Stochastic R_0 Matrix Linear Complementarity Problems*

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Abstract

We consider the expected residual minimization formulation of the stochastic R_0 matrix linear complementarity problem. We show that the involved matrix being a stochastic R_0 matrix is a necessary and sufficient condition for the solution set of the expected residual minimization problem to be nonempty and bounded. Moreover, local and global error bounds are given for the stochastic R_0 matrix linear complementarity problem. A stochastic approximation method with acceleration by averaging is applied to solve the expected residual minimization problem.

Key words. Stochastic linear complementarity problem, R_0 matrix, Expected residual minimization

AMS subject classifications. 90C33, 90C15

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a subset of $\mathbb{R}^m$, and $\mathcal{F}$ is a σ-algebra generated by $\{\Omega \cap U : U \text{ is an open set in } \mathbb{R}^m\}$. We consider the stochastic linear complementarity problem (SLCP):

$$x \geq 0, \ M(\omega)x + q(\omega) \geq 0, \ \ x^T(M(\omega)x + q(\omega)) = 0,$$

where $M(\omega) \in \mathbb{R}^{n\times n}$ and $q(\omega) \in \mathbb{R}^n$ for $\omega \in \Omega$. We denote this problem by SLCP$(M(\omega), q(\omega))$ for short. Throughout this paper, we assume that $M(\omega)$ and $q(\omega)$ are measurable functions of ω with the following property:

$$E\{\|M(\omega)^T M(\omega)\|\} < \infty \quad \text{and} \quad E\{\|q(\omega)\|^2\} < \infty,$$

where E stands for the expectation. If Ω only contains a single realization, then the SLCP reduces to the standard LCP. For the standard LCP, much effort has been made in developing theoretical analysis for the existence of a solution, numerical methods for finding a solution,

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and applications in engineering and economics [4, 5, 7]. On the other hand, in many practical applications, some data in the LCP cannot be known with certainty. The SLCP is aimed at a practical treatment of the LCP under uncertainty. However, only a little attention has been paid on the SLCP in the literature.

In general, there is no x satisfying the SLCP$(M(\omega), q(\omega))$ for almost all $\omega \in \Omega$. A deterministic formulation for the SLCP provides a decision vector which is optimal in a certain sense. Different deterministic formulations may yield different solutions that are optimal in different senses.

Gürkan,Özge and Robinson [10] considered the *Expected Value* (EV) method for stochastic variational inequalities. When applied to the SLCP$(M(\omega), q(\omega))$, the EV method uses the expected function of the random function $M(\omega)x + q(\omega)$ and solves the deterministic problem

$$x \geq 0, \quad E\{M(\omega)x + q(\omega)\} \geq 0, \; x^T E\{M(\omega)x + q(\omega)\} = 0.$$

In [10], a simulation-based algorithm is presented to find a solution of this problem.

Recently, Chen and Fukushima [2] proposed a new deterministic formulation called the *Expected Residual Minimization* (ERM) method, which is to find a vector $x \in \mathbb{R}^n_+$ that minimizes the expected residual of the SLCP$(M(\omega), q(\omega))$, i.e.,

$$\min_{x \in \mathbb{R}^n_+} E\{\|\Phi(x, \omega)\|^2\}, \tag{1}$$

where $\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is defined by

$$\Phi(x, \omega) = \begin{pmatrix} \phi([M(\omega)x]_1 + q_1(\omega), x_1) \\ \vdots \\ \phi([M(\omega)x]_n + q_n(\omega), x_n) \end{pmatrix},$$

and $[x]_i$ denotes the ith component of the vector x. Here $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an NCP function which has the property

$$\phi(a, b) = 0 \iff a \geq 0, \, b \geq 0, \, ab = 0.$$

Various NCP functions have been studied for solving complementarity problems [5]. In this paper, we will concentrate on the "min" function

$$\phi(a, b) = \min(a, b).$$

Similar results can be obtained for other NCP functions, such as the Fischer-Burmeister (FB) function [8], which have the same growth behavior as the "min" function.

Let ERM$(M(\cdot), q(\cdot))$ denote problem (1) and define

$$G(x) = \int_\Omega \|\Phi(x, \omega)\|^2 dF(\omega), \tag{2}$$

where $F(\omega)$ is the distribution function of ω. Then ERM$(M(\cdot), q(\cdot))$ is rewritten as

$$\text{minimize } G(x) \qquad \text{subject to } x \geq 0. \tag{3}$$

Recall that an $n \times n$ matrix A is called an R_0 matrix if

$$x \geq 0, \; Ax \geq 0, \; x^T Ax = 0 \qquad \Longrightarrow \qquad x = 0.$$

It is known [4, Theorem 3.9.23] that the solution set of the standard LCP(A, b)

$$x \geq 0, \;\; Ax + b \geq 0, \;\; x^T(Ax + b) = 0$$

is bounded for every $b \in \mathbb{R}^n$, if and only if A is an R_0 matrix. In addition, when A is a P_0 matrix, the LCP(A, b) has a nonempty solution set if and only if A is an R_0 matrix [4, Theorem 3.9.22]. Example 1 in [2] shows that the solution set of LCP$(M(\bar{\omega}), q(\bar{\omega}))$ being nonempty and bounded for some $\bar{\omega} \in \Omega$ does not imply that the ERM$(M(\cdot), q(\cdot))$ has a solution. The following results on the existence of a solution of ERM$(M(\cdot), q(\cdot))$ are given in [2].

(i) If $M(\cdot)$ is continuous in ω and there is an $\bar{\omega} \in \Omega$ such that $M(\bar{\omega})$ is an R_0 matrix, then the solution set of ERM$(M(\cdot), q(\cdot))$ is nonempty and bounded.

(ii) When $M(\omega) \equiv M$, the solution set of ERM$(M(\cdot), q(\cdot))$ is nonempty and bounded for any $q(\cdot)$ if and only if M is an R_0 matrix.

In this paper, we substantially extend and refine the results established in [2]. In particular, we introduce the concept of a stochastic R_0 matrix and show that $M(\cdot)$ being a stochastic R_0 matrix is a necessary and sufficient condition for the solution set of ERM$(M(\cdot), q(\cdot))$ to be nonempty and bounded. Moreover, we will extend the local and global error bound results for the R_0 matrix LCP given by Mangasarian and Ren [14] to the stochastic R_0 matrix LCP in the ERM formulation.

Throughout the paper, the norm $\|\cdot\|$ denotes the Euclidean norm and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$. For a given vector $x \in \mathbb{R}^n$, we denote $I(x) = \{i : x_i = 0\}$ and $J(x) = \{i : x_i \neq 0\}$. For vectors $x, y \in \mathbb{R}^n$, $\min(x, y)$ denotes the vector with components $\min(x_i, y_i)$, $i = 1, \cdots, n$.

The remainder of the paper is organized as follows: In Section 2, the definition and some properties of a stochastic R_0 matrix are given. In Section 3, we show a necessary and sufficient condition for the existence of a minimizer of the ERM problem with an arbitrary $q(\cdot)$ is that $M(\cdot)$ is a stochastic R_0 matrix. In Section 4, the differentiability of G is considered. Some optimality conditions and error bounds of the ERM problem are given in Section 5. In Section 6, we first present a method to generate test problems of the stochastic R_0 matrix LCP. Next, we use a stochastic approximation method [1,12] with acceleration by averaging [16] to solve the general ERM problem, and use Newton-type method to solve the ERM problem with $M(\omega) \equiv M$. Preliminary numerical results show that the ERM formulation has various advantages.

2 Stochastic R_0 matrix

A stochastic R_0 matrix is formally defined as follows.

Definition 1 *$M(\cdot)$ is called a stochastic R_0 matrix if*

$$x \geq 0, \;\; M(\omega)x \geq 0, \;\; x^T M(\omega)x = 0, \quad a.e. \implies x = 0.$$

If Ω only contains a single realization, then the definition of a stochastic R_0 matrix reduces to that of an R_0 matrix.

Let G be defined by (2). We call $x^* \in \mathbb{R}^n_+$ a *local solution* of the ERM$(M(\cdot),\ q(\cdot))$, if there is $\gamma > 0$ such that $G(x) \geq G(x^*)$ for all $x \in \mathbb{R}^n_+ \cap B(x^*, \gamma) := \{x : \|x - x^*\| \leq \gamma\}$, and call x^* a *global solution* of ERM$(M(\cdot),\ q(\cdot))$, if $G(x) \geq G(x^*)$ for all $x \in \mathbb{R}^n_+$.

Theorem 1 *The following statements are equivalent.*

(i) *$M(\cdot)$ is a stochastic R_0 matrix.*

(ii) *For any $x \geq 0\,(x \neq 0)$, at least one of the following two conditions is satisfied:*

(a) $P\{\omega : [M(\omega)x]_i \neq 0\ \} > 0$ *for some* $i \in J(x)$;

(b) $P\{\omega : [M(\omega)x]_i < 0\} > 0$ *for some* $i \in I(x)$.

(iii) *ERM($M(\cdot), q(\cdot)$) with $q(\omega) \equiv 0$ has zero as its unique global solution.*

Proof. The proof is given in the order (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): It is easy to see that zero is a global solution of ERM($M(\cdot), q(\cdot)$) with $q(\omega) \equiv 0$, since $G(x) \geq 0$ for all $x \in \mathbb{R}^n_+$ and $G(0) = 0$. Now we show the uniqueness of the solution. Let $\bar{x} \in \mathbb{R}^n_+$ be an arbitrary vector such that $G(\bar{x}) = 0$. By the definition of G, we have

$$\Phi(\bar{x}, \omega) = \min(M(\omega)\bar{x}, \bar{x}) = 0, \quad \text{a.e.},$$

which implies

$$\bar{x} \geq 0, \quad M(\omega)\bar{x} \geq 0, \quad \bar{x}^T M(\omega)\bar{x} = 0, \quad \text{a.e.}$$

By the definition of a stochastic R_0 matrix, we deduce $\bar{x} = 0$.

