in (0,\bar{\varepsilon}) \times (N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \cap \mathcal{S})$. Since $\Phi(0,x,\mathbf{u},\mathbf{y}) = 0$, we have, by (3.13) and the mean value theorem, that

$$\begin{aligned} 0 &= \Phi(\varepsilon, x, \mathbf{u}(\varepsilon,x), \mathbf{y}(\varepsilon,x)) - \Phi(0,x,\mathbf{u},\mathbf{y}) \\ &= \nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon}, x, \tilde{\mathbf{u}}, \tilde{\mathbf{y}})^T \begin{pmatrix} \mathbf{u}(\varepsilon,x) - \mathbf{u} \\ \mathbf{y}(\varepsilon,x) - \mathbf{y} \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{0} \\ \tilde{\theta}^1 \\ \vdots \\ \mathbf{0} \\ \tilde{\theta}^L \end{pmatrix}, \end{aligned} \tag{3.16}$$

where $\mathbf{u} < \tilde{\mathbf{u}} < \mathbf{u}(\varepsilon,x)$, $\mathbf{y} < \tilde{\mathbf{y}} < \mathbf{y}(\varepsilon,x)$, $0 < \tilde{\varepsilon} < \varepsilon$, $\mathbf{0}$ is the r-dimensional zero vector and $\tilde{\theta}^l = (\tilde{\theta}^l_1, \cdots, \tilde{\theta}^l_J)^T$, where

$$\tilde{\theta}^l_j = \frac{4\tilde{\varepsilon}}{\sqrt{((\tilde{y}_l)_j + h^l_j(x,\tilde{u}_l))^2 + 4\tilde{\varepsilon}^2}}, \quad j = 1,2,\cdots,J, \ l = 1,2,\cdots,L.$$

It is clear that $0 < \tilde{\theta}^l_j < 2$. Note that (3.14) holds even if the values of $\tilde{\varepsilon}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$ in different rows of $\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon},x,\tilde{\mathbf{u}},\tilde{\mathbf{y}})$ are different, we have, by (3.16) and (3.14), that

$$\begin{aligned} \left\| \begin{pmatrix} \mathbf{u}(\varepsilon,x) - \mathbf{u} \\ \mathbf{y}(\varepsilon,x) - \mathbf{y} \end{pmatrix} \right\| &\leq \varepsilon \, ||\nabla_{(\mathbf{u},\mathbf{y})}\Phi(\tilde{\varepsilon},x,\tilde{\mathbf{u}},\tilde{\mathbf{y}})^{-1}|| \left\| \begin{pmatrix} 0 \\ \tilde{\theta}^1 \\ \vdots \\ 0 \\ \tilde{\theta}^L \end{pmatrix} \right\| \\ &\leq 2\sqrt{LJ}\bar{\tau} \, \varepsilon, \end{aligned}$$

where the last inequality follows from the fact that $||\tilde{\theta}^l|| \leq 2\sqrt{J}$ for $l = 1,2,\cdots,L$. This immediately yields the desired result. The proof is complete. ∎

Proposition 3.6 *Suppose that there exists a vector $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \in \mathcal{S}$ such that for every $l = 1,2,\cdots,L$, $\bar{A}_l$ is negative definite and the LICQ is satisfied at $\bar{u}_l$ for problem $Q_l(\bar{x})$. Then there exists an $\bar{\varepsilon} > 0$ such that the feasible set $\mathcal{F}(\varepsilon^2,\delta)$ of problem $\tilde{P}(\varepsilon^2,\delta)$ is nonempty for any $0 < \varepsilon < \bar{\varepsilon}$ and $\delta = \varepsilon$.*

Proof. By Proposition 3.5, there exist two positive numbers $\bar{\varepsilon}$ and $\bar{\tau}$ and a continuous function $(\mathbf{u}(\cdot,\bar{x}), \mathbf{y}(\cdot,\bar{x})) : [0,\bar{\varepsilon}) \to \mathcal{R}^r \times \mathcal{R}^J$ such that for any $0 < \varepsilon < \bar{\varepsilon}$,

$$\Phi(\varepsilon^2, \bar{x}, \mathbf{u}(\varepsilon^2,\bar{x}), \mathbf{y}(\varepsilon^2,\bar{x})) = 0 \tag{3.17}$$

and

$$||\mathbf{u}(\varepsilon^2, \bar{x}) - \bar{\mathbf{u}}|| \leq 2\sqrt{LJ}\bar{\tau}\varepsilon^2.$$

