Monte Carlo Sampling and Penalty Method for Stochastic Mathematical Programs with Complementarity Constraints and Recourse*

Gui-Hua Lin† and Masao Fukushima‡

June 26, 2005

**Abstract.** In this paper, we consider a new formulation for stochastic mathematical programs with complementarity constraints and recourse. We show that the new formulation is equivalent to a smooth semi-infinite program. Then, we propose a Monte Carlo sampling and penalty method for solving the problem. Comprehensive convergence analysis and numerical examples are included as well.

**Key words.** Stochastic mathematical program with complementarity constraints, here-and-now, recourse, semi-infinite programming, Monte Carlo sampling method, penalization.

# 1 Introduction

Recently, stochastic mathematical programs with equilibrium constraints (SMPECs) have been receiving much attention in the optimization world [1, 9, 10, 11, 13, 16, 18, 19, 20]. In particular, Lin et al. [9] introduced two kinds of SMPECs: One is the *lower-level wait-and-see* model, in which the upper-level decision is made before a random event is observed, while a lower-level decision is made after a random event is observed. The other is the *here-and-now* model that requires us to make all decisions before a random event is observed. Lin and Fukushima [10, 11, 13] suggested a smoothing penalty method and a regularization method, respectively, for a special class of here-and-now problems. Shapiro and Xu [18, 19, 20] discussed the sample average approximation and implicit programming approaches for the lower-level wait-and-see problems. In addition, Birbil et al. [1] considered an SMPEC in which both the objective and constraints involve expectations.

In [9], the here-and-now problem is formulated as follows:

$$\begin{aligned} \min \quad & \mathbb{E}[f(x,y,\omega)+d^T z(\omega)] \\ \text{s.t.} \quad & g(x)\le 0,\ h(x)=0, \\ & 0\le y\perp (F(x,y,\omega)+z(\omega))\ge 0, \\ & z(\omega)\ge 0,\quad \omega\in\Omega\ \text{ a.e.}, \end{aligned} \tag{1.1}$$

*This work was supported in part by the Scientific Research Grant-in-Aid from Japan Society for the Promotion of Science.

†Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China. Current address: Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan. E-mail: ghlin@amp.i.kyoto-u.ac.jp.

‡Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan. E-mail: fuku@amp.i.kyoto-u.ac.jp.

where $f : \Re^{n+m} \times \Omega \to \Re$, $g : \Re^n \to \Re^{s_1}$, $h : \Re^n \to \Re^{s_2}$, and $F : \Re^{n+m} \times \Omega \to \Re^m$ are functions, $\mathbb{E}$ means expectation with respect to the random variable $\omega \in \Omega$, the symbol $\perp$ means the two vectors are perpendicular to each other, "a.e." is the abbreviation for "almost everywhere" under the given probability measure, $z(\omega)$ is a recourse variable, and $d \in \Re^m$ is a constant vector with positive elements. Moreover, $x$ denotes the upper-level decision, $y$ represents the lower-level decision, and both the decisions $x$ and $y$ need to be made at once, before $\omega$ is observed.

Lin et al. [9, 11, 13] considered the case where the function $F$ is affine and the underlying sample space $\Omega$ is discrete and finite. In this paper, we consider a general case, i.e., $F$ is nonlinear and $\Omega$ is a compact subset of $\Re^l$. A general strategy for SMPECs with infinitely many samples is to discretize the problem by some kind of sampling selection methods, which means the approximation problems are still MPECs. The strategy of this paper is, in contrast, to solve some standard nonlinear programs as approximations of the original SMPEC.

We suppose that all functions involved are continuous and, particularly, $f$ and $F$ are continuously differentiable with respect to $(x, y)$, $g$ and $h$ are continuously differentiable with respect to $x$. The main contributions of the paper can be stated as follows. In problem (1.1), $d^T z(\omega)$ actually serves as a penalty term caused by the possible violation of the complementarity constraint $0 \le y \perp F(x, y, \omega) \ge 0$. In this paper, we consider another penalty formulation of stochastic mathematical program with complementarity constraints and recourse:

$$\begin{array}{ll} \min & \mathbb{E}[f(x,y,\omega) + \sigma \|z(\omega)\|^2] \\ \text{s.t.} & g(x) \le 0,\ \ h(x) = 0, \\ & 0 \le y \perp (F(x,y,\omega) + z(\omega)) \ge 0, \\ & z(\omega) \ge 0, \quad \omega \in \Omega \ \ \text{a.e.}, \end{array} \tag{1.2}$$

where $\sigma > 0$ is a weight constant. We can show that problem (1.2) is equivalent to

$$\begin{array}{ll} \min & \mathbb{E}[f(x,y,\omega) + \sigma \|u(x,y,\omega)\|^2] \\ \text{s.t.} & g(x) \le 0,\ \ h(x) = 0,\ \ y \ge 0, \\ & y \circ F(x,y,\omega) \le 0, \quad \omega \in \Omega \ \ \text{a.e.}, \end{array} \tag{1.3}$$

where $u : \Re^{n+m} \times \Omega \to \Re^m$ is defined by

$$u(x,y,\omega) := \max\{-F(x,y,\omega),\ 0\} \tag{1.4}$$

and $\circ$ denotes the Hadamard product, i.e., $y \circ F(x,y,\omega) := (y_1 F_1(x,y,\omega), \cdots, y_m F_m(x,y,\omega))^T$. See the appendix for a proof of the equivalence between (1.2) and (1.3). Note that problem (1.3) does no longer contain complementarity constraints and recourse variables. However, problem (1.3) is actually a semi-infinite programming problem with a large number of complementarity-like constraints and hence it is generally very complicated. We will employ a Monte Carlo sampling method and a penalty technique to get approximations of problem (1.3) and investigate the limiting behavior of optimal solutions and stationary points of the approximations.

The following notations are used in the paper. For any vectors $a$ and $b$ of the same dimension, both $\max\{a, b\}$ and $\min\{a, b\}$ are understood to be taken componentwise. For a given function

$c : \Re^s \to \Re^{s'}$ and a vector $t \in \Re^s$, $\nabla c(t)$ is the transposed Jacobian of $c$ at $t$ and $\mathcal{I}_c(t) := \{i \mid c_i(t) = 0\}$ stands for the active index set of $c$ at $t$.

# 2 Optimality Conditions

We consider problem (1.3). In the literature on semi-infinite programming, it is often assumed that there are a finite number of active constraints at a solution (see e.g. [8]). However, the above assumption does not hold in problem (1.3) in general. For example, if $y_i = 0$ for some index $i$, there must be infinitely many active constraints at the point. This indicates that problem (1.3) is difficult to deal with than an ordinary semi-infinite programming problem. We define the stationarity for problem (1.3) as follows.

**Definition 2.1** *We say $(x^*, y^*)$ is stationary to (1.3) if there exist Lagrangian multiplier vectors $\alpha^* \in \Re^{s_1}, \beta^* \in \Re^{s_2}, \gamma^* \in \Re^m$, and a Lagrangian multiplier function $\delta^* : \Omega \to \Re^m$ such that*

$$\begin{aligned} 0 &= \mathbb{E}[\nabla_x f(x^*, y^*, \omega) - 2\sigma \nabla_x F(x^*, y^*, \omega) u(x^*, y^*, \omega)] \\ &\qquad + \nabla g(x^*)\alpha^* + \nabla h(x^*)\beta^* + \mathbb{E}[\nabla_x (y^* \circ F(x^*, y^*, \omega))\delta^*(\omega)], \end{aligned} \tag{2.1}$$

$$\begin{aligned} 0 &= \mathbb{E}[\nabla_y f(x^*, y^*, \omega) - 2\sigma \nabla_y F(x^*, y^*, \omega) u(x^*, y^*, \omega)] \\ &\qquad - \gamma^* + \mathbb{E}[\nabla_y (y^* \circ F(x^*, y^*, \omega))\delta^*(\omega)], \end{aligned} \tag{2.2}$$

$$0 \le \alpha^* \perp -g(x^*) \ge 0, \tag{2.3}$$

$$0 \le \gamma^* \perp y^* \ge 0, \tag{2.4}$$

$$\beta^* : \text{free}, \quad h(x^*) = 0, \tag{2.5}$$

$$0 \le \delta^*(\omega) \perp -y^* \circ F(x^*, y^*, \omega) \ge 0, \quad \omega \in \Omega \ \text{ a.e.} \tag{2.6}$$