(iii) $\Rightarrow$ (ii): Suppose (ii) does not hold, that is, there exists a nonzero $x^0 \geq 0$ such that

$$\begin{aligned} P\{\omega :\ [M(\omega)x^0]_i = 0\} &= 1 \text{ for all } i \in J(x^0), \\ P\{\omega :\ [M(\omega)x^0]_i \geq 0\} &= 1 \text{ for all } i \in I(x^0). \end{aligned}$$

Then it follows from $q(\omega) \equiv 0$ that $G(x^0) = 0$. Moreover, it is easy to see that for any $\lambda > 0$, λx^0 is a solution of ERM($M(\cdot), 0$), i.e., zero is not the unique solution of ERM($M(\cdot), 0$).

(ii) $\Rightarrow$ (i): Assume that there exists $x \neq 0$ such that $x \geq 0$, $M(\omega)x \geq 0$ and $x^T M(\omega)x = 0$, a.e. Then, since $x^T M(\omega)x = 0$, we have for almost all ω, $[M(\omega)x]_i = 0$ for all $i \in J(x)$ and $[M(\omega)x]_i \geq 0$ for all $i \in I(x)$. This contradicts (ii). ∎

For $\nu > 0$, let us denote $B_\Omega(\bar{\omega}, \nu) := \{\omega\ :\ \|\omega - \bar{\omega}\| < \nu\}$ and

$$\text{supp}\Omega := \{\bar{\omega} \in \Omega :\ \int_{B_\Omega(\bar{\omega},\nu)\cap\Omega} dF(\omega) > 0 \text{ for any } \nu > 0\}.$$

Here suppΩ is called the support set of Ω. When Ω consists of countable discrete points, i.e., $\Omega = \{\omega_1, \cdots, \omega_i, \cdots\}$ and $P(\omega_i) = p_i > 0$ for all i, we have supp$\Omega = \Omega$. In the case that there is a density function ρ such that $dF(\omega) = \rho(\omega)d\omega$, we have supp$\Omega = \bar{S}$, where $\bar{S}$ is the closure of set $S = \{\omega \in \Omega :\ \rho(\omega) > 0\}$.

Corollary 2 *Suppose that $M(\omega)$ is a continuous function of ω. Then $M(\cdot)$ is a stochastic R_0 matrix if and only if for any $x \geq 0\,(x \neq 0)$, at least one of the following two conditions is satisfied:*

(a) *there exists $\bar{\omega} \in$ suppΩ such that $[M(\bar{\omega})x]_i \neq 0$ for some $i \in J(x)$;*

(b) *there exists $\bar{\omega} \in$ suppΩ such that $[M(\bar{\omega})x]_i < 0$ for some $i \in I(x)$.*

Proof. By the continuity of $M(\omega)$ and the definition of suppΩ, conditions (a) and (b) in this corollary imply (a) and (b) in Theorem 1 (ii), respectively. ∎

Corollary 3 *Suppose that $M(\omega)$ is a continuous function of ω and $M(\bar{\omega})$ is an R_0 matrix for some $\bar{\omega} \in$ suppΩ. Then $M(\cdot)$ is a stochastic R_0 matrix.*

The following example shows that the condition that $M(\cdot)$ is a stochastic R_0 matrix is weaker than the condition that $M(\omega)$ is continuous in ω and there is an $\bar{\omega} \in \mathrm{supp}\Omega$ such that $M(\bar{\omega})$ is an R_0 matrix.

Example 1 Let

$$M(\omega) = \begin{pmatrix} -2\omega & \omega - |\omega| & 0 \\ 0 & \omega + |\omega| & -2\omega \\ 0 & 0 & 0 \end{pmatrix},$$

where $\omega \in \Omega = [-0.5, 0.5]$ and ω is uniformly distributed on Ω. Clearly, for $\omega < 0$, $M(\omega) = \begin{pmatrix} -2\omega & 2\omega & 0 \\ 0 & 0 & -2\omega \\ 0 & 0 & 0 \end{pmatrix}$. Then $x = (1,1,0)^T$ satisfies $M(\omega)x = 0$. On the other hand, for $\omega > 0$, $M(\omega) = \begin{pmatrix} -2\omega & 0 & 0 \\ 0 & 2\omega & -2\omega \\ 0 & 0 & 0 \end{pmatrix}$. Then $x = (0,1,1)^T$ satisfies $M(\omega)x = 0$. In this example, there is no $\omega \in \Omega$ such that $M(\omega)$ is an R_0 matrix. However $M(\cdot)$ is a stochastic R_0 matrix as verified by Theorem 1 (ii). For any $x \geq 0$ with $x \neq 0$, if $x_1 \neq 0$, then for any $\omega > 0$, $[M(\omega)x]_1 = -2\omega x_1 < 0$. If $x_1 = 0$ but $x_2 \neq 0$, then for any $\omega < 0$, $[M(\omega)x]_1 = 2\omega x_2 < 0$. If only $x_3 \neq 0$, then for any $\omega > 0$, $[M(\omega)x]_2 = -2\omega x_3 < 0$.

The following proposition shows a relation between $M(\cdot)$ and $\bar{M} := E\{M(\omega)\}$.

Proposition 4 *If $\bar{M}$ is an R_0 matrix, then $M(\cdot)$ is a stochastic R_0 matrix.*

Proof. If $M(\cdot)$ were not a stochastic R_0 matrix, then by Theorem 1 (ii), there exists $x \geq 0$ such that $x \neq 0$ and, for almost all ω, $[M(\omega)x]_i = 0$ for $i \in J(x)$ and $[M(\omega)x]_i \geq 0$ for $i \in I(x)$. Therefore, $[\bar{M}x]_i = 0$ for $i \in J(x)$ and $[\bar{M}x]_i \geq 0$ for $i \in I(x)$. This is impossible, since $\bar{M}$ is an R_0 matrix. ∎

This proposition implies that for any given $\tilde{M}$, if $\tilde{M}$ is an R_0 matrix, then $M(\cdot) = \tilde{M} + M_0(\cdot)$ with $E\{M_0(\omega)\} = 0$ is a stochastic R_0 matrix. The converse of this proposition is not true. The next proposition gives a way to construct a stochastic R_0 matrix $M(\cdot)$ from a given $\tilde{M}$ which is not necessarily an R_0 matrix. Let

$$\Xi(M) := \{x : x \geq 0, x \neq 0,\ [Mx]_i = 0, i \in J(x) \text{ and } [Mx]_i \geq 0, i \in I(x)\}. \tag{4}$$

Obviously, if $\Xi(\tilde{M}) = \emptyset$, then $\tilde{M}$ is an R_0 matrix, and hence, by Proposition 4, $M(\cdot) = \tilde{M} + M_0(\cdot)$ with $E\{M_0(\omega)\} = 0$ is a stochastic R_0 matrix.

Proposition 5 *Let $\tilde{M}$ and $M_0(\cdot)$ be such that $\Xi(\tilde{M}) \neq \emptyset$ and $E\{M_0(\omega)\} = 0$. Suppose that for any $x \in \Xi(\tilde{M})$, at least one of the following two conditions is satisfied:*

(1) *For some $i \in J(x)$, $E\{([M_0(\omega)x]_i)^2\} > 0$;*

(2) *For some $i \in I(x)$, $P\{\omega : [M_0(\omega)x]_i < -b\} > 0$ for any $b > 0$.*

Then $M(\cdot) = \tilde{M} + M_0(\cdot)$ is a stochastic R_0 matrix.

Proof. For $x \in \Xi(\tilde{M})$, these two conditions imply that the conditions in Theorem 1 (ii) hold for $M(\cdot)$. For $x \notin \Xi(\tilde{M})$, the same conditions also hold trivially. So $M(\cdot)$ is a stochastic R_0 matrix. ∎

This proposition suggests a way to obtain a stochastic R_0 matrix $M(\cdot)$ from an arbitrary matrix $\tilde{M}$. Specifically, we can construct a simple stochastic matrix $M_0(\cdot)$ such that $\tilde{M} + M_0(\cdot)$ is a stochastic R_0 matrix, as illustrated in the following example.

Example 2 We consider the following matrix [2,11]:

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -6 & -3 \\ -1 & -1 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $\tilde{M}$ is not an R_0 matrix, and $\Xi(\tilde{M}) = \{x : x = (0,0,\lambda,\alpha,\beta)^T, \lambda > 0, \alpha, \beta \geq 0, \lambda - 6\alpha - 3\beta \geq 0\}$. Let ω_0 be a random variable whose distribution is $\mathcal{N}(0,1)$. Let

$$M_0(\omega_0) = \begin{pmatrix} 0 & 0 & 0.5\omega_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.5\omega_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then for any $b > 0$, $P\{\omega_0 : [M_0(\omega_0)x]_1 < -b\} > 0$ holds for any $x \in \Xi(\tilde{M})$. Hence, by Proposition 5, $\tilde{M} + M_0(\cdot)$ is a stochastic R_0 matrix.

The following proposition shows that the sum of an R_0 matrix $M(\cdot)$ and a matrix $M_1(\cdot)$ with $E\{M_1(\omega_1)\} = 0$ yields a stochastic R_0 matrix.