Hence,

$$||u_l(\varepsilon^2, \bar{x}) - \bar{u}_l|| \leq 2\sqrt{LJ}\bar{\tau}\varepsilon^2, \ \ l = 1, 2, \cdots, L. \tag{3.18}$$

Since $g^l(\bar{x}, u_l)$ has continuous first-derivatives $\nabla_{u_l} g^l(\bar{x}, u_l)$ for every $l = 1, 2, \cdots, L$, it is clear that $g^l(\bar{x}, u_l)$ is locally Lipschitz with respect to the variable u_l. Therefore, there exists a positive constant $\bar{M}$ such that for $l = 1, 2, \cdots, L$,

$$|g^l(\bar{x}, u_l(\varepsilon^2, \bar{x})) - g^l(\bar{x}, \bar{u}_l)| \leq \bar{M} \ ||u_l(\varepsilon^2, \bar{x}) - \bar{u}_l||,$$

which in turn implies

$$\begin{aligned} G(\bar{x}, \mathbf{u}(\varepsilon^2, \bar{x})) \quad & \leq G(\bar{x}, \bar{\mathbf{u}}) + \bar{M} \sum_{l=1}^{L} p_l \ ||u_l(\varepsilon^2, \bar{x}) - \bar{u}_l|| \\ & \leq 2\bar{M}\sqrt{LJ}\bar{\tau}\varepsilon^2, \end{aligned} \tag{3.19}$$

where the last inequality follows from (3.18) and the fact that $G(\bar{x}, \bar{\mathbf{u}}) \leq 0$. By (3.19), we can take $\bar{\varepsilon}$ small enough such that $2\bar{M}\sqrt{LJ}\bar{\tau}\bar{\varepsilon} \leq 1$. Hence, for any $0 < \varepsilon < \bar{\varepsilon}$, we have

$$G(\bar{x}, \mathbf{u}(\varepsilon^2, \bar{x})) \leq \varepsilon.$$

This formula, together with (3.17), implies $\bar{x} \in \mathcal{F}(\varepsilon^2, \varepsilon)$ for $0 < \varepsilon < \bar{\varepsilon}$. We obtain the desired result and complete the proof of the proposition. ∎

Assumption C. For every $\varepsilon > 0$ and $l = 1, 2, \cdots, L$, there are vectors $u_l(\varepsilon^2, x)$ and $y_l(\varepsilon^2, x)$ such that

$$\Phi_l(\varepsilon^2, x, u_l(\varepsilon^2, x), y_l(\varepsilon^2, x)) = 0.$$

The vector $u_l(\varepsilon^2, x)$ is unique and continuous with respect to x for every $l = 1, 2, \cdots, L$.

Theorem 3.1 *Suppose that for any $x \in \mathcal{R}^n$ and every $l = 1, 2, \cdots, L$, the LICQ is satisfied at every $u_l \in P_l(x)$ for problem $Q_l(x)$, where*

$$P_l(x) := \{u \in \mathcal{R}^r \mid \exists \ j \in \{1, 2, \cdots, J\} \ s.t. \ \ h_j^l(x, u) = 0\}.$$

Suppose further that there exists a vector $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$ such that the conditions in Proposition 3.6 hold and the level set

$$\{x \in R^n \mid f(x) \leq f(\bar{x})\}$$

is bounded. Then, under Assumption C, there exists a positive number $\bar{\varepsilon}$ such that problem $\tilde{P}(\varepsilon^2, \varepsilon)$ is solvable for any $0 < \varepsilon < \bar{\varepsilon}$.

Proof. Firstly, by Propositions 3.6, there exists an $\bar{\varepsilon} > 0$ such that $\bar{x} \in \mathcal{F}(\varepsilon^2, \varepsilon)$ for any $0 < \varepsilon < \bar{\varepsilon}$, which implies $\mathcal{F}(\varepsilon^2, \varepsilon)$ is nonempty. Furthermore, we may prove that for any fixed $\varepsilon \in (0, \bar{\varepsilon})$, $\mathcal{F}(\varepsilon^2, \varepsilon)$ is closed from Assumption C and the given condition that $g^l(\cdot, \cdot)$, $h^l(\cdot, \cdot)$, $l = 1, 2, \cdots, L$, are continuous. In fact, for a sequence of feasible points $\{x^k\} \subset \mathcal{F}(\varepsilon^2, \varepsilon)$ with a limit point $\hat{x}$, there exist $\mathbf{u}(\varepsilon^2, x^k)$ and $\mathbf{y}(\varepsilon^2, x^k)$ such that $\Phi(\varepsilon^2, x^k, \mathbf{u}(\varepsilon^2, x), \mathbf{y}(\varepsilon^2, x)) = 0$ and $G(x^k, \mathbf{u}(\varepsilon^2, x^k)) \leq \varepsilon$. Since $\Phi(\varepsilon^2, x^k, \mathbf{u}(\varepsilon^2, x^k), \mathbf{y}(\varepsilon^2, x^k)) = 0$, we have that for every $l = 1, 2, \cdots, L$,

