The Traffic Equilibrium Problem with Nonadditive Costs and Its Monotone Mixed Complementarity Problem Formulation *

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Abstract. Various models of traffic equilibrium problems (TEPs) with nonadditive route costs have been proposed in the last decade. However, equilibria of those models are not easy to obtain because the variational inequality problems (VIPs) derived from those models are not monotone in general. In this paper, we consider a TEP whose route cost functions are nonadditive disutility functions of time (with money converted to time). We show that the TEP with the disutility functions can be reformulated as a monotone Mixed Complementarity Problem (MCP) under appropriate conditions. We then establish the existence and uniqueness results for an equilibrium of the TEP. Numerical experiments are carried out using various sample networks with different disutility functions for both the single-mode case and the case of two different transportation modes in the network.

Keywords. Traffic Equilibrium Problem, Variational Inequality Problem, Mixed Complementarity Problem, Monotone, Nonadditive Cost.

1 Introduction

In the study of the traffic equilibrium problem (TEP), the researchers have presented various formulations in which many different assumptions are made to represent the "real" traffic conditions (Aashtiani and Magnanti (1981), Chen, et al. (1999), Dafermos (1980)). One of the standard assumptions used is that *the route costs faced by the users in the network are* ***additive***. That is, *the route costs are simply the sum of the arc costs for all the arcs on the route being considered.*

There are many situations, however, where this additivity assumption on the route costs is inappropriate. Gabriel and Bernstein (1997) discussed some of the situations where nonadditive route costs occur. They claimed that almost all toll and fare schemes being implemented around the world are nonadditive. For example, the different pricing policies

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such as congestion pricing and the collection of emission fees add to the nonadditivity of travel costs. Moreover, different individuals have different valuations of time, which contributes to the nonadditivity of route costs.

Although nonadditivity is important in presenting a more realistic view of the traffic situation, it causes a difficulty in the analysis and computation of an equilibrium, which are usually done by formulating the TEP as the variational inequality problem (VIP). The VIP is generally stated as follows (Facchinei and Pang, 2003): Find a vector $x \in K$ such that

$$(y - x)^T G(x) \geq 0, \quad \forall y \in K, \tag{1.1}$$

where K is a nonempty closed convex subset of $\Re^n$ and $G : K \to \Re^n$ is a continuous function. Special cases of the VIP include the Nonlinear Complementarity Problem (NCP) and the Mixed Complementarity Problem (MCP). The NCP is the VIP with $K = \Re^n_+ \equiv \{x \in \Re^n | x_i \geq 0, i = 1, \ldots, n\}$ and the MCP is the VIP with $K = \{x \in \Re^n | a_i \leq x_i \leq b_i$, $i = 1, \ldots, n\}$, where $a_i \in \Re \cup \{-\infty\}$, $b_i \in \Re \cup \{+\infty\}$, $a_i \leq b_i$, $i = 1, \ldots, n$. We denote the NCP with the function G by NCP(G) and the MCP with the function G and the set K by MCP(G,K).

In the last decades, the VIP has been studied extensively. The monotonicity of G particularly plays an important role in the existence and uniqueness of solutions of VIP. Moreover, the monotonicity is also important for solution methods for VIP to work efficiently. We recall that a function $G : \Re^n \to \Re^n$ is called

(i) **monotone** if $(x - y)^T \big(G(x) - G(y)\big) \geq 0, \forall x, y \in \Re^n$; and

(ii) **strictly monotone** if $(x - y)^T \big(G(x) - G(y)\big) > 0, \forall x, y \in \Re^n (x \neq y)$.

We also say that VIP or NCP or MCP is monotone if G is monotone. Most of the existing results for the VIP rely on the assumption that the function G involved satisfies certain conditions such as strong or strict monotonicity (Facchinei and Pang, 2003).

A VIP equivalent to the TEP with additive costs may usually be formulated as a monotone VIP (Facchinei and Pang, 2003). However, a VIP derived from the TEP with nonadditive costs does not immediately possess monotonicity unless restrictive assumptions are made or a certain reformulation is introduced.

Lo and Chen (2000) considered a special case of the TEP with nonadditive cost functions. Specifically, they introduced a route-specific cost structure, where the route cost is assumed to be the sum of the travel time and an additional charge which is route-specific (a specific travel cost, possibly in the form of toll, is added only to a particular route in the network). This additional cost is only incurred by travelers on that route. They showed that the equivalent NCP becomes monotone. However, they reported that other users of the network (not necessarily using this route) are affected by this added route cost when they share a common link with the route with the added cost. Moreover, the route cost function they considered was very simple, hence not so realistic. In order to solve the TEP, they converted the NCP formulation into an equivalent optimization problem by using a merit function.

Gabriel and Bernstein (1997) proposed a more general route cost function. They also used some assumptions on the route costs in order to ensure monotonicity of their formulation. However, as will be shown in Section 2.3, those assumptions imply that the cost function is an affine function of time. In their work, they proposed a merit function approach to solve the NCP formulation of the TEP with nonadditive costs. Their method was based on transforming the NCP first into a problem of finding a zero of a system of nonsmooth equations. The problem can be solved by using an existing method when the NCP is monotone.

In this paper, we modify the model presented by Gabriel and Bernstein (1997) by introducing a disutility function. We show that the equivalent VIP can be transformed into a monotone MCP, and then give the existence and uniqueness results for the proposed model.

