An Optimal Design of Collateralized Mortgage Obligation with PAC-Companion Structure Using Dynamic Cash Reserve

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Abstract

This paper presents a model for optimally designing a *collateralized mortgage obligation* (CMO) with a *planned amortization class* (PAC)-companion structure using dynamic cash reserve. In this structure, the mortgage pool's cash flow is allocated by rule to the two bond classes such that PAC bondholders receive substantial prepayment protection, that protection being provided by the companion bondholders. The structure we propose provides greater protection to the PAC bondholders than current structures during periods of rising interest rates when this class of bondholders faces greater extension risk. We do so by allowing a portion of the cash flow from the collateral to be reserved to meet the PAC's scheduled cash flow in subsequent periods. The greater protection is provided by the companion bondholders exposure to interest loss. To tackle this problem, we transform the problem of designing the optimal PAC-companion structure into a standard stochastic linear programming problem which can be solved efficiently. Moreover, we present an extended model by considering the quality of the companion bond and by relaxing the PAC bondholder shortfall constraint. Based on numerical experiments through Monte Carlo simulation, we show the utility of the proposed model.

Keywords: Stochastic programming; Finance; Linear programming; Simulation
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1 Introduction

The residential mortgage market in most countries is typically the largest debt sector. The development of a strong housing finance market requires the participation of investors who are willing to hold residential mortgage loans as either whole loans or in the form of a security. The best developed housing finance market in the world is that of the United States because of the ability of investment bankers in conjunction with government agencies to create securities backed by a pool of residential mortgage loans that are more appealing for institutional investors to hold rather whole loans. These securities, referred to as *mortgage-backed securities* (MBS), were first issued in 1969 with the process of creating these securities referred to as securitization. More specifically, the MBS issued at the time were mortgage passthrough securities, securities where the cash flow of the pool of residential mortgage loans (amortization, prepayments, and interest after servicing expenses and any guarantee fees) is prorated among the certificate holders. The uncertainty about the cash flow to investors in a mortgage passthrough security is due to prepayments (i.e., payments in excess of the principal payment due to amortization) (Fabozzi, 1995).

The major problem with investing in a mortgage passthrough security is that because of prepayments and the associated uncertainty about the cash flow, these securities do not provide an effective matching of assets and liabilities for institutional investors. In fact, it is fair to say that there are few institutional investors who found mortgage passthrough securities attractive from an asset-liability perspective, thereby limiting the institutional appeal of these securities. The problem was that there was considerable uncertainty about the security's average life. It could be shorter than expected when the security was purchased (a risk referred to as contraction risk) or longer than expected (a risk referred to as extension risk).

To deal with this problem, in the early 1980s investment bankers created a different type of MBS. Instead of an MBS that distributes the cash flow from the pool of mortgage loans on a pro rata basis, the cash flow is distributed according to rules for principal (both amortization and prepayments) and interest to different bond classes (tranches) in the structure. This type of MBS is called a *collateralized mortgage obligation* (CMO). The initial structures created provided some bond classes with reduced risk of one type of prepayment (extension or contraction) with the protection provided by another bond class that realized a reduction in the risk of the other type of prepayment risk.

The key innovation in the CMO market was the creation of a *planned amortization class* (PAC) and companion bond class. In a structure with a PAC and a companion bond class, the PAC bondholders are provided significant protection against prepayment risk (both extension and contraction risk). The protection provided to the PAC bondholders was supported by the companion bondholders (hence the reference to these types of bonds as "support bonds"). The rule for the distribution of principal to the PAC bondholders is as follows. There is a schedule of principal payments that must be made to the bondholders each month. The PAC bondholders are given priority over the support bondholders with respect to the payment of the scheduled PAC amount (Fabozzi and Ramsey, 1999).

In the creation of a CMO, the issuer seeks to maximize the proceeds from the sale of the bond classes backed by the pool of mortgages. Equivalently, the issuer seeks to minimize the weighted average yield at which it offers the bond classes. When creating a CMO with a PAC-companion structure, the issuer would like to maximize the amount of the PAC bonds offered because they can be sold at a lower yield than no-PAC bonds due to their reduced prepayment risk; in contrast, the companion bonds must offer a higher yield to compensate for the substantial prepayment risk. The trade off is that for a given pool of mortgages, the greater the percentage of the PAC bonds, the less prepayment protection the PAC bonds are afforded.

In the United States, a PAC-companion structure is created by providing an upper and lower collar prepayment rate. The prepayment rates are based on a prepayment benchmark developed in the 1980s by the Public Security Association (now the Bond Market Association). However, it is possible to create a scheduled payment (i.e., the PAC schedule) based on some other type of benchmark and allow for cash reserves to protect the PAC bondholders against prepayment risk. Kutsuna et al. (2004), based on a prepayment model developed by Schwartz and Torous' prepayment model (1989), present a new method of optimally designing a CMO with PAC-companion structure using cash reserves. If there are principal payments in excess of the scheduled amount in a period due to prepayments generated by the collateral, then the maximum cash that does not exceed the upper bound of cash reserve is reserved, and finally, if any, the amount that exceeds the upper bound of cash reserve is distributed to the companion bondholders. If there is a shortage in the PAC scheduled cash flow generated by the collateral, the PAC bondholders would realize a shortfall whose total amount is kept in a reasonable range this bondholders can accept. Numerical experiments using Monte Carlo simulation confirmed the feasibility and validity of their approach.

In this paper, we propose an alternative method for optimally designing a CMO with a PAC-companion structure. We do so by introducing a dynamic decision process for the upper bound of the cash reserve, which is different from Kutsuna et al. (2004). Strictly speaking, the amount of cash reserve is highly correlated with the past cash flow and the future prepayment behavior. Hence, it will be more reasonable to assume that the amount of the cash reserve is time-varying and dependent on the actual nature of the prepayment cash flow. In contrast with the fixed upper bound assumed by Kutsuna et al. (2004), we will investigate the dynamic approach that takes into account past information about cash flow and future payment process.

The structure presented here reduces the extension risk for the PAC bondholders, a particular concern to investors during a period of rising interest rates. While the structure does not expose the companion bondholders to risk of loss of principal, the increased risk faced by the companion bondholders is the risk that there will be a loss in the interest paid to them. The model makes this potential loss as small as possible.

Another contribution of the model proposed in this paper is that the structurer (designer) of the CMO can try to improve the quality of the companion bond by making the absolute deviation of this bond's cash flow small under the condition that the scheduled principal payment for the PAC bondholders can be satisfied.

Specifically, we transform the design problem of a CMO into a standard stochastic linear programming problem which maximizes the objective function formulated by the CMO structurer under PAC bondholder shortfall and cash reserve constraints. That is, the structurer aims at maximizing the amount of the PAC bond under the condition that the sum of the cash flow from the mortgagors and the cash reserve brought forward from the previous term can cover the PAC bond with high certainty. Moreover, it is possible to perform sensitivity analysis with various risk aversion levels. This approach may provide a new efficient and flexible vehicle to optimize and price MBS and its derivatives.

In summary, we formulate the following optimization problem:

$$\begin{aligned} \max \quad & W(\mathbf{a}) - \lambda R(\mathbf{v}) \\ \text{s.t.} \quad & L(\mathbf{C}, \mathbf{V}, \mathbf{a}) \leq U_L, \\ & \mathbf{a} \geq 0, \ \mathbf{v} \geq 0, \end{aligned}$$

where λ is a coefficient used to tradeoff the value of the PAC bond $W(\mathbf{a})$ and the total cash reserve $R(\mathbf{v})$, and U_L is a given threshold for the total PAC bondholder shortfall $L(\mathbf{C}, \mathbf{V}, \mathbf{a})$.

Based on these considerations, the organization of the paper is as follows. In Section 2 we propose a generic mathematical programming model for designing a CMO with PAC-companion structure which allows a dynamic cash reserve at each term due on repayment. In Section 3 we transform the model in Section 2 into a standard stochastic linear programming problem which can be solved efficiently by the simplex method or the interior-point method. To improve the tractability of the model, we present in Section 4 an extended model which may serve as a more reasonable model for the issuer of the CMO. In Section 5 we show some numerical experiments with Monte Carlo simulation about the basic model and the extended model. In Section 6 we conclude with a brief discussion of future research.

2 Modelling A CMO with PAC-Companion Structure

In this section, we address the generic targets along with the constraint equalities and inequalities of a CMO structurer which will be integrated into a mathematical programming problem later.

