Computing Bounds on Risk-Neutral Measures from the Observed Prices of Call Options *

Michi NISHIHARA,† Mutsunori YAGIURA,‡ Toshihide IBARAKI§

Abstract

This paper derives, in closed forms, upper and lower bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of European call options, based only on the no-arbitrage assumption. The computed bounds from real data show that the gap between the upper and lower bounds is large near the underlying asset price but gets smaller away from the underlying asset price. Since the bounds can be easily computed and visualized, they could be practically used by investors in various ways.

Keywords: Option pricing, Risk-neutral measure, Semi-infinite programming, Risk-neutral cumulative distribution function

1 Introduction

One of the most important issues in financial economics is option pricing, i.e., to derive an appropriate price of a derivative security, from the well-known fundamental assumption that the market is no-arbitrage (i.e., intuitively, the value of a portfolio can not be increased without any risk of the loss). Under the no-arbitrage assumption, a derivative price must become the same as the value of a portfolio that replicates the derivative, assuming that such a hedging portfolio exists. In addition to the no-arbitrage assumption, existing option pricing methods assume other conditions described by stochastic differential equations for prices of risky assets. A

*November 22, 2005 Technical Report 2005-14.

†Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan, michi@amp.i.kyoto-u.ac.jp

‡Department of Computer Science and Mathematical Informatics, Graduate School of Information Science, Nagoya University, Furocho, Chikusaku, Nagoya 464-8603, Japan, yagiura@nagoya-u.jp

§Department of Informatics, School of Science and Technology, Kwansei Gakuin University, Sanda 669-1337, Japan, ibaraki@ksc.kwansei.ac.jp

typical approach, the Black-Scholes model introduced in [5] and [13], assumes a geometric Brownian motion for the risky stock price. By this assumption, every derivative can be replicated by a portfolio consisting of the risk-free bond and the underlying stock, and therefore has the unique price equal to the price of the hedging portfolio. However, precisely speaking, a stock price in the actual market does not obey the geometric Brownian motion. For example, a log-price of a stock displays a heavy tailed distribution, which is different from the Gaussian distribution. However, it is hard to find a stochastic differential equation that perfectly describes the dynamics of real asset prices.

Thus, a natural question that arises is to derive a derivative price range from the observed prices of other derivatives, based only on the no-arbitrage assumption without assuming any stochastic model for the dynamics of asset prices. This question has been studied in [6], [10] and [12]. They derived upper and lower bounds on option prices consistent with the given mean and (co)variance of the underlying asset prices under a risk-neutral measure. Bertsimas and Popescu [3] have shown that the question can be treated in the framework of an SILP (semi-infinite linear programming problem). In particular, they reduced the problem to an LP, assuming that the payoff functions are continuous and piecewise linear functions over a compact support. This approach is also related to the study of the implied binomial and trinomial models proposed in [7], [8] and [15] in the sense that both use the observed prices of derivatives.

In this paper, we investigate the problem of finding bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call options, based only on the no-arbitrage assumption. By considering this special case, we can analytically derive the bounds on risk-neutral measures, which saves us from computing the numerous corresponding LPs as discussed in [3]. We then compute the bounds from Nikkei-225 option data in Japan. To derive the risk-neutral measure implied from the real data is important, because the risk-neutral measure plays a decisive role in pricing financial securities, and it represents market's view of risk. Actually, several studies such as [9] and [11] have investigated this problem, but they restrict their attention to a class of densities with heavy tails and the Garch model, respectively.

This paper is organized as follows. Section 2 explains the problem formulation and the results obtained in [1]. Section 3 describes our main result, that is, the bounds on risk-neutral measures in closed forms. Section 4 illustrates computational results obtained from Nikkei-225 option data in Japan.

2 The Problem Formulation

This section introduces the problem and describes some results obtained in [1], [2] and [3] for future use.

We consider the problem of finding bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call options, based only on the no-arbitrage assumption. By the Fundamental Theorem in option pricing (see a standard textbook such as p.133 – p.153 in [4]), the no-arbitrage assumption is equivalent to the existence of a risk-neutral measure Q which is equivalent to the observed probability measure P. Let $S(t)$ and $r(t)$ denote the price of the underlying asset at time t and the risk-free rate, respectively. Formally, $S(t)$ is an $\mathcal{F}_t$ adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. This intuitively means that for each t (≥ 0), the value of $S(t)$ is observed (determined) at time t. Let us observe n (≥ 1) types of European call options on the same underlying asset, with the same exercise date and different strikes. For $i = 1, 2, \ldots, n$, let q_i denote the observed prices of European call options with exercise date T and strikes k_i at time 0. Without loss of generality, we assume $0 \leq k_1 < k_2 < \cdots < k_n$ in the rest of this paper.