This paper is organized as follows. In the next section, we provide an overview of the important concepts used in this paper, namely, the traffic equilibrium principle, the MCP formulation of the TEP, and the nonadditive travel costs. The proposed TEP and its monotone MCP reformulation are presented in Section 3. We also establish the existence and uniqueness results in this section. Computational results for TEPs with different disutility functions and various networks to compare our reformulation to the original VIP formulation are given in Section 4. We give a brief conclusion in Section 5.

2 Preliminaries

In this section, we introduce some important concepts used in this paper. We also present the general MCP formulation of the TEP.

2.1 Traffic Equilibrium Principle

In what follows we consider the formulation of the traffic equilibrium problem (TEP) with nonadditive route costs. Throughout our discussion, we consider a network $\mathcal{G} = (\mathcal{A},\mathcal{N})$, where $\mathcal{A}$ is the set of arcs (with cardinality $n_{\mathcal{A}}$) and $\mathcal{N}$ is the set of nodes (with cardinality $n_{\mathcal{N}}$). We denote by W the set of origin-destination (OD) pairs in $\mathcal{G}$ (with cardinality n_W). For every OD pair $w \in W$, there corresponds the set R_w of routes connecting the OD pair w. We denote by R the set of all routes (with cardinality n_R), i.e., $R = \bigcup_{w\in W} R_w$. We assume that the network $\mathcal{G}$ is connected, that is, there exists a route between each pair of nodes.

The **Wardrop user equilibrium principle** states that *users of the traffic network will choose the route having the minimum cost between each OD pair, and through this process, the routes that are used will have equal costs; moreover, routes with costs higher than the minimum will have no flow.*

The cost experienced by a person using route r is denoted by C_r. In general, route costs can be a function of the entire vector of route flows. The demand associated with each OD pair w, denoted by D_w, is a function of the vector of minimum OD travel costs.

Mathematically, the Wardrop equilibrium principle, together with the condition imposed on the travel demand function, can be written as

$$0 \leq C_r(F) - u_w \perp F_r \geq 0, \; \forall r \in R_w, \; w \in W, \tag{2.1}$$

$$\sum_{r \in R_w} F_r = D_w(u), \; \forall w \in W, \tag{2.2}$$

$$u_w \geq 0, \; \forall w \in W. \tag{2.3}$$

where $F \in R_+^{n_R}$ is the vector of route flows F_r, u_w is the minimal route cost for the OD pair w, and $u \in R_+^{n_W}$ is the vector with components u_w. The notation "$x \perp y$" means that vectors x and y are orthogonal and thus (2.1) implies $(C_r(F) - u_w)F_r = 0$ for all $r \in R_w, w \in W$.

Here, (2.2) means that the travel demand must be satisfied, while (2.3) indicates that the minimum travel costs must be nonnegative.

2.2 MCP Formulation of the TEP

The traffic user equilibrium problem is to find a vector pair (F,u) of route flows and minimum route costs such that conditions (2.1) – (2.3) are satisfied. When the travel cost and the demand functions $C_r(F)$ and $D_w(u)$ are nonnegative, and for each OD pair $w \in W$,

$$\Big[\sum_{r \in R_w} F_r C_r(F) = 0, F \geq 0 \Big] \Longrightarrow [F_r = 0, \; \forall \, r \in R_w], \tag{2.4}$$

then conditions (2.1) – (2.3) are equivalent to the NCP(H) with the function H defined by

$$H(F, u) \equiv \begin{pmatrix} C(F) - \Gamma u \\ \Gamma^T F - D(u) \end{pmatrix}, \tag{2.5}$$

where $C(F)$ is the vector of route costs $C_r(F)$, and $\Gamma = (\Gamma_{rw})$ is the route-OD pair incidence matrix whose entries are given by

$$\Gamma_{rw} = \begin{cases} 1 & \text{if} \quad r \in R_w \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1. *If the route cost function C_r is positive, then since for each $w \in W$, F_r is positive for some $r \in R_w$, we have $C_r(F) - u_w = 0$ from (2.1), hence, $u_w = C_r(F) > 0$. Thus, the NCP(H) can be rewritten as*

$$\begin{aligned} 0 \leq C(F) - \Gamma u \perp F \geq 0, \\ \Gamma^T F - D(u) = 0, \end{aligned}$$

which is the MCP(H,L) with the set L defined by

$$L = \Re_+^{n_R} \times \Re^{n_W}. \tag{2.5$'$}$$

Various approaches for solving the MCP have been proposed. Those solution methods include the generalized Newton's method (Jiang and Qi, 1997), the smoothing method (Chen, 2000) and the regularization method (Facchinei and Kanzow, 1999). Another method is to reformulate the VIP as a minimization problem by the introduction of a merit function (Gabriel and Bernstein, 1997). Convergence results for these approaches have been established under the key assumption of monotonicity on H.

2.3 Nonadditive travel costs

Previous studies on the TEP focused on the assumption that *the cost on route r is simply the sum of the costs on each arc a comprising the route r*, that is ,

$$C_r(F) = \sum_{a\in\mathcal{A}} \delta_{ar} t_a(f) \text{ for all } r \in R_w, w \in W, \tag{2.7}$$

where δ_{ar} are the elements of the arc-route incidence matrix Δ, i.e.,

$$\delta_{ar} = \begin{cases} 1 & \text{if route } r \text{ passes through link } a \\ 0 & \text{otherwise,} \end{cases}$$

and $t_a(f)$ is the travel time on arc $a \in \mathcal{A}$, f_a is the flow on arc $a \in \mathcal{A}$, and f is the vector of arc flows.