2.1 Parameters and Variables

We first define some parameters and variables which will be used throughout the paper.
T: maturity of the final payout of the longest maturity loan in the mortgage pool;
$\bar{x}_t(t = 0, 1, ..., T)$: remaining principal balance at time t without prepayments;
$\pi_t(t = 0, 1, ..., T-1)$: the probability that a mortgagor will prepay at time $t+1$ conditional on no prepayments up to time t;
$X_t(t = 0, 1, ..., T)$: actual remaining principal balance at time t (random variable);
$C_t(t = 1, ..., T)$: cash flow at time t (includes amortization, prepayments, and interest [In this paper we assume there are no service and guarantee fees.]) (random variable);
$V_t(t = 1, ..., T)$: actual cash reserve at time t (random variable);
$a_t(t = 1, ..., T)$: scheduled PAC cash flow at time t (decision variable);
$A_t(t = 1, ..., T)$: actual PAC cash flow at time t (random variable);
$b_t(t = 1, ..., T)$: payment to the companion bondholders after paying the PAC bondholders at time t (dependent variable);
$B_t(t = 1, ..., T)$: actual payment to the companion bondholders at time t (random variable);
$v_t(t = 1, ..., T)$: upper bound of the cash reserve at time t (decision variable).
We also use the vector notations $\mathbf{C}_t = (C_1, ..., C_t)$, $\mathbf{C} = \mathbf{C}_T$, $\mathbf{V}_t = (V_1, ..., V_t)$, $\mathbf{V} = \mathbf{V}_T$, $\mathbf{a}_t = (a_1, ..., a_t)$, $\mathbf{a} = \mathbf{a}_T$, $\mathbf{A}_t = (A_1, ..., A_t)$, $\mathbf{A} = \mathbf{A}_T$, $\mathbf{v}_t = (v_1, ..., v_t)$ and $\mathbf{v} = \mathbf{v}_T$.

2.2 Basic Framework and Optimization Problem

Given a maturity T, a mortgage loan $\bar{x}_0$ is specified by a fixed coupon rate c and a regular cash flow y (i.e., sum of the regularly scheduled principal repayment and interest from the mortgage pool). If the mortgagors do not prepay, they will remit y at each due payment date until maturity. Thus we have the relation

$$x_0 = \sum_{t=1}^{T} \frac{y}{(1+c)^t},$$

i.e.,

$$y = x_0 \cdot \frac{c(1+c)^T}{(1+c)^T - 1}.$$

Then, the remaining principal balance $\bar{x}_t$ at each time $t \in \{0, 1, ..., T\}$ without prepayments is computed as the discounted future cash flows at the initial coupon rate, that is,

$$\bar{x}_t = \sum_{k=t+1}^{T} \frac{y}{(1+c)^{k-t}} = \bar{x}_0 \cdot \frac{(1+c)^T - (1+c)^t}{(1+c)^T - 1}.$$

Effectively, the mortgagors own an American call option on the underlying mortgage with a time-varying exercise price $\bar{x}_t$. The lower the current interest rate is, the lower the refinancing rate, and hence the more valuable the call option.

As to the PAC bondholders, they can receive a scheduled cash flow at each time t to a certain extent since the payment for the PAC bonds is by rule allocated out of the sum of repayment from mortgagors and cash reserve from the previous period. Suppose there were no prepayments and the PAC's scheduled cash flow is a_t at time t. Then the holders of the companion bond would receive $b_t = y - a_t$.

However, due to prepayments motivated by the fluctuation of refinancing rate and exogenous factors, the cash flow from mortgagors is not known with certainty, in contrast to amortization (ignoring defaults). From this perspective, for each $t = 1, ..., T$, we have the following relations with $V_0 = V_T = 0$:

$$\begin{aligned}
A_t(\mathbf{a}_t, \mathbf{v}_{t-1}, \mathbf{C}_t) &= \min\{a_t, C_t + V_{t-1}(\mathbf{a}_{t-1}, \mathbf{v}_{t-1}, \mathbf{C}_{t-1})\}, \\
V_t(\mathbf{a}_t, \mathbf{v}_t, \mathbf{C}_t) &= \min\{v_t, C_t + V_{t-1}(\mathbf{a}_{t-1}, \mathbf{v}_{t-1}, \mathbf{C}_{t-1}) - A_t(\mathbf{a}_t, \mathbf{v}_{t-1}, \mathbf{C}_t)\}, \\
B_t(\mathbf{a}_t, \mathbf{v}_t, \mathbf{C}_t) &= C_t + V_{t-1}(\mathbf{a}_{t-1}, \mathbf{v}_{t-1}, \mathbf{C}_{t-1}) - A_t(\mathbf{a}_t, \mathbf{v}_{t-1}, \mathbf{C}_t) - V_t(\mathbf{a}_t, \mathbf{v}_t, \mathbf{C}_t).
\end{aligned}$$

For simplicity, we will abbreviate $A_t(\mathbf{a}_t, \mathbf{v}_{t-1}, \mathbf{C}_t)$, $V_t(\mathbf{a}_t, \mathbf{v}_t, \mathbf{C}_t)$ and $B_t(\mathbf{a}_t, \mathbf{v}_t, \mathbf{C}_t)$ as A_t, V_t and B_t, respectively.

Needless to say, the PAC bonds are more preferred in terms of lower prepayment risk than the corresponding companion bond, since the PAC's scheduled cash flow remains within a relatively regular range according to a preset schedule. Thus, it can be offered at a lower yield. So the first target of the CMO structurer is to maximize the amount of the PAC bonds issued subject to other considerations. Let $\dot{r}$ be the coupon rate for the PAC bond to be issued. Then the present value of the PAC bonds is defined as

$$W(\mathbf{a}) = \sum_{t=1}^{T} \frac{a_t}{(1+\dot{r})^t} = \sum_{t=1}^{T} \alpha_t \cdot a_t,$$

where $\alpha_t = \frac{1}{(1+\dot{r})^t}$ is the discount factor at time t.

At the same time, the trustee for the *special purpose vehicle* (SPV) created to purchase the collateral, issue the bonds, and distribute the cash flow according to the priority rules specified, will be accumulating a cash reserve over time. The cash reserve comes from a portion of the principal payments that are in excess of the scheduled principal paid to the PAC bondholders. One economic consequence of the cash reserve for the companion bondholders is the postponement of the receipt of principal. Assuming no defaults as we do in this paper, the principal owed to the companion bondholders will be delayed but ultimately paid out.

The risk from the perspective of the companion bondholders is that the interest earned on the cash reserve for period t will be less than the coupon rate to be paid to them. As a result, while the companion bondholders will eventually recover their entire principal due, there may be an interest loss during the postponement period. The factors that will affect that risk are changes in the level of interest rates and changes in the shape of the yield curve over the life of the CMO. Accordingly, the second goal of the structurer should be to decrease the amount of the cash reserve. However, how to set the upper bound of the cash reserve for the CMO structurer is complicated due to the uncertainty of prepayments made by the mortgagors in the mortgage pool. The upper bound will depend on the expectation of the next period's repayment from the mortgagors, the scheduled amount for the PAC bondholders, and the past information about prepayments. If the mortgagors prepay at a high rate in the early life of the mortgage loan, then the prepayment speed in later periods must be considerably slower. Moreover, if the estimated prepayments occur at a high rate in the next period, the structurer need not design much cash reserve since the cash flow from the mortgagors can meet the schedule for the PAC bondholders. However, since appropriate data in respect to the design of the cash reserve in a CMO structure are not available in the emerging mortgage markets, it is difficult to grasp the actual distribution of cash flows. In this paper, to develop a general model, we assume that the upper bound of the cash reserve is represented as

$$v_t = \tau_t - \xi_t \pi_t X_t,$$

where τ_t and $\xi_t (t = 0, 1, ..., T-1)$ are parameters determined by the structurer of the CMO. This model incorporates the following rule: The higher the estimated prepayment $\pi_t X_t$ at time $t+1$, the lower the upper bound v_t at time t. Thus, the total cash reserve function for the structurer can be defined as

$$R(\mathbf{v}) = \mathbf{E}\left[\sum_{t=1}^{T-1} \frac{v_t}{(1+\dot{r})^t}\right] = \mathbf{E}\left[\sum_{t=1}^{T-1} \alpha_t(\tau_t - \xi_t \pi_t X_t)\right], \tag{1}$$

where $\mathbf{E}(\cdot)$ denotes the expectation.

Now, we consider a CMO with a PAC-companion structure that operates under the following rules:

1. Pay the PAC bonds as scheduled.

2. Set the upper bound of the cash reserve in accordance with the pattern of the prior prepayments, and reserve the maximum amount of cash that does not exceed the upper bound.

