Approximating minimum cost multigraphs of specified edge-connectivity under degree bounds*

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Abstract

In this paper, we consider the problem of constructing a minimum cost graph with a specified edge-connectivity under a degree constraint. For a set V of vertices, let $r : \binom{V}{2} \to \mathbb{Z}_+$ be a connectivity demand, $a : V \to \mathbb{Z}_+$ be a lower capacity, $b : V \to \mathbb{Z}_+$ be an upper capacity and $c : \binom{V}{2} \to \mathbb{Q}_+$ be a metric edge cost. The problem (V, r, a, b, c) asks to find a minimum cost multigraph $G = (V, E)$ with no self-loops such that $\lambda(u, v) \geq r(u, v)$ for each pair $u, v \in V$ and $a(v) \leq d(v) \leq b(v)$ for each $v \in V$, where $\lambda(u, v)$ (resp., $d(v)$) denotes the local-edge-connectivity between u and v (resp., the degree of v) in G. We show several conditions on functions r, a, b and c for which the above problem admits an approximation algorithm. For example, we give a $(2 + 1/\lfloor k/2 \rfloor)$-approximation algorithm to (V, r, a, b, c) with $r(u, v) \geq 2$, $u, v \in V$ and a uniform $b(v)$, $v \in V$, where $k = \min_{u,v \in V} r(u, v)$. To design the algorithms in this paper, we use our new results on edge-splitting and detachment, which are graph transformations to split vertices while preserving edge-connectivity.

Keywords: edge-connectivity, degree bound, edge-splitting, detachment

1 Introduction

In this paper, we let $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the set of nonnegative integers, rational numbers and real numbers, respectively. Let graph $G = (V, E)$ stand for an undirected multigraph with no self-loop unless stated otherwise. Let $d(v; G)$ (or $d(v)$) denote the degree of v in G, and $\lambda(u, v; G)$ (or $\lambda(u, v)$) denote the local-edge-connectivity between two vertices u and v in G. For a function $r : \binom{V}{2} \to \mathbb{Z}_+$, a graph G with a vertex set V is called *r-edge-connected* when $\lambda(u, v; G) \geq r(u, v)$ holds for each pair $u, v \in V$. If $r(u, v) = k$ for all $u, v \in V$, then an r-edge-connected graph may be called k-edge-connected. In this paper, we consider the problem of constructing a minimum cost graph with a specified edge-connectivity under a degree constraint, which is formulated as follows: Given a set V of vertices, a connectivity demand $r : \binom{V}{2} \to \mathbb{Z}_+$, a lower capacity $a : V \to \mathbb{Z}_+$, an upper capacity $b : V \to \mathbb{Z}_+$ and an edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$, the problem asks to find a minimum cost multigraph $G = (V, E)$ with no self-loops such that

$$\lambda(u, v; G) \geq r(u, v) \text{ for each pair } u, v \in V$$

and

$$a(v) \leq d(v; G) \leq b(v) \text{ for each } v \in V.$$

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We denote a problem instance consisting of the above inputs by (V, r, a, b, c). For a function h and a constant $\ell \in \mathbb{Q}_+$, we denote $h \le \ell$ (resp., $h \ge \ell$) to mean that the value of h is always less than or equal to ℓ (resp., greater than or equal to ℓ), where $h = \ell$ means $h \le \ell$ and $h \ge \ell$.

This problem includes a wide range of classically fundamental problems such as the minimum spanning tree problem $(V, 1, 1, +\infty, c)$, the steiner tree problem $(V, \{0, 1\}, 0, +\infty, c)$, the steiner network problem without edge capacity constraints $(V, r, 0, +\infty, c)$, the traveling salesman problem $(V, \{1, 2\}, 2, 2, c)$ and so on. Each of these problems is theoretically important and has been studied extensively from the view points of computational complexity and polyhedral structure. The first problem admits a polynomial time algorithm for any cost function c, but the last three are all NP-hard. In particular, the traveling salesman problem is inapproximable within a constant factor unless $P = NP$, and is shown to be approximable within a factor of 1.5 if c is metric by N. Christofides [1]. K. Jain [7] proved that the steiner network problem $(V, r, 0, +\infty, c)$ is approximable within factor of 2 based on a primal-dual method. G. Robins and A. Zelikovsky [10] gave a 1.55-approximation algorithm for the steiner tree problem $(V, \{0, 1\}, 0, +\infty, c)$. S. P. Fekete et al. [2] proved that problem $(V, 1, 0, b \ge 2, c)$ is approximable within a factor of $2 - \min_{v \in V, d(v;T)>2} \frac{b(v)-2}{d(v;T)-2}$, where T is a minimum spanning tree. Although the problem (V, r, a, b, c) is a natural framework as a generalization of the above problems, a few results on this problem setting have been obtained so far. A. Frank [3] solved the problem of augmenting a given graph to an r-edge-connected graph by adding a smallest number of new edges under lower and upper bounds on degrees. This implies that problem $(V, r, a, b, 1)$ admits a polynomial time algorithm. Moreover, an extended result by A. Frank [3] suggests that (V, r, a, b, c) is polynomially solvable in a special case where cost $c(e)$ for each edge $e = uv$ is given by $w(u) + w(v)$ for some vertex weight $w : V \to \mathbb{Q}_+$.

In this paper, we consider problem (V, r, a, b, c) with a metric cost c, and show several conditions on functions r, a, b and c for which the problem admits an approximation algorithm. To design most algorithms proposed in this paper, we use edge-splitting and detachment, which are graph transformations that split vertices while preserving edge-connectivity. Use of such operations to design a minimum cost graph is new. Only edge-splitting has been used to solve the edge-connectivity augmentation problem. However, the way of using edge-splitting in this paper is different from those methods in the edge-connectivity augmentation.

The paper is organized as follows. Section 2 derives new results on edge-splitting and detachment, which will be the basis of our algorithms in this paper. It also considers the relation between that result and parsimonious property of the steiner network problem. Section 3 deals with the case of $b = +\infty$, i.e., each degree is bounded only from below, and Section 4 considers the case of $a = 0$, i.e., each degree is bounded only from above. Section 5 shows an approximation algorithm for problem $(V, r \ge 2, a, \ell, c)$. Section 6 considers the case of $a = b$, i.e., each degree is specified, and gives an approximation algorithm for problem $(V, 1 \le r \le 2, a, a, c)$. This section also shows an approximation algorithm for constructing a minimum cost strongly connected spanning digraph whose in-degrees and out-degrees are specified.

2 Edge-splitting and detachment

Edge-splitting is an operation to replace edges $e = us$ and $f = vs$ by a new edge uv. The resulting graph by splitting a pair $\{e, f\}$ of edges in a graph G is denoted by G^{ef}. Note that splitting a pair $\{e = us, f = vs\}$ of edges decreases only the degree of vertex s while keeping degrees of the other vertices. Also in a metric cost $c : \binom{V}{2} \to \mathbb{Q}_+$, splitting a pair $\{e = us, f = vs\}$ of edges does not increase the cost of the graph since $c(us) + c(vs) \ge c(uv)$ holds by the triangle inequality. However, this may decrease the local-edge-connectivity between some pairs of vertices. Mader [4, 8] proved that, for a designated vertex $s \in V$, there always exists a pair of edges such that the local-edge-connectivity between every pair of vertices in $V - \{s\}$ is preserved (except for the case where $d(s) = 3$ or a cut-edge is incident to s).

Theorem 1 (Mader) *Let $G = (V, E)$ be a connected graph and $s \in V$ be a vertex with $d(s) \neq 3$. If no cut-edge is incident to s, then there is at least one pair $\{e, f\}$ of edges incident to s such that $\lambda(x, y; G^{ef}) = \lambda(x, y; G)$ for each $x, y \in V - s$.*

Edge-splitting has been used as a useful technique for solving many connectivity problem. In particular, it plays a key role to solve the edge-connectivity augmentation problem (see [3]). However, the above result on edge-splitting has not been used to design a minimum cost graph even if a cost function is metric. Moreover, in Mader's theorem, the local-edge-connectivity between the designated vertex s and other vertices is not taken into account. Our algorithms discussed in the following sections utilize edge-splitting in order to make solutions satisfy degree constraints. For this, we need to preserve the local-edge-connectivity between s and other vertices as well. We now show the following slightly stronger result which also preserves the local-edge-connectivity between s and the other vertices (up to $d(s) - 2$). The proof is given in Appendix A .