Although the additivity assumption is convenient, there are various situations in which the route costs in the network are no longer additive. A particular case of a nonadditive route cost model considers both time and money in the formulation. Moreover, different individuals normally have different values for time. Hence, the additivity assumption is no longer appropriate for such a case. A detailed discussion on various situations where route costs are nonadditive can be found in Gabriel and Bernstein (1997).

Gabriel and Bernstein (1997) and Larsson, et al. (2002) presented two different formulations of the nonadditive route cost functions:

(i) Gabriel and Bernstein (1997):

$$C_r(F) = \varphi_r\big(\sum_{a\in A} \delta_{ar} t_a(f)\big) + \eta_1 \sum_{a\in A} \delta_{ar} t_a(f) + \Lambda_r(F),\ \forall r \in R_w, w \in W, \tag{2.8}$$

where $\eta_1 > 0$ is the time-based operating costs factor (e.g., gasoline consumption), φ_r is a function which converts time into money, and $\Lambda_r(F)$ is the route-specific financial costs (e.g., tolls) which are allowed to vary in cost according to route flows.

(ii) Larsson et al. (2002):

$$C_r(F) = \sum_{a\in A} \delta_{ar} t_a(f) + \phi_r(m_r),\ \forall r \in R_w, w \in W, \tag{2.9}$$

where m_r is the monetary outlay (e.g., route-specific financial cost which is allowed to vary according to route) and the function ϕ_r converts money into time.

In Gabriel and Bernstein (1997) the route cost function is based on money ("money-based"), while in the formulation of Larsson et al. (2002), the route cost is expressed in terms of time ("time-based"). There has been no clear explanation as to which formulation is better, or as to why the route costs should be represented as such. It has been noted by Bernstein and Wynter (2000), however, that even if one chooses $\phi_r = \varphi_r^{-1}$ in (2.9), this will not make the two formulations equivalent.

We point out that, although the route cost function in Gabriel and Bernstein (1997) is a general form of the route cost function, the assumptions they used to establish its monotonicity are somewhat restrictive. They assumed that there exists a function $\alpha : \Re^{n_R} \to \Re$ such that $\varphi_r'(\omega_r) = \alpha(\omega) \geq 0$, for all $r = 1, \ldots, n_R$, where ω is the vector of route travel times, i.e., $\omega = \Delta^T t(\Delta F)$. This assumption implies that $\alpha(\omega)$ is a constant independent of ω and hence the function φ_r must be affine. To see this, consider ω and $\overline{\omega}$ such that $\overline{\omega}_r = \omega_r$ for all r except for some $\overline{r}$, and $\overline{\omega}_{\overline{r}} = \omega_{\overline{r}} + \delta$. Then $\alpha(\omega) = \varphi_{\overline{r}}'(\omega_{\overline{r}}) = \alpha(\overline{\omega})$. This holds for all δ and for any $\overline{r}$. Therefore, $\alpha(\omega)$ must be constant, and hence, $\varphi_r(\omega)$ is affine.

In our proposed model, we will present a route cost function that can deal with both linear and nonlinear cases by introducing a particular disutility function, and show its monotonicity.

3 TEP with Disutility Functions and Its Monotone MCP Reformulation

In this section, we propose a new formulation of the TEP with nonadditive costs that can be reformulated as a monotone MCP. We then establish the existence and uniqueness result for an equilibrium of this reformulation.

3.1 TEP Model with Disutility Function

We consider a special case of the "time function" given in the form

$$T_r(F) = \sum_{a \in \mathcal{A}} \delta_{ar} t_a(f) + g_r(\Lambda_r), \ \forall r \in R_w, w \in W, \tag{3.1}$$

where Λ_r is the route toll (assumed to be fixed) and g_r is a function that converts money into time. Next, we introduce a disutility function U_w for each OD pair $w \in W$.

We propose the following new route cost function:

$$C_r(F) = U_w\big(T_r(F)\big) = U_w\big(\sum_{a \in \mathcal{A}} \delta_{ar} t_a(f) + g_r(\Lambda_r)\big), \ \forall r \in R_w, w \in W. \tag{3.2}$$

Note that when each disutility function U_w is the identity function, the route cost function (3.2) reduces to the route cost function (2.9) proposed by Larsson et al. (2002). Also, when $g_r(\Lambda_r)$ and $\Lambda_r(F)$ are absent, (3.2) becomes equivalent to (2.8) by letting $U_w(\omega_r) = \varphi_r(\omega_r) + \eta_1 \omega_r$. The model (2.8) may describe more realistic situations than (3.2). However, in (2.8), φ_r must be affine to ensure the monotonicity of the equivalent MCP as pointed out in Section 2.3. We stress that the disutility function U_w includes both linear and nonlinear cases. Moreover, formulation (3.2) can be used to deal with the multimodal TEP where different modes (such as trucks, cars, etc.) use different disutility functions.

In what follows, we make use of (3.2) in order to obtain a monotone MCP reformulation of the TEP.

3.2 A Monotone MCP Reformulation

In this subsection, we present a monotone MCP equivalent to the TEP with (3.2). In the succeeding discussions, we assume that the functions D_w, U_w and C_r are continuous.

We also assume the following conditions for our purpose.