Note that, for any $(x, y) \in \Re^{n+m}$ and any $\omega \in \Omega$,

$$\nabla_x (y \circ F(x, y, \omega)) = \nabla_x F(x, y, \omega) \mathrm{diag}(y_1, \cdots, y_m), \tag{2.7}$$

$$\nabla_y (y \circ F(x, y, \omega)) = \nabla_y F(x, y, \omega) \mathrm{diag}(y_1, \cdots, y_m) + \mathrm{diag}(F_1(x, y, \omega), \cdots, F_m(x, y, \omega)). \tag{2.8}$$

# 3 Monte Carlo Sampling and Penalty Approximations

Let $\phi : \Omega \to \Re$ be a function. The Monte Carlo sampling estimate for $\mathbb{E}[\phi(\omega)]$ is obtained by taking independently and identically distributed random samples $\{\omega_1, \cdots, \omega_k\}$ from $\Omega$ and letting $\mathbb{E}[\phi(\omega)] \approx \frac{1}{k}\sum_{\ell=1}^k \phi(\omega_\ell)$. The strong law of large numbers guarantees that this procedure converges with probability one (abbreviated by "w.p.1" below), i.e.,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^k \phi(\omega_\ell) = \mathbb{E}[\phi(\omega)] := \int_\Omega \phi(\omega) d\zeta(\omega) \qquad \text{w.p.1}, \tag{3.1}$$

where $\zeta(\omega)$ is the distribution function of $\omega$. See [15, 17] for more details about the Monte Carlo sampling method.

Applying the above method and using a penalty technique, we obtain the problem

$$\begin{aligned} \min \quad & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x,y,\omega_\ell)+\sigma\|u(x,y,\omega_\ell)\|^2+\rho_k\,\|y\circ v(x,y,\omega_\ell)\|^2\Big) \\ \text{s.t.} \quad & g(x)\le 0,\ h(x)=0,\ y\ge 0, \end{aligned} \tag{3.2}$$

which is a smooth approximation of problem (1.3). Here, $\rho_k > 0$ is a penalty parameter, $u : \Re^{n+m}\times\Omega\to\Re^m$ is defined by (1.4), and $v:\Re^{n+m}\times\Omega\to\Re^m$ is given by

$$v(x,y,\omega) := \max\{F(x,y,\omega),\ 0\}. \tag{3.3}$$

Problem (3.2) is neither a semi-infinite program nor an MPEC and it is generally not difficult to deal with in practice.

# 4 Convergence Analysis

In this section, we investigate convergence properties of the Monte Carlo sampling and penalty method. Throughout, we denote by $\mathcal{F}$ the feasible region of problem (3.2) and for each $k$, we let $\{\omega_1,\cdots,\omega_k\}$ be independently and identically distributed random samples drawn from $\Omega$.

## 4.1 Limiting behavior of optimal solutions

We first study the convergence of optimal solutions of problems (3.2).

**Theorem 4.1** *Suppose that both $f$ and $F$ are Lipschitz continuous in $(x,y)$ with Lipschitz constants independent of $\omega$ and $\lim_{k\to\infty}\rho_k\to+\infty$. Assume that $(x^k,y^k)$ solves problem (3.2) for each $k$ and the sequence $\{(x^k,y^k)\}$ is bounded. Let $(x^*,y^*)$ be an accumulation point of $\{(x^k,y^k)\}$. Then $(x^*,y^*)$ is an optimal solution of problem (1.3) with probability one.*

**Proof.** Without loss of generality, we suppose $\lim_{k\to\infty}(x^k,y^k)=(x^*,y^*)$.

(a) We first prove that $(x^*,y^*)$ is almost surely feasible to (1.3). It is obvious that $(x^*,y^*)$ satisfies the constraints of problem (3.2). Therefore, it is sufficient to show that there holds

$$y^*\circ F(x^*,y^*,\omega)\le 0,\quad \omega\in\Omega\ \text{ a.e.} \tag{4.1}$$

with probability one. In fact, since $(x^k,y^k)$ is an optimal solution of problem (3.2) and $(x^*,0)$ is a feasible point of (3.2), we have

$$\begin{aligned} & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^k,y^k,\omega_\ell)+\sigma\|u(x^k,y^k,\omega_\ell)\|^2+\rho_k\|y^k\circ v(x^k,y^k,\omega_\ell)\|^2\Big) \\ \le\ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^*,0,\omega_\ell)+\sigma\|u(x^*,0,\omega_\ell)\|^2\Big). \end{aligned}$$

It then follows from the Lipschitz continuity of $f$ and the boundedness of the functions $f(x^*,0,\cdot)$ and $u(x^*,0,\cdot)$ on $\Omega$ that the sequence $\left\{\frac{\rho_k}{k}\sum_{\ell=1}^{k}\|y^k\circ v(x^k,y^k,\omega_\ell)\|^2\right\}$ is bounded. As a result,

$\left\{\frac{\rho_k}{k}\sum_{\ell=1}^k (y_i^k)^2 \left(F_i(x^k, y^k, \omega_\ell) + u_i(x^k, y^k, \omega_\ell)\right)^2\right\}$ is bounded for each $i$ and, since $y^k \geq 0$ and $F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell) = v(x^k, y^k, \omega_\ell) \geq 0$ for every $k$ and $\ell$, $\left\{\frac{\rho_k}{k}\sum_{\ell=1}^k (y^k)^T (F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell))\right\}$ is also bounded. Noting that $\lim_{k\to\infty} \rho_k = +\infty$, we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{\ell=1}^k (y^k)^T \Big(F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell)\Big) = 0. \tag{4.2}$$

Moreover, by the assumptions of the theorem, there exists a constant $\kappa > 0$ such that

$$\|F(x, y, \omega) - F(x', y', \omega)\| \leq \kappa(\|x - x'\| + \|y - y'\|) \tag{4.3}$$

holds for any $x, x' \in \Re^n$, $y, y' \in \Re^m$, and $\omega \in \Omega$. Therefore, for any $k$ and $\ell$, we have

$$\begin{aligned} & \|(F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell)) - (F(x^*, y^*, \omega_\ell) + u(x^*, y^*, \omega_\ell))\| \\ \leq\ & 2\|F(x^k, y^k, \omega_\ell) - F(x^*, y^*, \omega_\ell)\| \\ \leq\ & 2\kappa\,(\|x^k - x^*\| + \|y^k - y^*\|) \end{aligned}$$

and then

$$\begin{aligned} & \lim_{k\to\infty} \left|\frac{1}{k}\sum_{\ell=1}^k (y^k)^T \Big((F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell)) - (F(x^*, y^*, \omega_\ell) + u(x^*, y^*, \omega_\ell))\Big)\right| \\ \leq\ & \lim_{k\to\infty} 2\kappa\,\|y^k\|\,(\|x^k - x^*\| + \|y^k - y^*\|) \\ =\ & 0. \end{aligned} \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{1}{k}\sum_{\ell=1}^k (y^k)^T \Big(F(x^k, y^k, \omega_\ell) + u(x^k, y^k, \omega_\ell)\Big) \\ &= \lim_{k\to\infty} \frac{1}{k}\sum_{\ell=1}^k (y^k)^T \Big(F(x^*, y^*, \omega_\ell) + u(x^*, y^*, \omega_\ell)\Big) \\ &= \int_\Omega (y^*)^T (F(x^*, y^*, \omega) + u(x^*, y^*, \omega_\ell))d\zeta(\omega) \qquad \text{w.p.1}, \end{aligned} \tag{4.5}$$

where the last equality follows from (3.1). Noting that both $(y^*)^T(F(x^*, y^*, \cdot) + u(x^*, y^*, \cdot))$ and $(y^*)^T u(x^*, y^*, \cdot)$ are nonnegative on $\Omega$, we obtain (4.1) from (4.5) immediately.