Theorem 2 *Let $G = (V, E)$ be a connected graph and $s \in V$ be a vertex with $d(s) \neq 3$. Moreover let*

$$r(x, y) = \begin{cases} \lambda(x, y; G) & \text{for } x, y \in V - s, \\ \min\{d(s) - 2, \lambda(s, y; G)\} & \text{for } x = s \text{ and } y \in V - s. \end{cases}$$

If no cut-edge is incident to s, then there is at least a pair $\{e, f\}$ of edges incident to s such that $\lambda(x, y; G^{ef}) = r(x, y)$ for each $x, y \in V$, where edges $e = us$ and $f = vs$ can be chosen so that $u \neq v$ unless s is adjacent to only one vertex. No new cut-edge will be created after splitting $\{e, f\}$.

This theorem has a close relationship to the *parsimonious property* of the steiner network problem, which tells that a LP relaxation of the steiner network problem

$$\begin{array}{lll} \text{minimize} & \sum_{e \in \binom{V}{2}} c(e)x(e) & \\ \text{subject to} & \sum_{e \in \delta(X)} \geq \max_{u \in X, v \in V - X} r(u, v) & \text{for each } X \subset V,\ X \neq V, \\ & x(e) \in \mathbb{R}_+ & \text{for each } e \in \binom{V}{2}, \end{array}$$

is equivalent to

$$\begin{array}{lll} \text{minimize} & \sum_{e \in \binom{V}{2}} c(e)x(e) & \\ \text{subject to} & \sum_{e \in \delta(X)} \geq \max_{u \in X, v \in V - X} r(u, v) & \text{for each } X \subset V,\ X \neq V, \\ & \sum_{e \in \delta(\{v\})} x(e) = \max_{u \in V - v} r(u, v) & \text{for each } v \in V, \\ & x(e) \in \mathbb{R}_+ & \text{for each } e \in \binom{V}{2}, \end{array}$$

where $\delta(X)$ denotes a set of edges whose one end vertex is in X and the other is in $V - X$. In fact, M. X. Goemans and D. J. Bertsimas [5] proved this property by showing that every *eulerian* graph admits a pair $\{e, f\}$ of edges incident to s such that G^{ef} is r-edge-connected, which is a weaker of Theorem 2. From Theorem 2, we can derive an integer pragramming version of the parsimonious property for the steiner network problem.

Corollary 1 *If a cost function $c : \binom{V}{2} \to \mathbb{Q}_+$ is metric, the steiner network problem $(V, r, 0, +\infty, c)$ has an optimal solution such that the degree of each vertex v is $\max_{u \in V - v} r(u, v)$ or $\max_{u \in V - v} r(u, v) + 1$.*

Proof: Let G be an optimal solution for $(V, r, 0, +\infty, c)$, and suppose $d(v; G) > \max_{u\in V-v} r(u, v)+1$ for a vertex $v \in V$. From Theorem 2, we can obtain another r-edge connected graph G' with $d(v; G') = d(v; G) - 2 \geq \max_{u\in V-v} r'(u, v)$ by splitting an appropriate pair of edges incident to v, and since c is metric, cost of G' is at most that of G. Hence, by repeating this operation, we can obtain another optimal solution such that the degree of each vertex v is $\max_{u\in V-v} r(u, v)$ or $\max_{u\in V-v} r(u, v) + 1$. □

Detachment is an extension of edge-splitting. For a vertex s in a graph G, a *degree specification* $g(s)$ for s consists of a set V_s of new vertices and a function $\rho : V_s \to \mathbb{Z}_+$ such that $\sum_{s'\in V_s} \rho(s') = d(s; G)$. Let $E(s; G)$ be the set of edges incident to s. A $g(s)$-detachment G' of G at s is a graph obtained from G by replacing s with vertices in V_s changing the end vertex of each edge $us \in E(s; G)$ from s to a vertex $s' \in V_s$ so that $d(s'; G') = \rho(s')$ holds for each $s' \in V_s$. In other words, G is regained from G' by contracting V_s into a single vertex s.

Corollary 2 *For a vertex $s \in V$ with $d(s; G) \geq 4$ which has no incident cut-edge in a connected graph $G = (V, E)$, and degree specification $g(s) = (V_s = \{s_1, s_2\}, \{\rho(s_1) = d(s; G)-2, \rho(s_2) = 2\}$, let*

$$r(x, y) = \begin{cases} \lambda(x, y; G) & \text{if } \{x, y\} \cap \{s_1, s_2\} = \emptyset, \\ \min\{\lambda(s, z; G), \rho(s_i)\} & \text{if } \{s_i\} = \{x, y\} \cap \{s_1, s_2\} \text{ and } \{z\} = \{x, y\} - \{s_1, s_2\}, \\ 2 & \text{if } \{x, y\} = \{s_1, s_2\}. \end{cases}$$

Then there is an r-edge-connected $g(s)$-detachment of G at s. Two edges incident to s_2 in it are not parallel unless s is adjacent to only one vertex in G

Proof: By Theorem 2, G has a splittable pair $\{e = us, f = vs\}$. That is, $\lambda(x, y; G^{e,f}) \geq \lambda(x, y; G)$ holds for all pairs $x, y \in V - s$, and $\lambda(s, y; G^{e,f}) \geq \min\{d(s) - 2, \lambda(s, y; G)\}$ holds for all $y \in V - s$. Let G' be a graph obtained from $G^{e,f}$ by regarding s as s_1 and by replacing edge uv with two edges us_2 and vs_2 introducing a new vertex s_2. Observe that $\lambda(x, y; G') = \lambda(x, y; G^{e,f}) \geq r(x, y)$ for vertices x, y with $\{x, y\}\cap\{s_2\} = \emptyset$. We first show that $\lambda(s_1, s_2; G') \geq 2$. Assume $\lambda(s_1, s_2; G') \leq 1$; there is a minimal subset X with $s_2 \in X \subseteq (V - s) \cup \{s_2\}$ and $d(X; G') \leq 1$. By the minimality, X induces a connected component. Suppose h is an edge whose one end vertex is in X and the other is in $V \cup \{s_2\} - (X \cup \{s\})$, i.e., $d(X; G') = 1$. Then removing h disconnects X from the other vertices in G' but does not do so in G. This implies h is a new cut-edge, a contradiction. If $d(X; G') = 0$, then $\lambda(s_1, v; G') = \lambda(s_1, v; G^{e,f}) = 0$ for each $v \in X - s_2$, which contradicts the splittability of $\{e, f\}$. Hence $\lambda(s_1, s_2; G') \geq 2$ holds. Finally we show that $\lambda(s_2, y; G') \geq r(s_2, y) = \min\{\lambda(s, y; G), \rho(s_2) = 2\}$ holds. Assume $\lambda(s_2, y; G') < \min\{\lambda(s, y; G), \rho(s_2) = 2\}$. Then $\lambda(s_2, y; G') \leq 1$. Then by $\lambda(s_1, s_2; G') \geq 2$, there is a subset $Y \in V - s$ with $y \in Y$ and $d(Y; G') = \lambda(s_2, y; G') \leq 1$. This also implies that $\lambda(s_1, y; G') \leq d(Y; G') = \lambda(s_2, y; G') < \min\{\lambda(s, y; G), \rho(s_2) = 2\}$, which is a contradiction to the splittability of $\{e, f\}$. Therefore G' is a desired $g(s)$-detachment of G. □

The next corollary is immediate from Corollary 2.

Corollary 3 *For a vertex $s \in V$ which has no incident cut-edge in a connected graph $G = (V, E)$, and a degree specification $g(s) = (V_s = \{s_1, \ldots, s_p\}, \rho)$ with $\rho(s_1) = d(s; G)-2(p-1) \geq 2$ and $\rho(s_i) = 2$, $s_i \in V_s - s_1$, let*

$$r(x, y) = \begin{cases} \lambda(x, y; G) & \text{if } \{x, y\} \cap V_s = \emptyset, \\ \min\{\lambda(s, z; G), \rho(s_i)\} & \text{if } \{s_i\} = \{x, y\} \cap V_s \text{ and } \{z\} = \{x, y\} - V_s, \\ 2 & \text{if } \{x, y\} \subseteq V_s. \end{cases}$$

Then there is an r-edge-connected $g(s)$-detachment of G at s.