Proposition 6 *Let $\omega = (\omega_0, \omega_1)$ and $\hat{M}(\omega) = M(\omega_0) + M_1(\omega_1)$, where $M(\cdot)$ is a stochastic R_0 matrix, $E\{M_1(\omega_1)\} = 0$, and $M(\omega_0)$ is independent of $M_1(\omega_1)$. Then $\hat{M}(\cdot)$ is a stochastic R_0 matrix.*

Proof. If $\tilde{M} := E\{M(\omega_0)\}$ is an R_0 matrix, then from $E\{M_1(\omega_1)\} = 0$ and Proposition 4, $M(\cdot) + M_1(\cdot)$ is a stochastic R_0 matrix. Otherwise, let $M_0(\omega_0) = M(\omega_0) - \tilde{M}$ and choose any $x \in \Xi(\tilde{M})$. Suppose that the first condition of Proposition 5 holds for $M_0(\omega_0)$. Since $M(\omega_0)$ is independent of $M_1(\omega_1)$, we have

$$E\{([(M_0(\omega_0) + M_1(\omega_1))x]_i)^2\} = E\{([M_0(\omega_0)x]_i)^2\} + E\{([M_1(\omega_1)x]_i)^2\} > 0$$

for some $i \in J(x)$. Now, suppose that the second condition of Proposition 5 holds for $M_0(\omega_0)$, i.e., $P\{\omega_0 : [M_0(\omega_0)x]_i < -b\} > 0$ for some $i \in I(x)$. Note that

$$\begin{aligned} & P\{\omega : [(M_0(\omega_0) + M_1(\omega_1))x]_i < -b\} \\ & \geq P\{(\omega_0, \omega_1) : [M_0(\omega_0)x]_i < -b \text{ and } [M_1(\omega_1)x]_i \leq 0\} \\ & = P\{\omega_0 : [M_0(\omega_0)x]_i < -b\} P\{\omega_1 : [M_1(\omega_1)x]_i \leq 0\}. \end{aligned}$$

Since $E\{[M_1(\omega_1)x]_i\} = 0$, we have $P\{\omega_1 : [M_1(\omega_1)x]_i \leq 0\} > 0$. Thus, we have

$$P\{\omega : [(M_0(\omega_0) + M_1(\omega_1))x]_i < -b\} > 0,$$

i.e., the second condition of Proposition 5 also holds for $M_0(\omega_0 + M_1(\omega_1)$. Since

$$\hat{M}(\omega) = M(\omega_0) + M_1(\omega_1) = \tilde{M} + M_0(\omega_0) + M_1(\omega_1),$$

Proposition 5 ensures that $\hat{M}(\cdot)$ is a stochastic R_0 matrix. ■

3 Boundedness of solution set

In this section, the boundedness of the solution set of the ERM problem (3) is studied.

Theorem 7 *Let $q(\cdot)$ be arbitrary. Then $G(x) \to \infty$ as $\|x\| \to \infty$ with $x \in \mathbb{R}^n_+$ if and only if $M(\cdot)$ is a stochastic R_0 matrix.*

Proof. First we prove the "if" part. For simplicity, we denote $|x| = (|x_1|, \cdots, |x_n|)^T$ and $\mathrm{sign}(x) = (\mathrm{sign}(x_1), \cdots, \mathrm{sign}\ (x_n))^T$ for a vector x, where

$$\mathrm{sign}(x_i) = \begin{cases} 1, & x_i > 0, \\ 0, & x_i = 0, \\ -1, & x_i < 0. \end{cases}$$

Note that for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} 2\min(a,b) &= a + b - \mathrm{sign}(a-b)(a-b) \\ &= (1 - \mathrm{sign}(a-b))a + (1 + \mathrm{sign}(a-b))b \end{aligned}$$

and

$$\begin{aligned} 4(\min(a,b))^2 &= a(1-\mathrm{sign}(a-b))^2 a + b(1+\mathrm{sign}(a-b))^2 b + 2b(1-\mathrm{sign}^2(a-b))a \\ &= 2a(1-\mathrm{sign}(a-b))a + 2b(1+\mathrm{sign}(a-b))b. \end{aligned}$$

For any $x \in \mathbb{R}^n$ and $\omega \in \Omega$, we define the diagonal matrix

$$D(x,\omega) = \mathrm{diag}(\mathrm{sign}(M(\omega)x + q(\omega) - x)).$$

Then we have

$$\begin{aligned} \|\Phi(x,\omega)\|^2 &= \frac{1}{2}[(M(\omega)x + q(\omega))^T (I - D(x,\omega))(M(\omega)x + q(\omega)) \\ &\quad + x^T (I + D(x,\omega))x]. \end{aligned} \tag{5}$$

Consider an arbitrary $x \geq 0$ with $\|x\| = 1$. Suppose condition (a) in Theorem 1 (ii) holds. Choose $i \in J(x)$ such that $P\{\omega : [M(\omega)x]_i \neq 0\} > 0$. Then there exists a sufficiently large $K > 0$ such that $P\{\omega : [M(\omega)x]_i \neq 0,\ |q_i(\omega)| \leq K\} > 0$.

First consider the case where $P\{\omega : [M(\omega)x]_i < x_i,\ |q_i(\omega)| \leq K\} > 0$. Let

$$\Omega_1 := \{\omega : [M(\omega)x]_i < (1-\delta)x_i,\ |q_i(\omega)| \leq K\},$$

where $\delta > 0$. Then we have $P\{\Omega_1\} > 0$ whenever δ is sufficiently small. Moreover, for any sufficiently large $\lambda > 0$, $\mathrm{sign}(\lambda[M(\omega)x]_i + q_i(\omega) - \lambda x_i) = -1$ for any $\omega \in \Omega_1$. Therefore, by (2) and (5), we have

$$G(\lambda x) \geq \int_{\Omega_1} (\lambda [M(\omega)x]_i + q_i(\omega))^2 dF(\omega) \to \infty \text{ as } \lambda \to \infty. \tag{6}$$

Next, consider the case where $P\{\omega : [M(\omega)x]_i > x_i, |q_i(\omega)| \leq K\} > 0$. Let

$$\Omega_2 := \{\omega : [M(\omega)x]_i > (1+\delta)x_i, |q_i(\omega)| \leq K\}.$$

Then we have $P(\Omega_2) > 0$ for a sufficiently small $\delta > 0$. Moreover, for any sufficiently large $\lambda > 0$, $\text{sign}(\lambda[M(\omega)x]_i + q_i(\omega) - \lambda x_i) = 1$ for $\omega \in \Omega_2$. Hence we have

$$G(\lambda x) \geq \int_{\Omega_2} (\lambda x_i)^2 dF(\omega) \to \infty \text{ as } \lambda \to \infty. \tag{7}$$

Finally consider the case where $P\{\omega : [M(\omega)x]_i = x_i, |q_i(\omega)| \leq K\} > 0$. Let

$$\Omega_3 := \{\omega : [M(\omega)x]_i = x_i, |q_i(\omega)| \leq K\}.$$

Then we have

$$G(\lambda x) \geq \int_{\Omega_3} \{(\lambda x_i + q_i(\omega))^2 1_{\{q_i(\omega)<0\}} + (\lambda x_i)^2 1_{\{q_i(\omega)\geq 0\}}\} dF(\omega) \to \infty \text{ as } \lambda \to \infty. \tag{8}$$

Combining (6)–(8), we see that $G(\lambda x) \to \infty$ as $\lambda \to \infty$.

Now, suppose condition (b) in Theorem 1 (ii) holds. Choose $i \in I(x)$ such that $P\{\omega : [M(\omega)x]_i < 0\} > 0$. Let

$$\Omega_4 := \{\omega : \ [M(\omega)x]_i < -\delta, \ |q_i(\omega)| < K\}.$$

Then we have $P\{\Omega_4\} > 0$ for any sufficiently small $\delta > 0$ and sufficiently large $K > 0$. Moreover, for any $\lambda > 0$ large enough, $\lambda[M(\omega)x]_i + q_i(\omega) < 0$ for $\omega \in \Omega_4$. Thus we have

$$(1 - \text{sign}(\lambda[M(\omega)x]_i + q_i(\omega)))(\lambda[M(\omega)x]_i + q_i(\omega))^2 = 2(\lambda[M(\omega)x]_i + q_i(\omega))^2,$$

which yields

$$G(\lambda x) \geq \int_{\Omega_4} (\lambda[M(\omega)x]_i + q_i(\omega))^2 dF(\omega) \to \infty \text{ as } \lambda \to \infty.$$

Since x is an arbitrary nonzero vector such that $x \geq 0$, we deduce from the above arguments that $G(x) \to \infty$ as $\|x\| \to \infty$ with $x \geq 0$, provided the statement (ii) in Theorem 1 holds.