3. If cash flow is available, pay the companion bond if the payment exceeds the upper bound of the cash reserve.

Based on the above rules, the structurer determines the amount of the PAC's scheduled cash flow and the upper bound of the cash reserve. Therefore, at each time t, the first and the most important requirement for the structurer is to ensure that the PAC's scheduled cash flow satisfies the following condition:

$$C_t + V_{t-1} \geq a_t \quad \text{with high probability.} \tag{2}$$

When inequality (2) is violated, there is a PAC bondholder shortfall that the PAC bondholders will realize. As a measure of this shortfall, we use the difference between the right-hand side and left-hand side in inequality (2). More specifically, the PAC bondholder shortfall function L_t is defined by

$$\begin{aligned} L_t(\mathbf{C}_t, \mathbf{V}_{t-1}, \mathbf{a}_t) &= \max\{0, a_t - C_t - V_{t-1}\} \\ &= a_t - A_t \\ &\geq 0. \end{aligned} \tag{3}$$

Then since (3) is rewritten as

$$A_t = a_t - L_t,$$

we may replace A_t by L_t in V_t and B_t, and get

$$\begin{aligned} V_t &= \min\{v_t, C_t + V_{t-1} + L_t - a_t\}, \\ B_t &= C_t + V_{t-1} + L_t - a_t - V_t. \end{aligned}$$

Moreover, the total PAC bondholder shortfall function is defined as

$$L(\mathbf{C}, \mathbf{V}, \mathbf{a}) = \mathbf{E}\left[\sum_{t=1}^{T} \frac{L_t}{(1+\dot{r})^t}\right] = \mathbf{E}\left[\sum_{t=1}^{T} \alpha_t \cdot L_t\right]. \tag{4}$$

Of course, for the PAC bondholders, it is not acceptable that the total PAC bondholder shortfall exceeds a certain threshold, and hence the following condition is imposed:

$$L(\mathbf{C}, \mathbf{V}, \mathbf{a}) \leq U_L,$$

where U_L is a given threshold value.

2.3 Mortgage Prepayment Model

Unlike the standard PAC-companion structure created in United States where the PSA prepayment benchmark is used to establish the prepayment collars in creating the PAC bond schedule, the structure we are describing can be used in any country interested in creating this product. This means that in any country, the structurer can adopt our optimal design using a specified prepayment model to simulate the cash flows. To illustrate the CMO structure we propose in this paper, we employ the Schwartz and Torous' prepayment model (1989).

Their prepayment model represents an empirical estimation of the mortgagors' refinancing decision. It tries to explain prepayments from the observed actual prepayment behaviors and relates the prepayment rate to the measurable factors suggested by their economic theory of prepayments. Given the heterogeneous mortgagors, Schwartz and Torous specify the prepayment rate π_t involving several factors. For simplicity, we adopt the same prepayment model as that of Kutsuna et al. (2004), which is calculated as follows:

$$\begin{aligned}
\pi_t &= \hat{\pi}_t \cdot \exp\{\beta_1 k_t + \beta_2 l_t + \beta_3 m_t\}, \\
\hat{\pi}_t &= b\frac{w\nu(wt)^{\nu-1}}{1+(wt)^{\nu}}, \\
k_t &= r_0 - r_{t-s}, \\
l_t &= (k_t)^3, \\
m_t &= \ln\frac{x_t}{\bar{x}_t},
\end{aligned}$$

where
$\beta_i (i = 1, 2, 3), b, w, \nu$: constant parameters;
$\hat{\pi}_t$: baseline function;
s: time lagged factor (in this paper $s = 0$);
k_t: interest rate spread between the coupon rate and the refinancing rate;
l_t: accelerating effect;
m_t: burnout effect.

As to the simulation generating cash flows for each sample path, interested readers may refer to Kutsuna et al. (2004), where the parameters are set as

$$\begin{aligned}
b &= 1.5, \\
w &= 0.083, \\
\nu &= 1.74, \\
\beta_1 &= 34.2, \\
\beta_2 &= 0, \\
\beta_3 &= 0.3.
\end{aligned}$$

2.4 Mathematical Model

We now present the following stochastic optimization problem:

$$\begin{aligned}
(P_0)\quad \max_{\Theta} \quad & W(\mathbf{a}) - \lambda\mathbf{E}\left[\sum_{t=1}^{T-1} \alpha_t(\tau_t - \xi_t\pi_t X_t)\right] \\
\text{s.t.} \quad & \mathbf{E}\left[\sum_{t=1}^{T} \alpha_t L_t\right] \le U_L, \\
& L_t = \max\{0, a_t - C_t - V_{t-1}\}, \qquad (5) \\
& V_t = \min\{\tau_t - \xi_t\pi_t X_t, C_t + V_{t-1} + L_t - a_t\}, \qquad (6) \\
& \tau_t - \xi_t\pi_t X_t \ge 0, \quad t = 1, ..., T, \\
& \mathbf{a} \ge 0,\ V_0 = V_T = 0,
\end{aligned}$$

where the variables are $\Theta = (\mathbf{a}, \boldsymbol{\tau}, \boldsymbol{\xi}, \mathbf{L}, \mathbf{V})$ with $\boldsymbol{\tau} = (\tau_1, ..., \tau_{T-1})$, $\boldsymbol{\xi} = (\xi_1, ..., \xi_{T-1})$, $\mathbf{L} = (L_1, ..., L_T)$ and $\mathbf{V} = (V_1, ..., V_T)$.

This problem is to maximize the objective function that takes into account the amount of the PAC bonds and the cash reserve under the constraint that the total PAC bondholder shortfall does not exceed a preset threshold U_L. Of course, there exist many other targets and constraints for the structurer, which will be included in the model subsequently.

3 Linear Programming Model

The mathematical model (P_0) presented in the previous section involves the constraints (5) and (6) that are not easy to deal with directly. First note that problem (P_0) can be written as the following program:

$$
\begin{aligned}
(P_1) \quad \max_{\Theta} \quad & W(\mathbf{a}) - \lambda \mathbf{E}\Big[\sum_{t=1}^{T-1} \alpha_t(\tau_t - \xi_t \pi_t X_t)\Big] \\
\text{s.t.} \quad & \mathbf{E}\Big[\sum_{t=1}^{T} \alpha_t L_t\Big] \le U_L, \\
& L_t \ge 0, \\
& L_t \ge a_t - C_t - V_{t-1}, \\
& L_t = 0 \quad \text{or} \quad L_t \ge a_t - C_t - V_{t-1}, & (7) \\
& V_t \le \tau_t - \xi_t \pi_t X_t, \\
& V_t \le C_t + V_{t-1} + L_t - a_t, \\
& V_t = \tau_t - \xi_t \pi_t X_t \quad \text{or} \quad V_t = C_t + V_{t-1} + L_t - a_t, & (8) \\
& \tau_t - \xi_t \pi_t X_t \ge 0, \ t = 1, ..., T, \\
& \mathbf{a} \ge 0, \ V_0 = V_T = 0.
\end{aligned}
$$

The conditions (7) and (8) comprise the complementary constraints that make the problem quite intractable (Luo, Pang and Ralph, 1996). However, it will be shown shortly that we can replace the complementarity constraints (7) and (8) with the nonnegative constraints on

V_t. More specifically, we consider the following problem:

$$
\begin{aligned}
(P_2) \quad \max_{\Theta} \quad & W(\mathbf{a}) - \lambda \mathbf{E}\Big[\sum_{t=1}^{T-1} \alpha_t(\tau_t - \xi_t \pi_t X_t)\Big] \\
\text{s.t.} \quad & \mathbf{E}\Big[\sum_{t=1}^{T} \alpha_t L_t\Big] \le U_L, \\
& L_t \ge 0, \\
& L_t \ge a_t - C_t - V_{t-1}, \qquad (9) \\
& V_t \le \tau_t - \xi_t \pi_t X_t, \qquad (10) \\
& V_t \le C_t + V_{t-1} + L_t - a_t, \qquad (11) \\
& \tau_t - \xi_t \pi_t X_t \ge 0, \qquad (12) \\
& V_t \ge 0, \ t = 1, ..., T, \qquad (13) \\
& \mathbf{a} \ge 0, \ V_0 = V_T = 0.
\end{aligned}
$$

It can be seen that constraint (9) is derived from constraints (11) and (13), constraint (12) is derived from constraints (10) and (13). Thus, by removing those redundant constraints, we have the following problem:

$$
\begin{aligned}
(P_3) \quad \max_{\Theta} \quad & W(\mathbf{a}) - \lambda \mathbf{E}\Big[\sum_{t=1}^{T-1} \alpha_t(\tau_t - \xi_t \pi_t X_t)\Big] \\
\text{s.t.} \quad & \mathbf{E}\Big[\sum_{t=1}^{T} \alpha_t L_t\Big] \le U_L, \qquad (14) \\
& V_t \le C_t + V_{t-1} + L_t - a_t, \qquad (15) \\
& 0 \le V_t \le \tau_t - \xi_t \pi_t X_t, \\
& L_t \ge 0, \ a_t \ge 0, \ t = 1, ..., T, \\
& V_0 = V_T = 0.
\end{aligned}
$$

Problem (P_3) is a stochastic linear programming problem which can be solved efficiently by the well known interior-point methods or simplex method (Vanderbei, 1996). Now it remains to show that problem (P_1) is equivalent to problem (P_3).

Theorem 1 If problem (P_3) has an optimal solution, then constraint (14) is active at any optimal solution.