We now introduce a *global detachment* of a graph $G = (V, E)$. Let a degree specification g on V consist of a family $\{V_v \mid v \in V\}$ and a function $\rho : \cup_{v \in V} V_v \to \mathbb{Z}_+$ such that $\sum_{v' \in V_v} \rho(v') = d(v; G)$ for all $v \in V$. A g-detachment G' of G is a graph obtained from G by replacing each $v \in V$ with vertices in V_v changing the end vertex of each edge $uv \in E(v; G)$ from v to a vertex $v' \in V_v$ so that $d(v'; G') = \rho(v')$ holds for each $v' \in V_v$. Hence G is obtained from G' by contracting each V_v into a single vertex v.

Corollary 4 *For a graph $G = (V, E)$, let g be a degree specification such that $\rho(v_1) = d(v; G) - 2(p_v - 1) \geq 2$ and $\rho(v_i) = 2$ $(i = 2, \ldots, p_v)$ hold for $V_v = \{v_1, \ldots, v_{p_v}\}$, where $|V_v| = 1$ if $d(v; G) \leq 3$ or a cut-edge is incident to v. Let*

$$r(x, y) = \begin{cases} \min\{\lambda(u, v; G), \rho(x), \rho(y)\} & \text{if } x \in V_u,\ y \in V_v \text{ with } u \neq v, \\ 2 & \text{otherwise.} \end{cases}$$

Then there is an r-edge-connected g-detachment of G. Furthermore, if $|V| \geq 3$ and $\rho(v_1)$ is uniform for all $v \in V$, then two edges incident to $v_i \in V_v - \{v_1\}$ are not parallel for each $v \in V$.

Proof: In a $g(s)$-detachment of G, there is no new cut-edge as stated in Theorem 2. Hence we can adopt Corollary 3 consecutively for each vertex in G and this gives an r-edge-connected g-detachment of G as required.

Then, let us consider the case where $|V| \geq 3$ and $\rho(v_1) = \rho$ for all $v \in V$. Let $s \in V$ be a vertex with $d(s; G) > \rho$. If s is adjacent to only one vertex (say w), then w has another neighbor in $V - s$. Hence $d(w; G) > d(s; G) > \rho$. This implies that if G has a vertex with degree of more than ρ, G also has a vertex such that the number of its neighbors is at least 2. Hence by repeating to separate such a vertex, we can obtain an r-edge-connected g-detachment of G □

3 Problem with lower capacity

In this section, we consider the problem $(V, r, a, +\infty, c)$. We remark that assuming that edge cost c is metric does not lose the generality as in the steiner network problem, as observed in [5]. For a given cost function c which is not metric, we define a new edge cost $c'(e)$ for each $e \in \binom{V}{2}$ to be the cost of a shortest path between u and v. Then c' is metric, and for any feasible solution E to the problem, we obtain another feasible solution E' with $c(E') \leq c'(E)$ by replacing each edge in E by the edges in the corresponding shortest path. Hence we can reduce each instance to an instance with a metric edge cost. However, we do not assume that c is metric in this section because analysis of our algorithm does not require the triangle inequality.

We construct a feasible solution to this problem $(V, r, a, +\infty, c)$ in two phases. The first phase finds a graph $G_1 = (V, E_1)$ which satisfies only the connectivity requirement, i.e., solve $(V, r, 0, +\infty, c)$. For an $\ell \in \mathbb{Z}_+$ with $a \geq \ell$, an approximate solution G_1 to $(V, r, \ell, +\infty, c)$ also suffices in the first phase. The second phase constructs a graph $G_2 = (V, E_2)$ which satisfies only the lower degree bound, i.e., solve $(V, 0, a, +\infty, c)$, which is the problem called the *a-edge cover problem* and is known to be solved in a polynomial time [9]. Let $G = (V, E_1 \cup E_2)$ be a solution to the original problem $(V, r, a, +\infty, c)$. The approximation factor of this algorithm is the sum of the approximation factors from these two phases. Therefore we have the following theorem.

Theorem 3 *Suppose that there is an α-approximation algorithm for $(V, r, 0, +\infty, c)$. Then problem $(V, r, a, +\infty, c)$ is approximable within a factor of $1 + \alpha$. If there is an α'-approximation algorithm for $(V, r, \ell, +\infty, c)$ with $\ell \in \mathbb{Z}_+$, then problem $(V, r, a, +\infty, c)$ is approximable within a factor of $1 + \alpha'$.*

As mentioned in Section 1, $(V, r, 0, +\infty, c)$ is equivalent to the steiner network problem, which is approximable within a factor of $\alpha = 2$ [7]. Therefore $(V, r, a, +\infty, c)$ is approximable within a factor of 3. For $r = 1$, $(V, 1, 0, +\infty, c)$ is equivalent to the steiner tree problem, a subproblem of the steiner network problem, which is shown to be approximable within a factor of $\alpha = 1.55$ [10]. Hence $(V, 1, a, +\infty, c)$ is approximable within a factor of 2.55. Furthermore, $(V, 1, 1, +\infty, c)$ is the minimum spanning tree problem, which can be solved in polynomial time. Then $(V, 1, a \geq 1, +\infty, c)$ is approximable within a factor of 2.

In the following, we investigate the polyhedral aspect of approximating $(V, 1, a \geq 1, +\infty, c)$. An LP relaxation of this problem is given as

$$\begin{array}{llll} c(\mathrm{LP}) = & \text{minimize} & \sum_{e \in \binom{V}{2}} c(e)x(e) & \\ & \text{subject to} & x(\delta(\{v\})) \geq a(v) & \text{for each } v \in V, \\ & & x(\delta(P)) \geq |P| - 1 & \text{for each partition } P \text{ of } V, \\ & & x \in \mathbb{R}_+^E, & \end{array} \tag{1}$$

where a partition P denotes a set of disjoint nonempty subsets $V_1, \ldots, V_m$ whose union is V, $\delta(P) = \{e = uv \in \binom{V}{2} \mid u \in V_i, v \in V_j, i \neq j\}$, and $x(F) = \sum_{e \in F} x(e)$ for a set F of edges. Let $c(\mathrm{LP})$ denote the minimum cost of the above LP.

Theorem 4 *Let T be a minimum spanning tree and C be a minimum cost a'-edge cover, where $a'(v) = \max\{0, a(v) - d(v; T)\}$, $v \in V$. Then $c(T) + c(C) \leq 2c(\mathrm{LP})$ if $a \geq 1$.*

Proof: First, we show that $c(T) \leq c(\mathrm{LP})$. A connector is defined as a set E of edges such that a graph (V, E) is connected. The connector polytope, i.e., the convex hull of incidence vectors of connectors is known [11] to be represented by

$$\begin{array}{ll} 0 \leq x(e) \leq 1 & \text{for each } e \in \binom{V}{2}, \\ x(\delta(P)) \geq |P| - 1 & \text{for each partition } P \text{ of } V. \end{array} \tag{2}$$

We note that inequality $x(e) \leq 1$ is not necessary in a minimization problem with respect to a nonnegative cost $c \in \mathbb{Q}_+^{\binom{V}{2}}$. Hence the connector polytope contains the feasible region of (1), which implies $c(T) \leq c(\mathrm{LP})$.

Next, we show that $c(C) \leq c(\mathrm{LP})$. Note that the incidence vector of C achieves the minimum cost over all vectors $x \in \mathbb{R}_+^{\binom{V}{2}}$ such that

$$\begin{array}{ll} x(e) \geq 0 & \text{for each } e \in \binom{V}{2}, \\ x(\delta(\{v\})) \geq a'(v) & \text{for each } v \in V, \\ x(E[U]) + x(\delta(U)) \geq \left\lceil \frac{a'(U)}{2} \right\rceil & \text{for each } U \subseteq V \text{ such that } a'(U) \text{ is odd,} \end{array} \tag{3}$$

where $E[U]$ denotes a set of edges whose both end vertices are in U [11]. First and second inequalities in (3) are implied by constraints in (1) apparently. It holds

$$2x(E[U]) + 2x(\delta(U)) \geq 2x(E[U]) + x(\delta(U)) \geq a(U),$$

from (1), which implies

$$x(E[U]) + x(\delta(U)) \geq \frac{a(U)}{2} \geq \left\lceil \frac{a'(U)}{2} \right\rceil.$$

Therefore the third inequality of (3) is also lead by constraints of (1). This means that (3) is a relaxation of the feasible region of (1). Hence we have $c(C) \leq c(\mathrm{LP})$. □

Let us introduce two integer polyhedra. One is a convex hull of incidence vectors of connectors, which is called a *connector polytope* while a *connector* is a set E of edges such that a graph (V, E) is connected [11]. The other is a convex hull of incidence vectors of a-edge covers, which is called *a-edge cover polyhedron* [11]. Theorem 4 implies the following useful corollary.