Let us turn to proving the "only if" part. Suppose that $M(\cdot)$ is not a stochastic R_0 matrix, i.e., there exists $x \geq 0$ with $x \neq 0$ such that $[M(\omega)x]_i = 0$ for all $i \in J(x)$ and $[M(\omega)x]_i \geq 0$ for all $i \in I(x)$, a.e. For any $\lambda > 0$, from (2) and (5), we have

$$\begin{aligned} G(\lambda x) = \frac{1}{2}\sum_{i=1}^{n} E\{(1 - \ &\text{sign}(\lambda[M(\omega)x]_i + q_i(\omega) - \lambda x_i))(\lambda[M(\omega)x]_i + q_i(\omega))^2 \\ &+ (1 + \text{sign}(\lambda[M(\omega)x]_i + q_i(\omega) - \lambda x_i))(\lambda x_i)^2\}. \end{aligned} \tag{9}$$

The ith term of the right-hand side of (9) with $x_i \neq 0$ equals

$$\begin{aligned} &E\{(1 - \text{sign}(q_i(\omega) - \lambda x_i))q_i(\omega)^2 + (1 + \text{sign}(q_i(\omega) - \lambda x_i))(\lambda x_i)^2\} \\ &\qquad = 2E\{q_i(\omega)^2 1_{\{q_i(\omega)\leq \lambda x_i\}} + (\lambda x_i)^2 1_{\{q_i(\omega)>\lambda x_i\}}\} \leq 2E\{q_i(\omega)^2\}, \end{aligned}$$

while the ith term of the right-hand side of (9) with $x_i = 0$ equals

$$\begin{aligned} &E\{(1 - \text{sign}(\lambda[M(\omega)x]_i + q_i(\omega)))(\lambda[M(\omega)x]_i + q_i(\omega))^2\} \\ &\qquad = 2E\{(\lambda[M(\omega)x]_i + q_i(\omega))^2 1_{\{\lambda[M(\omega)x]_i < -q_i(\omega)\}}\} \leq 2E\{q_i(\omega)^2\}, \end{aligned}$$

where the last inequality follows from $0 > \lambda[M(\omega)x]_i + q_i(\omega) \geq q_i(\omega)$, implying $(\lambda[M(\omega)x]_i + q_i(\omega))^2 \leq q_i(\omega)^2$. So, we obtain

$$G(\lambda x) \leq E\{\|q(\omega)\|^2\} \text{ for any } \lambda > 0.$$

Since $x \geq 0$ with $x \neq 0$, this particularly implies that G is bounded above on a nonnegative ray in $\mathbb{R}^n_+$. This completes the proof of the "only if " part. ■

The solution set of ERM$(M(\cdot), q(\cdot))$ may be bounded even if $M(\cdot)$ is not a stochastic R_0 matrix. It depends on the distribution of $q(\omega)$, as shown in the following two propositions.

Proposition 8 *If $M(\cdot)$ is not a stochastic R_0 matrix, $P\{\omega : q_i(\omega) > 0\} > 0$ for some $i \in J(x)$, and $P\{\omega : q_i(\omega) \geq 0\} = 1$ for all $i \in I(x)$, where $x \neq 0$ is any nonnegative vector at which the conditions (a) and (b) in Theorem 1 (ii) fail to hold, then the solution set of ERM$(M(\cdot), q(\cdot))$ is bounded.*

Proof. Note that

$$G(0) = E\{\|\Phi(0,\omega)\|^2\} = \sum_{i=1}^{n} E\{q_i(\omega)^2 1_{\{q_i(\omega)<0\}}\}. \tag{10}$$

For any nonnegative vector $x \neq 0$ satisfying the conditions (a) and (b) in Theorem 1 (ii), the proof of Theorem 7 indicates that

$$G(\lambda x) \to \infty \text{ as } \lambda \to 0. \tag{11}$$

Let $x \neq 0$ be any nonnegative vector which does not satisfy the conditions (a) and (b) in Theorem 1 (ii), i.e., $[M(\omega)x]_i = 0$ for $i \in J(x)$, and $[M(\omega)x]_i \geq 0$ for $i \in I(x)$, a.e. Then by (5), we have

$$\begin{aligned} G(\lambda x) &= \sum_{i \in J(x)} E\{[(1 - \text{sign}(q_i(\omega) - \lambda x_i))q_i(\omega)^2 + (1 + \text{sign}(q_i(\omega) - \lambda x_i))(\lambda x_i)^2]/2\} \\ &= \sum_{i \in J(x)} \{E\{q_i(\omega)^2\} - E\{1_{\{q_i(\omega)-\lambda x_i>0\}}[q_i(\omega)^2 - (\lambda x_i)^2]\}\}, \end{aligned} \tag{12}$$

where the first equality follows from the assumption that $P\{\omega : q_i(\omega) \geq 0\} = 1$ for $i \in I(x)$ and hence $[M(\omega)x]_i + q_i(\omega) \geq 0$, a.e., for $i \in I(x)$. Note that

$$\begin{aligned} 0 &\leq E\{1_{\{q_i(\omega)-\lambda x_i>0\}}[q_i(\omega)^2 - (\lambda x_i)^2]\} = E\{1_{\{q_i(\omega)>\lambda x_i\}}[q_i(\omega)^2 - (\lambda x_i)^2]\} \\ &\leq E\{1_{\{q_i(\omega)>\lambda x_i\}}q_i(\omega)^2\} \to 0 \text{ as } \lambda \to \infty, \end{aligned}$$

which together with (12) implies

$$\lim_{\lambda\to\infty} G(\lambda x) = \sum_{i \in J(x)} E\{q_i(\omega)^2\}. \tag{13}$$

On the other hand, for any nonzero $x \geq 0$, we have

$$\sum_{i=1}^{n} E\{q_i(\omega)^2 1_{\{q_i(\omega)<0\}}\} = \sum_{i \in J(x)} E\{q_i(\omega)^2 1_{\{q_i(\omega)<0\}}\} < \sum_{i \in J(x)} E\{q_i(\omega)^2\}, \tag{14}$$

where the equality follows from the assumption that $P\{\omega : q_i(\omega) \geq 0\} = 1$ for all $i \in I(x)$ and the strict inequality follows from the assumption that $P\{\omega : q_i(\omega) > 0\} > 0$ for some $i \in J(x)$. Combining (10), (13), and (14), we have

$$G(0) < \lim_{\lambda \to +\infty} G(\lambda x). \tag{15}$$

Let $\Lambda := \{x \in \mathbb{R}^n_+ : G(x) \leq G(0)\}$. From (11) and (15), we have $\sup_{x \in \Lambda} \|x\| < +\infty$. Since any solution belongs to Λ, this implies that the solution set is bounded. ■

Proposition 9 *If $M(\cdot)$ is not a stochastic R_0 matrix and, for any i, $P\{\omega : -b \leq q_i(\omega) < 0\} = 1$ for some $b > 0$ and $P\{\omega : q_i(\omega) \neq 0$ and $[M(\omega)x^0]_i = 0\} = 0$, where $x^0 \neq 0$ is any nonnegative vector at which the conditions (a) and (b) in Theorem 1 (ii) fail to hold, then the solution set of ERM($M(\cdot), q(\cdot)$) is empty or unbounded.*

Proof. Let $x^0 \neq 0$ be any nonnegative vector which does not satisfy the conditions (a) and (b) in Theorem 1 (ii). From (2) and (5), we have

$$\begin{aligned} G(\lambda x^0) = \sum_{i=1}^n E\{[(1 - \mathrm{sign}(\lambda[M(\omega)x^0]_i + q_i(\omega) - \lambda x_i^0))(\lambda[M(\omega)x^0]_i + q_i(\omega))^2 \\ + (1 + \mathrm{sign}(\lambda[M(\omega)x^0]_i + q_i(\omega) - \lambda x_i))(\lambda x_i^0)^2]/2\}. \end{aligned} \tag{16}$$

For every $i \in J(x^0)$, we have $[M(\omega)x^0]_i = 0$ and $q_i(\omega) = 0$, a.e., and hence the ith term of the right-hand side of (16) is zero for any $\lambda > 0$. For every $i \in I(x^0)$, we have $[M(\omega)x^0]_i \geq 0$ and $q_i(\omega) < 0$, a.e., which implies

$$\begin{aligned} &E\{(1 - \mathrm{sign}(\lambda[M(\omega)x^0]_i + q_i(\omega)))(\lambda[M(\omega)x^0]_i + q_i(\omega))^2\} \\ &= 2E\{(\lambda[M(\omega)x^0]_i + q_i(\omega))^2 1_{\{\lambda[M(\omega)x^0]_i < -q_i(\omega), [M(\omega)x^0]_i > 0\}}\} \\ &\quad + 2E\{q_i^2(\omega) 1_{\{[M(\omega)x^0]_i = 0\}}\}. \end{aligned} \tag{17}$$

By assumption, the second term on the right-hand side of (17) is zero for any $\lambda > 0$, and

$$\begin{aligned} &E\{(\lambda[M(\omega)x^0]_i + q_i(\omega))^2 1_{\{\lambda[M(\omega)x^0]_i < -q_i(\omega), [M(\omega)x^0]_i > 0\}}\} \\ &\leq b^2 P\{\omega : 0 < \lambda[M(\omega)x^0]_i < b\} \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Therefore, we obtain

$$\lim_{\lambda \to +\infty} G(\lambda x^0) = 0,$$

but for any $x \in \mathbb{R}^n_+$, $G(x) \geq 0$. So for any $\gamma > 0$, the level set $\Lambda_\gamma := \{x : G(x) \leq \gamma\}$ is unbounded, which means the solution set is unbounded if it is not empty. ■

From Theorem 7, we have the following necessary and sufficient condition for the solution set of ERM($M(\cdot), q(\cdot)$) to be bounded for any $q(\cdot)$.

Theorem 10 *The solution set of ERM($M(\cdot), q(\cdot)$) is nonempty and bounded for any $q(\cdot)$ if and only if $M(\cdot)$ is a stochastic R_0 matrix.*

4 Differentiability of G

The objective function G of ERM($M(\cdot), q(\cdot)$) is, in general, not convex. If G is differentiable at x, then $\min(\nabla G(x), x) = 0$ implies that x is a stationary point of ERM($M(\cdot), q(\cdot)$). The differentiability of G is studied in [2] for the special case where $M(\omega) \equiv M$, $q(\omega) = \bar{q} + T\omega$ with $M \in \mathbb{R}^{n\times n}$, $\bar{q} \in \mathbb{R}^n$, $T \in \mathbb{R}^{n\times m}$ being constants and T having at least one nonzero element in each row.