$$\begin{cases} \nabla_{u_l} g^l(x^k, u_l(\varepsilon^2, x^k)) - \nabla_{u_l} h^l(x^k, u_l(\varepsilon^2, x^k)) y_l(\varepsilon^2, x^k) = 0, \\ \phi_{\varepsilon^2}((y_l(\varepsilon^2, x^k))_j, -h^l_j(x^k, u_l(\varepsilon^2, x^k))) = 0, \ \ j = 1, 2, \cdots, J. \end{cases} \tag{3.20}$$

We claim that $\{\mathbf{y}(\varepsilon^2, x^k)\}$ is bounded. Otherwise, there exists an index l_0 such that $||y_{l_0}(\varepsilon^2, x^k)|| \to \infty$. Then, by dividing every equality for index l_0 in (3.20) by $||y_{l_0}(\varepsilon^2, x^k)||$ and letting $k \to \infty$, we obtain

$$\begin{cases} \nabla_{u_{l_0}} h^{l_0}(\hat{x}, \hat{u}_{l_0}) \hat{y}_{l_0} = 0, \\ (\hat{y}_{l_0})_j \geq 0, h^{l_0}_j(\hat{x}, \hat{u}_{l_0}) \leq 0, \\ (\hat{y}_{l_0})_j h^{l_0}_j(\hat{x}, \hat{u}_{l_0}) = 0, \ \ j = 1, 2, \cdots, J, \end{cases} \tag{3.21}$$

where $\hat{u}_{l_0} = u_{l_0}(\varepsilon^2, \hat{x})$ and $0 \neq \hat{y}_{l_0} = ((\hat{y}_{l_0})_1, (\hat{y}_{l_0})_2, \cdots, (\hat{y}_{l_0})_J)^T \in U$. From the last equality in (3.21), we have $(\hat{y}_{l_0})_j = 0$ for all j such that $h^{l_0}_j(\hat{x}, \hat{u}_{l_0}) \neq 0$. Hence, the first equality in (3.21) can be rewritten as

$$\sum_{j \in \mathcal{I}_{h^{l_0}}(\hat{x}, \hat{u}_{l_0})} (\hat{y}_{l_0})_j \nabla_{u_{l_0}} h^{l_0}_j(\hat{x}, \hat{u}_{l_0}) = 0. \tag{3.22}$$

It is clear that the set $\{(\hat{y}_{l_0})_j \mid j \in \mathcal{I}_{h^{l_0}}(\hat{x}, \hat{u}_{l_0})\}$ contains a non-zero element and $\hat{u}_{l_0} \in P_{l_0}(\hat{x})$. Hence, by the given condition that the LICQ is satisfied at $\hat{u}_{l_0}$ for problem $Q_{l_0}(\hat{x})$, we deduce that

$$\nabla_{u_{l_0}} h^{l_0}_j(\hat{x}, \hat{u}_{l_0}), \quad j \in \mathcal{I}_{h^{l_0}}(\hat{x}, \hat{u}_{l_0})$$

are linearly independent. This contradicts (3.22). Since $\{\mathbf{y}(\varepsilon^2, x^k)\}$ is bounded, without loss of generality, we assume that $\mathbf{y}(\varepsilon^2, x^k) \to \hat{\mathbf{y}}$. Since $\Phi(\varepsilon^2, x^k, \mathbf{u}(\varepsilon^2, x^k), \mathbf{y}(\varepsilon^2, x^k)) = 0$ and $G(x^k, \mathbf{u}(\varepsilon^2, x^k)) \leq \varepsilon$, letting $k \to \infty$ yields $\Phi(\varepsilon^2, \hat{x}, \mathbf{u}(\varepsilon^2, \hat{x}), \hat{\mathbf{y}}) = 0$ and $G(\hat{x}, \mathbf{u}(\varepsilon^2, \hat{x})) \leq \varepsilon$, which implies $\hat{x} \in \mathcal{F}(\varepsilon^2, \varepsilon)$. Hence, we obtain the desired result from the continuity of f. The proof is complete. ∎