(b) Let $(x, y)$ be an arbitrary feasible solution of problem (1.3). It is obvious that $(x, y)$ is feasible to problem (3.2). Moreover, if $y_i > 0$ for some $i$, there must hold $F_i(x, y, \omega) \leq 0$ for almost all $\omega \in \Omega$ and so $u_i(x, y, \omega) = -F_i(x, y, \omega)$ for almost all $\omega \in \Omega$. This means

$$y \circ v(x, y, \omega) = y \circ (F(x, y, \omega) + u(x, y, \omega)) = 0, \quad \omega \in \Omega \ \text{ a.e.} \tag{4.6}$$

Since $(x^k, y^k)$ is an optimal solution of problem (3.2), we have almost surely that

$$\begin{aligned}
& \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x,y,\omega_\ell)+\sigma\|u(x,y,\omega_\ell)\|^2\Big) \\
= \ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x,y,\omega_\ell)+\sigma\|u(x,y,\omega_\ell)\|^2+\rho_k\|y\circ v(x,y,\omega_\ell)\|^2\Big) \\
\geq \ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^k,y^k,\omega_\ell)+\sigma\|u(x^k,y^k,\omega_\ell)\|^2+\rho_k\|y^k\circ v(x^k,y^k,\omega_\ell)\|^2\Big) \\
\geq \ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^k,y^k,\omega_\ell)+\sigma\|u(x^k,y^k,\omega_\ell)\|^2\Big).
\end{aligned}$$

As a result, we have

$$\begin{aligned}
& \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^*,y^*,\omega_\ell)+\sigma\|u(x^*,y^*,\omega_\ell)\|^2\Big)-\frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x,y,\omega_\ell)+\sigma\|u(x,y,\omega_\ell)\|^2\Big) \\
\leq \ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^*,y^*,\omega_\ell)+\sigma\|u(x^*,y^*,\omega_\ell)\|^2\Big)-\frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^k,y^k,\omega_\ell)+\sigma\|u(x^k,y^k,\omega_\ell)\|^2\Big) \\
\leq \ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(|f(x^*,y^*,\omega_\ell)-f(x^k,y^k,\omega_\ell)| \\
& +\sigma\|u(x^*,y^*,\omega_\ell)-u(x^k,y^k,\omega_\ell)\|\,(\|u(x^*,y^*,\omega_\ell)\|+\|u(x^k,y^k,\omega_\ell)\|)\Big) \qquad \text{w.p.1.} \qquad (4.7)
\end{aligned}$$

Note that $f$ is Lipschitz continuous in $(x, y)$ with Lipschitz constants independent of $\omega$. This yields

$$\lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k}|f(x^*,y^*,\omega_\ell)-f(x^k,y^k,\omega_\ell)|=0. \tag{4.8}$$

On the other hand, it follows from (1.4) and (4.3) that

$$\begin{aligned}
\|u(x^*,y^*,\omega_\ell)-u(x^k,y^k,\omega_\ell)\| &\leq \|F(x^*,y^*,\omega_\ell)-F(x^k,y^k,\omega_\ell)\| \\
&\leq \kappa\,(\|x^k-x^*\|+\|y^k-y^*\|), \qquad \ell=1,\cdots,k.
\end{aligned}$$

By the boundedness of the sequence $\Big\{\frac{1}{k}\sum_{\ell=1}^{k}(\|u(x^*,y^*,\omega_\ell)\|+\|u(x^k,y^k,\omega_\ell)\|)\Big\}$, we have

$$\lim_{k\to\infty}\frac{\sigma}{k}\sum_{\ell=1}^{k}\|u(x^*,y^*,\omega_\ell)-u(x^k,y^k,\omega_\ell)\|\Big(\|u(x^*,y^*,\omega_\ell)\|+\|u(x^k,y^k,\omega_\ell)\|\Big)=0. \tag{4.9}$$

Letting $k\to+\infty$ in (4.7) and taking (4.8)–(4.9) and (3.1) into account, we obtain

$$\mathbb{E}[f(x^*,y^*,\omega)+\sigma\|u(x^*,y^*,\omega)\|^2]\leq\mathbb{E}[f(x,y,\omega)+\sigma\|u(x,y,\omega)\|^2] \qquad \text{w.p.1,}$$

which indicates that $(x^*, y^*)$ is an optimal solution of (1.3) with probability one. ∎

We next discuss the existence conditions of solutions of problem (3.2). Let $F$ be affine with respect to $(x, y)$ and given by

$$F(x,y,\omega):=N(\omega)x+M(\omega)y+q(\omega), \tag{4.10}$$

where $N:\Omega\to\Re^{m\times n}$, $M:\Omega\to\Re^{m\times m}$, and $q:\Omega\to\Re^{m}$ are all continuous.

**Definition 4.1** *Suppose that $\bar{M}$ is an $m \times m$ matrix. We call $\bar{M}$ an $R_0$-matrix if*

$$y \geq 0,\ \bar{M}y \geq 0,\ y^T\bar{M}y = 0 \quad \Longrightarrow \quad y = 0.$$

It is well-known that any P-matrix must be an $\mathrm{R}_0$-matrix [4]. We have the following result.

**Lemma 4.1** *Let $\{M_k\} \subset \Re^{m\times m}$ be convergent to $\bar{M} \in \Re^{m\times m}$ and $\bar{M}$ be an $R_0$-matrix. Then, there exists an integer $k_0 > 0$ such that $M_k$ is an $R_0$-matrix for every $k \geq k_0$.*

**Theorem 4.2** *Suppose that the set $\mathcal{X} := \{x \in \Re^n \mid g(x) \leq 0, h(x) = 0\}$ is nonempty and bounded, the function $f$ is bounded below, and $\lim_{k\to\infty} \rho_k \to +\infty$. Let $F$ be defined by (4.10) and $\bar{M} := \int_\Omega M(\omega)d\zeta(\omega)$ be an $R_0$-matrix. We then have the following statements almost surely.*

(i) *Problem (3.2) has at least one optimal solution when $k$ is sufficiently large.*

(ii) *Let $(x^k, y^k)$ be a solution of (3.2) for each $k$ sufficiently large. Then the sequence $\{(x^k, y^k)\}$ is bounded.*

**Proof.** (i) For each $k$, let $M_k := \frac{1}{k}\sum_{\ell=1}^k M(\omega_\ell)$. It then follows from (3.1) that $\bar{M} = \lim_{k\to\infty} M_k$ with probability one. Since $\bar{M}$ is an $\mathrm{R}_0$-matrix, by Lemma 4.1, there exists an integer $k_0 > 0$ such that $M_k$ is an $\mathrm{R}_0$-matrix for every $k \geq k_0$.

Let $k \geq k_0$ be fixed and suppose $M_k$ is an $\mathrm{R}_0$-matrix. It is easy to see that $\mathcal{F}$ is a nonempty and closed set and the objective function of problem (3.2) is bounded below on $\mathcal{F}$. Then, there exists a sequence $\{(x^j, y^j)\} \subseteq \mathcal{F}$ such that

$$\begin{aligned} &\lim_{j\to\infty} \frac{1}{k}\sum_{\ell=1}^k \Big( f(x^j, y^j, \omega_\ell) + \sigma\|u(x^j, y^j, \omega_\ell)\|^2 + \rho_k\|y^j \circ v(x^j, y^j, \omega_\ell)\|^2 \Big) \\ = \ &\inf_{(x,y)\in\mathcal{F}} \frac{1}{k}\sum_{\ell=1}^k \Big( f(x, y, \omega_\ell) + \sigma\|u(x, y, \omega_\ell)\|^2 + \rho_k\|y \circ v(x, y, \omega_\ell)\|^2 \Big). \end{aligned} \tag{4.11}$$

Since $f$ is bounded below and $\rho_k$ is a positive constant, it follows from (4.11) that the sequences

$$\Big\{\frac{1}{k}\sum_{\ell=1}^k \|u(x^j, y^j, \omega_\ell)\|^2\Big\}_{j=0,1,\cdots} \quad \text{and} \quad \Big\{\frac{1}{k}\sum_{\ell=1}^k \|y^j \circ v(x^j, y^j, \omega_\ell)\|^2\Big\}_{j=0,1,\cdots}$$

are bounded. This implies that

$$\Big\{\frac{1}{k}\sum_{\ell=1}^k u(x^j, y^j, \omega_\ell)\Big\}_{j=0,1,\cdots} \text{and } \Big\{\frac{1}{k}\sum_{\ell=1}^k (y^j)^T \Big(N(\omega_\ell)x^j + M(\omega_\ell)y^j + q(\omega_\ell) + u(x^j, y^j, \omega_\ell)\Big)\Big\}_{j=0,1,\cdots}$$

are also bounded. Note that the latter sequence can be rewritten as

$$\Big\{(y^j)^T\Big(\frac{1}{k}\sum_{\ell=1}^k N(\omega_\ell)x^j + M_k y^j + \frac{1}{k}\sum_{\ell=1}^k q(\omega_\ell) + \frac{1}{k}\sum_{\ell=1}^k u(x^j, y^j, \omega_\ell)\Big)\Big\}_{j=0,1,\cdots}. \tag{4.12}$$