Since the price of European option with payoff $\phi(S(T))$ is expressed as

$$E^Q[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\phi(S(T))] \tag{1}$$

for a risk-neutral measure Q, a risk-neutral measure Q implied by the observed prices q_i must satisfy

$$E^Q[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\max\{S(T) - k_i, 0\}] = q_i \qquad (i = 1, 2, \ldots, n). \tag{2}$$

Here, E^Q denotes the mean under the probability measure Q. Note that the payoff of the call option is defined by $\max\{S(T) - k_i, 0\}$, because the holder of the option receives $S(T) - k_i$ by exercising the option on the exercise date T. Thus, the problem of finding the supremum and the infimum on values of risk-neutral cumulative distribution functions of $S(T)$ at each a (≥ 0) can be formulated as follows: For each $a \geq 0$,

$$\begin{array}{lll} \text{maximize}_{\,Q(\sim P)} & Q[S(T) \in [0, a]] & \\ \text{(or minimize)} & & \\ & & \\ \text{subject to} & E^Q[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\max\{S(T) - k_i, 0\}] = q_i & (i = 1, 2, \ldots, n), \end{array} \tag{3}$$

where Q moves over the set of probability measures on $(\Omega, \mathcal{F})$ such that Q is equivalent to the observed probability measure P ($Q \sim P$ in problem (3) denotes that Q is a probability measure equivalent to P). If we could derive the optimal values of problem (3) for all $a \geq 0$, the upper and lower bound functions can be obtained as functions of $a \geq 0$. Note that the obtained bounds may not be *tight* in the following sense: It is likely that no single risk-neutral probability measure Q gives the upper (or lower) bound function for all $a \geq 0$, though for any fixed $a \geq 0$ there exists a Q that attains the bound at a. To determine the bounds on risk-neutral cumulative distribution functions is a fundamental question, because every European option on $S(T)$ can be priced via (1), from the implied risk-neutral measure.

Since no particular dynamics of $S(t)$ under P is assumed, the equivalence $Q \sim P$ in problem (3) does not add any restriction. Thus, by taking μ as a distribution of $S(T)$ under Q, the price of call option with exercise date T and strike a is given by

$$\Psi_\mu(a) = \int_{R_+} \mathrm{e}^{-\int_0^T r(t)\mathrm{d}t} \max\{x - a, 0\} \mathrm{d}\mu(x). \tag{4}$$

Then, problem (3) can be reduced to the following SILP to determine a probability measure μ on the Borel space $(R_+, \mathcal{B}(R_+))$: For each $a \geq 0$,

$$\begin{array}{lll} \text{maximize}_{\ \mu \in \Delta(R_+)} & \mu([0,a]) & \\ \text{(or minimize)} & & \\ \text{subject to} & \displaystyle\int_{R_+} \mathrm{e}^{-\int_0^T r(t)\mathrm{d}t} \max\{x - k_i, 0\} \mathrm{d}\mu(x) = q_i & (i = 1, \ldots, n), \end{array} \tag{5}$$

where R_+ denotes $[0, \infty)$, and $\Delta(R_+)$ denotes the set of probability measures on the Borel space $(R_+, \mathcal{B}(R_+))$. This problem is a concrete and solvable problem compared with the abstract problem (3) on the probability space $(\Omega, \mathcal{F})$.

Problem (5) is a special case of the problems investigated in [3], because $\mu([0,a])$ is the same as $\int_{R_+} 1_{[0,a]}\mu(x)$, where $1_{[0,a]}(x)$ denotes the defining function of the set $[0,a]$. Although it involves the discontinuous payoff function $1_{[0,a]}(x)$, for a fixed $a \geq 0$, problem (5) can be reduced to an LP by using the same dual technique proposed in [3]. In Section 3 of this paper, however, we derive the infimum and the supremum of problem (5) as functions of a (≥ 0) in closed forms. In other words, we can compute the upper and lower bounds without actually solving the numerous LPs. From the dual viewpoint revealed in [14], problem (5) is equivalent to finding the minimum costs necessary to super-hedge a binary option with payoff $1_{\{S(T) \in [0,a]\}}(S(T))$ with a buy-and-hold portfolio including the given call options.