In this section, we will give a condition for the function G to be differentiable under a general setting. The continuity of $M(\cdot)$ and $q(\cdot)$ is not assumed.

Definition 2 *We say that the strict complementarity condition holds at x with probability one if*

$$P\{\omega : [M(\omega)x]_i + q_i(\omega) = x_i\} = 0, \quad i = 1, \cdots, n.$$

Obviously this definition is a generalization of the strict complementarity condition for the LCP. The proof for the differentiability of G at x under the strict complementarity condition with probability one is not trivial.

For any fixed ω, if $[M(\omega)x]_i + q_i(\omega) - x_i \neq 0$ for all i, then $\|\Phi(x,\omega)\|^2$ is differentiable at x and

$$\nabla_x \|\Phi(x,\omega)\|^2 = M(\omega)^T(I - D(x,\omega))(M(\omega)x + q(\omega)) + (I + D(x,\omega))x.$$

To simplify the notation, we define

$$f(x,\omega) := M(\omega)^T(I - D(x,\omega))(M(\omega)x + q(\omega)) + (I + D(x,\omega))x. \tag{18}$$

Theorem 11 *The function $g(x) := \int_\Omega f(x,\omega)dF(\omega)$ is continuous at x if the strict complementarity condition holds at x with probability one.*

Proof. We will show that $\|g(x+h) - g(x)\| \to 0$ as $h \to 0$. Since

$$\begin{aligned} f(x,\omega) - f(x+h,\omega) = {} & (M(\omega)^T(I - D(x,\omega))M(\omega) + I + D(x,\omega))h \\ & + M(\omega)^T(D(x+h,\omega) - D(x,\omega))(M(\omega)(x+h) + q(\omega)) \\ & - (D(x+h,\omega) - D(x,\omega))(x+h), \end{aligned}$$

there exist some constants c_1, $c_2 > 0$ such that

$$\begin{aligned} \|g(x+h) - g(x)\| &= \|\int_\Omega [f(x+h,\omega) - f(x,\omega)]dF(\omega)\| \\ &\le c_1\|h\| + c_2 \int_\Omega \|D(x+h,\omega) - D(x,\omega)\|dF(\omega). \end{aligned}$$

Then we just need to show that

$$\int_\Omega \|D(x+h,\omega) - D(x,\omega)\|dF(\omega) \to 0 \ \text{ as } \ h \to 0.$$

Note that

$$\{\omega : \|D(x+h,\omega) - D(x,\omega)\| \neq 0\} \subset \cup_{i=1}^n \{A_i \cup B_i\},$$

where

$$\begin{aligned} A_i &:= \{\omega : [M(\omega)x]_i + q_i(\omega) - x_i \ge 0, \ [M(\omega)(x+h)]_i + q_i(\omega) - x_i - h_i \le 0\}, \\ B_i &:= \{\omega : [M(\omega)x]_i + q_i(\omega) - x_i \le 0, \ [M(\omega)(x+h)]_i + q_i(\omega) - x_i - h_i \ge 0\}. \end{aligned}$$

For any $\varepsilon > 0$, since the strict complementarity condition holds at x with probability one, there is a $\delta > 0$ such that

$$P\{\omega : |[M(\omega)x]_i + q_i(\omega) - x_i| < \delta\} < \varepsilon/2. \tag{19}$$

Let

$$C_i := \{\omega : [M(\omega)x]_i + q_i(\omega) - x_i \geq \delta,\ [M(\omega)(x+h)]_i + q_i(\omega) - x_i - h_i \leq 0\}.$$

Then, we have

$$\begin{aligned} A_i &\subset C_i \cup \{\omega : |[M(\omega)x]_i + q_i(\omega) - x_i| < \delta\}, \\ C_i &\subset \{\omega : [M(\omega)h]_i - h_i \leq -\delta\}. \end{aligned}$$

Applying a similar procedure to B_i, we have

$$P\{A_i \cup B_i\} \leq P\{\omega : |[M(\omega)h]_i - h_i| \geq \delta\} + P\{\omega : |[M(\omega)x]_i + q_i(\omega) - x_i| < \delta\}.$$

By the Chebychev inequality, there is an $h_0 > 0$ such that for any h with $\|h\| < h_0$,

$$P\{\omega : |[M(\omega)h]_i - h_i| \geq \delta\} < \varepsilon/2.$$

This together with (19) implies that g is continuous at x. ■

Theorem 12 *If the strict complementarity condition holds at any x in an open set $U \subset \mathbb{R}^n$ with probability one, then G is Fréchet differentiable at $x \in U$ and*

$$\nabla G(x) = \int_\Omega f(x,\omega)dF(\omega). \tag{20}$$

Proof. First, we will show that for almost all ω, $\mu\{x \in U :\ [M(\omega)x]_i + q_i(\omega) = x_i\} = 0$ for any i, where μ is Lebesgue measure. If it were no true, then for some i

$$P\{\omega : \mu\{x \in U :\ [M(\omega)x]_i + q_i(\omega) = x_i\} > 0\} > 0,$$

which implies

$$\int_\Omega \int_U 1_{\{[M(\omega)x]_i + q_i(\omega) = x_i\}} dx dF(\omega) > 0. \tag{21}$$

But from the assumption and the Fubini Theorem [9], we obtain

$$\int_\Omega \int_U 1_{\{[M(\omega)x]_i + q_i(\omega) = x_i\}} dx dF(\omega) = \int_U \int_\Omega 1_{\{[M(\omega)x]_i + q_i(\omega) = x_i\}} dF(\omega) dx = 0.$$

This contradicts (21), and hence for almost all ω, $\mu\{x \in U :\ [M(\omega)x]_i + q_i(\omega) = x_i\} = 0$ for any i.

Note that, for any $\omega \in \Omega$, $\|\Phi(x,\omega)\|^2$ is locally Lipschitz and hence absolutely continuous with respect to x. For any (x,ω) such that $[M(\omega)x]_i + q_i(\omega) \neq x_i$, $\|\Phi(x,\omega)\|^2$ is differentiable with respect to x. Therefore by the Fundamental Theorem of Calculus for Lebesgue Integrals [9], for any x, we have

$$\|\Phi(x + h_i e_i, \omega)\|^2 - \|\Phi(x,\omega)\|^2 = \int_0^{h_i} [f(x + s e_i, \omega)]_i ds \tag{22}$$

for almost all ω, where $e_i = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0)^T$. Thus

$$G(x+h) - G(x) = \sum_{i=1}^{n} \int_{\Omega} (\|\Phi(x + \sum_{k=i}^{n} h_k e_k, \omega)\|^2 - \|\Phi(x + \sum_{k=i+1}^{n} h_k e_k, \omega)\|^2) dF(\omega). \tag{23}$$

By (22), (23) and the Fubini Theorem, we deduce that

$$\int_{\Omega} \int_{0}^{h_i} [f(y + se_i, \omega)]_i ds dF(\omega) = \int_{0}^{h_i} \int_{\Omega} [f(y + se_i, \omega)]_i dF(\omega) ds$$

for any i and $y \in B(x, \|h\|) \subset U$, and hence

$$\begin{aligned}
&G(x+h) - G(x) - h^T \int_{\Omega} f(x, \omega) dF(\omega) \\
&= \sum_{i=1}^{n} \int_{0}^{h_i} \int_{\Omega} [f(x + \sum_{k=i+1}^{n} h_k e_k + se_i, \omega)]_i dF(\omega) ds - \sum_{i=1}^{n} \int_{0}^{h_i} \int_{\Omega} [f(x, \omega)]_i dF(\omega) ds \\
&= \sum_{i=1}^{n} \int_{0}^{h_i} \int_{\Omega} ([f(x + \sum_{k=i+1}^{n} h_k e_k + se_i, \omega)]_i - [f(x, \omega)]_i) dF(\omega) ds \\
&= \sum_{i=1}^{n} \int_{0}^{h_i} (g_i(x + \sum_{k=i+1}^{n} h_k e_k + se_i) - g_i(x)) ds,
\end{aligned}$$

where g is defined in Theorem 11. From Theorem 11, for any $\varepsilon > 0$, there exists a sufficiently small $h_0 > 0$ such that for any h with $\|h\| < h_0$,

$$|G(x+h) - G(x) - h^T \int_{\Omega} f(x, \omega) dF(\omega)| < \varepsilon \|h\|,$$

which implies

$$\frac{|G(x+h) - G(x) - h^T \int_{\Omega} f(x, \omega) dF(\omega)|}{\|h\|} \to 0 \quad \text{as} \quad \|h\| \to 0.$$

Therefore, G is Fréchet differentiable at x and (20) holds. ■

REMARK. When $M(\omega) \equiv M$ and $q(\omega) = \bar{q} + T\omega$, if $[T\omega]_i$ has no mass at any point for each i, i.e., $P\{\omega : [T\omega]_i = a\} = 0$ for any $a \in \mathbb{R}$, then $P\{\omega : [M(\omega)x]_i + q_i(\omega) = x_i\} = 0$, $i = 1, \cdots, n$. Therefore, if T has at least one nonzero element in each row [2], then for all $x \in \mathbb{R}^n$, the strict complementarity condition holds with probability one, and G is differentiable in $\mathbb{R}^n_+$. This indicates that the result shown in Theorem 12 contains the results established in [2].