4 Algorithm and Its Convergence Analysis

In this section, we further consider problem $\tilde{P}$ in the case where Ω is a finite discrete set. From the discussion in the previous sections, problem (1.4) is equivalent to problem $\tilde{P}$. Furthermore, if $\tilde{P}$ is solvable, then there exists a positive number $\bar{\varepsilon}$ such that problem $P(\varepsilon^2, \varepsilon)$ is solvable for any $0 < \varepsilon < \bar{\varepsilon}$ under suitable conditions. Since $\tilde{P}(\varepsilon^2, \varepsilon)$ is a smooth approximation to the nonsmooth problem $\tilde{P}$, we may obtain a solution of problem $\tilde{P}$ by solving a sequence of smooth problems $\tilde{P}(\varepsilon^2, \varepsilon)$. Now we present a smoothing implicit programming approach for solving problem $\tilde{P}$:

Algorithm 4.1 *Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\varepsilon_k \downarrow 0$.*
For $k = 1, 2, \cdots$, find a solution x^k of the problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ s.\ t. & G(x, \mathbf{u}(\varepsilon_k^2, x)) \leq \varepsilon_k, \end{array} \tag{4.23}$$

where $\mathbf{u}(\varepsilon_k^2, x) = (u_1(\varepsilon_k^2, x), \cdots, u_L(\varepsilon_k^2, x))$, together with $\mathbf{y}(\varepsilon_k^2, x) = (y_1(\varepsilon_k^2, x), \cdots, y_L(\varepsilon_k^2, x))$, satisfies the system

$$\Phi_l(\varepsilon_k^2, x, u_l(\varepsilon_k^2, x), y_l(\varepsilon_k^2, x)) = 0, \qquad l = 1, 2, \cdots L. \tag{4.24}$$

Let $u_l^k = u_l(\varepsilon_k^2, x^k)$, $y_l^k = y_l(\varepsilon_k^2, x^k)$, $l = 1, 2, \cdots L$, and

$$\mathbf{u}^k = \begin{pmatrix} u_1^k \\ \vdots \\ u_L^k \end{pmatrix}, \qquad \mathbf{y}^k = \begin{pmatrix} y_1^k \\ \vdots \\ y_L^k \end{pmatrix}.$$

Note that problem (4.23) is a smooth optimization problem. Under Assumption C, Algorithm 4.1 is well-defined. Now we investigate the limiting behavior of a sequence of optimal solutions of (4.23). To this end, we make the following assumption throughout the rest of this section.

Assumption D. The sequence $\{(x^k, \mathbf{u}^k, \mathbf{y}^k)\}$ generated by Algorithm 4.1 is convergent to a point $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$.

Recall that $\tilde{F}$ denotes the feasible region of problem $\tilde{P}$ and the set $\mathcal{S}$ is defined by (3.11). We define the set-valued mapping $S : \tilde{\mathcal{F}} \to \mathcal{R}^{(r+J)L}$ by

$$S(x) := \left\{ (\mathbf{u}, \mathbf{y}) \in \mathcal{R}^{(r+J)L} \mid (x, \mathbf{u}, \mathbf{y}) \in \mathcal{S} \right\}.$$

Definition 4.1 *Let $\bar{x} \in \tilde{\mathcal{F}}$ and $(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \in S(\bar{x})$. We say that the set-valued mapping S is stable at $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$ if, for any neighborhood $N(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ of $(\bar{\mathbf{u}}, \bar{\mathbf{y}})$, there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $S(x) \cap N(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \neq \emptyset$ for any $x \in N(\bar{x}) \cap \tilde{\mathcal{F}}$.*

Theorem 4.1 *Let $\{(x^k, \mathbf{u}^k, \mathbf{y}^k)\}$ be a sequence generated by Algorithm 4.1. Then the limit point $\bar{x}$ of $\{x^k\}$ lies in $\tilde{\mathcal{F}}$. Moreover, suppose that for every $l = 1, 2, \cdots, L$, $\bar{A}_l$ is negative definite, and the LICQ is satisfied at $\bar{u}_l$ for problem $Q_l(\bar{x})$, and the set-valued mapping S is stable at $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$. Then $\bar{x}$ is a local optimal solution of problem $\tilde{P}$.*