Moreover, by the boundedness of the set $\mathcal{X}$, the sequence $\{x^j\}$ is bounded. On the other hand, it is obvious from the feasibility of $(x^j, y^j)$ in (3.2) and the definition of $u$ that, for each $j$,

$$y^j \geq 0, \qquad \frac{1}{k}\sum_{\ell=1}^k N(\omega_\ell)x^j + M_k y^j + \frac{1}{k}\sum_{\ell=1}^k q(\omega_\ell) + \frac{1}{k}\sum_{\ell=1}^k u(x^j, y^j, \omega_\ell) \geq 0. \tag{4.13}$$

Suppose the sequence $\{y^j\}$ is unbounded. Taking a subsequence if necessary, we assume that

$$\lim_{j\to\infty} \|y^j\| = +\infty, \quad \lim_{j\to\infty} \frac{y^j}{\|y^j\|} = \bar{y}, \quad \|\bar{y}\| = 1. \tag{4.14}$$

Then, dividing (4.12) and (4.13) by $\|y^j\|^2$ and $\|y^j\|$, respectively, and letting $j \to +\infty$, we obtain

$$0 \le \bar{y} \perp M_k \bar{y} \ge 0.$$

Since $M_k$ is an $R_0$-matrix, we have $\bar{y} = 0$. This contradicts (4.14) and hence $\{y^j\}$ is bounded.

Therefore, $\{(x^j, y^j)\}$ is bounded. Since $\mathcal{F}$ is closed, we see from (4.11) that any accumulation point of $\{(x^j, y^j)\}$ must be an optimal solution of (3.2). This completes the proof of (i).

(ii) Let $(x^k, y^k)$ be a solution of (3.2) for each sufficiently large $k$. The boundedness of $\{x^k\}$ follows from the boundedness of the set $\mathcal{X}$ immediately. We next prove that $\{y^k\}$ is almost surely bounded. To this end, we choose a vector $\bar{x} \in \mathcal{X}$ arbitrarily. Then, $(\bar{x}, 0)$ is feasible to problem (3.2). Since $(x^k, y^k)$ is an optimal solution of (3.2), we have

$$\begin{aligned} & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(x^k, y^k, \omega_\ell) + \sigma\|u(x^k, y^k, \omega_\ell)\|^2 + \rho_k\|y^k \circ v(x^k, y^k, \omega_\ell)\|^2\Big) \\ \le\ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(\bar{x}, 0, \omega_\ell) + \sigma\|u(\bar{x}, 0, \omega_\ell)\|^2\Big) \end{aligned} \tag{4.15}$$

and, by the definitions (1.4) and (4.10),

$$\frac{1}{k}\sum_{\ell=1}^{k} N(\omega_\ell)x^k + \frac{1}{k}\sum_{\ell=1}^{k} M(\omega_\ell)y^k + \frac{1}{k}\sum_{\ell=1}^{k} q(\omega_\ell) + \frac{1}{k}\sum_{\ell=1}^{k} u(x^k, y^k, \omega_\ell) \ge 0, \quad y^k \ge 0. \tag{4.16}$$

It follows from (4.15) that

$$\begin{aligned} 0 \ \le\ & \frac{\sigma}{k}\sum_{\ell=1}^{k}\|u(x^k, y^k, \omega_\ell)\|^2 + \frac{\rho_k}{k}\sum_{\ell=1}^{k}\|y^k \circ v(x^k, y^k, \omega_\ell)\|^2 \\ \le\ & \frac{1}{k}\sum_{\ell=1}^{k}\Big(f(\bar{x}, 0, \omega_\ell) - f(x^k, y^k, \omega_\ell)\Big) + \frac{\sigma}{k}\sum_{\ell=1}^{k}\|u(\bar{x}, 0, \omega_\ell)\|^2. \end{aligned}$$

Since $f$ is bounded below, we have from (3.1) that

$$\Big\{\frac{1}{k}\sum_{\ell=1}^{k}\Big(f(\bar{x}, 0, \omega_\ell) - f(x^k, y^k, \omega_\ell)\Big)\Big\} \quad \text{and} \quad \Big\{\frac{\sigma}{k}\sum_{\ell=1}^{k}\|u(\bar{x}, 0, \omega_\ell)\|^2\Big\}$$

are almost surely bounded. In consequence, the sequences

$$\Big\{\frac{1}{k}\sum_{\ell=1}^{k}\|u(x^k, y^k, \omega_\ell)\|^2\Big\} \quad \text{and} \quad \Big\{\frac{\rho_k}{k}\sum_{\ell=1}^{k}\|y^k \circ v(x^k, y^k, \omega_\ell)\|^2\Big\}$$

are almost surely bounded. By Cauchy inequality, we have

$$\Big(\sum_{\ell=1}^{k} u_i(x^k, y^k, \omega_\ell)\Big)^2 \le k\sum_{\ell=1}^{k}\Big(u_i(x^k, y^k, \omega_\ell)\Big)^2, \qquad i = 1, \cdots, m$$

for each $k$ and hence

$$\begin{aligned}\Big\|\frac{1}{k}\sum_{\ell=1}^{k}u(x^k,y^k,\omega_\ell)\Big\|^2 &= \frac{1}{k^2}\sum_{i=1}^{m}\Big(\sum_{\ell=1}^{k}u_i(x^k,y^k,\omega_\ell)\Big)^2\\ &\le \frac{1}{k}\sum_{i=1}^{m}\sum_{\ell=1}^{k}\Big(u_i(x^k,y^k,\omega_\ell)\Big)^2 = \frac{1}{k}\sum_{\ell=1}^{k}\|u(x^k,y^k,\omega_\ell)\|^2. \end{aligned}\tag{4.17}$$

Similarly, we have

$$\begin{aligned}&\Big|\frac{1}{k}\sum_{\ell=1}^{k}(y^k)^T\Big(N(\omega_\ell)x^k+M(\omega_\ell)y^k+q(\omega_\ell)+u(x^k,y^k,\omega_\ell)\Big)\Big|^2\\ =\;& \frac{1}{k^2}\Big|\sum_{i=1}^{m}\sum_{\ell=1}^{k}y_i^k v_i(x^k,y^k,\omega_\ell)\Big|^2 \le \frac{m}{k^2}\sum_{i=1}^{m}\Big(\sum_{\ell=1}^{k}y_i^k v_i(x^k,y^k,\omega_\ell)\Big)^2\\ \le\;& \frac{m}{k}\sum_{i=1}^{m}\sum_{\ell=1}^{k}\Big(y_i^k v_i(x^k,y^k,\omega_\ell)\Big)^2 = \frac{m}{k}\sum_{\ell=1}^{k}\|y^k\circ v(x^k,y^k,\omega_\ell)\|^2. \end{aligned}\tag{4.18}$$

It follows from (4.17) and (4.18) that both $\left\{\frac{1}{k}\sum_{\ell=1}^{k}u(x^k,y^k,\omega_\ell)\right\}$ and

$$\left\{\frac{1}{k}\sum_{\ell=1}^{k}(y^k)^T\Big(N(\omega_\ell)x^k+M(\omega_\ell)y^k+q(\omega_\ell)+u(x^k,y^k,\omega_\ell)\Big)\right\}\tag{4.19}$$

are almost surely bounded. Suppose that the sequence $\{y^k\}$ is unbounded with probability one. Taking a subsequence if necessary, we assume that

$$\lim_{k\to\infty}\|y^k\|=+\infty,\quad \lim_{k\to\infty}\frac{y^k}{\|y^k\|}=\bar{y},\quad \|\bar{y}\|=1.\tag{4.20}$$