Let $F_{q_i}(s)$ be the distribution function of $q_i(\omega)$, i.e., $F_{q_i}(s) = P\{\omega : q_i(\omega) \le s\}$. Suppose $M(\omega) \equiv M$. Then, we have

$$\begin{aligned}
G(x) &= \int_{-\infty}^{+\infty} \sum_{i=1}^{n} (\min([Mx]_i + s, x_i)^2 dF_{q_i}(s) \\
&= \sum_{i=1}^{n} \int_{-\infty}^{[(I-M)x]_i} ([Mx]_i + s)^2 dF_{q_i}(s) + \sum_{i=1}^{n} x_i^2 (1 - F_{q_i}([(I-M)x]_i)).
\end{aligned} \tag{24}$$

It is shown in [2] that for some special distribution functions, $G(x)$ can be computed without using discrete approximation. The following proposition shows that, under some conditions, we can also compute $\nabla G(x)$ without using discrete approximation.

Proposition 13 *If $M(\omega) \equiv M$ and $F_{q_i}(s)$ is a continuous function for all i, then*

$$\nabla G(x) = 2M^T H(x)x + 2(I - H(x))x - 2M^T v(x), \tag{25}$$

where

$$H(x) := \mathrm{diag}(F_{q_1}([(I-M)x]_1), \cdots, F_{q_n}([(I-M)x]_n),$$
$$v(x) := (\int_{-\infty}^{[(I-M)x]_1} F_{q_1}(s)ds, \cdots, \int_{-\infty}^{[(I-M)x]_n} F_{q_n}(s)ds)^T.$$

Proof. If $M(\omega) \equiv M$ and $F_{q_i}(s)$ is continuous for all i, then $P\{\omega : q_i(\omega) = a\} = 0$ for any $a \in \mathbb{R}$, and hence $P\{\omega : [Mx]_i + q_i(\omega) = x_i\} = 0$ for each $x \in \mathbb{R}^n_+$. Then, by Theorem 12, $G(x)$ is differentiable at any $x \in \mathbb{R}^n_+$ and

$$\begin{aligned}
\nabla G(x) &= \int_\Omega [M^T(I - D(x,\omega))(Mx + q(\omega)) + (I + D(x,\omega))x]dF(\omega) \\
&= M^T[\int_\Omega (I - D(x,\omega))dF(\omega)Mx + \int_\Omega (I - D(x,\omega))q(\omega)dF(\omega)] \\
&\quad + (\int_\Omega (I + D(x,\omega))dF(\omega))x \\
&= 2M^T H(x)Mx + 2M^T R(x) + 2(I - H(x))x,
\end{aligned} \tag{26}$$

where

$$R(x) := (\int_{-\infty}^{[(I-M)x]_1} sdF_{q_1}(s), \cdots, \int_{-\infty}^{[(I-M)x]_n} sdF_{q_n}(s))^T.$$

By integration by parts, we have

$$\int_{-\infty}^{[(I-M)x]_i} sdF_{q_i}(s) = [(I-M)x]_i F_{q_i}([(I-M)x]_i) - \int_{-\infty}^{[(I-M)x]_i} F_{q_i}(s)ds.$$

This implies that

$$R(x) = (\int_{-\infty}^{[(I-M)x]_1} sdF_{q_1}(s), \cdots, \int_{-\infty}^{[(I-M)x]_n} sdF_{q_n}(s))^T = H(x)(I-M)x - v(x).$$

Combining this with (26), we have the desired formula (25). ■

From (25), we see that the smoothness of $G(\cdot)$ depends on the smoothness of $F_{q_i}(\cdot)$, $i = 1, \cdots, n$. If for all i, $F_{q_i}(\cdot)$ is differentiable at $[(I-M)x]_i$ and $\rho_i(\cdot)$, the derivative of $F_{q_i}(\cdot)$, is continuous at $[(I-M)x]_i$, then the Hessian matrix of $G(x)$ can be written as

$$\begin{aligned}
\nabla^2 G(x) &= 2M^T H(x)M + 2(M^T S(x) + S(x)M) - 2M^T S(x)M + 2(I - S(x) - H(x)) \\
&= 2M^T H(x)M + 2(I - H(x)) - 2(I-M)^T S(x)(I-M),
\end{aligned} \tag{27}$$

where

$$S(x) := \mathrm{diag}(x_1\rho_1([(I-M)x]_1), \cdots, x_n\rho_n([(I-M)x]_n)).$$

5 Optimality conditions and error bounds

In numerical algorithms, residual functions play an important role in terminating iterations and verifying accuracy of a computed solution. The following theorem shows the basic properties of the residual function defined by

$$r(x) = \| \min(\nabla G(x), x)\|.$$

Theorem 14 *Suppose that the strict complementarity condition holds at any x in an open set U with probability one. Then the following statements are true.*

(1) *If $\bar{x} \in U$ is a local solution of ERM($M(\cdot), q(\cdot)$), then $r(\bar{x}) = 0$.*

(2) *If $G(\cdot)$ is twice continuously differentiable at $\bar{x} \in \mathbb{R}^n_+$ where $r(\bar{x}) = 0$ and the Hessian matrix $\nabla^2 G(\bar{x})$ is positive definite, then there are an open set $\bar{U} \subset U$ and a constant $\tau > 0$ such that $\bar{x}$ is a unique local solution of ERM($M(\cdot), q(\cdot)$) in $\bar{U}$, and for all $x \in \bar{U}$*

$$\|x - \bar{x}\| \le \tau r(x). \tag{28}$$

Proof. From Theorem 12, we can write the first order optimality condition for the ERM problem (3) as $r(x) = 0$. Now we show (28). Since $G(\cdot)$ is twice continuously differentiable at $\bar{x}$ and $\nabla^2 G(\bar{x})$ is positive definite, there is an open set $\bar{U} \subset U$ such that $\nabla G(x)$ is a locally Lipschitz continuous and uniform P function in $\bar{U}$. Applying Proposition 6.3.1 in [5] to the nonsmooth equation $\min(x, \nabla G(x)) = 0$, we obtain (28). ■

Corollary 15 *If $\Omega = \{\omega_1, \cdots, \omega_N\}$, and the strict complementarity condition holds at $x \in \mathbb{R}^n_+$ with probability one, then $r(x) = 0$ implies x is a local solution of ERM($M(\cdot), q(\cdot)$).*

Proof. Since the strict complementarity condition holds at $x \in \mathbb{R}^n_+$ with probability one, by (5), for each i, $\|\Phi(x, \omega_i)\|^2$ is twice continuously differentiable and

$$\nabla^2 G(x) = \sum_{i=1}^{N} [M(\omega_i)^T (I - D(x, \omega_i))M(\omega_i) + (I + D(x, \omega_i))].$$

Since the Hessian matrix $\nabla^2 G(x)$ is positive semidefinite, and $G(x)$ is a quadratic function in $B(x, \nu)$ for a sufficiently small $\nu > 0$, x is a local solution. ■

Now we consider error bounds for the case where G is not necessarily differentiable. Let

$$s(x) = G(x) - \min_{x \in \mathbb{R}^n_+} G(x).$$

When $\Omega = \{\omega_1, \cdots, \omega_N\}$, we can write

$$G(x) = \sum_{j=1}^{N} \sum_{i=1}^{n} |\min([M(\omega_j)x + q(\omega_j)]_i, x_i)|^2 p(\omega_j),$$

where $p(\omega_j)$ is the probability of ω_j. Clearly, there exist finitely many convex polyhedra such that G is a convex quadratic function on each polyhedron, i.e., G is a piecewise convex quadratic function. By Theorem 2.5 in [13], we have the following local error bound result:

Proposition 16 *If $\Omega = \{\omega_1, \cdots, \omega_N\}$, then there exist constants $\tau > 0$ and $\varepsilon > 0$ such that for any $x \in \mathbb{R}^n_+$ with $s(x) \le \varepsilon$*

$$\|x - x^*(x)\| \le \tau s(x)^{1/2},$$

where $x^*(x)$ is a global solution of ERM($M(\cdot), q(\cdot)$) closest to x under the norm $\|\cdot\|$.

Let us denote $s_\gamma(x) = s(x)^\gamma$ for $\gamma > 0$. For a general continuous distribution of ω, G may not be a piecewise convex function. The following example shows that the function s_γ provides a local error bound for the ERM$(M(\cdot), q(\cdot))$ with various values of γ depending on the distribution of ω.

Example 3 Consider the SLCP$(M(\omega), q(\omega))$ with

$$M(\omega) \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \quad q(\omega) = \begin{pmatrix} -1 \\ -1 \\ \omega \end{pmatrix},$$

where ω is a random variable with supp$\Omega \subset [-1, 0]$. It is easy to check that $M(\omega)$ is an R_0 matrix. For any ω, the solution set of LCP$(M(\omega), q(\omega))$ is $\{(x_1, 1 - x_1, 0)^T : x_1 \in [-\omega, 1]\} \cup \{(-\frac{1}{2}\omega + \frac{1}{2}, 0, \frac{1}{2}\omega + \frac{1}{2})^T\}$. Let $\rho(\omega)$ be the density function of ω. We consider the following two cases: $\rho(\omega) \equiv 1$ and $\rho(\omega) = 2(\omega + 1)$. Clearly, $x^* = (1, 0, 0)^T$ is the unique global solution of ERM$(M(\cdot), q(\cdot))$ and $r(x^*) = 0$ for these two cases. But for any $x = (x_1, 1 - x_1, 0)^T$ with $x_1 \in [0, 1]$, if $\rho(\omega) \equiv 1$, then $s(x) = (1 - x_1)^{3/2}/\sqrt{3}$, but if $\rho(\omega) = 2(\omega + 1)$, then $s(x) = (1 - x_1)^2/\sqrt{6}$. Noticing that $\|x - x^*\| = \sqrt{2}(1 - x_1)$, we have $\|x - x^*\| \le \tau s_\gamma(x)$, where γ depends on the distribution of ω. So the general form of local error bound for ERM$(M(\cdot), q(\cdot))$ with continuous random variables is difficult to obtain unless the information on the distribution of ω is known.