Formulation (5) permits any deterministic function as the risk-free rate $r(t)$, but we assume $r(t) = 0$ in the rest of this paper. In this case, the above price $\Psi_\mu(a)$ of (4) becomes

$$\Psi_\mu(a) = \int_{R_+} \max\{x - a, 0\} \mathrm{d}\mu(x). \tag{6}$$

If $r(t) \neq 0$, we just replace q_i $(i = 1, 2, \ldots, n)$ with $e^{\int_0^T r(t)\mathrm{d}t} q_i$. In most cases, not only the prices of the call options but also the underlying asset price $S(0)$ itself is observed. In this case, we have only to put $k_1 = 0$ and take $S(0)$ as q_1, as the underlying asset price is equal to the price of the call option with strike 0, i.e., $S(0) = \Psi_\mu(0)$. If $S(0)$ is observed the results in this paper can be also applied to European put options, because the prices of the corresponding European call options can be derived from $S(0)$ and the prices of put options via the put-call parity (e.g., see p.123 in [4]), which is deduced only from the no-arbitrage assumption.

We note that the results in this paper can also be applied to the modified problems, in which $S(T)$ in problem (3) are replaced with the maximum asset price $\max_{0\le t\le T} S(t)$ and the average asset prices $1/T \int_0^T S(t)\mathrm{d}t$, by taking μ as distributions of $\max_{0\le t\le T} S(t)$ and $1/T \int_0^T S(t)\mathrm{d}t$, respectively.

Now, we describe the result derived in [1] before explaining our results. The following condition will be assumed in the subsequent analysis:

Condition A The observed prices of European call options q_i with strikes k_i (where $0 \le k_1 < k_2 < \cdots < k_n$) satisfy

$$\begin{aligned} & q_1 \ge q_2 \ge \cdots \ge q_n \ge 0, \\ & \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{n+1}, \end{aligned}$$

where $\alpha_i = (q_i - q_{i-1})/(k_i - k_{i-1}) \quad (i = 2, \ldots, n)$, $\alpha_1 = -1$ and $\alpha_{n+1} = 0$. If there exists an m $(< n)$ such that $q_m = q_{m+1}$, then $q_m = q_{m+1} = \cdots = q_n = 0$.

Condition A tells that the piecewise linear price function obtained by connecting points (k_i, q_i) $(i = 1, 2, \ldots, n)$ is convex and monotonically decreasing (see Figure 1). The following proposition proved in [1] shows that Condition A is a necessary and sufficient condition for the existence of a risk-neutral measure μ.

Proposition 1 *At least one probability measure μ on $(R_+, \mathcal{B}(R_+))$ exists such that*

$$\Psi_\mu(k_i) = q_i \qquad (i = 1, \ldots, n)$$

if and only if Condition A holds, where Ψ_μ is defined by (6).

Remark 1 *Condition A is usually observed to hold on real data when the trade volume is large. We will discuss this in Section 4* (*see Figure 3*).

3 Bounds on Risk-neutral Measures

This section derives the optimal values of problem (5) in closed forms, for both versions of maximizing and minimizing the objective function. We then discuss potential applications of the results.