Note that the sequences $\{x^k\}$ and $\left\{\frac{1}{k}\sum_{\ell=1}^{k}u(x^k,y^k,\omega_\ell)\right\}$ are bounded and

$$\lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k}M(\omega_\ell)=\bar{M},\quad \lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k}N(\omega_\ell)=\int_\Omega N(\omega)d\zeta(\omega),\quad \lim_{k\to\infty}\frac{1}{k}\sum_{\ell=1}^{k}q(\omega_\ell)=\int_\Omega q(\omega)d\zeta(\omega)$$

with probability one. Dividing (4.16) and (4.19) by $\|y^k\|$ and $\|y^k\|^2$, respectively, and letting $k\to+\infty$, we obtain $0\le\bar{y}\perp\bar{M}\,\bar{y}\ge 0$. Since $\bar{M}$ is an $\mathrm{R}_0$-matrix, we have $\bar{y}=0$ with probability one. This contradicts (4.20) and hence, the sequence $\{y^k\}$ is almost surely bounded. This completes the proof of (ii). ■

## 4.2 Limiting behavior of stationary points

In general, it is difficult to obtain an optimal solution, whereas computation of stationary points is relatively easy. Therefore, it is important to study the limiting behavior of stationary points of problems (3.2). To this end, for any $\epsilon>0$, we denote

$$\mathcal{E}_\epsilon := \Big\{(x,y)\in\Re^{n+m}\;\Big|\; g(x)\le 0,\ h(x)=0,\ y\ge 0,\ y_iF_i(x,y,\omega)\le\epsilon \text{ for each } i \text{ and } \omega\in\Omega \text{ a.e.}\Big\}$$

We can show that, for any fixed $(\tilde{x},\tilde{y})\in\mathcal{F}\setminus\mathcal{E}_\epsilon$, there exists a constant $\tilde{\rho}>0$ such that $(\tilde{x},\tilde{y})$ cannot serve as an optimal solution of problem (3.2) for any $\rho_k\ge\tilde{\rho}$. In fact, since $(\tilde{x},\tilde{y})\notin\mathcal{E}_\epsilon$, there are an index $\tilde{i}$ and a subset $\tilde{\Omega} \subseteq \Omega$ with nonzero measure such that $\tilde{y}_{\tilde{i}} F_{\tilde{i}}(\tilde{x}, \tilde{y}, \omega) > \epsilon$ holds for every $\omega \in \tilde{\Omega}$. It follows that

$$\frac{1}{k} \sum_{\ell=1}^{k} \Big( f(\tilde{x}, \tilde{y}, \omega_\ell) + \sigma \|u(\tilde{x}, \tilde{y}, \omega_\ell)\|^2 + \rho_k \, \|\tilde{y} \circ v(\tilde{x}, \tilde{y}, \omega_\ell)\|^2 \Big) \longrightarrow +\infty \quad \text{as} \quad \rho_k \to +\infty. \tag{4.21}$$

Note that $(\tilde{x}, 0)$ is feasible to (3.2) for any $\rho_k$. By (4.21), the value of the objective function of (3.2) at $(\tilde{x}, \tilde{y})$ must be larger than the value at $(\tilde{x}, 0)$ when $\rho_k$ is large enough. This means that $(\tilde{x}, \tilde{y})$ cannot serve as an optimal solution of problem (3.2) when $\rho_k$ is sufficiently large.

Therefore, when the penalty parameter $\rho_k$ is chosen sufficiently large, we may expect to get a point $(x^k, y^k) \in \mathcal{E}_\epsilon$ by solving (3.2) for a given $\epsilon > 0$. Noting that

$$0 \le a \perp b \ge 0 \quad \Longleftrightarrow \quad \min\{a,\ b\} = 0,$$

we define the approximate stationarity for (1.3) as follows.

**Definition 4.2** *Let $\epsilon > 0$. We say $(x^*, y^*)$ is an $\epsilon$-stationary point of problem (1.3) if there exist Lagrangian multiplier vectors $\alpha^* \in \Re^{s_1}, \beta^* \in \Re^{s_2}, \gamma^* \in \Re^m$, and a Lagrangian multiplier function $\delta^* : \Omega \to \Re^m$ satisfying (2.1)-(2.5) and*

$$|\min\{\delta_i^*(\omega),\ -y_i^* F_i(x^*, y^*, \omega)\}| \le \epsilon, \qquad \forall\, i,\ \omega \in \Omega \ \text{ a.e.} \tag{4.22}$$

The main result can be stated as follows.

**Theorem 4.3** *Suppose $\nabla_{(x,y)} f, F, \nabla_{(x,y)} F$ are all Lipschitz continuous in $(x, y)$ with Lipschitz constants independent of $\omega$ and $\lim_{k\to\infty} \rho_k \to \bar{\rho}$, where $\bar{\rho} > 0$ is a sufficiently large number. Let $(x^k, y^k)$ be a Karush-Kuhn-Tucker point of (3.2) for each $k$ and $(x^k, y^k) \in \mathcal{E}_\epsilon$ for each $k$ large enough, where $\epsilon > 0$ is a given scalar. Suppose that $(x^*, y^*)$ is an accumulation point of $\{(x^k, y^k)\}$ and the system $\{g(x) \le 0,\ h(x) = 0\}$ satisfies the Mangasarian-Fromovitz constraint qualification at $x^*$. Then $(x^*, y^*)$ is $\epsilon$-stationary to (1.3) with probability one.*

**Proof.** Without loss of generality, we suppose that $\lim_{k\to\infty} (x^k, y^k) = (x^*, y^*)$. It is obvious that $(x^*, y^*) \in \mathcal{E}_\epsilon$. Since $(x^k, y^k)$ is a Karush-Kuhn-Tucker point of problem (3.2), there must exist Lagrangian multiplier vectors $\alpha^k \in \Re^{s_1}, \beta^k \in \Re^{s_2}$, and $\gamma^k \in \Re^m$ such that

$$\begin{aligned} 0 \;=\;& \frac{1}{k} \sum_{\ell=1}^{k} \Big( \nabla_x f(x^k, y^k, \omega_\ell) - 2\sigma \nabla_x F(x^k, y^k, \omega_\ell) u(x^k, y^k, \omega_\ell) \\ &+ 2\rho_k \nabla_x F(x^k, y^k, \omega_\ell) \mathrm{diag}(y_1^k, \cdots, y_m^k)(y^k \circ v(x^k, y^k, \omega_\ell)) \Big) + \nabla g(x^k)\alpha^k + \nabla h(x^k)\beta^k, \end{aligned} \tag{4.23}$$

$$\begin{aligned} 0 \;=\;& \frac{1}{k} \sum_{\ell=1}^{k} \Big( \nabla_y f(x^k, y^k, \omega_\ell) - 2\sigma \nabla_y F(x^k, y^k, \omega_\ell) u(x^k, y^k, \omega_\ell) \\ &+ 2\rho_k \Big( \nabla_y F(x^k, y^k, \omega_\ell) \mathrm{diag}(y_1^k, \cdots, y_m^k) \\ &+ \mathrm{diag}(v_1(x^k, y^k, \omega_\ell), \cdots, v_m(x^k, y^k, \omega_\ell)) \Big) (y^k \circ v(x^k, y^k, \omega_\ell)) \Big) - \gamma^k, \end{aligned} \tag{4.24}$$

$$0 \ \le\ \alpha^k \ \perp\ -g(x^k) \ \ge\ 0, \tag{4.25}$$

$$\beta^k \ :\ \text{free}, \qquad h(x^k) = 0, \tag{4.26}$$

$$0 \ \le\ \gamma^k \ \perp\ y^k \ \ge\ 0. \tag{4.27}$$

We next show that there exist multiplier vectors $\alpha^* \in \Re^{s_1}, \beta^* \in \Re^{s_2}, \gamma^* \in \Re^m$, and a multiplier function $\delta^* : \Omega \to \Re^m$ such that there hold (2.1)-(2.5) and (4.22).