Theorem 17 *Let $M(\cdot)$ be a stochastic R_0 matrix. Then for any $\varepsilon > 0$, there exists $\tau > 0$ such that for each $x \in \mathbb{R}^n_+$ with $s(x) > \varepsilon$*

$$\|x - x^*(x)\| \le \tau s(x)^{1/2},$$

where $x^*(x)$ is a global solution of ERM($M(\cdot), q(\cdot)$) closest to x under the norm $\|\cdot\|$.

Proof. If the assertion were not true, then for any positive integer k, there exists x^k with $s(x^k) > \varepsilon$ such that

$$\|x^k - x^*(x^k)\| > ks(x^k)^{1/2} > k\varepsilon^{1/2}.$$

Since $M(\cdot)$ is a stochastic R_0 matrix, by Theorem 10, the global solution set of ERM$(M(\cdot), q(\cdot))$ is nonempty and bounded. Therefore $\|x^k\| \to \infty$ as $k \to \infty$, and

$$\frac{s(x^k)^{1/2}}{\|x^k\|} \le \frac{\|x^k - x^*(x^k)\|}{k\|x^k\|} \to 0 \text{ as } k \to \infty. \tag{29}$$

Let $\{x^{n_k}/\|x^{n_k}\|\}$ be a convergent subsequence of $\{x^k/\|x^k\|\}$. Note that

$$\begin{aligned} \lim_{k\to\infty} \frac{s(x^{n_k})^{1/2}}{\|x^{n_k}\|} &= \lim_{k\to\infty} \frac{(G(x^{n_k}) - \min_{x\in\mathbb{R}^n_+} G(x))^{1/2}}{\|x^{n_k}\|} = \lim_{k\to\infty} \frac{G(x^{n_k})^{1/2}}{\|x^{n_k}\|} \\ &= \lim_{k\to\infty} \Big(\int_\Omega \sum_{i=1}^n |\min(\frac{[M(\omega)x^{n_k}]_i + q_i(\omega)}{\|x^{n_k}\|}, \frac{x_i^{n_k}}{\|x^{n_k}\|})|^2 dF(\omega)\Big)^{1/2}. \end{aligned}$$

Since for any x with $\|x\| = 1$

$$\int_\Omega \sum_{i=1}^n |\min([M(\omega)x]_i + q_i(\omega), x_i)|^2 dF(\omega) \le \int_\Omega (\|M(\omega)\|^2 + \|q(\omega)\|^2) dF(\omega) + 1 < \infty,$$

by the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \frac{s(x^{n_k})^{1/2}}{\|x^{n_k}\|} = (\int_\Omega \sum_{i=1}^n |\min([M(\omega)\hat{x}]_i, \hat{x}_i)|^2 dF(\omega))^{1/2} < \infty,$$

where $\hat{x}$ is an accumulation point of $\{x^{n_k}/\|x^{n_k}\|\}$. This, together with (29) and $s(x^{n_k})^{1/2}/\|x^{n_k}\| \ge 0$, yields

$$\int_\Omega \sum_{i=1}^n |\min([M(\omega)\hat{x}]_i, \hat{x}_i)|^2 dF(\omega) = 0,$$

which implies that $\hat{x}$ is a solution of the ERM($M(\cdot), 0$). Since $\|\hat{x}\| = 1$, this contradicts the assumption that $M(\cdot)$ is a stochastic R_0 matrix from Theorem 1 (iii). ■

6 Numerical results

In this section, we first report numerical results for two examples of the stochastic R_0 matrix LCP in the ERM formulation. We constructed the test problems by the following procedure:

1. Set $M_0(\omega) = 0$. Generate an R_0 matrix $\tilde{M} \in \mathbb{R}^{n\times n}$ and go to step 3, or generate a matrix $\tilde{M}$ with $\Xi(\tilde{M}) \ne \emptyset$, where $\Xi(\tilde{M})$ is defined by (4).
2. Let V be the set of vertices of $\tilde{\Xi}(\tilde{M}) := \Xi(\tilde{M}) \cap \{x : \sum_{i=1}^n x_i = 1\}$.
   - 2.1. Choose $x \in V$.
   - 2.2. Generate a random vector $m(\omega) \in \mathbb{R}^n$ such that $E\{m(\omega)\} = 0$ and either of the following two conditions holds:
     - a) The support set of $m_i(\omega)$ is unbounded if $i \in I(x)$;
     - b) $E\{m_i(\omega)^2\} \ne 0$ if $i \in J(x)$.
   - 2.3. Choose index i arbitrarily from $J(x)$ (since $x \ne 0$, we must have $J(x) \ne \emptyset$). Add the vector $m(\omega)$ to the ith row of $M_0(\omega)$. Remove x from the set V. If $V = \emptyset$, then go to step 3.
   - 2.4. For all $y \in V$, do the following: If either of the conditions a) and b) in step 2.2 is satisfied with x replaced by y, and the vector $m(\omega)$ has not been added to any row of $M_0(\omega)$ corresponding to the index set $J(y)$, then choose an index $j \in J(y)$ and add $m(\omega)$ to the jth row of $M_0(\omega)$. Remove y from the set V.
   - 2.5. If there is any $x \in V$ such that neither of the conditions a) and b) in step 2.2 is satisfied, then choose such x and go to step 2.2. Otherwise, go to step 3.
3. Generate a random matrix $M_1(\omega)$ such that $E\{M_1(\omega)\} = 0$ and all random variables in $M_1(\omega)$ are independent of those in $M_0(\omega)$.
4. Let $M(\omega) = \tilde{M} + M_1(\omega)$ if $\tilde{M}$ is an R_0 matrix. Otherwise, let $M(\omega) = \tilde{M} + M_0(\omega) + M_1(\omega)$.

By Proposition 5 and Proposition 6, $M(\cdot)$ is a stochastic R_0 matrix.

Let the measure of feasibility of $x \in \mathbb{R}^n_+$ with tolerance $\varepsilon \geq 0$ be defined by

$$rel_\varepsilon(x) = P\{\omega : [M(\omega)x]_i + q_i(\omega) \geq -\varepsilon,\ i = 1, \cdots, n\}. \tag{30}$$

In the first example, we use $\tilde{M}$ and $M_0(\omega_0)$ given in Example 2, and

$$M_1(\omega') = \begin{pmatrix} 0 & 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & 0 & -0.4 - 0.4 \ln \omega_2 \\ 0 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & -2\sqrt{3}\omega_3 & -2\sqrt{3}\omega_3 \\ 0 & 0.4 + 0.4 \ln \omega_2 & 0 & -3\omega_4 & 3\omega_4 \end{pmatrix},$$

where $\omega' = (\omega_1, \cdots, \omega_4)^T$ with the distributions of ω_1, ω_2, ω_3, ω_4 being $\mathcal{U}[-0.8, 0.8]$, $\mathcal{U}[0,1]$, $\mathcal{N}(0,1)$ and $\mathcal{N}(0,1)$, respectively. Let $\omega = (\omega_0, \omega_1, \cdots, \omega_4)^T$ and $M(\omega) = \tilde{M} + M_0(\omega_0) + M_1(\omega')$. By Proposition 5 and Proposition 6, $M(\cdot)$ is a stochastic R_0 matrix.

We set $q(\omega) = \tilde{q} + q_0(\omega)$, where $\tilde{q}$ is a constant vector and

$$q_0(\omega) = (0.1\omega_0, 0.1\omega_0, 0, -2\sqrt{3}\omega_3, -3\omega_4)^T.$$

Note that $E\{q_0(\omega)\} = 0$. We consider three different cases for $\tilde{q}$, i.e.,

$$\tilde{q}^1 = (2, 3, 100, -180, -162)^T,\ \tilde{q}^2 = (-5, -5, 0, 10, 10)^T,\ \tilde{q}^3 = (-5, -5, -5, -5, -5)^T.$$

The deterministic LCP$(\tilde{M}, \tilde{q}^i)$, $i = 1, 2, 3$, have a unique solution $(36, 18, 0, 0.25, 0.5)^T$, multiple solutions $(0, 0, \lambda, 0, 0)^T$ with $\lambda \geq 5$, and no feasible solution, respectively.