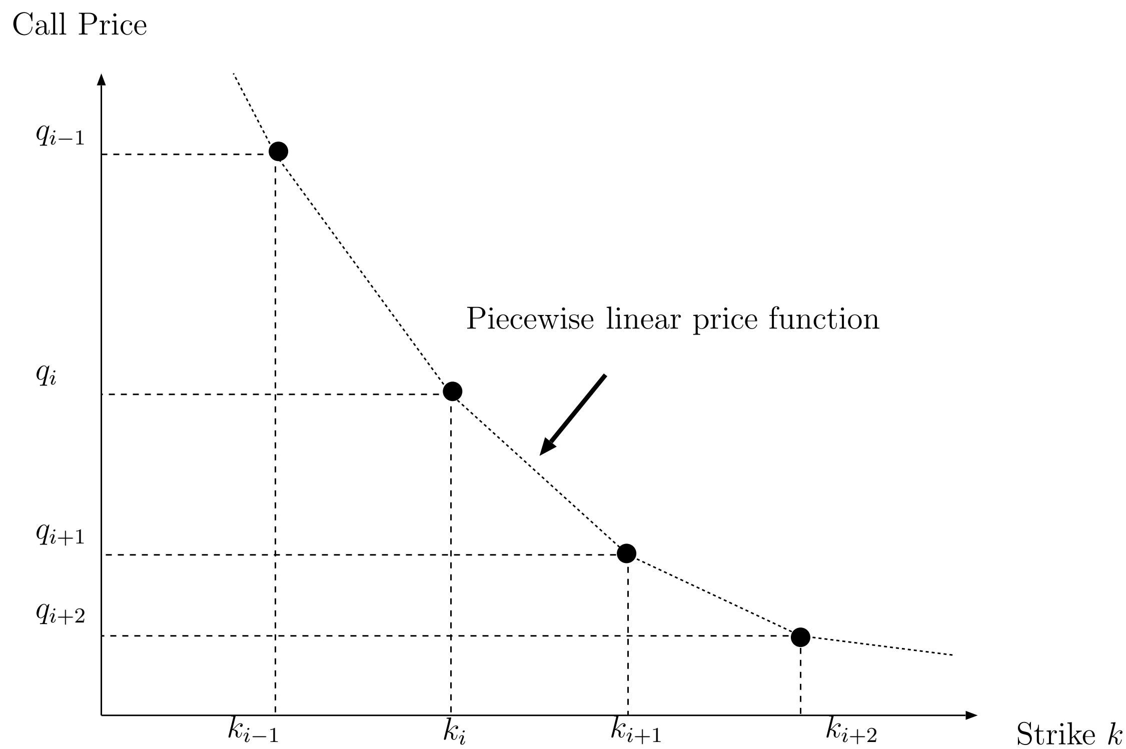

Figure 1: Convexity of call option prices.

Let $f_{\max}(a)$ and $f_{\min}(a)$ denote the optimal values of problem (5) to maximize and to minimize, respectively. First, we introduce the following notations:

$$\begin{aligned}
&\beta_i = q_i - \alpha_i k_i \qquad (i = 1, 2, \ldots, n),\\
&\beta_{n+1} = q_n,\\
&l_i = \frac{\beta_i - \beta_{i+2}}{\alpha_{i+2} - \alpha_i} \qquad (i = 1, 2, \ldots, n-1),\\
&l_n = -\frac{\beta_n}{\alpha_n},
\end{aligned}$$

where α_i $(i = 1, 2, \ldots, n+1)$ are defined in Condition A. Figure 2 illustrates the meaning of these quantities. The following proposition, giving close forms of $f_{\max}(a)$ and $f_{\min}(a)$, is our main theoretical result. With this proposition, we no longer need to solve the corresponding LP for each a, as proposed in [3].

Proposition 2 *For strikes k_i $(i = 1, 2, \ldots, n)$, let q_i (i.e., the prices of call options with payoff $\max\{S(T) - k_i, 0\}$) be given. If prices q_i satisfy Condition A and $q_n > 0$, then $f_{\max}(a)$ and $f_{\min}(a)$ are expressed as follows:*

$$f_{\max}(a) = \begin{cases} 1 + \alpha_2 & (0 \leq a < k_1) \\ 1 + \alpha_i + \dfrac{q_{i+1} - \alpha_i k_{i+1} - \beta_i}{k_{i+1} - a} & (k_i \leq a < l_i, i = 1, 2, \ldots, n-1) \\ 1 + \alpha_{i+2} & (l_i \leq a < k_{i+1}, i = 1, 2, \ldots, n-1) \\ 1 & (k_n \leq a), \end{cases} \tag{7}$$

$$f_{\min}(a) = \begin{cases} 0 & (0 \leq a < k_1) \\ 1 + \alpha_{i+2} + \dfrac{\beta_{i+2} + \alpha_{i+2} k_i - q_i}{a - k_i} & (l_i \leq a < k_{i+1}, i = 1, 2, \ldots, n-1) \\ 1 + \alpha_i & (k_i \leq a < l_i, i = 1, \ldots, n) \\ 1 - \dfrac{q_n}{a - k_n} & (l_n \leq a). \end{cases} \tag{8}$$

(Proof)
Assume that Condition A and $q_n > 0$ hold. Let $\Psi_\mu : R_+ \mapsto R_+$ be given by (6). The following equality was proved in [1]:

$$\begin{aligned} \Psi'_\mu(a+) &= -\mu((a, \infty]) \\ &= -1 + \mu([0, a]), \end{aligned} \tag{9}$$

where $\Psi'_\mu(a+)$ denotes the right derivative of Ψ_μ at a. By (9) and the definition of $f_{\max}$ and $f_{\min}$ (i.e., optimal values of problem (5)) we have

$$\begin{aligned} f_{\max}(a) &= \sup_{\mu \in U} \mu([0, a]) \\ &= 1 + \sup_{\mu \in U} \Psi'_\mu(a+). \end{aligned} \tag{10}$$

