Duality in Option Pricing Based on Prices of Other Derivatives*

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Abstract

We clarify a financial meaning of duality in the semi-infinite programming problem which emerges in the context of determining a derivative price range based only on the no-arbitrage assumption and the observed prices of other derivatives. The interpretation links studies in the above context to studies in stochastic models.
Keywords: Option pricing, Hedging, Duality, Semi-infinite programming

1 Introduction

One of the most important issues in financial economics is to derive an appropriate price of a derivative security, which is called option pricing. Option pricing is based on the well-known fundamental assumption that the market is no-arbitrage, which intuitively means that we cannot increase a value of our portfolio without any risk. Under the no-arbitrage assumption, a derivative price must be the same as a value of a portfolio that replicates the derivative if such a hedging portfolio exists. In addition to the no-arbitrage assumption, many option pricing methods assume some stochastic differential equations for prices of risky assets. A typical approach, the Black-Sholes model introduced in [5] and [12] assumes a geometric Brownian motion for the risky stock price. By this assumption, every derivative can be replicated by a portfolio consisting of the risk-free bond and the underlying stock, and therefore has a unique price equal to the price of the hedging portfolio. However, it is well

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known that a stock price in the actual market does not obey the geometric Brownian motion. For example, a log-return of a stock displays a heavy tailed distribution different from a Gaussian distribution. It seems hard to find a stochastic differential equation that perfectly fits the dynamics of an asset price.

Thus, a natural question that arises is to derive a derivative price range based only on the no-arbitrage assumption and the observed prices of other derivatives without assuming any stochastic model for the dynamics of asset prices. This question has been studied in [6], [9] and [11]. They derived upper and lower bounds on option prices consistent with given mean and (co)variance of the underlying asset prices under a risk-neutral measure. Bertsimas and Popescu [3] showed that the question can be well treated in the framework of an SILP (semi-infinite linear programming problem). They furthermore showed that several problems are reducible to an SDP (semi-definite programming problem) by using duality in the SILP. By the same duality technique, Han et al. [10] investigated a case in which a derivative is written on multi-assets. While all studies mentioned above have treated the case of a single maturity, Bertsimas and Bushueva [1, 2] derived an option price range consistent with the prices of other derivatives with distinct maturities. This type of study is also related to a study of implied models proposed in [8], [7] and [14] in the sense that both studies use the observed prices of derivatives.

This paper gives a financial interpretation of duality of the SILP, which has been used only from the computational profit in the previous studies [3] and [10]. We show that the dual problem is related to a hedging strategy called a buy-and-hold hedging portfolio. This financial interpretation also explains the relationship between the approach based only on the no-arbitrage assumption and the observed prices of derivatives and the usual stochastic approach such as the B. S. model.

This paper is organized as follows. Section 2 gives a brief review of the results in [3], after introducing two financial market models and notations. Section 3 describes the financial interpretation of duality in the SILP.

2 Preliminaries

This section introduces two financial market models, and then gives a brief explanation for the previous results in [3]. We first introduce notations and two models which will be used throughout this paper.

Notation Let $T > 0$ and let m be a positive integer. Let Φ^T and F_i^T denote simple claims written on m risky assets with exercise date T and payoff functions ϕ and $f_i : R_+^m \mapsto R_+$, respectively. Prices of Φ^T and F_i^T at time t are $\Phi^T(t)$ and $F_i^T(t)$ respectively. Let $\Delta(R_+^m)$ denote the set which consists of all probability measures on the Borel set $(R_+^m, \mathcal{B}(R_+^m))$, and Q denotes a risk-neutral measure.

Model A Assume a no-arbitrage financial market which consists of m risky assets and one risk-free asset with constant risk-free rate $r(t) = 0$. The price process of m risky assets $S(t)$ is an m dimensional $\mathcal{F}_t$ adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t}, P)$.

Model B In addition to the assumptions in Model A, $S(t)$ follows stochastic differential equations under P such that

$$P\left(S(t) \in \{|x - a| < \epsilon\}\right) > 0 \quad (t, \epsilon > 0,\ a \in R_+^m).$$

We can take any deterministic function for the risk-free rate $r(t)$, but we assume $r(t) = 0$ without loss of generality. While Model A is based only on the no-arbitrage assumption, Model B includes conventional models such as the B. S. model assumed in many studies in option pricing.