Proof. First note that

$$\begin{array}{l} G(x^k, \mathbf{u}^k) \leq \varepsilon_k, \\ \Phi(\varepsilon_k^2, x^k, \mathbf{u}^k, \mathbf{y}^k) = 0 \end{array}$$

hold for all k. Letting $k \to \infty$, we have $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}}) \in \mathcal{S}$, which implies that $\bar{x}$ is a feasible solution of $\tilde{P}$. Moreover, by Proposition 3.5, there exist a positive number $\bar{\varepsilon}$, a neighborhood $N(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$ of $(\bar{x}, \bar{\mathbf{u}}, \bar{\mathbf{y}})$, a continuous function $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot)) : [0,\bar{\varepsilon}) \times \Pi_x N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \to \mathcal{R}^{(r+J)L}$, and a positive constant $\bar{\tau}$ such that, for any $(\varepsilon_k, x, \mathbf{u}, \mathbf{y}) \in (0,\bar{\varepsilon}) \times (N(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}}) \cap \mathcal{S})$,

$$\Phi(\varepsilon_k^2, x, \mathbf{u}(\varepsilon_k^2, x), \mathbf{y}(\varepsilon_k^2, x)) = 0 \tag{4.25}$$

and

$$||\mathbf{u}(\varepsilon_k^2, x) - \mathbf{u}|| \le 2\sqrt{LJ}\bar{\tau}\varepsilon_k^2, \quad ||\mathbf{y}(\varepsilon_k^2, x) - \mathbf{y}|| \le 2\sqrt{LJ}\bar{\tau}\varepsilon_k^2. \tag{4.26}$$

Then, in a similar way to the proof of Proposition 3.6, we can show that there exists a positive constant $\bar{M}$ such that

$$\begin{aligned} G(x, \mathbf{u}(\varepsilon_k^2, x)) &\le G(x, \mathbf{u}) + \bar{M}\ ||\mathbf{u}(\varepsilon_k^2, x) - \mathbf{u}|| \\ &\le 2\bar{M}\sqrt{LJ}\bar{\tau}\varepsilon_k^2 \le \varepsilon_k, \end{aligned}$$

for all k large enough. The above discussion shows that there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for any $x \in N(\bar{x}) \cap \tilde{\mathcal{F}}$, x is a feasible solution of (4.23) whenever k is large enough, since the set-valued mapping S is stable at $(\bar{x},\bar{\mathbf{u}},\bar{\mathbf{y}})$. Therefore, for any $x \in N(\bar{x}) \cap \tilde{F}$, the inequality

$$f(x^k) \le f(x)$$

holds for all k large enough. Letting $k \to \infty$, we have

$$f(\bar{x}) \le f(x),$$

which implies that $\bar{x}$ is a local optimal solution of problem $\tilde{P}$. The proof is complete. ∎

Theorem 4.2 *Let $\{(x^k, \mathbf{u}^k, \mathbf{y}^k)\}$ be a sequence generated by Algorithm 4.1. Suppose that, for every $(x,\mathbf{u},\mathbf{y}) \in \mathcal{S}$ and every $l = 1,2,\cdots,L$, A_l is negative definite, and the LICQ is satisfied at u_l for problem $Q_l(x)$, and the set-valued mapping S is stable at every $(x,\mathbf{u},\mathbf{y}) \in \mathcal{S}$. Then the limit point $\bar{x}$ of $\{x^k\}$ is a global optimal solution of problem $\tilde{P}$.*

Proof. Recall that $\bar{x}$ is a feasible solution of $\tilde{P}$. For an arbitrary positive number η, we define the set $\tilde{\mathcal{F}}_\eta$ by

$$\tilde{\mathcal{F}}_\eta = \left\{ x \in \tilde{\mathcal{F}} \mid ||x - \bar{x}|| \le \eta \right\}.$$

It is clear that $\tilde{\mathcal{F}}_\eta$ is a nonempty compact set. For any $\hat{x} \in \tilde{\mathcal{F}}_\eta$, there exists $(\hat{\mathbf{u}},\hat{\mathbf{y}})$ such that $(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}}) \in \mathcal{S}$. Since the conditions in Proposition 3.5 are satisfied at $(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}})$, in a similar way to the proof of Theorem 4.1, we can show that there exist a neighborhood $N(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}})$ of $(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}})$, two positive numbers $\hat{\varepsilon} = \hat{\varepsilon}(\hat{x})$ and $\hat{\tau} = \hat{\tau}(\hat{x})$, and a continuous function $(\mathbf{u}(\cdot,\cdot),\mathbf{y}(\cdot,\cdot)) : [0,\hat{\varepsilon}) \times \Pi_x N(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}}) \to \mathcal{R}^{(r+J)L}$ such that, for any $(\varepsilon, x, \mathbf{u}, \mathbf{y}) \in (0,\hat{\varepsilon}) \times (N(\hat{x},\hat{\mathbf{u}},\hat{\mathbf{y}}) \cap \mathcal{S})$,