Corollary 5 *Let T be a minimum spanning tree and $a'(v) = \max\{0, a(v) - d(v;T)\}$, $v \in V$. Minimizing over the Mincowski addition of connector polytope and a'-edge cover polyhedron gives a 2-approximate solution for the problem $(V, 1, a \geq 1, +\infty, c)$.*

For a general r, we can also obtain an analogous result with Theorem 4, i.e., the cost of a solution to (V, r, a, b, c) obtained by our algorithm can be bounded by $3c(\mathrm{LP})$ for an LP relaxation.

4 Problem with upper capacity

In this section, we discuss the approximability of problem $(V, r, 0, b, c)$. Notice that the problem has no feasible solution if there is a vertex $v \in V$ with $b(v) < \max_{u \in V-v} r(u, v)$. Therefore we assume that $b(v) \geq \max_{u \in V-v} r(u, v)$ for each $v \in V$. In addition, we can assume without loss of generality that $\sum_{v \in V} b(v)$ is even. In order to show this fact, let us assume that $\sum_{v \in V} b(v)$ is odd. For such b, any optimal solution $G = (V, E)$ to $(V, r, 0, b, c)$ has a vertex u^* with $d(u^*; G) < b(u^*)$ since $\sum_{v \in V} d(v; G)$ is even. Hence G is also optimal for $(V, r, 0, b', c)$, where $b'(u^*) = b(u^*) - 1$ and $b'(v) = b(v)$ for $v \in V - u^*$. Therefore, any approximation algorithm for instances with even $\sum_{v \in V} b(v)$ can be used to approximate those instances with odd $\sum_{v \in V} b(v)$; Apply the algorithm to at most $|V|$ instances each of which is obtained by decreasing $b(v)$ by 1 for a vertex $v \in V$, and then output the best of the obtained solutions.

Our algorithm for $(V, r \geq 2, 0, b, c)$ consists of the following three phases. The first phase finds an approximate solution G_1 to $(V, r \geq 2, 0, +\infty, c)$, where G_1 is an r-edge-connected graph. Let $V' = \{v \in V \mid |b(v) - d(v; G_1)| \text{ is odd}\}$, where $|V'|$ is even since $\sum_{v \in V} b(v)$ and $\sum_{v \in V} d(v; G_1)$ are even. The second phase computes a minimum cost 1-factor M on $|V'|$, (i.e., solves $(V', 0, 1, 1, c)$), and adds M to G_1 to obtain a graph G_2. At this point, there may be some vertices v that violate degree constraints (i.e., $d(v; G_2) > b(v)$). Note that $|b(v) - d(v; G_2)|$ is even for all $v \in V$. The third phase reduces the degree of each vertex v with $d(v; G_2)$ to at most $b(v)$. This can be done by computing a global g-detachment G of G_2 for a degree specification g such that, for each $v \in V$, $V_v = \{v_1, \ldots, v_{p_v}\}$, $\rho(v_1) = \min\{b(v), d(v; G_2)\}$, and $\rho(v_i) = 2$ $(v_i \in V_v - \{v_1\})$, where $|V_v| = 1$ if $d(v; G_2) \leq b(v)$. Note that G_2 has no cut-edge since it is an r-edge-connected graph with $r \geq 2$. By Corollary 4, we have a g-detachment G of G_2 which preserves the r-edge-connectivity. By neglecting all vertices $v_i \in V_v - \{v_1\}$ $(v \in V)$ (i.e., replacing edges uv_i, v_iu' with uu'), we obtain an r-edge-connected graph G' on V satisfying the degree constraint. The last process may create self-loops, which will be simply eliminated whenever created. Although this may further reduce the degree of a vertex v with $d(v; G_2) \leq b(v)$, the resulting graph G' remains feasible since $a = 0$.

Theorem 5 *Suppose that there is an α-approximation algorithm for $(V, r \geq 2, 0, +\infty, c)$. Problem $(V, r \geq 2, 0, b, c)$ is approximable within a factor of $\alpha + 1/\lfloor k/2 \rfloor$ for $k = \min_{u,v \in V} r(u, v)$.*

Proof: It suffices to show that M is at most $1/K$ times the optimal cost, where $K = \lfloor k/2 \rfloor$. Let G^* be an optimal solution, which is k-edge-connected. It is known that any k-edge-connected graph G^* contains K edge-disjoint spanning trees $\{T_1, \ldots, T_K\}$ [6]. Let $j = \arg\min_{1 \leq i \leq K} c(T_i)$. Then $c(T_j) \leq c(G^*)/K$. Observe that a spanning tree T_j has $|V'|/2$ edge-disjoint paths whose end vertices are V'. By shortcutting intermediate vertices in the paths, we can obtain a 1-factor on V' whose cost is at most $c(T_j) \leq c(G^*)/K$, as required. □

Since $(V, r, 0, +\infty, c)$ is approximable within factor of $\alpha = 2$ [7], $(V, r \geq 2, 0, b, c)$ is approximable within a factor of $2 + 1/\lfloor k/2 \rfloor$.

5 Problem with lower and upper capacities

We now consider the problem $(V, r \geq 2, a, b, c)$ with lower and upper capacities. In general, a detachment from a loop-less graph may give a self-loop. Hence our algorithm in the previous section cannot be applied to this general case. In this section, we show that the problem $(V, r \geq 2, a, b, c)$ is approximable if an upper bound is uniform, i.e., $b(v) = \ell$, $v \in V$ for some $\ell \in \mathbb{Z}_+$.

Theorem 6 *Suppose that there is an α-approximation algorithm for $(V, r \geq 2, a, +\infty, c)$. If $k = \min_{u,v \in V} r(u, v) \geq 2$ and $b(v) = \ell$, $v \in V$ for an $\ell \in \mathbb{Z}_+$, then $(V, r \geq 2, a, b = \ell, c)$ is approximable within a factor of $\alpha + 1/\lfloor k/2 \rfloor$.*

Proof: Let $G = (V, E)$ be an approximate solution of $(V, r, a, +\infty, c)$ and $V' = \{v \in V \mid a(v) = \ell \text{ and } |d(v; G) - a(v)| \text{ is odd}\}$. If $|V'|$ is odd, let $|V'|$ be even by adding a vertex u with $a(u) < \ell$ to V'. Such vertex u exists by the following reason; Suppose $a(v) = b(v) = \ell$ for all $v \in V$. If ℓ is even, then $V' = \{v \in V \mid d(v) \text{ is odd}\}$, which leads to the contradiction that $\sum_{v \in V} d(v)$ is odd. If ℓ is odd, $|V|$ must be even since otherwise problem is infeasible. Because $V' = \{v \in V \mid d(v) \text{ is even}\}$ and $|V'|$ is odd, the size of $V - V' = \{v \in V \mid d(v) \text{ is odd}\}$ is also odd, which leads to the above contradiction again. Hence we can let $|V'|$ be even. Then, compute a minimum cost 1-factor on V' and let it be M.

Let $G' = (V, E \cup M)$. Since $\lambda(u, v; G') \geq r(u, v) \geq 2$ for $u, v \in V$, there is no cut-edge in G'. Hence we can obtain an r'-edge-connected g-detachment of G' by Corollary 4, where $g(v) = \{V_v = \{v_1, \ldots, v_{p_v}\}, \{\rho(v_1) = \min\{d(v; G'), \ell\}, \rho(v_2) = 2, \ldots, \rho(v_{p_v}) = 2\}\}$, $p_v = (d(v; G') - \ell)/2$ if $d(v; G') > \ell$ and 1 otherwise, $r'(x, y) = r(u, v)$ for $x \in V_u$, $y \in V_v$, $u \neq v$ and $r'(x, y) = 2$ otherwise while two edges incident to v_i $(i \geq 2)$ are not parallel. Then splitting two edges incident to v_i $(i \geq 2)$ does not generate any self-loops. Therefore we can obtain a feasible solution for (V, r, a, b, c), whose cost is at most $c(E)$ and $c(M)$.

We can see that $c(E)$ is at most α times the optimal cost of (V, r, a, b, c) because the optimal solutions of (V, r, a, b, c) is feasible for $(V, r, a, +\infty, c)$. It also holds that $c(M)$ is at most $1/\lfloor k/2 \rfloor$ times the optimal cost as stated in Theorem 5. □

From the 3-approximability of $(V, r, a, +\infty, c)$ stated in Section 3, $(V, r \geq 2, a, b = \ell, c)$ is approximable within a factor of $3 + 1/\lfloor k/2 \rfloor$.