Assumption 1. *For all $w \in W$, the demand function D_w is always positive, $U_w : [0, \infty) \to [0, \infty)$ is a strictly increasing function such that $U_w(0) = 0$ and $\lim_{v\to\infty} U_w(v) = \infty$. Also, for each r, $T_r(F) > 0$ for all $F \geq 0$, and $g_r(\Lambda_r)$ in (3.1) is nonnegative.*

Assumption 1 holds in general, since most network users would prefer the shortest travel time, and hence the disutility function is strictly increasing.

Note that Assumption 1 implies that C_r defined by (3.2) is positive and thus we can reformulate the TEP with (3.2) as the following MCP(H,L):

$$U_w(T_r(F)) - u_w \geq 0,\ F_r \geq 0,\ \big(U_w(T_r(F)) - u_w\big)F_r = 0,\ \forall r \in R_w, w \in W,$$
$$\sum_{r\in R_w} F_r = D_w(u),\ \forall w \in W,$$

where

$$H(F, u) \equiv \begin{pmatrix} U(T(F)) - \Gamma u \\ \Gamma^T F - D(u) \end{pmatrix},$$

$U(T(F)) = \big(\dots, U_w(T_r(F)), \dots\big)^T$, and $L = R_+^{n_R} \times R^{n_W}$.

However, the above MCP formulation is not monotone in general. In what follows, we reformulate MCP(H,L) into an MCP with cost functions $T_r(F)$. We then show that this reformulation is monotone under appropriate conditions.

Proposition 3.1. *Suppose that Assumption 1 holds. Then MCP(H,L) is equivalent to MCP($\widetilde{H}$,L) with*

$$\widetilde{H}(F, v) \equiv \begin{pmatrix} T(F) - \Gamma v \\ \Gamma^T F - D(U(v)) \end{pmatrix} \tag{3.3}$$

and $U(v) = \big(\dots, U_w(v_w), \dots\big)^T$.

Proof. First we show that MCP(H,L) implies MCP($\widetilde{H}$,L). Let (F^*, u) be a solution of MCP(H,L). By Assumption 1, for each $w \in W$ there exists a unique $v_w \geq 0$ such that $U_w(v_w) = u_w$. If $F_r^* > 0$, then $U_w(T_r(F^*)) = u_w = U_w(v_w)$. Thus, $T_r(F^*) = v_w$, and hence $\big(T_r(F^*) - v_w\big)F_r^* = 0$. If $F_r^* = 0$, then $U_w(T_r(F^*)) \geq u_w = U_w(v_w)$. Since U_w is strictly increasing, we have $T_r(F^*) \geq v_w$ and $\big(T_r(F^*) - v_w\big)F_r^* = 0$. Moreover, $\sum_{r\in R_w} F_r - D_w(u) = \sum_{r\in R_w} F_r - D_w\big(U(v)\big) = 0$. Therefore, (F^*, v) is a solution of MCP($\widetilde{H}$,L).

To show that MCP($\widetilde{H}$,L) implies MCP(H,L), let (F^*, v) be a solution of MCP($\widetilde{H}$,L). Then, since, $F_r^* \geq 0$ and $\sum_{r\in R_w} F_r^* = D_w\big(U(v)\big)$, we can find, for each $w \in W$, a route $j_w \in R_w$ such that $F_{j_w}^* > 0$. For such $j_w \in R_w$, $T_{j_w}(F^*) = v_w$. Since a route cost function is assumed to be always positive, we have $T_{j_w}(F^*) > 0$ and $v_w > 0$.

Let $u_w = U_w(v_w)$. Since $U_w(v_w) > 0$, we have $u_w > 0, \forall w \in W$. To complete the proof, we need to show that $U_w(T_r(F^*)) - u_w \geq 0$ and $\big(U_w(T_r(F^*)) - u_w\big)F_r^* = 0$ for all $r \in R_w$.

Now, suppose $F_r^* > 0$. Then $T_r(F^*) = v_w$. This implies that $U_w(T_r(F^*)) = U_w(v_w) = u_w$. Hence, $U_w(T_r(F^*)) = u_w$ and $\big(U_w(T_r(F^*)) - u_w\big)F_r^* = 0$. If $F_r^* = 0$, then $T_r(F^*) \geq v_w$ and $U_w(T_r(F^*)) \geq U_w(v_w)$. Thus, $U_w(T_r(F^*)) - u_w \geq 0$ and $\big(U_w(T_r(F^*)) - u_w\big)F_r^* = 0$. Consequently, (F^*, u) is a solution of MCP(H,L). □

Having shown that MCP(H,L) is equivalent to MCP($\widetilde{H}$,L), in the succeeding discussions we focus our attention to MCP($\widetilde{H}$,L). Note that MCP(H,L) is not monotone in general. However, we can show that under the following additional assumption the MCP($\widetilde{H}$,L) becomes monotone.

Assumption 2. *There exist a nonincreasing function $d_w : \Re \to \Re$ and a strictly increasing function $\bar{t}_a : \Re \to \Re$ such that $D_w(u) = d_w(u_w)$ for each $w \in W$ and $t_a(f) = \bar{t}_a(f_a)$ for each $a \in \mathcal{A}$.*

Assumption 2 means that D_w is a nonincreasing function of u_w only for each $w \in W$, and t_a is an increasing function of arc flow f_a only for each $a \in \mathcal{A}$.