Since the market is no-arbitrage in both models, there exists a risk-neutral measure Q. By using Q, the price of Φ^T at time t must be expressed as

$$\Phi^T(t) = E^Q[\phi(S(T))|\mathcal{F}_t], \tag{1}$$

which follows from the Fundamental Theorem in option pricing (for instance, refer to p.133 – p.153 in [4]). In Model A, the problem of finding the supremum on prices of a simple claim Φ^T consistent with observed prices of F_i^T is described as follows:

$$\begin{array}{lll} \text{maximize }_{Q(\sim P)} & E^Q[\phi(S(T))] & \\ \text{subject to} & E^Q[f_i(S(T))] = q_i & (i = 1, 2, \ldots, n), \end{array} \tag{2}$$

where $Q \sim P$ means that Q is a probability measure equivalent to P. We can also consider the problem of finding the infimum on $\Phi^T(0)$ consistent with $F_i^T(0)$ by replacing *maximize* with *minimize* in (2). Since no particular dynamics of $S(t)$ under P is given, the property of equivalence restricts nothing. Problem (2) with respect to a probability measure Q on $(\Omega, \mathcal{F})$ can be reduced to the following SILP with respect to a probability measure μ on $(R_+^m, \mathcal{B}(R_+^m))$.

$$\begin{array}{lll} \text{maximize }_{\mu \in \Delta(R_+^m)} & \displaystyle\int_{R_+^m} \phi(x)\mathrm{d}\mu & \\ \text{subject to} & \displaystyle\int_{R_+^m} f_i(x)\mathrm{d}\mu = q_i & (i = 1, 2, \ldots, n). \end{array} \tag{3}$$

The dual of problem (3) is the following SILP:

$$\begin{array}{lll} \text{minimize }_{y \in R^{n+1}} & \displaystyle y_0 + \sum_{i=1}^n q_i y_i & \\ \text{subject to} & \displaystyle y_0 + \sum_{i=1}^n y_i f_i(x) - \phi(x) \ge 0 & (\forall x \in R_+^m). \end{array} \tag{4}$$

Problems (3) and (4) have the same optimal value if ϕ and f_i are continuous functions having a compact support. By using this duality in the SILP, Bertsimas and Popescu [3] reduced the problem of finding the supremum and the infimum on $\Phi^T(0)$ consistent with $F_i^T(0)$ in Model A to an SDP for $n = 1$, when ϕ and f_i are continuous piecewise polynomials with compact support. The problem for $n > 1$ has been reduced to solving a sequence of SDPs via the same duality in [10].

However, the previous studies have employed the dual problem only from the computational advantage and lack a financial interpretation of the duality. The next section describes our results which reveal financial importance of the duality in terms of a buy-and-hold hedging portfolio. The dual viewpoint gives another importance of the problem of finding a derivative price range based only on the no-arbitrage assumption and other derivative prices.

3 Financial Interpretation of Duality

This section clarifies the financial meaning of the duality between problems (3) and (4). We can actually show that problem (4) itself is a meaningful problem of finding the minimum investment cost of buy-and-hold super-hedging portfolios in Model B. We can also show that problem (4) finds an arbitrage buy-and-hold strategy if the observed prices of derivatives contradict the no-arbitrage assumption.

3.1 A buy-and-hold hedging portfolio

First, we explain a buy-and-hold portfolio before clarifying the meaning of the duality from the viewpoint of financial economics. We consider option pricing and hedging in Model B, which is a general approach. In Model B, buyers' price of a simple claim Φ^T and sellers' price of a simple claim Φ^T are usually defined as