6 Problem with degree specification

In this section, we deal with the case where the degree of each vertex $v \in V$ is prescribed, i.e., $a(v) = b(v)$ for all $v \in V$, to which we refer by (V, r, a, a, c). In the following, we first prove that $(V, 1 \leq r \leq 2, a, a, c)$ is approximable, and then show that the argument can be applied to the problem of finding a strongly connected spanning digraph under degree specification. We may denote $(V, 1 \leq r \leq 2, a, a, c)$ by $(V, \{1, 2\}, a, a, c)$. Notice that $(V, \{1, 2\}, 2, 2, c)$ is equivalent to the traveling salesman problem. First, we consider the feasibility of $(V, 1, a, a, c)$.

Theorem 7 *Problem $(V, 1, a, a, c)$ is feasible if and only if $A = \sum_{v \in V} a(v)$ is even, $A \geq 2(|V| - 1)$, and $a(v) \leq A/2$ for each $v \in V$.*

Proof: Necessity is trivial. We show the sufficiency. When $V = \{v_1, v_2\}$ (i.e., $|V| = 2$), theorem apparently holds because the conditions are equivalent to $a(v_1) = a(v_2)$. Therefore let us suppose $|V| \geq 3$. In addition, we assume $a(v) \geq 1$ for each $v \in V$. If $A > 2(|V|-1)$, then prepare an edge zw and let $a(z) := a(z) - 1$ and $a(w) := a(w) - 1$ for a vertex $z = \arg\max_{v \in V} a(v)$ and a vertex $w(\neq z)$ with $a(w) \geq 2$ (such a w exists since $\sum_{v \in V-z} a(v) = A - a(z) \geq A - A/2 = A/2 > |V|-1$). Obviously A remains even. We show that $a(v) \leq A/2$ still holds for each $v \in V$ after the above operation. For $v \in \{z, w\}$, the inequality holds because its both sides are decreased by 1. In order to prove the other case, let us suppose indirectly that there is a vertex $v \in V - \{z, w\}$ with $a(v) > A/2$ after the operation. It must hold $a(v) > (A-2)/2$ for the A before the operation. Then we would have $2 \leq a(w) \leq A - (a(z) + a(v)) \leq A - 2a(v) \leq 1$, a contradiction.

Hence we can assume without loss of generality that $A = 2(|V| - 1)$. From $|V| \geq 3$, $A = 2(|V| - 1) > |V|$ holds, indicating that there is a vertex $w \in V$ with $a(w) \geq 2$. Moreover, from $A = 2(|V| - 1) < 2|V|$, there is a vertex $z \in V$ with $a(z) = 1$. Then prepare an edge wz, set $a(w) := a(w) - 1$, and remove z from V. Also after this, all assumptions hold. Hence it is possible to obtain a connected multigraph by repeating this operation until $|V| = 2$ holds. □

In the following, we propose an algorithm for $(V, \{1, 2\}, a \geq 2, a, c)$. We suppose that $\sum_{v \in V} a(v)$ is even, and $a(v) \leq \frac{1}{2} \sum_{u \in V} a(u)$ for each $v \in V$, which are necessary for the existence of a feasible solution as explained in Theorem 7.

Our algorithm for $(V, \{1, 2\}, a \geq 2, a, c)$ consists of the following two phases. The first phase constructs a minimum cost a-factor F without any self-loops and a hamiltonian cycle H whose cost is at most 2 times the minimum cost of a minimum spanning tree while a-factor denotes a graph in which the degree of each vertex v is exactly $a(v)$. We know that F exists from Theorem 7, and that F and H can be obtained in a polynomial time (Theorem 7 indicates the connectivity of F, but we do not use it). Then, construct graph $G = (V, F \cup H)$, where $d(v; G) = a(v) + 2$ holds for each $v \in V$. The second phase decreases the degree of each vertex v by 2 by splitting a pair of edges incident to v without creating self-loops. Since c is assumed to be metric, this phase does not increase the cost of the graph. The remaining task is to show that such a sequence of edge-splittings exists.

Lemma 1 *Let F be an a-factor on V and H be a hamiltonian cycle on V. It is possible to obtain a 2-edge-connected loopless a-factor whose cost is at most $c(F) + c(H)$.*

Proof: Let $G = (V, F \cup H)$. We call a vertex s with $d(s; G) = a(s) + 2$ an *excess vertex*. Initially all vertices in G are excess vertices. Before finding a sequence of edge-splitting, we remove an arbitrary cycle $C \subseteq F$ from F; such C, which is possibly a pair of multiple edges, exists since $a \geq 2$. (Removing C can be regarded as a series of edge-splittings which ends up by removing a self-loop). Hence we know that the number of vertices which are not excess vertices after this preprocessing is at least two. We call such a vertex *initial non-excess vertex*. In the next, we repeat choosing an excess vertex s in the current graph and splitting a pair of edges incident to s until $d(v; G) = a(v)$ holds for every vertex $v \in V$. The key point of our algorithm is to maintain an edge set H as a Hamiltonian cycle that passes through all excess vertices (possibly together with some non-excess vertices). For this, after splitting a pair of edges incident to an excess vertex, we update two disjoint edge sets H and F so that the following four conditions hold during the sequence of splitting.

(1) G has no self-loops.

(2) H forms a hamiltonian cycle on the set V_H of vertices in H.

(3) V_H contains all excess vertices in G and all initial non-excess vertices.

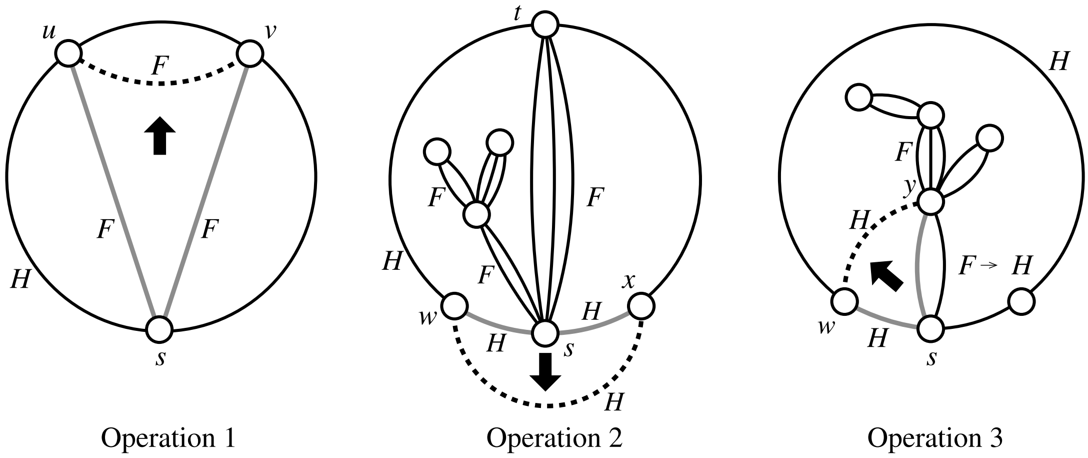

Figure 1: Three operations to split on s (undirected version)

(4) The set $F' \subseteq F$ of edges incident to a non-excess vertex in $V - V_H$ has the following properties: the graph $G' = (V, F')$ has no cut-edge or cycle of length > 2, and each of the components in G' contains exactly one vertex in V_H.

First, let us explain how to split edges incident to an excess vertex, inductively proving that the above conditions are maintained. Let $N(s; \mathcal{E})$ denote the set of vertices adjacent to a vertex s by edges in an edge set $\mathcal{E}$. Suppose that conditions (1)-(4) hold for the current graph $G = (V, F \cup H)$ (these conditions trivially hold for the initial graph $G = (V, F \cup H)$ with $V_H = V$ and $F' = \emptyset$). We then choose an arbitrary excess vertex $s \in V_H$ in the current graph $G = (V, F \cup H)$. Note that $|V_H| \geq 3$ by (3) and $|N(s; H)| \geq 2$ by (2). Moreover s also has at least two incident edges in F, since $d(s; G) = a(s) + 2 \geq 4$. We here distinguish three cases.

Case-1. $|N(s; F - F')| \geq 2$, i.e., there are two edges $us, vs \in F - F'$ with $u \neq v$ and $u, v \in V_H$: We split such a pair $\{us, vs\}$ into a new edge uv, generating no self-loop, and classify the new edge uv into F, without changing H and F' (see Operation 1 in Figure 1). Hence conditions (1)-(4) still hold after the operation.