Proof. We show this theorem by contradiction. Suppose there exists an optimal solution $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \bar{\mathbf{V}})$ of problem (P_3) satisfying

$$\mathbf{E}\Big[\sum_{t=1}^{T} \alpha_t \bar{L}_t\Big] < U_L.$$

We define δ_1, $\check{L}_1$, and $\check{a}_1$ as

$$\begin{aligned}\delta_1 &= U_L - \mathbf{E}\Big[\sum_{t=1}^{T}\alpha_t\bar{L}_t\Big] > 0,\\ \check{L}_1 &= \bar{L}_1 + \frac{\delta_1}{\alpha_1},\\ \check{a}_1 &= \bar{a}_1 + \frac{\delta_1}{\alpha_1}.\end{aligned}$$

Let $\check{\mathbf{L}} = (\check{L}_1, \bar{L}_2, ..., \bar{L}_T)$ and $\check{\mathbf{a}} = (\check{a}_1, \bar{a}_2, ..., \bar{a}_T)$. Then, constraint (14) becomes active at $(\check{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \check{\mathbf{L}}, \bar{\mathbf{V}})$, since

$$\begin{aligned}\mathbf{E}\Big[\sum_{t=1}^{T}\alpha_t\check{L}_t\Big] &= \mathbf{E}\Big[\alpha_1\check{L}_1 + \sum_{t=2}^{T}\alpha_t\bar{L}_t\Big]\\ &= \mathbf{E}\Big[\alpha_1\bar{L}_1 + \delta_1 + \sum_{t=2}^{T}\alpha_t\bar{L}_t\Big]\\ &= \mathbf{E}\Big[\sum_{t=1}^{T}\alpha_t\bar{L}_t\Big] + \delta_1 = U_L.\end{aligned}$$

Moreover, from

$$\bar{V}_1 \leq C_1 + \bar{V}_0 + (\bar{L}_1 + \frac{\delta_1}{\alpha_1}) - (\bar{a}_1 + \frac{\delta_1}{\alpha_1}),$$

we have

$$\bar{V}_1 \leq C_1 + \bar{V}_0 + \check{L}_1 - \check{a}_1,$$

and hence

$$\bar{V}_t \leq C_t + \bar{V}_{t-1} + \check{L}_t - \check{a}_t \quad (t = 1, ..., T),$$

which means $(\check{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \check{\mathbf{L}}, \bar{\mathbf{V}})$ satisfies constraint (15). Thus $(\check{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \check{\mathbf{L}}, \bar{\mathbf{V}})$ is feasible for problem (P_3). However, since

$$W(\check{\mathbf{a}}) - W(\bar{\mathbf{a}}) = \delta_1 > 0,$$

the objective value of $(\check{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \check{\mathbf{L}}, \bar{\mathbf{V}})$ is larger than that of $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \bar{\mathbf{V}})$, which contradicts the assumption that $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \bar{\mathbf{V}})$ is optimal. Therefore constraint (14) is active at the optimal solution of (P_3). The proof is complete.

Lemma 1 Let $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \bar{\mathbf{V}})$ be an optimal solution of problem (P_3). Then, $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is also an optimal solution of problem (P_3), where $\tilde{\mathbf{V}} = (\tilde{V}_0, ..., \tilde{V}_T)$ is recursively given by

$$\begin{aligned}\tilde{V}_0 &= 0,\\ \tilde{V}_t &= \min\{\bar{\tau}_t - \bar{\xi}_t\pi_t X_t, C_t + \tilde{V}_{t-1} + \bar{L}_t - \bar{a}_t\},\ t = 1, ...T-1,\\ \tilde{V}_T &= 0.\end{aligned}$$

Proof. First we show that $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is feasible to problem (P_3). Clearly $\tilde{V}_0 = \tilde{V}_T = 0$ is satisfied. For $t = 1$, since

$$\begin{aligned}\bar{V}_1 &\leq C_1 + \bar{L}_1 - \bar{a}_1,\\ \bar{V}_1 &\leq \bar{\tau}_1 - \bar{\xi}_1\pi_1 X_1,\end{aligned}$$

we have

$$\tilde{V}_1 = \min\{\bar{\tau}_1 - \bar{\xi}_1\pi_1 X_1, C_1 + \bar{L}_1 - \bar{a}_1\} \geq \bar{V}_1 \geq 0.$$

Assume $\tilde{V}_{t-1} \geq \bar{V}_{t-1}$ is true for each $t = 2, 3, ..., T-1$. Then we have

$$\begin{aligned} \tilde{V}_t &= \min\{\bar{\tau}_t - \bar{\xi}_t\pi_t X_t, C_t + \tilde{V}_{t-1} + \bar{L}_t - \bar{a}_t\} \\ &\geq \min\{\bar{\tau}_t - \bar{\xi}_t\pi_t X_t, C_t + \bar{V}_{t-1} + \bar{L}_t - \bar{a}_t\} \\ &\geq \bar{V}_t \geq 0. \end{aligned}$$

Therefore, $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is feasible for problem (P_3). Moreover, it is an optimal solution for (P_3) since its objective function value is equal to that of $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \bar{\mathbf{V}})$. This completes the proof.

Lemma 2 Let $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ be defined as in Lemma 1. Then at least one of the following inequalities is active for each t:

$$\bar{L}_t \geq 0, \quad \bar{L}_t \geq \bar{a}_t + \tilde{V}_t - C_t - \tilde{V}_{t-1}. \tag{16}$$

Proof. Assume the contrary. Then there exists $t_0 \in \{1, ..., T\}$ such that

$$\bar{L}_{t_0} > 0 \quad \text{and} \quad \bar{L}_{t_0} > \bar{a}_{t_0} + \tilde{V}_{t_0} - C_{t_0} - \tilde{V}_{t_0-1}.$$

We define

$$\begin{aligned} \delta_2 &= \min\{\bar{L}_{t_0}, \bar{L}_{t_0} - \bar{a}_{t_0} - \tilde{V}_{t_0} + C_{t_0} + \tilde{V}_{t_0-1}\} > 0, \\ \dot{a}_{t_0} &= \bar{a}_{t_0} + \delta_2 \\ &= \min\{\bar{a}_{t_0} + \bar{L}_{t_0}, \bar{L}_{t_0} - \tilde{V}_{t_0} + C_{t_0} + \tilde{V}_{t_0-1}\} \\ &> 0. \end{aligned}$$

Then we have

$$\dot{a}_{t_0} \leq \bar{L}_{t_0} - \tilde{V}_{t_0} + C_{t_0} + \tilde{V}_{t_0-1},$$

i.e.,

$$\tilde{V}_{t_0} \leq C_{t_0} + \tilde{V}_{t_0-1} + \bar{L}_{t_0} - \dot{a}_{t_0}.$$

Let $(\dot{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ be a vector obtained by replacing $\bar{a}_{t_0}$ in $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ with $\dot{a}_{t_0}$. Then, $(\dot{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is a feasible solution for problem (P_3). From the definition of $W(\mathbf{a})$, however, we have

$$W(\dot{\mathbf{a}}) - W(\bar{\mathbf{a}}) = \alpha_{t_0}\delta_2 > 0.$$

This implies that the objective function value of $(\dot{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is larger than that of $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$, which contradicts the assumption that $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is an optimal solution for problem (P_3). As a result, at least one of the inequalities in (16) is active at $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$. The proof is complete.

Lemma 3 Let $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ be defined as in Lemma 1. If there exists a t_1 such that $\bar{L}_{t_1} > 0$, then $\tilde{V}_{t_1} = 0$.

Proof. We prove the lemma by contradiction. Suppose $\tilde{V}_{t_1} > 0$. Note that, from Lemma 2, the following equality holds for each $t \in \{1, ..., T\}$,

$$\bar{L}_t = \max\{0, \bar{a}_t + \tilde{V}_t - C_t - \tilde{V}_{t-1}\},$$

which together with $\bar{L}_{t_1} > 0$ implies

$$\bar{L}_{t_1} = \bar{a}_t + \tilde{V}_t - C_t - \tilde{V}_{t-1}. \tag{17}$$

We define δ_3, $\hat{V}_{t_1}$, $\hat{L}_{t_1}$, and $\hat{L}_{t_1+1}$ as

$$\begin{aligned}
\delta_3 &= \min\{\tilde{V}_{t_1}, \bar{a}_{t_1} + \tilde{V}_{t_1} - C_{t_1} - \tilde{V}_{t_1-1}\} > 0, \\
\hat{V}_{t_1} &= \tilde{V}_{t_1} - \delta_3 \\
&= \max\{0, C_{t_1} + \tilde{V}_{t_1-1} - \bar{a}_{t_1}\} \\
&\geq 0, \\
\hat{L}_{t_1} &= \bar{L}_{t_1} - \delta_3 \\
&= \max\{\bar{L}_{t_1} - \tilde{V}_{t_1}, \bar{L}_{t_1} - \bar{a}_{t_1} - \tilde{V}_{t_1} + C_{t_1} + \tilde{V}_{t_1-1}\} \\
&= \max\{\bar{L}_{t_1} - \tilde{V}_{t_1}, 0\} \\
&\geq 0, \\
\hat{L}_{t_1+1} &= \bar{L}_{t_1} + \delta_3,
\end{aligned}$$

where the first inequality follows from (17) and the assumption $\tilde{V}_{t_1} > 0$.