$$\Phi(\varepsilon^2, x, \mathbf{u}(\varepsilon^2, x), \mathbf{y}(\varepsilon^2, x)) = 0 \tag{4.27}$$

and

$$G(x, \mathbf{u}(\varepsilon^2, x)) \le 2\hat{M}\sqrt{LJ}\hat{\tau}\varepsilon^2, \tag{4.28}$$

where $\hat{M}$ is given by

$$\hat{M} = \max\{\hat{M}^1(\hat{x}, \hat{u}_1), \cdots, \hat{M}^L(\hat{x}, \hat{u}_L)\},$$

and $\hat{M}^l(\hat{x}, \hat{u}_l)$ is a local Lipschitz constant of the function $g^l(\hat{x}, \cdot)$ at $\hat{u}_l$ for each $l = 1, 2, \cdots, L$. Moreover, there exists a neighborhood $N(\hat{x})$ of $\hat{x}$ such that (4.27) and (4.28) hold for any $(\varepsilon, x) \in (0, \hat{\varepsilon}) \times N(\hat{x})$, since the set-valued mapping S is stable at $(\hat{x}, \hat{\mathbf{u}}, \hat{\mathbf{y}})$. Since the family of neighborhoods

$$\mathcal{N} = \{N(\hat{x}) \mid \hat{x} \in \tilde{\mathcal{F}}_\eta\}$$

is an open covering of $\tilde{\mathcal{F}}_\eta$, there is a finite number of neighborhoods, say $N_1, N_2, \cdots, N_s$, in $\mathcal{N}$ such that $\{N_1, N_2, \cdots, N_s\}$ constitutes a covering of $\tilde{\mathcal{F}}\eta$. Accordingly, there exist constants $\hat{\varepsilon}_1, \hat{\varepsilon}_2, \cdots, \hat{\varepsilon}_s$, $\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_s$ and $\hat{M}_1, \hat{M}_2, \cdots, \hat{M}_s$, respectively. Thus, by setting

$$\begin{aligned} \varepsilon^* &= \min\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \cdots, \hat{\varepsilon}_s\}, \\ \tau^* &= \max\{\hat{\tau}_1, \hat{\tau}_2, \cdots, \hat{\tau}_s\}, \\ M^* &= \max\{\hat{M}_1, \hat{M}_2, \cdots, \hat{M}_s\}, \end{aligned}$$

we have $\varepsilon_k \le \varepsilon^*$ and $2M^*\sqrt{LJ}\tau^*\varepsilon_k \le 1$ for all k large enough, and hence, for every $x \in \tilde{\mathcal{F}}_\eta$,

$$\Phi(\varepsilon_k^2, x, \mathbf{u}(\varepsilon_k^2, x), \mathbf{y}(\varepsilon_k^2, x)) = 0$$

and

$$G(x, \mathbf{u}(\varepsilon_k^2, x)) \le \varepsilon_k.$$

This shows that for every $x \in \tilde{\mathcal{F}}_\eta$, x is a feasible solution of (4.23) whenever k is large enough. Therefore, by using similar arguments to the proof of Theorem 4.1, we can show that $\bar{x}$ is an optimal solution of the problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s. t.} & x \in \tilde{\mathcal{F}}_\eta. \end{array}$$

Since η is arbitrary, $\bar{x}$ is actually a global optimal solution of $\tilde{P}$. The proof is complete. ∎

5 Final Remarks

In this paper, we have reformulated the SGSIP problem as a nonlinear programming problem with stochastic complementarity constraints, and established some properties of smoothing approximations for the reformulated problem. Furthermore, we have presented a smoothing implicit programming algorithm (Algorithm 4.1) for solving the problem with finite discrete distribution.

To illustrate the assumptions and the theorems in this paper, we consider the following example.