Recall that the functions $\nabla_{(x,y)} f, F, \nabla_{(x,y)} F$ are Lipschitz continuous in $(x, y)$ with Lipschitz constants independent of $\omega$. Then, there exists a constant $\kappa > 0$ such that

$$\begin{aligned}
\|\nabla_{(x,y)} f(x, y, \omega) - \nabla_{(x,y)} f(x', y', \omega)\| &\leq \kappa\,(\|x - x'\| + \|y - y'\|\|), \\
\|F(x, y, \omega) - F(x', y', \omega)\| &\leq \kappa\,(\|x - x'\| + \|y - y'\|), \\
\|\nabla_{(x,y)} F(x, y, \omega) - \nabla_{(x,y)} F(x', y', \omega)\| &\leq \kappa\,(\|x - x'\| + \|y - y'\|)
\end{aligned}$$

hold for any $x, x' \in \Re^n, y, y' \in \Re^m$, and $\omega \in \Omega$. Therefore, for any $k$, we have

$$\begin{aligned}
& \lim_{k\to\infty} \Big\| \frac{1}{k} \sum_{\ell=1}^{k} \Big( \nabla_{(x,y)} f(x^k, y^k, \omega_\ell) - \nabla_{(x,y)} f(x^*, y^*, \omega_\ell) \Big) \Big\| \\
\leq\ & \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \Big\| \nabla_{(x,y)} f(x^k, y^k, \omega_\ell) - \nabla_{(x,y)} f(x^*, y^*, \omega_\ell) \Big\| \\
\leq\ & \lim_{k\to\infty} \kappa \Big( \|x^k - x^*\| + \|y^k - y^*\| \Big) \\
=\ & 0.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} f(x^k, y^k, \omega_\ell) &= \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} f(x^*, y^*, \omega_\ell) \\
&= \int_\Omega \nabla_{(x,y)} f(x^*, y^*, \omega)\, d\zeta(\omega) \qquad \text{w.p.1}, \qquad (4.28)
\end{aligned}$$

where the last equality follows from (3.1). Moreover, since

$$\begin{aligned}
& \lim_{k\to\infty} \Big\| \frac{1}{k} \sum_{\ell=1}^{k} \Big( \nabla_{(x,y)} F(x^k, y^k, \omega_\ell) u(x^k, y^k, \omega_\ell) - \nabla_{(x,y)} F(x^*, y^*, \omega_\ell) u(x^*, y^*, \omega_\ell) \Big) \Big\| \\
\leq\ & \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \Big\| \nabla_{(x,y)} F(x^k, y^k, \omega_\ell) \Big( u(x^k, y^k, \omega_\ell) - u(x^*, y^*, \omega_\ell) \Big) \\
& \qquad + \Big( \nabla_{(x,y)} F(x^k, y^k, \omega_\ell) - \nabla_{(x,y)} F(x^*, y^*, \omega_\ell) \Big) u(x^*, y^*, \omega_\ell) \Big\| \\
\leq\ & \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \Big( \|\nabla_{(x,y)} F(x^k, y^k, \omega_\ell)\| \, \|F(x^k, y^k, \omega_\ell) - F(x^*, y^*, \omega_\ell)\| \\
& \qquad + \|\nabla_{(x,y)} F(x^k, y^k, \omega_\ell) - \nabla_{(x,y)} F(x^*, y^*, \omega_\ell)\| \, \|u(x^*, y^*, \omega_\ell)\| \Big) \\
\leq\ & \lim_{k\to\infty} 2\kappa\, C \Big( \|x^k - x^*\| + \|y^k - y^*\| \Big) \\
=\ & 0,
\end{aligned}$$

where $C > 0$ is an upper bounded of $\{\nabla_{(x,y)} F(x^k, y^k, \omega_\ell)\}$ and $\{u(x^*, y^*, \omega_\ell)\}$, there holds

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} F(x^k, y^k, \omega_\ell) u(x^k, y^k, \omega_\ell) &= \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} F(x^*, y^*, \omega_\ell) u(x^*, y^*, \omega_\ell) \\
&= \int_\Omega \nabla_{(x,y)} F(x^*, y^*, \omega) u(x^*, y^*, \omega)\, d\zeta(\omega) \qquad (4.29)
\end{aligned}$$

with probability one. In a similar way, we can show that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\rho_k}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} F(x^k, y^k, \omega_\ell)\, \mathrm{diag}(y_1^k, \cdots, y_m^k)\, (y^k \circ v(x^k, y^k, \omega_\ell)) \\
= &\lim_{k\to\infty} \frac{\rho_k}{k} \sum_{\ell=1}^{k} \nabla_{(x,y)} F(x^*, y^*, \omega_\ell)\, \mathrm{diag}(y_1^*, \cdots, y_m^*)\, (y^* \circ v(x^*, y^*, \omega_\ell)) \\
= &\ \bar{\rho} \int_\Omega \nabla_{(x,y)} F(x^*, y^*, \omega)\, \mathrm{diag}(y_1^*, \cdots, y_m^*)\, (y^* \circ v(x^*, y^*, \omega))\, d\zeta(\omega) \qquad \text{w.p.1} \qquad (4.30)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\rho_k}{k} \sum_{\ell=1}^{k} \mathrm{diag}(v_1(x^k, y^k, \omega_\ell), \cdots, v_m(x^k, y^k, \omega_\ell))\, (y^k \circ v(x^k, y^k, \omega_\ell)) \\
= &\lim_{k\to\infty} \frac{\rho_k}{k} \sum_{\ell=1}^{k} \mathrm{diag}(v_1(x^*, y^*, \omega_\ell), \cdots, v_m(x^*, y^*, \omega_\ell))\, (y^* \circ v(x^*, y^*, \omega_\ell)) \\
= &\ \bar{\rho} \int_\Omega \mathrm{diag}(v_1(x^*, y^*, \omega), \cdots, v_m(x^*, y^*, \omega))\, (y^* \circ v(x^*, y^*, \omega))\, d\zeta(\omega) \qquad \text{w.p.1.} \qquad (4.31)
\end{aligned}$$

It follows from (4.23)-(4.24) and (4.28)-(4.31) that both $\{\, \nabla g(x^k)\alpha^k + \nabla h(x^k)\beta^k \,\}$ and $\{\gamma^k\}$ are almost surely bounded. We next show that $\{\alpha^k\}$ and $\{\beta^k\}$ are almost surely bounded. To this end, let

$$\tau_k := \sum_{i=1}^{s_1} \alpha_i^k + \sum_{j=1}^{s_2} |\beta_j^k|. \qquad (4.32)$$

Taking a subsequence if necessary, we may assume that the limits

$$\bar{\alpha} := \lim_{k\to\infty} \frac{\alpha^k}{\tau_k}, \qquad \bar{\beta} := \lim_{k\to\infty} \frac{\beta^k}{\tau_k} \qquad (4.33)$$

exist. It is obvious from (4.32) that

$$\sum_{i=1}^{s_1} \bar{\alpha}_i + \sum_{j=1}^{s_2} |\bar{\beta}_j| = 1. \qquad (4.34)$$

Suppose that $\{\alpha^k\}$ or $\{\beta^k\}$ is unbounded with probability one. There almost surely holds $\lim\limits_{k\to\infty} \tau_k = +\infty$. Dividing $(\nabla g(x^k)\alpha^k + \nabla h(x^k)\beta^k)$ by $\tau_k$ and taking a limit, we get

$$\nabla g(x^*)\bar{\alpha} + \nabla h(x^*)\bar{\beta} = 0 \qquad \text{w.p.1.} \qquad (4.35)$$

Note that, if $i \notin \mathcal{I}_g(x^*)$, then $g_i(x^k) < 0$ for all $k$ sufficiently large and hence, by (4.25) and (4.33), we have $\bar{\alpha}_i = 0$. Thus, (4.35) becomes

$$\sum_{i\in\mathcal{I}_g(x^*)} \bar{\alpha}_i \nabla g_i(x^*) + \nabla h(x^*)\bar{\beta} = 0 \qquad \text{w.p.1.}$$

This together with (4.34) almost surely contradicts the assumption that the system $\{g(x) \le 0,\ h(x) = 0\}$ satisfies the Mangasarian-Fromovitz constraint qualification at $x^*$. Therefore, both $\{\alpha^k\}$ and $\{\beta^k\}$ are bounded with probability one. Recalling that $\{\gamma^k\}$ is almost surely bounded, we may assume without loss of generality that the following limits exist:

$$\alpha^* := \lim_{k\to\infty} \alpha^k, \quad \beta^* := \lim_{k\to\infty} \beta^k, \quad \gamma^* := \lim_{k\to\infty} \gamma^k.$$