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T) = \sup\left\{ V(0) \mid V(t) : \begin{array}{c} \text{a value process of a self-financing} \\ \text{portfolio such that} V(T) \le \phi(S(T)) \end{array} \right\}$$

and

$$\mathrm{q}_{\mathrm{sell}}(\Phi^T) = \inf\left\{ V(0) \mid V(t) : \begin{array}{c} \text{a value process of a self-financing} \\ \text{portfolio such that } V(T) \ge \phi(S(T)) \end{array} \right\}$$

respectively, where a value process $V(t)$ is expressed as

$$V(t) = h_0(t) + h_1(t) \cdot S(t) \tag{5}$$

for $\mathcal{F}_t$ adopted processes $h_0(t)$ and $h_1(t)$. Here, $h_0(t)$ and $h_1(t)$ mean the amounts of the risk-free asset and the risky assets included in a portfolio. Generally, the following relationship holds:

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T) \le \Phi^T(0) \le \mathrm{q}_{\mathrm{sell}}(\Phi^T).$$

In a complete market both prices equalize, and we have

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T) = \mathrm{q}_{\mathrm{sell}}(\Phi^T) = \Phi^T(0).$$

Notice that $h_0(t)$ and $h_1(t)$ are usually continuously re-balanced in portfolios which realize $\mathrm{q}_{\mathrm{buy}}(\Phi^T)$ and $\mathrm{q}_{\mathrm{sell}}(\Phi^T)$. In contrast to the usual buyers' and sellers' prices mentioned above, we define buyers' and sellers' buy-and-hold hedging prices by restricting a portfolio to a buy-and-hold portfolio, which means a constant portfolio with time t. For simple claims F_i^T $(i = 1, 2, \ldots, n)$, we define buyers' buy-and-hold hedging prices $\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T)$ and sellers' buy-and-hold hedging prices $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ as follows:

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T) = \sup \left\{ V(0) \mid V(t) : \begin{array}{l} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } V(T) \leq \phi(S(T)) \end{array} \right\}, \quad (6)$$

$$\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T) = \inf \left\{ V(0) \mid V(t) : \begin{array}{l} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } V(T) \geq \phi(S(T)) \end{array} \right\}, \quad (7)$$

where a value process $V(t)$ is expressed as

$$V(t) = y_0 + \sum_{i=1}^{n} y_i F_i^T(t), \quad (8)$$

for some constants y_i $(i = 0, 1, \ldots, n)$. In particular we can take F_i^T $(i = 1, 2, \ldots, m)$ as risky assets themselves, which means $F_i^T(t) = S_i(t)$. In that case, we have

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T) \leq \mathrm{q}_{\mathrm{buy}}(\Phi^T) \leq \mathrm{q}_{\mathrm{sell}}(\Phi^T) \leq \mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T),$$

because we restrict the set of self-financing portfolios (5) to the set of buy-and-hold portfolios (8).

The sellers' price $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ means the minimum investment costs necessary to super-hedge the simple claim Φ^T with a buy-and-hold portfolio consisting of the risk-free asset and F_i^T, and hence is a favorable price for sellers of Φ^T. Conversely the buyers' price $\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T)$ is a favorable price for buyers. In the following subsection, we reveal the financial meaning of the duality in terms of buyers' and sellers' buy-and-hold hedging prices.

3.2 Financial interpretation of duality in the SILP

Now we give a financial interpretation of duality in problems (3) and (4) which arise as a problem of determining a derivative price range based only on the no-arbitrage assumption and the observed prices of other derivatives. The following proposition states the meaning of the dual problem (4).

Proposition 1 *Let the derivative prices satisfy*

$$F_i^T(0) = q_i \qquad (i = 1, 2, \ldots, n),$$

which are consistent with Model B. The optimal value in problem (4) *is equivalent to* $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ *in Model B. An optimal solution* $y^* \in R^{n+1}$ *in problem* (4) *gives an optimal buy-and-hold super-hedging portfolio for* Φ^T*.*

(Proof) By definition (7), we have

$$\begin{aligned}
\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T) &= \inf \left\{ V(0) \mid \{V(t)\} : \begin{array}{c} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } V(T) \geq \phi(S(T)) \end{array} \right\} \\
&= \inf \left\{ V(0) \mid \; y \in R^{n+1} : \; y_0 + \sum_{i=1}^{n} y_i F_i^T(T) \geq \phi(S(T)) \right\} \\
&= \inf \left\{ V(0) \mid y \in R^{n+1} : \; y_0 + \sum_{i=1}^{n} y_i f_i(x) \geq \phi(x) \ \forall x \in R_+^m \right\}.
\end{aligned}$$