Here, U is the set of probability measures that satisfy the constraints of problem (5). Similarly, we have

$$f_{\min}(a) = 1 + \inf_{\mu \in U} \Psi'_\mu(a+). \tag{11}$$

By Proposition 1, for a fixed a (≥ 0), there exists a probability measure $\mu \in U$ satisfying $q = \Psi_\mu(a)$ if and only if Condition A holds for the set of points consisting

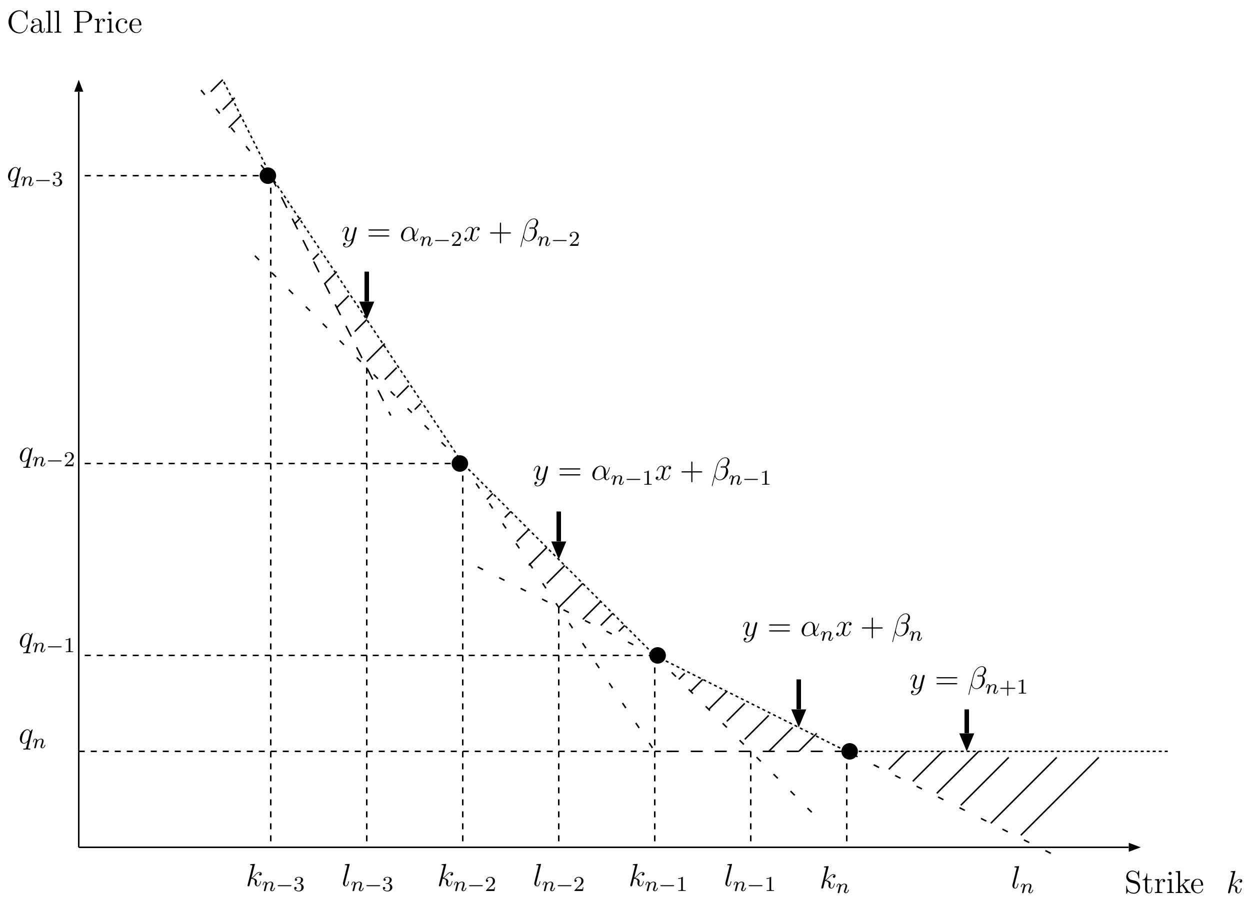

Figure 2: Meanings of α_i, β_i, l_i.