Let $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \hat{\mathbf{L}}, \hat{\mathbf{V}})$ be a vector obtained by substituting $\bar{V}_{t_1}$, $\bar{L}_{t_1}$, and $\bar{L}_{t_1+1}$ in $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ by $\hat{V}_{t_1}$, $\hat{L}_{t_1}$, and $\hat{L}_{t_1+1}$, respectively. In the following, we will prove $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \hat{\mathbf{L}}, \hat{\mathbf{V}})$ is feasible and even optimal for problem (P_3). First, it can be seen that

$$\hat{V}_{t_1} < \tau_{t_1} - \xi_{t_1}\pi_{t_1}X_{t_1}$$

is satisfied, since $\tilde{V}_{t_1} \leq \tau_{t_1} - \xi_{t_1}\pi_{t_1}X_{t_1}$ and $\tilde{V}_{t_1} > \hat{V}_{t_1}$. Since $\tilde{V}_{t_1} \leq C_{t_1} + \tilde{V}_{t_1-1} + \bar{L}_{t_1} - \bar{a}_{t_1}$, we have

$$(\tilde{V}_{t_1} - \delta_3) \leq C_{t_1} + \tilde{V}_{t_1-1} + (\bar{L}_{t_1} - \delta_3) - \bar{a}_{t_1},$$

i.e.,

$$\hat{V}_{t_1} \leq C_{t_1} + \tilde{V}_{t_1-1} + \hat{L}_{t_1} - \bar{a}_{t_1}.$$

Moreover, for $t = t_1 + 1$, the inequality $\tilde{V}_{t_1+1} \leq C_{t_1+1} + \tilde{V}_{t_1} + \bar{L}_{t_1+1} - \bar{a}_{t_1+1}$ yields

$$\tilde{V}_{t_1+1} \leq C_{t_1+1} + (\tilde{V}_{t_1} - \delta_3) + (\bar{L}_{t_1+1} + \delta_3) - \bar{a}_{t_1+1},$$

i.e.,

$$\hat{V}_{t_1+1} \leq C_{t_1+1} + \tilde{V}_{t_1} + \hat{L}_{t_1+1} - \bar{a}_{t_1+1}.$$

Furthermore, by the definition of α_t, we have $\alpha_{t_1} - \alpha_{t_1+1} > 0$, and hence we obtain

$$\mathbf{E}\left[\sum_{t=1}^{T} \alpha_t \hat{L}_t\right] = \mathbf{E}\left[\sum_{t=1}^{T} \alpha_t \bar{L}_t\right] - (\alpha_{t_1} - \alpha_{t_1+1})\delta_3 < U_L. \tag{18}$$

The above arguments indicate that $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \hat{\mathbf{L}}, \hat{\mathbf{V}})$ is feasible for problem (P_3). Moreover, it is optimal for problem (P_3), since its objective value is the same as that of $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$. However, (18) contradicts Theorem 1, which requires (18) to be active at any optimal solution of problem (P_3). This completes the proof of the lemma.

Now, we are in the position to establish the following theorem which relates problem (P_1) with problem (P_3).

Theorem 2 Let $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ be defined as in Lemma 1. Then $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is not only an optimal solution of problem (P_3) but also an optimal solution of problem (P_1).

Proof. In view of Lemma 1 and the fact that problem (P_1) and problem (P_3) have the same objective value at $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$, it only suffices to show that $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is feasible for problem (P_1). First, from Lemma 1, we obtain

$$\tilde{V}_t = \min\{\bar{\tau}_t - \bar{\xi}_t \pi_t X_t, C_t + \tilde{V}_{t-1} + \bar{L}_t - \bar{a}_t\},$$

which means constraint (8) is satisfied in problem (P_1).

Next from Lemma 2, at least one of the following inequalities is active for each t:

$$\bar{L}_t \geq 0, \quad \bar{L}_t \geq \bar{a}_t + \tilde{V}_t - C_t - \tilde{V}_{t-1},$$

which implies

$$\bar{L}_t = 0 \quad \text{or} \quad \bar{L}_t = \bar{a}_t + \tilde{V}_t - C_t - \tilde{V}_{t-1}, \ t = 1, ..., T.$$

Moreover, when $\bar{L}_t > 0$, we have $\tilde{V}_t = 0$ as shown in Lemma 3. Thus, we obtain

$$\bar{L}_t = 0 \quad \text{or} \quad \bar{L}_t = \bar{a}_t - C_t - \tilde{V}_{t-1}, \ t = 1, ..., T.$$

Hence constraint (7) is satisfied. Consequently, $(\bar{\mathbf{a}}, \bar{\tau}, \bar{\xi}, \bar{\mathbf{L}}, \tilde{\mathbf{V}})$ is feasible for problem (P_1). The proof is complete.

4 Extended Model

In this section, we extend the model presented in the previous section by considering a more practical objective function specified by the CMO structurer, in which the companion bond is paid more attention to, and by relaxing a tight constraint on the PAC bondholder shortfall L_t at each term t.

As mentioned in Section 2, the companion bondholders can obtain a deterministic cash flow $b_t = y - a_t$ at time t, if there are no prepayments. However, due to prepayments, i.e., due to the uncertainty of y at time t, the cash flow paid to the companion bondholders is unknown. Therefore, while the PAC's scheduled cash flow is satisfied, the structurer of the CMO should also improve the quality of the companion bond in terms of reducing prepayment risk since it plays an important role in the issuance of the CMO. One of the measures of the quality for the companion bond is the absolute deviation of its cash flows, which can be defined as

$$D(\mathbf{B}) = \mathbf{E}\left[\sum_{t=1}^{T} |B_t - b_t|\right], \tag{19}$$

which implies that the smaller the absolute deviation, the higher the quality of the companion bond.

In terms of the model proposed in the previous section, the PAC bondholders would face a shortfall L_t if the sum of the cash flow C_t from mortgagors and the cash reserve V_{t-1} from the previous period does not meet the PAC's scheduled cash flow a_t. Moreover, the total present value of the shortfall has an upper bound U_L. Therefore, if the scheduled cash flow is completely fulfilled, there is no shortfall for the PAC bondholders. From the perspective of the trustee, however, the CMO may not be well designed since the PAC bondholder shortfall constraint may restrain his efficient operation. As a matter of fact, when the scheduled cash flow for the PAC bondholders is satisfied, the trustee could have the right of deciding when and whether to execute the shortfall so long as the total shortfall meets the upper bound constraint. Mathematically, those constraints may be represented as

$$\begin{aligned} &\mathbf{E}\left[\sum_{t=1}^{T} \alpha_t L_t\right] \leq U_L, \\ &C_t + V_{t-1} + L_t \geq a_t. \end{aligned}$$

Now we propose the following model:

$$\begin{aligned} \max \quad & W(\mathbf{a}) - \lambda R(\mathbf{v}) - \rho \mathbf{E}\left[\sum_{t=1}^{T} |B_t - b_t|\right] && (20) \\ \text{s.t.} \quad & \mathbf{E}\left[\sum_{t=1}^{T} \alpha_t L_t\right] \leq U_L, \\ & C_t + V_{t-1} + L_t \geq a_t, && (21) \\ & a_t + B_t + V_t = C_t + V_{t-1} + L_t, && (22) \\ & V_t \leq \tau_t - \xi_t \pi_t X_t, && (23) \\ & \tau_t - \xi_t \pi_t X_t \geq 0, && (24) \\ & \mathbf{V} \geq 0, V_0 = V_T = 0, && (25) \\ & \mathbf{a} \geq 0,\ B_t \geq 0,\ L_t \geq 0,\ t = 1, ..., T, && (26) \end{aligned}$$

where $\lambda, \rho \geq 0$ are two risk aversion constants used to tradeoff the value of the PAC bond and the total cash reserve and the absolute deviation of the companion bond. Note that here L_t and B_t have slightly different implications from those in Section 2.

For a treatment of nonsmoothness of the absolute deviation in (20), we introduce artificial variables $(u_t)^+$ and $(u_t)^-$ that satisfy

$$\begin{aligned} |B_t - b_t| &= (u_t)^+ + (u_t)^-, \\ B_t - b_t &= (u_t)^+ - (u_t)^-, \\ (u_t)^+ &\geq 0, \quad (u_t)^- \geq 0. \end{aligned} \tag{27}$$

Here we omit the condition $(u_t)^+ \cdot (u_t)^- = 0$, since it is guaranteed to hold at any optimal solution of (27).