Theorem 3.2. *Suppose that Assumptions 1 and 2 hold. Then MCP($\widetilde{H}$,L) is monotone.*

Proof. Since $\bar{t}_a$ is an increasing function and $t_a(f) = \bar{t}_a(f_a)$ for each $a \in \mathcal{A}$ from Assumption 2, T is monotone. Also, since d_w is a nonincreasing function and $D_w(u) = d_w(u_w)$ for each $w \in W$ from Assumption 2, it follows that $-D\big(U(v)\big)$ is monotone. For any $(F_1, v_1)^T$, $(F_2, v_2)^T \in \Re^{n_R} \times \Re^{n_W}$, we have

$$
\begin{aligned}
&\Big(\widetilde{H}(F_1, v_1) - \widetilde{H}(F_2, v_2)\Big)^T \left(\begin{pmatrix} F_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} F_2 \\ v_2 \end{pmatrix} \right) \\
&= \Big(T(F_1) - T(F_2)\Big)^T (F_1 - F_2) - \Big(\Gamma v_1 - \Gamma v_2\Big)^T (F_1 - F_2) \\
&\quad + \Big(\Gamma^T (F_1 - F_2)\Big)^T (v_1 - v_2) - \Big(D(U(v)) - D(U(v))\Big)^T (v_1 - v_2) \\
&= \Big(T(F_1) - T(F_2)\Big)^T (F_1 - F_2) - \Big(D(U(v)) - D(U(v))\Big)^T (v_1 - v_2) \geq 0,
\end{aligned}
$$

where the last inequality follows from the monotonicity of $T(F)$ and $-D(U(v))$. Hence MCP($\widetilde{H}$,L) is monotone. □

Using a similar argument, we can show the following result.

Corollary 3.3. *If $D_w(u)$ is constant for each $w \in W$, then MCP($\widetilde{H}$,L) is monotone.* □

3.3 Existence and Uniqueness Results

In this subsection, we present some existence and uniqueness results for our proposed model (3.2).

The first result ensures that MCP($\widetilde{H}$,L) has a solution, i.e., our model has an equilibrium. To prove it, we make use of a result by Facchinei and Pang (2003).

Assumption 3. *The function C_r defined by (3.2) is nonnegative, and D_w is bounded above on the set $\{u \in \Re^{n_W} | u > 0\}$.*

Theorem 3.4. *Suppose Assumption 3 holds. Then MCP($\widetilde{H}$,L) has a nonempty bounded solution set. Moreover, if Assumptions 1 and 2 hold, the set of solutions is convex.*

Proof. Since C_r is nonnegative and D_w is bounded above by Assumption 3, it follows from Proposition 2.2.14 in Facchinei and Pang (2003) that MCP($\widetilde{H}$,L) has a solution. Next we show that the solution set is bounded. Let S be its solution set and let $S_F = \{F | (F, v) \in S\}$ and $S_v = \{v | (F, v) \in S\}$. Since $\sum_{r \in R_w} F_r = D_w(U(v))$, $\forall w \in W$ for all $(F, v) \in S$ and, by assumption, the demand function D is bounded, it follows that S_F is bounded. Moreover, we note that $T_r(F) \geq v_w \geq 0$, $\forall r \in R_w, w \in W$, from Assumption 1 and the definitions of MCP($\widetilde{H}$,L). Hence, S_v is bounded since S_F is bounded. Thus, the solution set S of MCP($\widetilde{H}$,L) is bounded.

Suppose that Assumptions 1 and 2 hold. Since MCP($\widetilde{H}$,L) is monotone by Theorem 3.2, it follows that the set of solutions is convex (Facchinei and Pang, 2003). □

Next we show that the set of solutions of MCP($\widetilde{H}$,L) is a singleton under the following assumption together with Assumption 1.

Assumption $2'$. *There exist a strictly decreasing function $d_w : \Re \to \Re$ such that $D_w(u) = d_w(u_w)$ for each $w \in W$. Moreover, $T(F) = (\ldots, t_a(F), \ldots)^T$ is a strictly monotone function.*

Under Assumption $2'$, both T and $-D(U(\cdot))$ are strictly monotone.

Theorem 3.5. *Suppose that Assumptions 1, $2'$ and 3 hold. Then MCP($\widetilde{H}$,L) has a unique solution.*

Proof. It follows from Theorem 3.4 that MCP($\widetilde{H}$,L) has a solution. To show that this solution is unique, let $x_1 = (F_1^T, v_1^T)^T$ and $x_2 = (F_2^T, v_2^T)^T$ be two solutions of MCP($\widetilde{H}$,L). Since x_1, x_2, $\widetilde{H}(x_1)$ and $\widetilde{H}(x_2)$ are nonnegative, from the complementarity conditions $x_1^T \widetilde{H}(x_1) = 0$ and $x_2^T \widetilde{H}(x_2) = 0$, we have

$$(x_1 - x_2)^T \big(\widetilde{H}(x_1) - \widetilde{H}(x_2)\big) \leq 0.$$

From the definition (3.3) of $\widetilde{H}$ and x, the above inequality can be rewritten as

$$(F_1 - F_2)^T \big(T(F_1) - \Gamma v_1 - T(F_2) + \Gamma v_2\big) + (v_1 - v_2)^T \big(\Gamma^T F_1 - D(U(v_1)) - \Gamma^T F_2 + D(U(v_2))\big) \leq 0,$$

which implies that

$$(F_1 - F_2)^T \big(T(F_1) - T(F_2)\big) + (v_1 - v_2)^T \big(- D(U(v_1)) + D(U(v_2))\big) \leq 0. \tag{3.4}$$