The last equality holds because $S(T)$ could be all vectors in R_+^m by the assumptions of Model B. By the right side of the last equality, the problem of finding $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ in Model B is equivalent to problem (4), and an optimal solution $y^* \in R^{n+1}$ in problem (4) gives an optimal buy-and-hold super-hedging portfolio for Φ^T if it exists. □

Remark 1 *The problem in which* minimizing *and* $\geq$ *in the constraint in problem* (4) *are replaced with* maximizing *and* $\leq$, *respectively, finds an optimal buy-and-hold under-hedging portfolio for* Φ^T *if it exists.*

By the duality between problems (3) and (4), $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ in Model B is larger than the supremum on $\Phi^T(0)$ in Model A. Furthermore, if f_i and ϕ are continuous and have compact supports, then $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ in Model B is equal to the supremum on $\Phi^T(0)$ in Model A. This is the financial interpretation of the duality which emerges in the context of option pricing based only on the no-arbitrage assumption and prices of other derivatives. The strong duality between problems (3) and (4) holds under weaker assumptions which do not require the functions to be continuous and have compact supports. For details, refer to [13].

Problem (4) gives an arbitrage buy-and-hold portfolio in the case that problem (3) is infeasible (i.e. observed prices $F_i^T(0) = q_i$ $(i = 1, 2, \ldots, n)$ contradict the no-arbitrage assumption of Model A).

Corollary 1 *Let the derivative prices satisfy*

$$F_i^T(0) = q_i \qquad (i = 1, 2, \ldots, n).$$

An optimal solution of the following problem gives an arbitrage buy-and-hold portfolio, if and only if the optimal value is less than 0 :

$$\begin{array}{lll} \text{minimize }_{y \in R^{n+1}} & y_0 + \sum_{i=1}^{n} q_i y_i & \\ \text{subject to} & y_0 + \sum_{i=1}^{n} y_i f_i(x) \geq 0 \qquad (\forall x \in R_+^m) & \\ & y_i \in [-1, 1] \qquad (i = 0, 1, \ldots, n). & \end{array} \tag{9}$$

Remark 2 *Problem* (9) *adds the extra constraints* $y_i \in [-1, 1]$ *to problem* (4) *for* $\phi = 0$, *so that the optimal value is always bounded. For an investment in the actual market, we must take the range of* y_i *as a volume to which we can trade* F_i^T *at the observed prices* q_i, *and restrict* y_i *to be integral multiples of a minimum trade unit.*

Proposition 1 shows that problem (4) itself is an important problem of finding the minimum investment costs of super-hedging buy-and-hold portfolios for Φ^T which consist of the risk-free asset and given derivatives F_i^T in Model B. This problem is meaningful especially for practical purpose, because in the actual market continuous hedging such as delta hedging has a problem of transaction costs. Since Corollary 1 enables us to make an arbitrage portfolio if it exists, it could be useful for a large investment company which can trade many kinds of derivative securities with the same maturity.

Our interpretation from the financial viewpoint also unveils the relationship between results in Model A and Model B. For instance, it is shown in [1] and [2] that $\Psi_\mu(k) = \int_{R_+} \max\{x - k, 0\} \mathrm{d}\mu$ $(k > 0)$ determines a unique risk-neutral measure μ. This has a dual relationship with the following proposition regarding buy-and-hold hedging in the B. S. model on p.123 in [4].

Proposition 2 *Assume the B.S. model that consists of a risk-free asset and a risky asset* S, *and let* $\phi : R_+ \mapsto R_+$ *be a continuous function with compact support. Then, a simple claim with payoff function* $\phi(S(T))$ *can be replicated with arbitrary precision using a buy-and-hold portfolio consisting of the risk-free asset and several call options.*