It can be seen that constraint (21) is derived from constraints (22), (25), and (26), and constraint (24) is derived from constraints (23) and (25). Thus, by removing those redundant constraints, we have the following linear programming problem:

$$\begin{aligned} (P_4) \quad \max_{\Theta} \quad & W(\mathbf{a}) - \lambda \mathbf{E}\left[\sum_{t=1}^{T-1} \alpha_t \cdot (\tau_t - \xi_t \pi_t X_t)\right] - \rho \mathbf{E}\left[\sum_{t=1}^{T} [(u_t)^+ + (u_t)^-\right] \\ \text{s.t.} \quad & \mathbf{E}\left[\sum_{t=1}^{T} \alpha_t L_t\right] \leq U_L, \\ & a_t + B_t + V_t = C_t + V_{t-1} + L_t, \\ & 0 \leq V_t \leq \tau_t - \xi_t \pi_t X_t, \\ & B_t - b_t = (u_t)^+ - (u_t)^-, \\ & (u_t)^+ \geq 0, \quad (u_t)^- \geq 0, \\ & a_t \geq 0, \ B_t \geq 0, \ L_t \geq 0, \ V_0 = V_T = 0, \ t = 1, ..., T, \end{aligned}$$

where $\Theta = (\mathbf{a}, \boldsymbol{\tau}, \boldsymbol{\xi}, \mathbf{B}, \mathbf{V}, \mathbf{L}, \mathbf{u}^+, \mathbf{u}^-)$, $\boldsymbol{\tau} = (\tau_1, ..., \tau_{T-1})$, $\boldsymbol{\xi} = (\xi_1, ..., \xi_{T-1})$, $\mathbf{B} = (B_1, ..., B_T)$, $\mathbf{V} = (V_1, ..., V_{T-1})$, $\mathbf{L} = (L_1, ..., L_T)$, $\mathbf{u}^+ = ((u_1)^+, ..., (u_T)^+)$ and $\mathbf{u}^- = ((u_1)^-, ..., (u_T)^-)$.

5 Numerical Experiments and Sensitivity Analysis

In this section, we present some results of numerical experiments with the proposed models. To value a CMO, we first describe the mortgage rate model, incentive prepayments, and cash flows. Then we consider the sensitivity of some special parameters on the structure of the CMO, such as cash reserve sensitivity factor and the tradeoff coefficient to the absolute deviation of the companion bond.

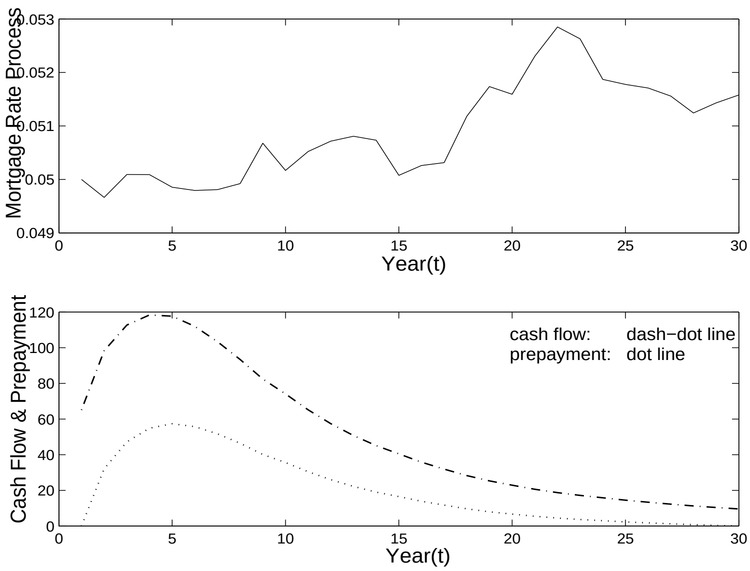

Figure 1: Sample paths of mortgage rate, cash flow and prepayment

5.1 Monte Carlo (MC) Simulation Method

For convenience, we will consider a 5% standard 30-year fixed-rate mortgage with original par value $\bar{x}_0 = 1000$. Of course, our model can be extended in a straightforward manner to other CMO mortgage designs.

First, we generate cash flows through Monte Carlo simulation. To model the mortgage rates, we adopt the CIR model (Cox, Ingersoll and Ross, 1985), which is given by

$$r_{t+\Delta t} = r_t + \kappa(\theta - r_t)\Delta t + \sigma\sqrt{r_t}\Delta B_t, \tag{28}$$

where $\Delta B_t \sim N(0, \Delta t)$. In the present paper, $\kappa = 0.20, \theta = 0.05, \sigma = 0.02$, and $\Delta t = 0.01$.

Once we generate J paths of T standard normal random numbers:

$$(\epsilon_1^j, \epsilon_2^j, ..., \epsilon_T^j), \quad j = 1, 2, ..., J, \tag{29}$$

each sequence in (29) gives a path of mortgage rates $(r_1^j, r_2^j, ..., r_T^j)$ via (28). Hence, we generate the corresponding prepayment factors $(\pi_0^j, \pi_1^j, ..., \pi_{T-1}^j)$ and in turn cash flows $(C_1^j, C_2^j, ..., C_T^j)$. In our simulation, we use $J = 1000$ sample paths for approximation.

As an illustration, a realization of the sample paths of the mortgage rates, prepayments, and cash flows is graphed in Figure 1. The upper panel displays a random mortgage rate generated by the CIR model and the lower panel displays the corresponding cash flows and prepayment behaviors based on the Schwartz-Torous prepayment model. This graph shows that the cash flows and prepayments in the first 10 years overwhelmingly dominate those in the later years. Moreover, the dominating prepayment behaviors in the first 10 years

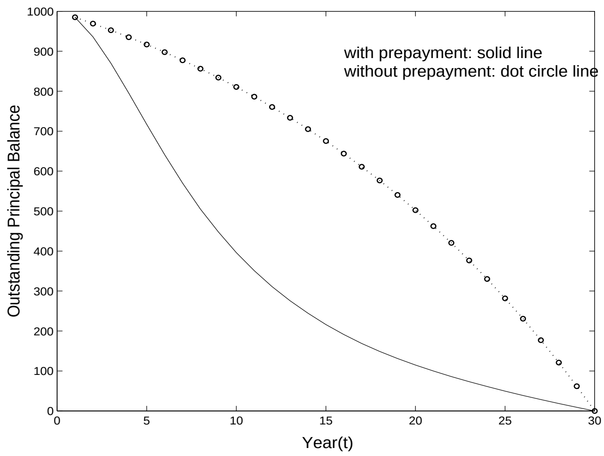

Figure 2: Outstanding repayment balance with and without prepayments

are closely related to the low level of the mortgage rates relative to the coupon rate for the assumed mortgage loans.

To demonstrate the necessity for the structurer to reassign cash flows from mortgagors and to develop derivatives, such as generic CMO, PAC, and other bond classes with schedule, for example, a *target amortization class* (TAC), Figure 2 provides a comparison of outstanding mortgage with and without prepayments. The significant prepayments in the early life of mortgage in both Figures 1 and 2, consistent with the conclusions in Stanton (1995), Kariya and Kobayashi (2000), and Kariya et al. (2002), also support the general validity of the Schwartz and Torous model for prepayments.

According to probability theory, we can approximate expressions (1), (4), and (19) as

$$R(\mathbf{v}) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T-1}\alpha_t(\tau_t - \xi_t \cdot \pi_t^j x_t^j),$$

$$L(\mathbf{C}, \mathbf{V}, \mathbf{a}) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\alpha_t L_t^j,$$

$$D(\mathbf{B}) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}|B_t^j - b_t|.$$

Therefore, using the results from the Monte Carlo simulation, we can solve the model (P_3), which is equivalent to the basic problem (P_1), and the extended model (P_4).

We implement our models with modelling language MatLab6.5 and linear programming package of SeDuMi1.05 (Sturm, 2001) on the platform of RedHat 8 with Intel Pentium 4 CPU 3.00GHz, 1.5GB memory. The computation time spent to solve a problem with 1000 sample paths is about 4 minutes.

Table 1: Sensitivity with respect to λ in the basic model ($U_L = 1$)

λ	$W(\mathbf{a})$	$R(\mathbf{v})$
0.00	987.1874	>100
0.10	974.8692	71.5790
0.20	968.2137	23.5035
0.40	964.5092	9.7962
0.60	962.5144	5.5418
0.80	960.7618	2.9489
1.00	960.6580	2.8441
5.00	956.4835	2.3764

5.2 Numerical Results of the Basic Model

In this subsection, we show experimental results of the basic model which does not take into account the quality of the companion bond. In Table 1, we report $W(\mathbf{a})$ and $R(\mathbf{v})$ at the optimal solution when $U_L = 1$ and λ varying from 0 to 5. Generally speaking, $W(\mathbf{a})$ and $R(\mathbf{v})$ decrease as λ increases with the other parameters being fixed. This is natural because a decrease in the sensitivity λ of the cash reserve allows more cash reserve, and increases the possibility of issuing more PAC bonds.