Case-2. $|N(s; F - F')| = 1$ and there are at least two parallel edges between s and the vertex $\{t\} = N(s; F - F')$: Since $|V_H| \geq 3$, $|N(s; H)| = 2$. Let $\{x, w\} = N(s; H)$. We split the pair $\{xs, ws\}$ of edges in H into a new edge wx, generating no self-loop, and classify the new edge wx into H (see Operation 2 in Figure 1). We can easily see that conditions (1)-(3) also hold after this operation. Hence let us see that condition (4) also holds. Before the splitting, s and t belong to different components G'_s and G'_t in the graph $G' = (V, F')$. After the splitting, s belongs to $V - V_H$ and all parallel edges ts now belong to F', implying that components G'_s and G'_t are merged by these parallel into one component, in which t is the only vertex in V_H. From this construction, we see that the new component satisfies the properties in (4). Hence condition (4) also holds after the splitting, as required.

Case-3. Neither Case 1 nor 2 holds: Then the number of edges incident to s in $F - F'$ is at most 1. Since $d(s; G) = 2 + a(s) \geq 4$, it must be $|N(s; F')| \geq 1$. Let $y \in N(s; F')$ (i.e., $y \in V - V_H$). Note that there are at least two parallel edges between s and y from condition (4). Let $w \in N(s; H)$ be a vertex adjacent to s. Then we split edges $sy \in F$ and $ws \in H$ into a new edge yw without creating a self-loop (see Operation 3 in Figure 1). We classify the new edge yw and an edge parallel to ys into H so that H remains to form a hamiltonian cycle containing all remaining excess vertices and the initial non-excess vertices, where the new V_H includes y after the operation. Hence condition (2) still holds. In the graph G', the component which contained s is divided into two components such that one of which contains s and the other contains y because parallel edges sy move from F' to $F - F'$ after the splitting. It is easy to see that both of new components satisfy the condition stated in (4). Hence condition (4) still holds. Conditions (1) and (3) trivially hold.

Therefore we can split a pair of edges incident to each excess vertex while maintaining the above four conditions. Condition (1) implies the resulting graph has no self-loops. We can see that it is 2-edge-connected from condition (2) and (4). □

We can derive the following theorem immediately from Lemma 1.

Theorem 8 *Problem* $(V, \{1, 2\}, a, a, c)$ *is approximable within a factor of* 3*.*

Proof: It is easy to see that $c(F)$ is at most the optimal cost. Since $c(H)$ is at most two times the cost of the minimum spanning tree and an optimal solutions contains a spanning tree as a subgraph, $c(H)$ is at most two time the optimal cost. Since the cost of the obtained solution is at most $c(F) + c(H)$, the approximation factor of the solution is at most 3 as required. □

The above algorithm indicates a necessary and sufficient condition for the feasibility of $(V, 2, a, a, c)$.

Corollary 6 *There is a 2-edge-connected graph* $G = (V, E)$ *such that* $d(v; G) = a(v)$ *for each* $v \in V$ *if and only if* $\sum_{v \in V} a(v)$ *is even and* $2 \leq a(v) \leq \frac{1}{2} \sum_{u \in V} a(u)$ *for each* $v \in V$*.*

If $a(v)$ is even for all $v \in V$, then we can obtain a hamiltonian cycle whose cost is at most 1.5 times the optimal cost. Then we can approximate the problem within a better factor.

Theorem 9 *If* $a(v)$ *is even for all* $v \in V$*, then problem* $(V, \{1, 2\}, a, a, c)$ *is approximable within a factor of* 2.5*.*

Proof: There is a 1.5-approximation algorithm to obtain a minimum cost hamiltonian cycle [1]. Hence it suffices to show that the minimum cost of hamiltonian cycles is at most the optimal cost of $(V, \{1, 2\}, a, a, c)$. Let G be an optimal solution of $(V, \{1, 2\}, a, a, c)$. Since G is connected and the degree of each vertex is even in G, it has an euler tour in it. If degree of a vertex is more than two in G, then split pairs of edges incident to the vertex while keeping 2-edge connectivity so that the degree is reduced to 2. Then we have obtained a hamiltonian cycle whose cost is at most the cost of G. This indicates that the cost of G is at least the minimum cost of hamiltonian cycles, as required. □

Before closing this section, we show that the above algorithm can be modified to approximate a digraph version of the problem. We denote the in-degree and the out-degree of a vertex v in a digraph G by $d_{\text{in}}(v; G)$ and $d_{\text{out}}(v; G)$, respectively. Given a set V of vertices, a symmetry metric edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$, an in-degree specification $a_{\text{in}} : V \to \mathbb{Z}_+$ and an out-degree specification $a_{\text{out}} : V \to \mathbb{Z}_+$, the problem asks to find a minimum cost strongly connected spanning graph $G = (V, A)$ which has in-degree $d_{\text{in}}(v; G) = a_{\text{in}}(v)$, out-degree $d_{\text{out}}(v; G) = a_{\text{out}}(v)$ for each $v \in V$, and no self-loop.

Theorem 10 *The problem of finding a minimum cost strongly connected spanning digraph with in-degree and out-degree specifications on a symmetric metric cost is approximable within a factor of* 3*.*

Proof: We can assume that if $a_{\rm in} \geq 1$ and $a_{\rm out} \geq 1$ (otherwise the instance is infeasible). The first phase computes a minimum cost loopless digraph $G = (V, A)$ with $d_{\rm in}(v; G) = a_{\rm in}(v)$ and $d_{\rm out}(v; G) = a_{\rm out}(v)$ for all $v \in V$. This can be done by reducing to the minimum cost bipartite matching problem [11]. Moreover compute a directed hamiltonian cycle H whose cost is at most 2 times the optimal cost.

The second phase reduces the in-degree and the out-degree of each vertex by 1. This can be done in a similar way as in Lemma 1. We describe how to reduce the in-degrees and the out-degrees of each vertex by 1 while keeping the strong connectivity, which is mentioned in the proof of Theorem 10. We call a vertex $v \in V$ an *excess vertex* (resp., *a non-excess vertex*) if $d_{\rm in}(s; G) = a_{\rm in}(s) + 1$ and $d_{\rm in}(s; G) = a_{\rm out}(s) + 1$ (resp., $d_{\rm in}(s; G) = a_{\rm in}(s)$ or $d_{\rm in}(s; G) = a_{\rm out}(s)$) in this problem. In the second phase, we first remove an arbitrary directed cycle $C \subseteq A$ from A. From the assumption that $a_{\rm in} \geq 1$ and $a_{\rm out} \geq 1$, this generates at least two initial non-excess vertices. After this preprocessing, we repeat choosing an excess vertex s in the current graph and splitting a pair of arcs entering and leaving s until there is no excess vertex. Again the key point of our algorithm is to maintain an arc set H as a hamiltonian cycle that passes through all excess vertices (possibly together with some non-excess vertices). For this, after splitting a pair of arcs incident to an excess vertex, we update two disjoint arc sets H and A so that the following four conditions hold during the sequence of splitting.

(1) G has no self-loops.

(2) H forms a hamiltonian cycle on the set V_H of vertices in H.

(3) V_H contains all excess vertices in G and all initial non-excess vertices.

(4) The set $A' \subseteq A$ of arcs incident to a non-excess vertex in $V - V_H$ has the following properties: the graph $G' = (V, A')$ contains no directed cycle of length > 2, and each of the undirected components in G' is a strong component and contains exactly one vertex in V_H

First, let us explain how to split arcs incident to an excess vertex, inductively proving that the above conditions are maintained. Let $N_{\rm in}(s; \mathcal{E})$ (resp., $N_{\rm out}(s; \mathcal{E})$) denote the set of vertices adjacent to a vertex s by arcs with the head (resp., tail) of s in an arc set $\mathcal{E}$. Suppose that conditions (1)-(4) hold for the current graph $G = (V, A \cup H)$ (these conditions trivially hold for the initial graph $G = (V, A \cup H)$ with $V_H = V$ and $A' = \emptyset$). We then choose an arbitrary excess vertex $s \in V_H$ in the current graph $G = (V, A \cup H)$. Note that $|N_{\rm in}(s; H)| \geq 1$ and $|N_{\rm out}(s; H)| \geq 1$ by (2), and $|V_H| \geq 3$ by (3). Moreover A contains at least an arc leaving s and one entering s, since $d_{\rm in}(s; G) = a_{\rm in}(s) + 1 \geq 2$ and $d_{\rm out}(s; G) = a_{\rm out}(s) + 1 \geq 2$. In what follows, we distinguish three cases.