Example. Let

$$\begin{cases} g^l(x,u_l) = \frac{1}{2}u_l^T B_l(x)u_l + C_l(x)^T u_l + d_l(x), \\ h_j^l(x,u_l) = p_{j,l}(x)^T u_l + q_{j,l}(x), \\ j=1,2,\cdots,J,\ l=1,2,\cdots,L, \end{cases}$$

where $B_l(x) : \mathcal{R}^n \to \mathcal{R}^{r\times r}$, $l=1,2,\cdots,L$, are $r\times r$ continuous negative definite symmetric matrix-valued functions, $C_l(x)$, $p_{j,l}(x) : \mathcal{R}^n \to \mathcal{R}^r$, $l=1,2,\cdots,L$, $j=1,2,\cdots,J$, are continuous vector-valued functions, $d_l(x)$, $q_{j,l}(x) : \mathcal{R}^n \to \mathcal{R}$, $l=1,2,\cdots,L$, $j=1,2,\cdots,J$, are continuous real-valued functions. Obviously, Assumption A holds. It is clear that the equation $\Phi_l(\varepsilon^2,x,u_l,y_l)=0$ can be written as

$$\begin{cases} B_l(x)u_l + C_l(x) - P_l(x)y_l = 0, \\ \phi_{\varepsilon^2}((y_l)_1, -h_1^l(x,u_l)) = 0, \\ \qquad\vdots \\ \phi_{\varepsilon^2}((y_l)_J, -h_J^l(x,u_l)) = 0, \end{cases}$$

where $P_l(x) = [p_{1,l}(x),\cdots,p_{J,l}(x)]$. Furthermore, we obtain

$$\begin{cases} \phi_{\varepsilon^2}\left((y_l)_1, -p_{1,l}(x)^T B_l(x)^{-1}(P_l(x)y_l - C_l(x)) - q_{1,l}(x)\right) = 0, \\ \qquad\vdots \\ \phi_{\varepsilon^2}\left((y_l)_J, -p_{J,l}(x)^T B_l(x)^{-1}(P_l(x)y_l - C_l(x)) - q_{J,l}(x)\right) = 0. \end{cases} \tag{5.29}$$

Write

$$w^l(x) = M_l(x)y_l + z^l(x),$$

where

$$M_l(x) = -P_l(x)^T B_l(x)^{-1} P_l(x), \quad z^l(x) = P_l(x)^T B_l(x)^{-1} C_l(x) - q_l(x)$$

and

$$q_l(x) = (q_{1,l}(x),\cdots,q_{J,l}(x))^T.$$

Then (5.29) can be further rewritten as

$$\begin{cases} M_l(x)y_l + z^l(x) - w^l(x) = 0, \\ \phi_{\varepsilon^2}((y_l)_1, w_1^l(x)) = 0, \\ \qquad\vdots \\ \phi_{\varepsilon^2}((y_l)_J, w_J^l(x)) = 0. \end{cases}$$

We discuss Assumption C in the following two cases.

(1) If $P_l(x)$ is nonsingular for any x, then $M_l(x)$ is a positive definite matrix. Hence, the equation $\Phi_l(\varepsilon^2,x,u_l,y_l)=0$ has a unique solution

$$\begin{cases} y_l(\varepsilon^2,x) = ((y_l(\varepsilon^2,x))_1,\cdots,(y_l(\varepsilon^2,x))_J)^T, \\ u_l(\varepsilon^2,x) = B_l(x)^{-1}(P_l(x)y_l(\varepsilon^2,x) - C_l(x)). \end{cases}$$

In particular, if

$$P_l(x)^T B_l(x)^{-1} P_l(x) = \mathrm{diag}(\lambda_{1,l}(x), \cdots, \lambda_{J,l}(x)),$$

where $\lambda_{j,l}(x) < 0$ for $j = 1, 2, \cdots, J$, $l = 1, 2, \cdots, L$, then, for every $l = 1, 2, \cdots, L$, the unique solution of equation $\Phi_l(\varepsilon^2, x, u_l, y_l) = 0$ is given by

$$\begin{cases} y_l(\varepsilon^2, x) = ((y_l(\varepsilon^2, x))_1, \cdots, (y_l(\varepsilon^2, x))_J)^T, \\ u_l(\varepsilon^2, x) = B_l(x)^{-1}(P_l(x) y_l(\varepsilon^2, x) - C_l(x)), \end{cases}$$

where

$$(y_l(\varepsilon^2, x))_j = \frac{-\bar{q}_{j,l}(x) - \sqrt{(\bar{q}_{j,l}(x))^2 - 4\lambda_{j,l}(x)\varepsilon^4}}{2\lambda_{j,l}(x)}$$

and

$$\bar{q}_{j,l}(x) = q_{1,l}(x) - p_{j,l}(x)^T B_l(x)^{-1} C_l(x), \quad l = 1, 2, \cdots, L.$$