The left panel of Figure 3 illustrates the actual mean of the cash reserve and the mean upper bound of the cash reserve, and the right panel graphs all sample paths of shortfall the PAC bondholders face at the optimal solution. Note that the actual mean of the cash reserve at time t is calculated as

$$\mathbf{E}[\tilde{\mathbf{V}}_t] = \frac{1}{J}\sum_{j=1}^{J} \tilde{V}_t^j,$$

where $\tilde{V}_t^j$ is given in Lemma 1. Clearly it is observed from the left panel of Figure 3 that the dynamic upper bound of the cash reserve is very close to the real reserve, particularly for the last 15 years. This implies that the dynamic strategy for the upper bound of the cash reserve adopted in this paper, to some extent, can fit well the simulated scenarios. Moreover, due to the significant prepayments occurring in the first half of the life of the mortgage pool, especially from the 5th to the 15th years, the amount of the actual cash reserve and its upper bound are much larger than those in other years.

Next, we define the mean present value of the actual cash flow for the PAC bondholders A_t^j, the real cash flow for the companion bondholders B_t^j, and the real cash reserve $\tilde{V}_t^j$, which are denoted by $\mathbf{E}[W_0(\mathbf{A})]$, $\mathbf{E}[W_0(\mathbf{B})]$, and $\mathbf{E}[W_0(\mathbf{V})]$, respectively, with the discount rate at

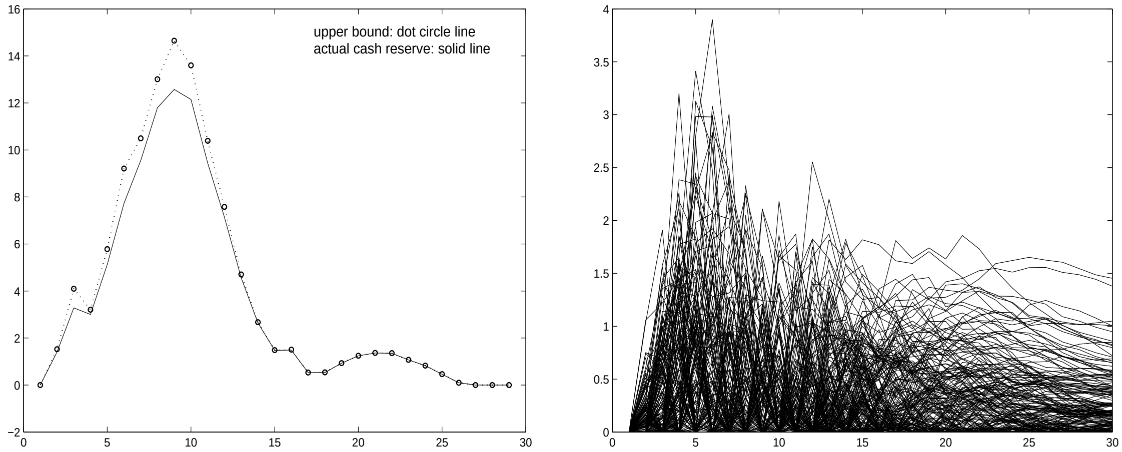

Figure 3: The upper bound of the cash reserve, the actual cash reserve (left), and the actual PAC bondholder shortfall (right) for the basic model at the optimal solution ($U_L = 1, \lambda = 0.1$).

Table 2: Real cash flow allocation for the basic model ($U_L = 1$)

λ	$\mathbf{E}[W_0(\mathbf{A})]$	$\mathbf{E}[W_0(\mathbf{B})]$	$\mathbf{E}[W_0(\mathbf{V})]$	$\mathbf{E}[W_0(\mathbf{A})]$+$\mathbf{E}[W_0(\mathbf{B})]$
0.00	986.1874	5.8161	169.9254	992.0036
0.10	973.8692	23.0658	64.3654	996.9350
0.20	967.2137	31.7259	22.2689	998.9396
0.40	963.5092	36.0445	9.3734	999.5536
0.60	961.5144	38.2279	5.4110	999.8106
0.80	959.7618	40.0985	2.9328	999.8603
1.00	959.6580	40.2074	2.8281	999.8653
5.00	955.4835	44.4144	2.1447	999.8979

period t being simplified as $(1+r_0)^{-t}$:

$$\mathbf{E}[W_0(\mathbf{A})] = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{A_t^j}{(1+r_0)^t},$$

$$\mathbf{E}[W_0(\mathbf{B})] = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{B_t^j}{(1+r_0)^t},$$

$$\mathbf{E}[W_0(\mathbf{V})] = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T-1}\frac{\tilde{V}_t^j}{(1+r_0)^t}.$$

Table 2 shows the values of $\mathbf{E}[W_0(\mathbf{A})]$, $\mathbf{E}[W_0(\mathbf{B})]$, $\mathbf{E}[W_0(\mathbf{V})]$, and $\mathbf{E}[W_0(\mathbf{A})]$+$\mathbf{E}[W_0(\mathbf{B})]$ at the optimal solution of the basic model where the total shortfall is fixed $U_L = 1$ and parameter λ varies from 0 to 5.

Both Tables 1 and 2 show that the the mean present value of the actual payment for the

PAC bonds decreases as λ increases with the other parameters fixed. This is natural because the structurer pays more attention to the cash reserve, i.e., the structurer makes the total cash reserve as small as possible, which in turn decreases the postponement of the receipt of principal paid to the companion bondholders. The fact that the sum of $\mathbf{E}[W_0(\mathbf{A})]+\mathbf{E}[W_0(\mathbf{B})]$ approaches x_0 when parameter λ tends to 5 also illustrates an increase in the sensitivity for the cash reserve decreases the postponement of principal receipt for the companion bondholders because the total amount of the cash reserve declines. In fact, this can be explained as follows: From the definition of B_t in Section 2, we have

$$A_t + B_t = C_t + \tilde{V}_{t-1} - \tilde{V}_t, \qquad (t = 1, ..., T),$$

which means

$$A_t^j + B_t^j = C_t^j + \tilde{V}_{t-1}^j - \tilde{V}_t^j, \qquad (t = 1, ..., T)$$

hold for each $j = 1, ..., J$. Hence,

$$\begin{aligned}
&\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{A_t^j}{(1+r_0)^t} + \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{B_t^j}{(1+r_0)^t} \\
&= \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{C_t^j}{(1+r_0)^t} + \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{\tilde{V}_{t-1}^j}{(1+r_0)^t} - \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{\tilde{V}_t^j}{(1+r_0)^t} \\
&= \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{C_t^j}{(1+r_0)^t} - \frac{r_0}{1+r_0}\cdot\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{\tilde{V}_t^j}{(1+r_0)^t} \\
&\approx x_0 - \frac{r_0}{1+r_0}\cdot\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{\tilde{V}_t^j}{(1+r_0)^t},
\end{aligned}$$

i.e.,

$$\mathbf{E}[W_0(\mathbf{A})] + \mathbf{E}[W_0(\mathbf{B})] \approx x_0 - \frac{r_0}{1+r_0}\mathbf{E}[W_0(\tilde{\mathbf{V}})],$$

where we use the initial and terminal conditions of cash reserve $V_0^j = V_T^j = 0$. Thus, the sum of $\mathbf{E}[W_0(\mathbf{A})]$ and $\mathbf{E}[W_0(\mathbf{B})]$ decreases as $\mathbf{E}[W_0(\tilde{\mathbf{V}})]$ increases. If there is no cash reserve, the present value of the cash flow paid to the holders of the PAC bonds and the companion bond is equal to the initial principal balance.

5.3 Numerical Results of the Extended Model

In this subsection, we report the numerical results for the extended model which incorporates the quality of the companion bond and relaxes the PAC bondholder shortfall constraint for each period.

Table 3 shows the present value of $W(\mathbf{a})$ and $R(\mathbf{v})$ at the optimal solution when $U_L = 1, \rho = 0$, and λ varies from 0 to 5. In view of Table 3, we emphasize that the extended model

Table 3: Sensitivity with respect to λ in the extended model ($U_L = 1, \rho = 0$)

λ	$W(\mathbf{a})$	$R(\mathbf{v})$
0.00	986.6716	>100
0.10	974.8292	69.1782
0.20	968.3741	22.4040
0.40	964.8793	9.4483
0.60	962.9691	5.3683
0.80	961.2563	2.8373
1.00	961.0620	2.6405
5.00	961.0254	2.5873

can achieve the same or even better results compared with the basic model. For example, the present value of the scheduled cash flow paid to the PAC bondholders at the optimal solution is very close to and always a little larger than that of the basic model, while the amount of the cash reserve is a bit lower, which is preferable for the reason presented in Section 2. In fact, it is imaginable that by relaxing the PAC bondholder shortfall constraint of the basic model, the trustee achieves more freedom to decide when and how to execute a PAC bondholder shortfall so long as the total shortfall for the PAC bondholders does not exceed the preset threshold. Figure 4 graphs the actual mean of the cash reserve, the actual mean of the upper bound of the cash reserve, and the actual PAC bondholder shortfall at the optimal solution when $U_L = 1, \lambda = 0.1$, and $\rho = 0$. Apparently, Figure 4 has many common characteristics with Figure 3, such as the shape and the trend of the actual cash reserve, the upper bound, and the actual PAC bondholder shortfall. However, the ranges of fluctuation of the actual mean of the cash reserve and of the upper bound of Figure 4 are significantly different from those of Figure 3. This provides a persuasive explanation that the extended model is superior to the basic model since both the general upper bound and the actual cash reserve are lower than those of the basic model. Furthermore, the persistently low level of the cash reserve in the extended model decreases the shortfall of the PAC bondholders. Another difference is that the actual mean of the cash reserve of the extended model is less closer to the mean upper bound from the 5th to the 15th years than that of other periods. This is mainly because the extended model does not rely on Lemma 2 and Lemma 3.