Case-1. $N_{\rm in}(s, A - A') \neq N_{\rm out}(s, A - A')$, $N_{\rm in}(s, A - A') \neq \emptyset$ and $N_{\rm out}(s, A - A') \neq \emptyset$, i.e., there are arcs $vs, su \in A - A'$ with $u \neq v$: We split such a pair $\{vs, su\}$ into vu, generating no self-loop, and classify new arc vu into A, without changing H and A' (see Operation 1 in Figure 2). Hence conditions (1)-(4) still hold after the operation.

Case-2. $N_{\rm in}(s, A - A') = N_{\rm out}(s, A - A') = \{t\}$, i.e., there are arcs $ts, st \in A$, where $t \in V_H$: We split a set $\{xs, sw\}$ of arcs in H and classify new arc xw into H (see Operation 2 in Figure 2). Since $|V_H| \geq 3$, it holds new arc xw is not a self-loop, which implies (1). It is easy to check that conditions (2) and (3) also hold. Moreover, the component containing s in G' before the splitting is connected to the component containing t by arcs ts and st, which are reclassified into A'. Since s is reclassified into V_A, the new component contains only one vertex $t \in V_H$ and becomes a strong component, as stated in (4). Hence we can see that condition (4) also holds after the splitting.

Case-3. Neither Case 1 nor 2 holds: It must hold either $N_{\rm in}(s, A') \geq 1$ or $N_{\rm out}(s, A') \geq 1$. Let $N_{\rm in}(s, A') \geq 1$ by symmetry and $y \in N_{\rm in}(s, A')$. From (4), there are both of arcs $ys, sy \in A'$.

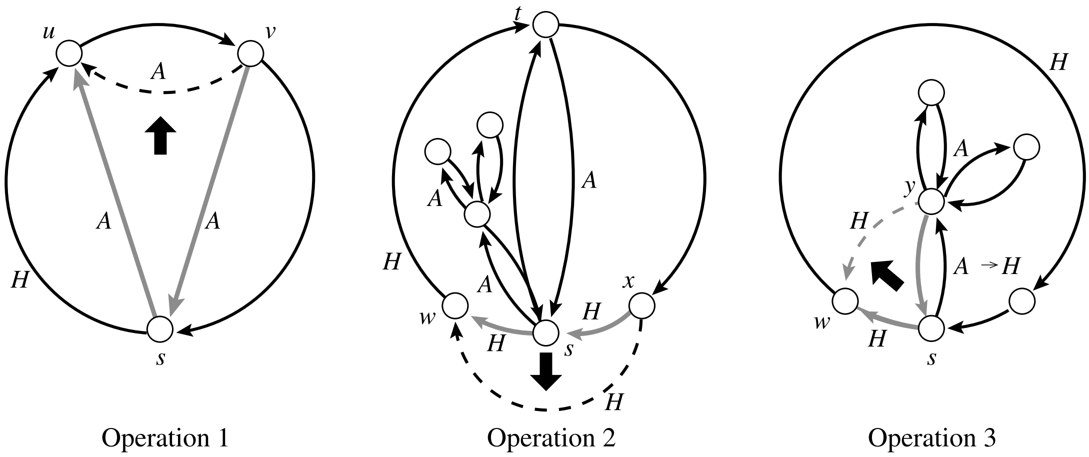

Figure 2: Three operations to split s (directed version)

Then, we split a pair of ys and an arc sw in H. New arc yw and ys are reclassified into H (see Operation 3 in Figure 2). After this, condition (2) holds because H remains to form a hamiltonian cycle over V_H containing y. The component containing s is separated into two strong components such that one of which contains s and the other contains y because arcs sy and ys move from A' to $A-A'$ after the operation. It is easy to see that both of new components satisfies the condition stated in (4). Hence condition (4) remains valid. Conditions (1) and (3) trivially hold.

Therefore, we can split a pair of arcs leaving and entering each vertex while maintaining the above four conditions, as required. □

7 Conclusion

We considered the problem (V, r, a, b, c) of finding a minimum cost undirected multigraph with a connectivity requirement under degree bounds. This framework contains a number of fundamental and important problems. To develop a unified treatment of the framework, we in this paper derived new results on edge-splitting and detachment, and showed several conditions on functions r, a, b and c for which the above problem admits an approximation algorithm.

We still have some open problems. One is to find a constant-factor approximation algorithm for (V, r, a, b, c) where b is not uniform. In this case, edge-splitting may generate a self-loop. Hence we may have to consider a method to split edges without generating self-loops. In Section 4, we considered to approximate $(V, r, 0, b, c)$ under the assumption that $b \geq 2$. However, a constant factor approximation for the case where $b(v)$ can be 1 is still open.

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A Proof of Theorem 2

Our proof is based on a shorter proof of Theorem 1 by A. Frank [4]. In what follows, we suppose that $d(s;G) \geq 4$, which does not lose the generality because $d(s;G) \in \{1,3\}$ does not meet the preconditions ($d(s;G)=1$ indicates existence of a cut-edge) and s has one splittable pair apparently if $d(s;G)=2$. For a vertex set $X \subseteq V$, let $d(X;G)$ (or $d(X)$) denotes the number of edges whose one end vertex is in X and the other end vertex is in $V-X$ in a graph G. Moreover, let

$$R(X) = \max_{u\in X, v\in V-X} r(u,v) \quad \text{and} \quad h(X) = d(X) - R(X).$$

Note that $h(X) \geq 0$ for all nonempty subsets $X \subset V$. A vertex set $X \subseteq V - s$ is called *tight* if $h(X) = 0$, and *dangerous* if $h(X) \leq 1$. A pair $\{su, sv\}$ of edges is called to be *splittable* if $\lambda(x,y;G^{su,sv}) \geq r(x,y)$ for each $x, y \in V$. This is equivalent to that no dangerous set X contains both u and v. For arbitrary $X, Y \subseteq V$, it is known that

$$d(X) + d(Y) = d(X\cap Y) + d(X \cup Y) + 2d(X,Y) \tag{4}$$

and

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X,Y), \tag{5}$$

hold, where $d(X,Y)$ (resp., $\bar{d}(X,Y)$) denotes the number of edges whose one end vertex is in $X-Y$ (resp., $X\cap Y$) and the other end vertex is in $Y-X$ (resp., $V-(X\cup Y)$). Furthermore, the following proposition is proven in [4].

Proposition 1 *For arbitrary $X, Y \subseteq V$, at least one of the following inequalities holds:*

$$R(X) + R(Y) \le R(X \cap Y) + R(X \cup Y), \tag{6}$$

$$R(X) + R(Y) \le R(X - Y) + R(Y - X). \tag{7}$$

From the above, we can also see the following fact.

Proposition 2 *For arbitrary $X, Y \subseteq V$, at least one of the following inequalities holds:*

$$h(X) + h(Y) \ge h(X \cap Y) + h(X \cup Y) + 2d(X, Y), \tag{8}$$

$$h(X) + h(Y) \ge h(X - Y) + h(Y - X) + 2\bar{d}(X, Y). \tag{9}$$

By the following claim, we can assume without loss of generality that the all tight sets are singletons.

Claim 1 *For a tight set T, let $G' = G/T$ be the graph obtained from G by contracting T into a single vertex. Moreover let $e' = u's$ and $f' = v's$ be edges in G' corresponding to $e = us$ and $f = vs$ in G, respectively. Then a pair $\{e, f\}$ is splittable in G if the corresponding pair $\{e', f'\}$ is splittable in G'.*

Proof: For a subset Z of vertices of G for which either $Z \subseteq V - T$ or $T \subseteq Z \subseteq V$, let Z' denote the subset of vertices of G' corresponding to Z. For such a Z, clearly $R(Z') \ge R(Z)$ and $d(Z') = d(Z)$. Therefore if Z is dangerous in G, then Z' is dangerous in G'.