Furthermore, we define the function $\delta^* : \Omega \to \Re^m$ by

$$\delta^*(\omega) := 2\,\bar{\rho}\, y^* \circ v(x^*, y^*, \omega).$$

Note that, by the definition (3.3), $(v_i(x^*, y^*, \omega))^2 = v_i(x^*, y^*, \omega)F_i(x^*, y^*, \omega)$ holds for each $i$ and $\omega$. It then follows that

$$\begin{aligned}
& \int_\Omega \mathrm{diag}(v_1(x^*, y^*, \omega), \cdots, v_m(x^*, y^*, \omega))\,(y^* \circ v(x^*, y^*, \omega))\,d\zeta(\omega) \\
= & \int_\Omega \mathrm{diag}(F_1(x^*, y^*, \omega), \cdots, F_m(x^*, y^*, \omega))\,(y^* \circ v(x^*, y^*, \omega))\,d\zeta(\omega).
\end{aligned}$$

In consequence, taking a limit in (4.23) and (4.24), we obtain (2.1) and (2.2) from (4.28)-(4.31) and (2.7)-(2.8) with probability one. Moreover, we have (2.3)-(2.5) from (4.25)-(4.27) immediately. In addition, it is obvious that $\delta_i^*(\omega) \geq 0$ for any $\omega \in \Omega$. Since $y_i^* F_i(x^*, y^*, \omega) \leq \epsilon$ for each $i$ and almost every $\omega \in \Omega$, we have (4.22) if $y_i^* F_i(x^*, y^*, \omega) \geq -\epsilon$. When $y_i^* F_i(x^*, y^*, \omega) < -\epsilon$, there must hold $F_i(x^*, y^*, \omega) < 0$ and hence $v_i(x^*, y^*, \omega) = 0$ by (3.3), which in turn implies $\delta_i^*(\omega) = 0$ by the definition of $\delta^*$. This indicates that the condition (4.22) is also valid.

Therefore, $(\alpha^*, \beta^*, \gamma^*, \delta^*(\cdot))$ satisfies (2.1)-(2.5) and (4.22) with probability one and hence $(x^*, y^*)$ is almost surely $\epsilon$-stationary to problem (1.3). ■

# 5 Numerical Examples

Given a mapping $F : \Re^m \times \Omega \to \Re^m$, we consider the stochastic complementarity problem

$$0 \leq y \perp F(y, \omega) \geq 0, \qquad \omega \in \Omega. \tag{5.1}$$

In [12], by introducing a recourse variable, we reformulate (5.1) as an SMPEC (1.1) and propose a smoothed penalty method for the case with a finite sample space. As an application of the new model (1.2), we consider the following SMPEC formulation of (5.1):

$$\begin{array}{ll}
\min & \mathbb{E}[\|z(\omega)\|^2] \\
\text{s.t.} & 0 \leq y \perp F(y, \omega) + z(\omega) \geq 0, \\
& z(\omega) \geq 0, \quad \omega \in \Omega.
\end{array} \tag{5.2}$$

Thus, we may employ the method proposed in Section 3 to solve problem (5.2).

**Example 5.1** Consider the stochastic complementarity problem (5.1) in which $\omega$ is uniformly distributed on $\Omega := [0, 1]$ and $F : \Re^3 \times \Omega \to \Re^3$ is given by

$$F(y, \omega) := \begin{pmatrix} y_1 - \omega y_2 + 3 - 2\omega \\ -\omega y_1 + 2y_2 + \omega y_3 - 2 - \omega \\ \omega y_2 + 3y_3 - 3 - \omega \end{pmatrix}.$$

Problem (5.1) has a unique solution $y^* = (0, 1, 1)$ for each $\omega \in \Omega$.

**Example 5.2** Consider the stochastic complementarity problem (5.1) in which $\omega$ is uniformly distributed on $\Omega := [0,1]$ and $F : \Re^2 \times \Omega \to \Re^2$ is given by

$$F(y,\omega) := \begin{pmatrix} y_1 + \omega y_2 - 2 + \omega \\ \omega y_1 + 2y_2 + 1 + \omega \end{pmatrix}.$$

This problem has no common solution for all $\omega \in \Omega$. Note that the SMPEC formulation (5.2) becomes

$$\begin{array}{ll} \min & \mathbb{E}[\|z(\omega)\|^2] \\ \text{s.t.} & 0 \le \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \perp \begin{pmatrix} y_1 + \omega y_2 - 2 + \omega \\ \omega y_1 + 2y_2 + 1 + \omega \end{pmatrix} + z(\omega) \ge 0, \\ & z(\omega) \ge 0, \quad \omega \in [0,1]. \end{array} \tag{5.3}$$

Let $(y_1, y_2, z(\cdot))$ be an arbitrary feasible point of (5.3). Since $\omega y_1 + 2y_2 + 1 + \omega > 0$ for $\omega \in [0,1]$, we have $y_2 = 0$. Let $z^*(\omega) := \binom{2-\omega-y_1}{0}$. Note that, if $y_1 \ne 0$, there must hold $y_1 - 2 + \omega + z_1(\omega) = 0$. It follows that $(y_1, y_2, z^*(\cdot))$ is feasible to problem (5.3) and $z(\omega) \ge z^*(\omega) \ge 0$ for any $\omega \in [0,1]$. Furthermore, we have

$$\mathbb{E}[\|z(\omega)\|^2] \ge \mathbb{E}[\|z^*(\omega)\|^2] = \int_0^1 (2-\omega-y_1)^2 d\omega = y_1^2 - 3y_1 + \tfrac{7}{3}.$$

Recall that $y_1 \le 2 - \omega$ for any $\omega \in [0,1]$. Therefore, we must have $y_1 \in [0,1]$. Thus, we obtain an optimal solution $(1, 0, z^*(\cdot))$ of problem (5.3) with $z^*(\omega) := \binom{1-\omega}{0}$.

Table 1: Computational Results for Examples 5.1-5.2

| Parameters | Example 5.1 | | Example 5.2 | |
|---|---|---|---|---|
| | $y^k$ | Obj | $y^k$ | Obj |
| $k = 10^2, \rho_k = 10^2$ | (0,1.0019,0.9489) | 6.8918e-004 | (1.1153,0) | 0.7905 |
| $k = 10^3, \rho_k = 10^3$ | (0,0.9996,0.9982) | 6.8476e-006 | (1.0329,0) | 0.4175 |
| $k = 10^4, \rho_k = 10^4$ | (0,0.9999,0.9985) | 1.3156e-006 | (1.0027,0) | 0.3242 |
| $k = 10^5, \rho_k = 10^5$ | (0,1.0000,0.9995) | 9.9033e-008 | (1.0007,0) | 0.3301 |

We apply the Monte Carlo sampling and penalty method to solve Examples 5.1 and 5.2. In our experiments, we set the initial values of $k$ and $\rho_k$ as $k := 10^2$ and $\rho_k := 10^2$, respectively. Then, we employed the random number generator `rand` in Matlab 6.5 to generate independently and identically distributed random samples $\{\omega_1, \cdots, \omega_k\}$ from $\Omega$ and we solved the subproblems

$$\min_{y \ge 0} \quad \frac{1}{k}\sum_{\ell=1}^{k} \Big( \|\max(-F(y,\omega_\ell),\ 0)\|^2 + \rho_k \, \|y \circ \max(F(y,\omega_\ell),\ 0)\|^2 \Big) \tag{5.4}$$

by the solver `fmincon` in Matlab 6.5 to get a point $y^k$. The initial point was chosen to be $(0, \cdots, 0)$ and the computed solution $y^k$ was used as the starting point in the next iteration. In addition, the parameters were updated by $k := 10k$ and $\rho_k := \min\{10\rho_k,\ \bar\rho\}$ with $\bar\rho = 10^5$. The computational results for Examples 5.1 and 5.2 are shown in Table 1, in which `Obj` denotes the values of the objective function of (5.4) at the current point. The results shown in the table reveal that the proposed method was able to solve the examples successfully.

## 6 Conclusions

We have presented a new formulation (1.2) for the SMPECs with recourse and shown that the new formulation is actually equivalent to a smooth semi-infinite programming problem. We then employed a Monte Carlo sampling method and a penalty technique to get some approximations to the problem. Under appropriate assumptions, we have established convergence of the proposed method. Recall that the sample space $\Omega$ is assumed to have infinitely many elements. Actually, if $\Omega$ has only a finite number of elements, we may present a similar method without resort to a Monte Carlo sampling approximation technique.