To demonstrate the effect of λ on the amount of the PAC bonds that can be issued, the cash reserve and the absolute deviation of the companion bond, Table 4 shows the values of $W(\mathbf{a})$, $R(\mathbf{v})$ and $D(\mathbf{B})$ at the optimal solution when $U_L = 1.0, \rho = 0.5$, and λ is changed from 0 to 5. In this situation, because of taking into account the quality of the companion bond, the amount of the PAC bonds as well as the cash reserve decreases generally as λ increases, while the absolute deviation of the companion bond $D(\mathbf{B})$ increases consistently. This is natural because when other parameters are fixed, an increase of λ implies that the structurer pays

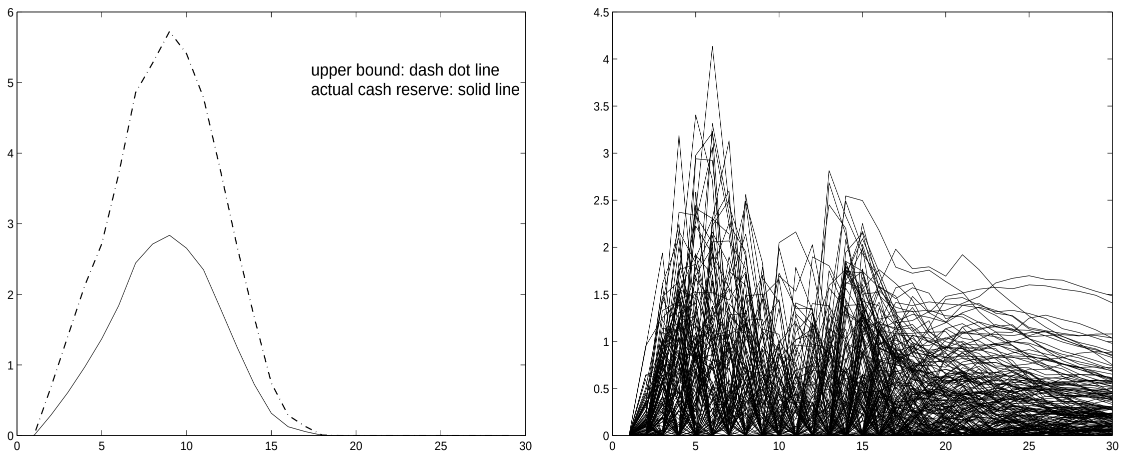

Figure 4: The upper bound of the cash reserve, the actual cash reserve (left), and the actual PAC bondholder shortfall (right) for the extended model at the optimal solution ($U_L = 1, \lambda = 0.1, \rho = 0$).

Table 4: Sensitivity with respect to λ in the extended model ($U_L = 1, \rho = 0.5$)

λ	$W(\mathbf{a})$	$R(\mathbf{v})$	$D(\mathbf{B})$
0.00	870.1363	>1000	444.2072
0.10	955.8715	692.7244	831.3734
0.20	969.7926	209.7110	984.3243
0.40	967.1359	32.2754	1072.6
0.60	963.9518	14.8819	1082.5
0.80	961.8528	7.8826	1087.9
1.00	961.5790	4.4925	1092.9
5.00	961.5549	4.3781	1092.1

more attention to the postponement of principal of receipt paid to the companion bondholders caused by the cash reserve, and hence decreases the possibility of taking on a shortfall for the PAC bondholders, and naturally affects the quality of the companion bond.

The effect of ρ on the amount of the PAC bonds that can be issued, the cash reserve, and the absolute deviation of the companion bond is reported in Table 5, where $U_L = 0, \lambda = 0.5$, and ρ is changed from 0 to 5. As expected, in this situation, the amount of the total cash reserve $R(\mathbf{v})$ significantly increases from 6.6412 to 139.9275, the absolute deviation of the companion bond consistently decreases as ρ increases, while the movement of the amount of the PAC bonds that can be issued stays at a high level.

Furthermore, in Figure 5 we show the actual mean of the cash reserve, the mean upper bound, and the actual PAC bondholder shortfall at the optimal solution when $U_L = 1, \lambda = 0.1$, and $\rho = 0.5$. In this case, there is a very strange but interesting phenomenon that the

Table 5: Sensitivity with respect to ρ in the extended model ($U_L = 1, \lambda = 0.5$)

ρ	$W(\mathbf{a})$	$R(\mathbf{v})$	$D(\mathbf{B})$
0.00	963.6553	6.6412	>1500
0.10	964.0379	7.5558	1094.9
0.20	964.2927	8.6110	1093.2
0.40	964.8102	16.8256	1082.1
0.60	964.8250	24.2560	1074.9
0.80	965.3639	61.8374	1047.2
1.00	963.2470	130.0867	1010.0
5.00	963.5471	139.9275	1007.2

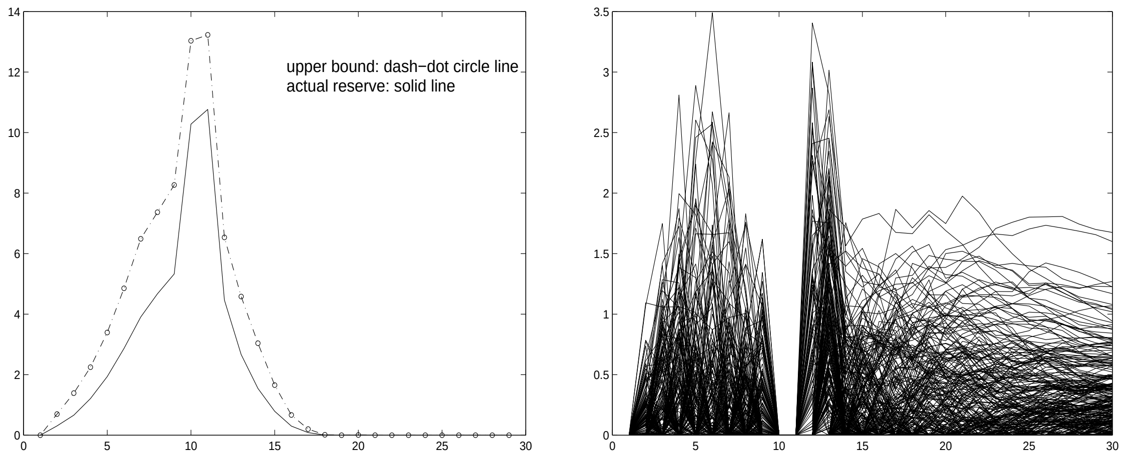

Figure 5: The upper bound of the cash reserve, the actual cash reserve (left), and the actual PAC bondholder shortfall (right) for the extended model at the optimal solution ($U_L = 1, \lambda = 0.1, \rho = 0.5$).

actual PAC bondholder shortfall is separated into two parts, before the 10th year and after the 11th year. The reason for this phenomenon is that, as seen in the left panel of Figure 5, the amount of the cash reserve during the 10th and 11th years is much larger than that of other years.

It should be mentioned that we omit some results for the basic model and the extended model when $\lambda > 5$ and $\rho > 5$ because the results for larger values of λ and ρ differ only slightly from those of $\lambda = 5$ and $\rho = 5$.

6 Conclusion

As the securitization of mortgage loans and the customization of securitized products continue to increase in countries throughout the world, optimization methods will be employed to design the optimal structure. In this paper, we propose a new model for optimally designing a CMO (a special type of MBS) with PAC-companion structure which assumes that a part of the cash flow from the underlying mortgage pool can be reserved to the next period. This is done to reduce the extension risk to the PAC bondholders but increase the risk of interest loss to the companion bondholders. We transform the design problem of the CMO structure into a standard stochastic linear programming problem which maximizes the objective function of the CMO structurer under PAC bondholder shortfall and cash reserve constraints. We then extend the model and find that the modified model yields a more desirable performance than the basic one.

Of course, there remains much room for enhancing the model. For example, we consider only the prepayment risk in this paper. It may be of interest in practice to incorporate the credit risk, i.e., default risk (Kau et al., 1992, 1995). As another example, in our model we use the Monte Carlo approach to generate the mortgage rate process, which has an inherent drawback that any two sample paths have no correlation. Therefore, the binomial tree simulation may be more plausible and preferred.

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