(2) In addition, suppose $M_l(x)$ is an R_0-matrix if $P_l(x)$ is singular. Since $M_l(x)$ is a positive semidefinite matrix, by [11, Corollary 3.9], the equation $\Phi_l(\varepsilon^2, x, u_l, y_l) = 0$ also has a unique solution. On the other hand, it is clear that

$$A_l(x) = \nabla^2_{u_l u_l} g^l(x, u_l) - \sum_{j=1}^{J} (y_l)_j \nabla^2_{u_l u_l} h^l_j(x, u_l) = B_l(x), \quad l = 1, 2, \cdots, L,$$

are negative definite. Consequently, by Lemma 3.1 (2), $\nabla_{(u_l, y_l)} \Phi_l(\varepsilon^2, x, u_l, y_l)$ is nonsingular for any $\varepsilon > 0$ and (x, u_l, y_l). Furthermore, since $\Phi_l(\varepsilon^2, x, u_l, y_l)$ is continuously differentiable with respect to (u_l, y_l) for any $\varepsilon > 0$, it follows from the Implicit Function Theorem [13, Theorem 5.2.4] that $y_l(\varepsilon^2, x)$ and $u_l(\varepsilon^2, x)$ are continuously differentiable. Therefore, Assumption C is satisfied. Furthermore, under certain conditions, Assumption B can be satisfied.

References

[1] J. R. Birge and F. Louveaux, *Introduction to Stochastic Programming,* Springer, New York, 1997.

[2] J. R. Birge and L. Qi, *Continuous Approximation Schemes for Solving Stochastic Programs,* Ann. Oper. Res. 56 (1995) 15–38.

[3] X. Chen, *Newton-type Methods for Stochastic Programming,* Math. Comput. Modeling 31 (2000) 89–98.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, New York, 1983.

[5] F. Facchinei, H. Jiang and L. Qi, *A Smoothing Method for Mathematical Programs with Equilibrium Constraints,* Math. Program. 85 (1999) 107–134.

[6] T. J. Graettinger and B. H. Krogh, *The Acceleration Radius: A Global Performance Measure for Robotic Manipulators,* IEE J, Robotics Automation 4 (1988) 60–69.

[7] G. Gürkan, A. Y. Özge and S. M. Robinson, *Sample-Path Solution of Stochastic Variational Inequalities*, Math. Program. 84 (1999) 313–333.

[8] R. Hettich and G. Still, *Semi-Infinite Programming Models in Robotics,* in: Parametric Optimization and Related Topics 2, Math. Res 62, J. Guddat et al., Eds., Akademie-Verlag, Berlin, 1991, 112–118.

[9] R. Hettich and G. Still, *Second-Order Optimality Conditions for Generalized Semi-Infinite Programming Problems,* Optimization 34 (1995) 195–211.

[10] H. T. Jogen, J. J. Ruckmann and O. Stein, *Generalized Semi-Infinite Optimization: A First Order Optimality Condition and Examples,* Math. Program. 83 (1998) 145–158.

[11] C. Kanzow, *Some Noninterior Continuation Methods for Linear Complementarity Problems,* SIAM J. Matrix Anal. Appl. 17 (1996) 851–868.

[12] W. Krabs, *On Time-Minimal Heating or Cooling of a Ball,* in: Internet. Ser. Numer. Math. 81, Birkhäuser, Basel, 1987, 121–131.

[13] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables,* Academic Press, New York, 1970.

[14] J. J. Ruckmann and A. Shapiro, *First-Order Condition in Generalized Semi-Infinite Programming,* J. Optim. Theory Appl. 10 (1999) 677–691.

[15] R. Schultz, L. Stougie and M. H. van der Vlerk, *Two-Stage Stochastic Integer Programming: A Survey,* Statist. Neerlandica 50 (1996) 404–416.

[16] O. Stein and G. Still, *On Optimality Conditions for Generalized Semi-Infinite Programming Problems,* J. Optim. Theory Appl. 104 (2000) 443–458.

[17] G. Still, *Generalized Semi-Infinite Programming: Theory and Methods,* European J. Oper. Res. 119 (1999) 301–313.

[18] G. Still, *Generalized Semi-Infinite Programming: Numerical Aspects,* Optimization 49 (2001) 223–242.

[19] R. J-B. Wets, *Stochastic Programming: Solution Techniques and Approximation Schemes,* in: Mathematical Programming: State-of-the-Art 1982, A. Bachem, M. Grötschel, and B. Korte, Eds., Springer-Verlag, Berlin, 1983, 560–603.