## References


[1] S.I. Birbil, G. Gürkan, and O. Listes, *Simulation-based solution of stochastic mathematical programs with complementarity constraints: Sample-path analysis*, Econometric Institute Report EI 2004-03, Erasmus University Rotterdam, 2004.

[2] J.R. Birge and F. Louveaux, Introduction to Stochastic Programming, Springer, New York, 1997.

[3] X. Chen and M. Fukushima, *Expected residual minimization method for stochastic linear complementarity problems*, Mathematics of Operations Research, to appear.

[4] R.W. Cottle, J.S. Pang, and R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, NY, 1992.

[5] F. Facchinei and J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Part I, Springer-Verlag, New York, 2003.

[6] G. Gürkan, A.Y. Özge and S.M. Robinson, *Sample-path solution of stochastic variational inequalities*, Mathematical Programming, 84 (1999), 313–333.

[7] A. Haurie and F. Moresino, *S-adapted oligopoly equilibria and approximations in stochastic variational inequalities, with application to option pricing*, Annals of Operations Research, 114 (2002), 183–201.

[8] R. Hettich and K.O. Kortanek, *Semi-infinite programming: Theory, methods, and applications*, SIAM Review, 35 (1993), 380–429.

[9] G.H. Lin, X. Chen and M. Fukushima, *Smoothing implicit programming approaches for stochastic mathematical programs with linear complementarity constraints*, Technical Report 2003-006, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2003.

[10] G.H. Lin, X. Chen and M. Fukushima, *Solving stochastic mathematical programs with equilibrium constraints via approximation and smoothing implicit programming with penaliza-*

*tion*, Technical Report 2005-003, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2005.

[11] G.H. Lin and M. Fukushima, *A class of stochastic mathematical programs with complementarity constraints: Reformulations and algorithms*, Journal of Industrial and Management Optimization, 1 (2005), 99–122.

[12] G.H. Lin and M. Fukushima, *New reformulations for stochastic complementarity problems*, Technical Report 2005-004, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan, 2005.

[13] G.H. Lin and M. Fukushima, *Regularization method for stochastic mathematical programs with complementarity constraints*, European Series of Applied and Industrial Mathematics (ESAIM): Control, Optimisation and Calculus of Variations, 11 (2005), 252–265.

[14] Z.Q. Luo, J.S. Pang, and D. Ralph, Mathematical Programs with Equilibrium Constraints, Cambridge University Press, Cambridge, United Kingdom, 1996.

[15] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, SIAM, Philadelphia, 1992.

[16] M. Patriksson and L. Wynter, *Stochastic mathematical programs with equilibrium constraints*, Operations Research Letters, 25 (1999), 159–167.

[17] A. Shapiro, *Monte Carlo sampling approach to stochastic programming*, European Series of Applied and Industrial Mathematics (ESAIM): Proceedings, 13 (2003), 65–73.

[18] A. Shapiro, *Stochastic mathematical programs with equilibrium constraints*, Preprint, School of Industrial and System Engineering, Georgia Institute of Technology, Atlanta, Georgia, USA, 2004.

[19] A. Shapiro and H. Xu, *Stochastic mathematical programs with equilibrium constraints, modeling and sample average approximation*, Preprint, School of Industrial and System Engineering, Georgia Institute of Technology, Atlanta, Georgia, USA, 2005.

[20] H. Xu, *An implicit programming approach for a class of stochastic mathematical programs with linear complementarity constraints*, SIAM Journal of Optimization, to appear.

**Appendix: Equivalence between problems (1.2) and (1.3).** *If* $(x^*, y^*)$ *solves problem (1.3), then* $(x^*, y^*, u(x^*, y^*, \cdot))$ *is an optimal solution of problem (1.2). Conversely, if* $(x^*, y^*, z^*(\cdot))$ *is an optimal solution of problem (1.2), then* $(x^*, y^*)$ *solves problem (1.3).*

**Proof.** (i) Suppose that $(x^*, y^*)$ is an optimal solution of (1.3). We then have from (1.4) that

$$F(x^*, y^*, \omega) + u(x^*, y^*, \omega) \geq 0, \qquad \forall \omega \in \Omega.$$

Note that, if $y_i^* > 0$ for some $i$, there must hold $F_i(x^*, y^*, \omega) \le 0$ for almost all $\omega \in \Omega$ and so $u_i(x^*, y^*, \omega) = -F_i(x^*, y^*, \omega)$ for almost all $\omega \in \Omega$. Therefore, we have

$$(y^*)^T (F(x^*, y^*, \omega) + u(x^*, y^*, \omega)) = 0, \quad \omega \in \Omega \text{ a.e.}$$

This indicates that $(x^*, y^*, u(x^*, y^*, \cdot))$ is feasible to problem (1.2). Let $(x, y, z(\cdot))$ be an arbitrary feasible point of problem (1.2). It then follows that, for almost every $\omega \in \Omega$,

$$z(\omega) - u(x, y, \omega) = \min\{F(x, y, \omega) + z(\omega),\ z(\omega)\} \ge 0$$

and hence $z(\omega) \ge u(x, y, \omega) \ge 0$. This implies that $\mathbb{E}[\, \|z(\omega)\|^2 - \|u(x, y, \omega)\|^2 \,] \ge 0$. On the other hand, it follows from the feasibility of $(x, y, z(\cdot))$ in problem (1.2) that

$$y \circ F(x, y, \omega) = -y \circ z(\omega) \le 0, \quad \omega \in \Omega \text{ a.e.},$$

and so the point $(x, y)$ is a feasible point of problem (1.3). Thus, we have from the optimality of $(x^*, y^*)$ in (1.3) that

$$\mathbb{E}[f(x, y, \omega) + \sigma\|u(x, y, \omega)\|^2] \ \ge \ \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|u(x^*, y^*, \omega)\|^2].$$

Therefore, there holds

$$\begin{aligned}
& \mathbb{E}[f(x, y, \omega) + \sigma\|z(\omega)\|^2] - \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|u(x^*, y^*, \omega)\|^2] \\
= \ & \mathbb{E}[f(x, y, \omega) + \sigma\|u(x, y, \omega)\|^2] - \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|u(x^*, y^*, \omega)\|^2] + \sigma\mathbb{E}[\|z(\omega)\|^2 - \|u(x, y, \omega)\|^2] \\
\ge \ & 0.
\end{aligned}$$

This indicates that $(x^*, y^*, u(x^*, y^*, \cdot))$ is an optimal solution of problem (1.2).

(ii) Suppose that $(x^*, y^*, z^*(\cdot))$ is an optimal solution of (1.2). Let $(x, y)$ be an arbitrary feasible point of (1.3). In a similar way to (i), we can show that $(x, y, u(x, y, \cdot))$ and $(x^*, y^*)$ are feasible to problems (1.2) and (1.3), respectively. Since $(x^*, y^*, z^*(\cdot))$ solves (1.2), there holds

$$\mathbb{E}[f(x, y, \omega) + \sigma\|u(x, y, \omega)\|^2] \ \ge \ \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|z^*(\omega)\|^2].$$

Moreover, similarly to (a), we have from (1.4) that $z^*(\omega) \ge u(x^*, y^*, \omega) \ge 0$ for almost all $\omega \in \Omega$ and hence $\mathbb{E}[\, \|z^*(\omega)\|^2 - \|u(x^*, y^*, \omega)\|^2 \,] \ge 0$. Therefore,

$$\begin{aligned}
& \mathbb{E}[f(x, y, \omega) + \sigma\|u(x, y, \omega)\|^2] - \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|u(x^*, y^*, \omega)\|^2] \\
= \ & \mathbb{E}[f(x, y, \omega) + \sigma\|u(x, y, \omega)\|^2] - \mathbb{E}[f(x^*, y^*, \omega) + \sigma\|z^*(\omega)\|^2] + \sigma\mathbb{E}[\|z^*(\omega)\|^2 - \|u(x^*, y^*, \omega)\|^2] \\
\ge \ & 0,
\end{aligned}$$

which implies that $(x^*, y^*)$ is an optimal solution of problem (1.3). ■