Since T and $-D_w(U(\cdot))$ are strictly monotone from Assumption $2'$, the inequality (3.4) implies that $F_1 = F_2$ and $v_1 = v_2$. Therefore, the solution set is a singleton. □

4 Numerical Results

In this section, we present our computational results. Under Assumption 1, we can easily verify that NCP(H) and NCP($\widetilde{H}$) are equivalent to MCP(H,L) and MCP($\widetilde{H}$,L), respectively. In our numerical experiments, we try to obtain an equilibrium solution of the TEP with (3.2) by solving NCP(H) and NCP($\widetilde{H}$) instead of MCP(H,L) and MCP($\widetilde{H}$,L). To solve NCPs, we use the Generalized Newton Method (GNM) of Jiang (1999). The main reason for choosing the GNM is due to the fact that our proposed model satisfies monotonicity properties, and the GNM has nice convergence properties under these conditions.

The numerical experiments consist of two parts. In the first part we check the validity of our model by comparing it with the traditional model with additive costs. Here, we use a network with two transportation modes. Our model uses different disutility functions for the different transportation modes.

In the second part of the experiments, we aim to find a solution for the two NCP formulations, namely, NCP(H) and NCP($\widetilde{H}$), in order to compare the two formulations.

The coding was done in Matlab 6.5. In our experiments, we used three different sample networks (Figures 1, 2 and 3). The network shown in Figure 1 is taken from Chen et al. (1999), the one shown in Figure 2 is taken from Yang (1997) and the one in Figure 3 is taken from Yang and Bell (1997).

Figure 1: The 7-arc Network A.

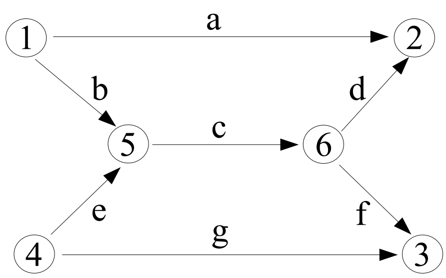
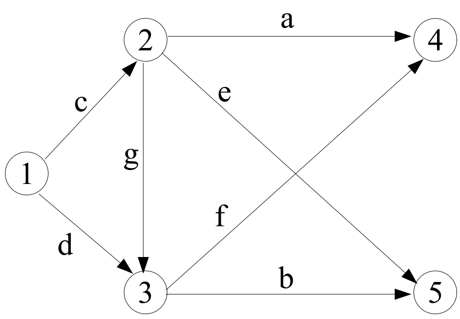


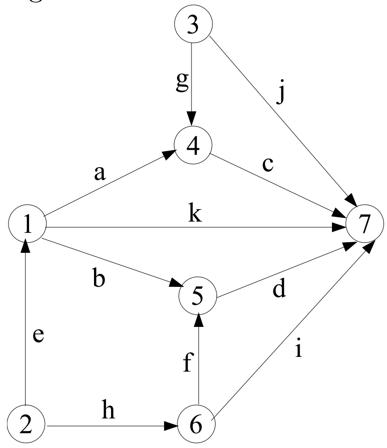
Figure 2: The 7-arc Network B.



The routes and OD pairs are given in Table 1. The demand function used is

$$D_w(u_w) = -b_w^1(exp(-b_w^2 u_w)),$$

Figure 3: The 11-arc Network.



where b_w^1 and b_w^2 are given in Table 2 (for the 2-mode case) and Table 4 (for the single-mode case). The arc cost function used is

$$t_a(f) = c_w^1(1.0 + 0.15(f/c_w^2)^4),$$

where c_w^1 and c_w^2 are given in Table 3 (for the 2-mode case) and Table 5 (for the single-mode case).

Table 1: Network routes and OD pairs.

Network	OD pair	Route
7-arc A	1-2	{a}, {b,c,d}
	1-3	{b,c,f}
	4-2	{c,d,e}
	4-3	{c,e,f}, {g}
7-arc B	1-4	{c,f,g}, {a,c}, {d,f}
	1-5	{b,c,g}, {c,e}, {b,d}
11-arc	1-7	{a,c}, {b,d}
	2-7	{h,i}, {b,d,e}, {e,k}, {a,c,e}
	3-7	{j}, {c,g}
	6-7	{i}, {d,f}

Table 2: Coefficients of the demand function of the 7-arc Network A for the 2-mode case.

Coefficients of the demand function	OD pair							
	MODE A				MODE B			
	1-2	1-3	4-2	4-3	1-2	1-3	4-2	4-3
b_w^1	400	400	400	400	400	400	400	400
b_w^2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05

Table 3: Coefficients of the arc cost function of the 7-arc Network A for the 2-mode case.

Coefficients of the demand function	Arc						
	a	b	c	d	e	f	g
c_w^1	60	10	5	8	12	5	70
c_w^2	200	300	700	300	300	300	200

Table 4: Coefficients of the demand function for the single-mode case.

Network	Coefficients of the demand function	
	b_w^1	b_w^2
7-arc A	600	0.04
	500	0.03
	500	0.05
	400	0.05
7-arc B	200	0.2
	220	0.2
11-arc	600	0.04
	500	0.03
	500	0.05
	400	0.05

4.1 Comparison of the Proposed Model and the Traditional Model

We have tested the validity of our proposed formulation. In this experiment, we compare our proposed model to that of the traditional model on the 7-arc Network A with two transportation modes.