Figure 1 illustrates Proposition 2. Here, v_1 and v_2 represent the values of the super-hedging and under-hedging portfolios for Φ^T at T consisting of call options F_i^T with payoff $f_i^T = \max\{S(T) - k_i, 0\}$, that is,

$$v_1(x) = V_1(T) \geq \phi(x) \quad (x \in R_+),$$

$$v_2(x) = V_2(T) \leq \phi(x) \quad (x \in R_+),$$

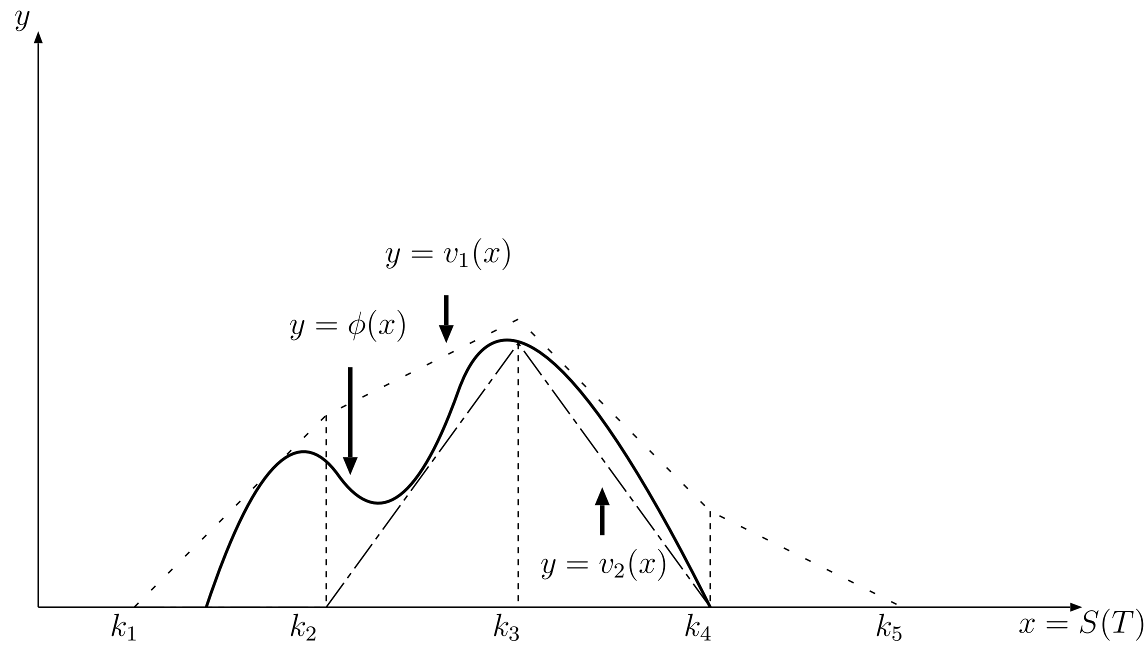

Figure 1: Buy-and-hold hedging portfolios

where V_j $(j = 1, 2)$ are of the form

$$V_j(t) = \sum_{i=1}^{n} y_{i,j} F_i^T(t)$$

with certain constants $y_{i,j}$ $(i = 1, 2, \ldots, n,\ j = 1, 2)$. The relationship $V_2(0) \leq \Phi^T(0) \leq V_1(0)$ always holds, and Proposition 2 shows that $V_j(0)$ can be made arbitrarily close to $\Phi^T(0)$ by letting $n \to +\infty$. Thus, the dual problem (4) could be more helpful to visualize the meaning than problem (3). As a special case of problem (3), the problem of determining a price range for a call option based on the observed prices of call options with other strikes has been fully investigated in [3]. From the dual viewpoint we can state that it is a problem of finding an optimal buy-and-hold hedging portfolio consisting of given call options.

4 Conclusion

This paper has investigated the duality in the semi-infinite linear programming problem which arises in the context of determining a derivative price range based only on the observed prices of other derivatives. and the no-arbitrage assumption

(Model A). A contribution of this paper is to give an interpretation of the duality from the viewpoint of financial economics and reveal another importance of studies in Model A.

We have actually clarified that the dual of a problem of finding the supremum on derivative prices with the observed prices of other derivatives in Model A is equivalent to the problem of finding the minimum investment costs of buy-and-hold super-hedging portfolios for the derivative in the usual financial market model (Model B). This problem is useful for investors because in the actual market rebalancing a hedging portfolio takes transaction costs. The interpretation links some previous studies in Model A to the results for Model B in terms of a buy-and-hold hedging portfolio.

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