For all $q_i(\omega)$, we can check that for any $x = (x_1, \cdots, x_5)^T \in \mathbb{R}^5_+$ with $x_i \neq 0, i = 1, 2$, the strict complementarity condition holds at x with probability one, and so $\nabla G(x)$ exists at these points. Hence we can use a stochastic approximation algorithm to find a minimizer of $G(x)$ in $\mathbb{R}^n_+$. The iterative formula is given by

$$x^{k+1} = \max(x^k - a_k f(x^k, \omega^k), 0), \tag{31}$$

where $f(x, \omega)$ is defined by (18), a_k is a stepsize satisfying $\sum_{k=1}^{\infty} a_k = \infty$ and $a_k \to 0$, and ω^k is the kth sample of ω. By the results of stochastic approximation algorithms [1,12], the generated sequence $\{x^k\}$ will converge to a connected set S such that every $\bar{x} \in S$ satisfies $\min(g(\bar{x}), \bar{x}) = 0$ with $g(x)$ defined in Theorem 11. If $\bar{x}_i \neq 0, i = 1, 2$, then by Theorem 12, $\nabla G(\bar{x}) = g(\bar{x})$. In this example, a_k is chosen as

$$a_k = \begin{cases} 0.003, & k \leq 10^4, \\ 0.0025, & 10^4 < k \leq 10^5, \\ 0.002, & 10^5 < k \leq 5 \times 10^5, \\ \frac{1}{k^{0.6}}, & 5 \times 10^5 < k \leq 2 \times 10^6. \end{cases}$$

When $k \geq 5 \times 10^5$, we use the averaging technique proposed by [16] to accelerate the convergence.

The stochastic approximation algorithm is a local optimization algorithm. To avoid being trapped in a local minimum, for each $\tilde{q}^i$, $i = 1, 2, 3$, we executed 36 times simulation from different initial points $x^0 = (10l,\ 10l',\ 0,\ 0,\ 0)^T$, $l, l' \in \{0,\ 1,\ \cdots,\ 5\}$.

For each $\tilde{q}^i$, the information on the last iterate $x^{(2 \times 10^6)}$ obtained by (31) is shown in Table 1. The columns labeled as "$G(x)$" and "$r(x)$" show the respective values obtained by the Monte Carlo method with 10^6 samples. The row labeled as "average" shows the average of the values

Table 1: Simulation results for the first example where $E\{M(\omega)\}$ is not an R_0 matrix.

	x_1	x_2	x_3	x_4	x_5	$G(x)$	$r(x)$
$\tilde{q}^1 = (2,\ 3,\ 100,\ -180,\ -162)^T$							
min	39.5439	23.298	0	0.2079	0.345	7.2405	0.0115
max	40.1396	23.5793	0.0096	0.3804	0.5413	7.5741	0.6486
average	39.7865	23.4563	0.0014	0.2610	0.4635	7.413	0.15826
$\tilde{x}$	36	18	0	0.25	0.5	197.0299	12025
$\tilde{q}^2 = (-5,\ -5,\ 0,\ 10,\ 10)^T$							
min	0.0004	0.0044	11.4092	0	0	1.8518	0
max	0.0030	0.0154	11.6959	0	0	1.9037	0.002
average	0.0008	0.0068	11.5410	0	0	1.8747	4.44×10^{-5}
$\tilde{x}$	0	0	5	0	0	3.1428	0.5718
$\tilde{q}^3 = (-5,\ -5,\ -5,\ -5,\ -5)^T$							
min	0.004	1.3915	5.7993	0.0005	0.1137	51.6515	0.0440
max	0.0377	1.5017	5.896	0.0186	0.2343	51.9433	5.2424
average	0.011	1.4347	5.8414	0.003	0.1555	51.734	1.2536

Table 2: $rel_\varepsilon(\%)$ for the first example

ε	0.0	0.1	0.2	0.5	1
$\tilde{x}$	0.18	5.81	29.71	32.36	34.17
average	30.84	71.9	90.07	94.88	95.18

obtained from 36 different initial points. The rows labeled as "min" and "max" indicate the interval of those values.

Recall that the EV method [10] solves the deterministic LCP$(\tilde{M}, \tilde{q})$. Let $\tilde{x}$ be a solution of LCP$(\tilde{M}, \tilde{q})$. For $\tilde{q}^2 = (-5, -5, 0, 10, 10)^T$, since there are multiple solutions $(0, 0, \lambda, 0, 0)^T$ with $\lambda \geq 5$, $G(x)$ and $r(x)$ are evaluated at $\tilde{x} = (0, 0, 5, 0, 0)^T$. There is no feasible solution of LCP$(\tilde{M}, \tilde{q})$ with $\tilde{q}^3 = (-5, -5, -5, -5, -5)^T$.

By using the Monte Carlo method with 10^6 samples, we evaluated the measure of feasibility rel_ε defined by (30) for the case of $\tilde{q}^1 = (2, 3, 100, -180, -162)^T$, at $\tilde{x} = (36, 18, 0, 0.25, 0.5)^T$ and at the last iterates obtained by the iterative formula (31) from 36 different initial points. The results are presented in Table 2. The row labeled as "average" shows the average values of rel_ε obtained at the 36 last iterates.

As the second example, we consider the case where $M(\omega) \equiv \tilde{M}$ is a P matrix and $q(\omega)$ has continuous distribution. In this case, the EV formulation LCP$(\tilde{M}, \tilde{q})$ has a unique solution $\tilde{x}$. The objective function G of the ERM formulation is twice continuously differentiable and the values of $G(x)$, $\nabla G(x)$, and $\nabla^2 G(x)$ can be computed by (24), (25), (27), respectively, without resorting to stochastic approximation.

Let $q(\omega) = \tilde{q} + q_0(\omega)$, where $\tilde{q} = E\{q(\omega)\}$, $q_0(\omega) = B\omega$, $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathcal{N}(0, I)$, $B \in \mathbb{R}^{n \times 3}$ is 100% dense, and the elements of B are randomly generated with the uniform distribution $\mathcal{U}(0, 5)$. We use Example 4.4 of [3] to generate $\tilde{M}$ and $\tilde{q}$. First, we randomly generate 100% dense $A \in \mathbb{R}^{n \times n}$ and $\bar{q} \in \mathbb{R}^n$ whose elements are uniformly distributed in $(-5, 5)$. Then we use the QR decomposition of A to get an upper triangular matrix N, and obtain a triangular matrix $\tilde{M}$ by replacing the diagonal elements of N by their absolute values.

Table 3: Simulation results for the second example where $E\{M(\omega)\}$ is a P matrix

	$n = 20$		$n = 50$		$n = 100$	
	$r(x)$	$G(x)$	$r(x)$	$G(x)$	$r(x)$	$G(x)$
$\tilde{x}$	97.39	180.77	168.04	427.58	307.42	823.8
$\bar{x}$	6.17×10^{-8}	75.78	5.14×10^{-8}	167.07	1.34×10^{-7}	293.72
$\|\tilde{x} - \bar{x}\|$	7.51		21.03		38.18	

Table 4: $rel_\varepsilon(\%)$ for the second example

	$n = 20$			$n = 50$			$n = 100$		
ε	0	1	5	0	1	5	0	1	5
$\tilde{x}$	25.07	33.87	66.15	20.72	29.3	61.52	18.56	27.09	59.34
$\bar{x}$	38.63	49.55	80.49	27.12	37.13	72.25	22.08	31.93	67.23

We first use Lemke's method [6] to find a solution $\tilde{x}$ of LCP$(\tilde{M}, \tilde{q})$, and then take $\tilde{x}$ as an initial point to find a local solution $\bar{x}$ of the ERM formulation by applying the semismooth Newton method [5] to the equation $\min(\nabla G(x), x) = 0$.

The numerical experiments were carried out for $n = 20$, 50 and 100. For each n, we generated 100 problems and solved them by the above-mentioned procedure. The figures presented in Table 3 are the average of the results obtained in this manner.

The measures rel_ε of feasibility at $\tilde{x}$ and $\bar{x}$ obtained by the EV method and the ERM method, respectively, are presented in Table 4.

7 Final remark

This paper proves that a necessary and sufficient condition for the ERM$(M(\cdot), q(\cdot))$ having a nonempty and bounded solution set is that $M(\cdot)$ is a stochastic R_0 matrix. Proposition 4 shows that if the matrix $E\{M(\omega)\}$ is an R_0 matrix, then $M(\cdot)$ is a stochastic R_0 matrix. Moreover, the first example in Section 6 shows that there are many cases where $M(\cdot)$ is a stochastic R_0 matrix, but $E\{M(\omega)\}$ is not an R_0 matrix, and the EV formulation LCP$(E\{M(\omega)\}, E\{q(\omega)\})$ either has no solution or has an unbounded solution set. Therefore, the condition for the ERM$(M(\cdot), q(\cdot))$ having a nonempty and bounded solution set is weaker than the condition for the EV formulation having a nonempty and bounded solution set. Furthermore, when ERM$(M(\cdot), q(\cdot))$ has a solution $\bar{x}$ and LCP$(E\{M(\omega)\}, E\{q(\omega)\})$ has a solution $\tilde{x}$, the residuals always satisfy

$$G(\bar{x}) = E\{\| \min(M(\omega)\bar{x} + q(\omega), \bar{x})\|^2\} \leq E\{\| \min(M(\omega)\tilde{x} + q(\omega), \tilde{x})\|^2\} = G(\tilde{x}).$$

The first example in Section 6 shows that $G(\bar{x})$ can be much smaller than $G(\tilde{x})$. Moreover, the values of rel_ε shown in Table 2 reveal that, for each tolerance level $\varepsilon \geq 0$, the number of ω^i at which $M(\omega^i)\bar{x} + q(\omega^i) < -\varepsilon$ holds is much less than the number of ω^i at which $M(\omega^i)\tilde{x}+q(\omega^i) < -\varepsilon$ holds. The second example shows that a local solution $\bar{x}$ of ERM$(M(\cdot), q(\cdot))$ may conveniently be obtained from a solution $\tilde{x}$ of the EV formulation. Since $\bar{x}$ generally has better reliability than $\tilde{x}$, we may recommend the ERM method to those decision makers who do not want to take high risk of violating the conditions $M(\omega)x + q(\omega) \geq 0$.

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