In this experiment, we suppose that both modes have the same OD pairs (Table 1), set of routes (Table 1) and demand functions (Table 2). For the traditional model, we use the same route cost functions for both modes A and B, that is, $C_r(F) = T_r(F)$. On the other hand, for the proposed model we use different route cost functions for mode A and for mode B, that is, $C_r^A(F) = T_r(F)$ and $C_r^B(F) = T_r(F) + 0.001(T_r(F))^2$, respectively.

The results are shown in Table 6. In the table, $F_r^A, r = 1, \ldots, 6$, stand for the route flows corresponding to mode A, and $F_r^B, r = 1, \ldots, 6$, stand for the route flows corresponding to mode B. The results show that, compared to the route flows for the traditional TEP model, there is a significant difference in the route flows of the two modes for our proposed model. As expected, the routes with lower travel costs (i.e., lower disutility function values) have higher route flows (in the case of mode A), while routes with higher disutility function values have lesser flows (in the case of mode B).

Table 5: Coefficients of the arc cost function for the single-mode case.

Network	Arc	Coefficients of the Arc Cost Function	
		c_w^1	c_w^2
7-arc A	a	60	50
	b	10	2
	c	5	6
	d	8	2
	e	12	13
	f	5	6
	g	70	60
7-arc B	a	6	15
	b	4	15
	c	3	30
	d	5	30
	e	6	15
	f	4	15
	g	1	15
11-arc	a	6	200
	b	5	200
	c	6	200
	d	7	200
	e	6	100
	f	1	100
	g	5	150
	h	10	150
	i	11	200
	j	11	200
	k	15	200

4.2 Comparison of NCP(H) and NCP($\widetilde{H}$) Formulations

We have also compared the NCP formulations of the TEP, namely, NCP(H) and NCP($\widetilde{H}$). The networks are tested using nonlinear link cost functions, an elastic demand function and various disutility functions. Here we introduce two disutility functions, namely,

(i) $U_w(T_r(F)) = (T_r(F))^2$; and

(ii) $U_w(T_r(F)) = T_r(F) + 0.01(T_r(F))^2$

for the route cost functions on each network.

The computational results are shown in Tables 7 and 8. In these tables, "NETWORK" stands for the sample network used, the columns NCP(H) and NCP($\widetilde{H}$) under "RESID-

Table 6: Route flows of 7-arc Network A for the 2-mode case.

Route Flow	MODEL	
	Traditional	Proposed
F_1^A	0.0000	0.0000
F_2^A	75.8216	76.8721
F_3^A	101.9756	103.2007
F_4^A	144.9559	146.1842
F_5^A	104.7306	105.5160
F_6^A	0.0000	0.0000
F_1^B	0.0000	0.0000
F_2^B	75.8216	72.8016
F_3^B	101.9756	99.4810
F_4^B	144.9559	143.2517
F_5^B	104.7306	101.8342
F_6^B	0.0000	0.0000

UAL" respectively show the values of the residuals for the two NCP formulation. The residual is defined as $r(x) = |x^T H(x)| + \sum_{i=1}^{n_R+n_W} \min\{0, x_i\} + \sum_{i=1}^{n_R+n_W} \min\{0, H_i(x)\}$ and it is computed in order to evaluate the quality of the solutions. Therefore, the residuals should be as small as possible; a value very close to zero is ideal.

Table 7: Residuals when $U_w(T_r(F)) = (T_r(F))^2$.

NETWORK	initial point of GNM	RESIDUAL	
		NCP(H)	NCP($\widetilde{H}$)
7-arc A	(0,0,0,0,0,0,0,0,0,0)	7.047E-003	1.1479E-005
	(1,1,1,1,1,1,1,1,1,1)	6.645E-003	1.7912E-005
	(10,10,10,10,10,10,10,10,10,10)	3.499E-003	2.5162E-005
7-arc B	(0,0,0,0,0,0,0,0)	2.1822E-009	8.0737E-006
	(1,1,1,1,1,1,1,1)	1.9485E-009	2.9617E-008
	(10,10,10,10,10,10,10,10)	1.7703E-009	2.9615E-008
11-arc	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	6.5077E-009	1.4218E-006
	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	9.8253E-007	2.9473E-009
	(10,10,10,10,10,10,10,10,10,10,10,10,10,10,10)	7.5163E-007	6.5297E-007

We have also tested our proposed reformulation for the case where there are two different transportation modes in the network. In this example, we use Network A (Figure 1). The results for this case are shown in Table 9.

In both cases, the results reveal that our proposed reformulation NCP($\widetilde{H}$) success-

Table 8: Residuals when $U_w(T_r(F)) = T_r(F) + 0.01(T_r(F))^2$.