Suppose $\{e, f\}$ is not splittable in G. We prove the claim by indicating that $\{e', f'\}$ is not splittable in G'. There is a dangerous subset X containing both u and v. If $Z := X \cup T$ is dangerous in G, then we are done because Z' is dangerous in G', implying that $\{e', f'\}$ is not be splittable in G', either. Hence let us consider the case where Z is not dangerous, i.e., $h(X \cup T) \ge 2$. For such X and T, (8) cannot hold since otherwise we would have

$$0 + 1 \ge h(T) + h(X) \ge h(X \cap T) + h(X \cup T) \ge 0 + 2.$$

Hence (9) must hold. In this case, we have

$$0 + 1 \ge h(T) + h(X) \ge h(T - X) + h(X - T) + 2\bar{d}(X, T) \ge 0 + 0 + 2\bar{d}(X, T),$$

from which $2\bar{d}(X, T) = 0$ and $h(X - T) \le 1$ follow. By $\bar{d}(X, T) = 0$, we have $u, v \in W := X - T$ while $h(X - T) \le 1$ means that W is dangerous in G. Then W' remains dangerous in G' showing that $\{e', f'\}$ is not splittable in G', as required. □

Claim 2 *Suppose that every tight set consists of one element. Then*

$$r(x, y) = \begin{cases} \min\{d(x), d(y)\} & \text{for } x, y \in V - s, \\ \min\{d(x), d(s) - 2\} & \text{for } x \in V - s \text{ and } y = s. \end{cases}$$

Proof: Let x, y be two distinct vertices in V. Observe that $\lambda(x, y) \le \min\{d(x), d(y)\}$ and that there is a subset $X \subseteq V - s$ such that $|\{x, y\} \cap X| = 1$ and $\lambda(x, y) = d(X)$. If $x, y \in V - s$ (or one of x and y is s and $\lambda(x, s) \ge d(s) - 2$), then $r(x, y) = \lambda(x, y) = d(X)$ holds and X is a tight set, which consists of a single vertex, implying that $\min\{d(x), d(y)\} = d(X) = r(x, y)$. Assume that one of x and y (say y) is s and $\lambda(x, s) > d(s) - 2$. In this case, $r(x, s) = \min\{d(s) - 2, \lambda(x, s)\} = \min\{d(s) - 2, d(x)\}$ since $d(s) - 2 < \lambda(x, s) \le d(x)$. □

Let S denote the set of neighbors of s and $t \in S$ be a vertex of the minimum degree.

Claim 3 *$R(X-t) \geq R(X)$ holds for every set $X \subseteq V$ with $t \in X$ and $|S \cap X| \geq 2$.*

Proof: For any $u \in S \cap (X-t)$, $d(u) \geq d(t)$ holds by the choice of t. $R(X) = r(v,z)$ for some $v \in X$, $z \in V-X$. If $v \neq t$, then $R(X-t) \geq r(v,z) = R(X)$, as required. If $v = t$ and $z = s$, then by Claim 2 we have

$$R(X) = r(t,s) = \min\{d(t), d(s)-2\} \leq \min\{d(u), d(s)-2\} = r(u,s) \leq R(X-t).$$

If $v = t$ and $z \neq s$, then by Claim 2 we have

$$R(X) = r(t,z) = \min\{d(t), d(z)\} \leq \min\{d(u), d(z)\} = r(u,z) \leq R(X-t),$$

as required. □

Assume that $\{st, su\}$ is not splittable for every vertex $u \in S-t$ (otherwise we are done). Let $\mathcal{L}$ be a minimal family of dangerous set containing t so that $\cup_{X \in \mathcal{L}} X \supseteq S$ holds. First, we show that $|\mathcal{L}| = 2$.

Claim 4 *$|\mathcal{L}| \neq 1$.*

Proof: Let $\mathcal{L} = \{X\}$ and $R(X) = r(u,v)$, where $u \in X$ and $v \in V-X$. First assume $v \neq s$. Then $d(X, V-X-s) = d(V-X-s) \geq \lambda(u,v) = r(u,v) = R(X)$. Since

$$d(X) = d(s) + d(X, V-X-s) \geq 4 + d(X, V-X-s) \geq 4 + R(X),$$

X is not dangerous, a contradiction. Next assume $v = s$. Then $R(X) = r(u,s) \leq d(s)-2$, and hence $d(X) \geq d(s) \geq R(X)+2$, a contradiction. □

Claim 5 *For every two members X and Y of $\mathcal{L}$, $|X-Y| = |Y-X| = 1$ and $\bar{d}(X,Y) = 1$.*

Proof: First we show that (9) holds for every two members X and Y of $\mathcal{L}$. Assume that (8) holds (otherwise (9) holds). By the minimality of $\mathcal{L}$, $h(X \cup Y) \geq 2$. Therefore

$$1+1 \geq h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y) \geq 0+2$$

and hence $h(X \cap Y) = 0$ follows, that is, $X \cap Y$ is tight. Hence $X \cap Y = \{t\}$. Since $X-Y = X-t$ and $Y-X = Y-t$, and by Claim 3, it holds $R(X) \leq R(X-Y)$ and $R(Y) \leq R(Y-X)$. Therefore, by (5), $h(X)+h(Y) \geq h(X-Y) + h(Y-X) + 2\bar{d}(X,Y)$, that is, (9) holds.

Then we have

$$1+1 \geq h(X) + h(Y) \geq h(X-Y) + h(Y-X) + 2\bar{d}(X,Y) \geq 0+0+2.$$

Hence $\bar{d}(X,Y) = 1$ and $h(X-Y) = h(Y-X) = 0$. Since both $X-Y$ and $Y-X$ are tight, it holds $|X-Y| = |Y-X| = 1$ from the assumption. □

Claim 6 *$|\mathcal{L}| \leq 2$.*

Proof: Assume $|\mathcal{L}| \geq 3$, and choose three members X_1, X_2, X_3 of $\mathcal{L}$. Let $M = X_1 \cap X_2 \cap X_3$. From Claim 5 and from the minimality of $\mathcal{L}$, it follows that $X_i = M \cup \{x_i\}$ for $1 \leq i \leq 3$ and $\bar{d}(X_i, X_j) = 1$ $(1 \leq i < j \leq 3)$. This implies that the only edge leaving M is st. That is to say, st is a cut edge, contradicting the assumption. □

Claim 7 *If $|\mathcal{L}| = 2$, then there is a splittable pair $\{e = us, f = vs\}$ of edges incident to s such that $u \neq v$.*

Proof: For $\mathcal{L} = \{X, Y\}$, let $X - Y = \{x\}$ and $Y - X = \{y\}$. Because pair $\{xs, ys\}$ of edges is not splittable, there is a dangerous set Z for which $x, y \in Z$, where $t \notin Z$ holds by the minimality of $\mathcal{L}$. Let $M = (X \cap Y) - Z$. Note that $R(Z) + 1 \geq d(Z) \geq d(s) + d(M) - 2$ because $d(s, M) = 1$ and $d(M, V - (X \cup Y \cup \{s\})) = 0$ by Claim 5.

Let us consider the case where $R(Z) = r(s, v)$ with $v \in Z$. Then $d(s) - 2 \geq R(Z)$. Hence it holds

$$d(s) - 1 \geq R(Z) + 1 \geq d(Z) \geq d(s) + d(M) - 2.$$

Therefore $1 \geq d(M)$, which implies that edge st is a cut edge incident to s, a contradiction.

Next, suppose that $R(Z) = r(u, v)$ with $u \in M$ and $v \in Z$. Then $d(M) \geq R(Z)$ and hence

$$d(M) + 1 \geq R(Z) + 1 \geq d(Z) \geq d(s) + d(M) - 2.$$

Therefore we have $3 \geq d(s)$, which contradicts the assumption that $d(s) \geq 4$.

In the end, suppose that $R(Z) = r(u, v)$ with $u \in Z$ and $v \in V - (Z \cup M \cup \{s\})$. Then

$$d(Z) = d(s) - 1 + d(Z, V - (Z \cup M \cup \{s\})) \geq 3 + d(Z, V - (Z \cup M \cup \{s\})).$$

Since it holds that $d(Z, V - (Z \cup M + s)) \geq r(u, v) = R(Z)$, we have $d(Z) \geq R(Z) + 3$, which contradicts the fact that Z is dangerous. □

From Claims 4, 7 and 6, we have proven that there is at least one splittable pair. In addition, edges in such a pair are not parallel if $|S| \geq 2$.

Finally we show that edge-splitting by a splittable pair does not generate a new cut-edge. For a splittable pair $\{e = us, f = vs\}$ of edges in G, assume that $G^{e,f}$ contains a new cut-edge $e' = zw$. If $e' = zw$ is an existing edge in G, then $1 = \lambda(z, w; \lambda(z, w; G^{e,f}) \geq r(z, w)$, implying that $\lambda(z, w; G) = 1$ by the definition of $r(z, w)$, contradicting that zw was not a cut-edge in G. Next assume that $e' = zw$ is a new edge in $G^{e,f}$, i.e., $zw = uv$. In this case, $G^{e,f} - e'$ is not connected and has a component not containing s, implying that s and this component was joined by a cut-edge, a contradiction to the assumption.

This completes the proof of Theorem 2. □