NETWORK	initial point of GNM	RESIDUAL	
		NCP(H)	NCP($\widetilde{H}$)
7-arc A	(0,0,0,0,0,0,0,0,0,0)	3.9814E+004	6.1748E-013
	(1,1,1,1,1,1,1,1,1,1)	3.8881E+004	4.3999E-014
	(10,10,10,10,10,10,10,10,10,10)	1.2173E-008	1.7146E-011
7-arc B	(0,0,0,0,0,0,0,0)	8.7901E-006	5.0343E-009
	(1,1,1,1,1,1,1,1)	9.8626E-007	4.2881E-010
	(10,10,10,10,10,10,10,10)	9.1695E-007	1.1765E-012
11-arc	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	3.6525E-006	1.5524E-006
	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	3.6775E-006	1.0126E-008
	(10,10,10,10,10,10,10,10,10,10,10,10,10,10,10)	2.0176E-006	1.6949E-008

Table 9: Residuals for the 7-arc Network A for the case when there are 2 modes of transportation using the routes in the network.

NETWORK	initial point of GNM	RESIDUAL	
		NCP(H)	NCP($\widetilde{H}$)
$U_w(T_r(F)) = (T_r(F))^2$	(0,0,0,0,0,0,0,0,0,0)	2.3817E-002	2.2658E-006
	(1,1,1,1,1,1,1,1,1,1)	8.8063E-004	3.2301E-005
	(10,10,10,10,10,10,10,10,10,10)	4.2729E-002	3.3028E-005
$U_w(T_r(F)) = T_r(F) + 0.01(T_r(F))^2$	(0,0,0,0,0,0,0,0,0,0)	1.9057E+004	7.9972E-009
	(1,1,1,1,1,1,1,1,1,1)	1.8794E+004	7.9962E-009
	(10,10,10,10,10,10,10,10,10,10)	9.1208E-010	7.9978E-009

fully yields an equilibrium of the original TEP as evident by the computed residual for each formulation (see for example, in Table 8 for the 7-arc A Network and Table 9 for $U_w(T_r(F)) = T_r(F) + 0.01(T_r(F))^2$). However, we have a difficulty in obtaining an equilibrium of the TEP by solving NCP(H) as it lacks the monotonicity.

5 Conclusions

In this paper, we have formulated the TEP with nonadditive route costs by introducing a disutility function, then presented its monotone MCP reformulation. For this reformulation, we have established the existence and uniqueness of the equilibrium of the proposed model. Moreover, we have shown through numerical experiments that our new

MCP reformulation is useful in identifying an equilibrium of the TEP.

Extending our monotone MCP formulation for the TEP with nonadditive costs to the more general multiclass network or cost functions will be another important topic to explore.

References

Aashtiani, H.Z., Magnanti, T.L., 1981. Equilibria on a congested transportation network. SIAM Journal on Algebraic and Discrete Methods 2, 213–226.

Bernstein, D., Gabriel, S. A., 1997. Solving the nonadditive traffic equilibrium problem. In: Pardalos, P.M., Hearn, D.W., Hager, W.W., (Eds.), Lecture Notes in Economics and Mathematical Systems: Proceedings of the Network Optimization Conference. Springer, New York, 72–102.

Bernstein, D., Wynter, L., 2000. Issues of uniqueness and convexity in non-additive bi-criteria equilibrium models. In: Paper Presented at the 8th Meeting of the EURO Working Group on Transportation, Rome Jubilee 2000 Conference, La Sapienza, Rome, Italy.

Chen, M., Bernstein, D. H., Chien, S. I. J., Mouskos, K., 1999. Simplified formulation of the toll design problem. Transportation Research Record 1667, 88–95.

Chen, X., 2000. Smoothing methods for complementarity problems and their applications: A survey. Journal of the Operations Research Society of Japan 43, 32–47.

Dafermos, S., 1980. Traffic equilibrium and variational inequalities. Transportation Science 14(1), 42–54.

Facchinei, F., Kanzow, C., 1999. Beyond monotonicity in regularization methods for complementarity problems. SIAM Journal of Control and Optimization 37, 1150–1161.

Facchinei, F., Pang, J.S., 2003. Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer-Verlag, New York.

Ferris, M.C., Kanzow, C., 1998. Complementarity and related problems: A survey. Mathematical Programming Technical Report, 98-17.

Gabriel, S. A., Bernstein, D., 1997. The traffic equilibrium problem with nonadditive path costs. Transportation Science 31(4), 337–348.

Jiang, H., 1999. Global convergence analysis of the generalized Newton and Gauss-Newton methods of the Fischer-Burmeister equation for the complementarity problem. Mathematics of Operations Research 24(3), 529–543.

Jiang, H., Qi, L., 1997. A new nonsmooth equations approach to the nonlinear complementarity problems. SIAM Journal of Control and Optimization 35(1), 178–193.

Larsson, T., Lindberg, P. O., Lundgren, J., Patriksson, M., Rydergren, C., 2002. On traffic equilibrium models with a nonlinear time/money relation. In: M. Patriksson and M. Labbé Eds.), Transportation Planning – State of the Art. Kluwer Academic Publishers, Boston, MA., 19–31.

Lo, H.K., Chen, A., 2000. Traffic equilibrium problem with route-specific costs: Formulation and algorithms. Transportation Research Part B 34, 493–513.

Moré, J.J., 1996. Global methods for nonlinear complementarity problems. Mathematics of Operations Research 21(3), 589–614.

Patriksson, M., 2004. Algorithms for computing traffic equilibria. Networks and Spatial Economics 4, 23–38.

Yang, H., 1997. Sensitivity analysis for the elastic-demand network equilibrium problem with applications. Transportation Research Part B 31(1), 55–70.

Yang, H., Bell, M.G.H., 1997. Traffic restraint, road pricing and network equilibrium. Transportation Research Part B 31(4), 303–314.