of (a,q) and (k_i,q_i) $(i=1,2,\ldots,n)$. Thus, we have

$$\sup_{\mu\in U}\Psi_\mu(a)=\begin{cases}\alpha_1 a+\beta_1 & (0\le a<k_1)\\ \alpha_{i+1}a+\beta_{i+1} & (k_i\le a<k_{i+1}, i=1,2,\ldots,n-1)\\ q_n & (k_n\le a)\end{cases} \tag{12}$$

and

$$\inf_{\mu\in U}\Psi_\mu(a)=\begin{cases}\alpha_2 a+\beta_2 & (k<k_1)\\ \alpha_i a+\beta_i & (k_i\le a<l_i, i=1,2,\ldots,n)\\ \alpha_{i+2}a+\beta_{i+2} & (l_i\le a<k_{i+1}, i=1,2,\ldots,n-1)\\ 0 & (l_n\le a),\end{cases} \tag{13}$$

from the fact that the piecewise linear function connecting (a,q) and (k_i,q_i) $(i=1,2,\ldots,n)$ is convex and decreasing. In Figure 2, the hatched regions between the upper dotted line and the lower dotted lines illustrate the area of points (a,q) between (12) and (13). Extending this results to all points a (≥ 0), we see that the price function $\Psi_\mu(a)$ must be a convex and decreasing function contained in the hatched regions. Conversely, we can show, by modifying Proposition 1 as in [1], that there exists a $\mu\in U$ such that $\Psi_\mu(a)=g(a)$ $(a\ge 0)$ for any convex and decreasing function $g(a)$ $(a\ge 0)$ in the hatched regions. Thus, by considering the right derivatives of all convex and decreasing functions in the hatched regions in Figure 2, for $k_i\le a<l_i$ $(i\le n-1)$, we have

$$\begin{aligned}\sup_{\mu\in U}\Psi'_\mu(a+) &= \frac{q_{i+1}-(\alpha_i a+\beta_i)}{k_{i+1}-a} && (14)\\ &= \alpha_i+\frac{q_{i+1}-\alpha_i k_{i+1}-\beta_i}{k_{i+1}-a}\\ \inf_{\mu\in U}\Psi'_\mu(a+) &= \alpha_i. && (15)\end{aligned}$$

Here, the right-hand side of (14) is the gradient of the line connecting two points (k_{i+1},q_{i+1}) and $(a,\inf_{\mu\in U}\Psi_\mu(a))$, and the right-hand side of (15) is the gradient of the lower dotted line for $k_i\le k\le l_i$ in Figure 2. For $l_i\le a<k_{i+1}$ $(i\le n-1)$, we have

$$\begin{aligned}\sup_{\mu\in U}\Psi'_\mu(a+) &= \alpha_{i+1} && (16)\\ \inf_{\mu\in U}\Psi'_\mu(a+) &= \frac{\alpha_{i+2}a+\beta_{i+2}-q_i}{a-k_i} && (17)\\ &= \alpha_{i+2}+\frac{\beta_{i+2}+\alpha_{i+2}k_i-q_i}{a-k_i},\end{aligned}$$

where, the right-hand side of (16) is the gradient of the lower dotted line for $l_i \le k \le k_{i+1}$ in Figure 2, and the right-hand side of (17) is the gradient of the line connecting two points (k_i, q_i) and $(a, \inf_{\mu \in U} \Psi_\mu(a))$. For the cases of $a < k_1$ and $k_n \le a$ we can derive the supremum and the infimum on the right derivatives in (10) and (11) by a similar geometric consideration. The resulting functions are given as (7) and (8) in this proposition. □

Remark 2 *In the above proposition, we assumed $q_n > 0$ for the practical reason that, in the actual market, no call option can be traded at price 0. However similar results can be obtained even if $q_n = 0$ is allowed.*

Remark 3 *Figure 4 illustrates the functions $f_{\max}(a)$ and $f_{\min}(a)$ for some given data (as will be discussed in Section 4).*

4 Computational Results

We compute the bounds of Proposition 2 from the data of Nikkei-225 options, which are most popular in the option market of Japan. Then, the underlying asset price $S(t)$ is the Nikkei-225 price at time t, and we took as q_i the closing prices of the options with strike k_i on the day 4 weeks before the exercise date (i.e., $t = 0$ on this day and $t = T$ on the exercise date). We set the risk-free rate as $r = 0$, as the maturity is only 4 weeks. For $k_1 = 0$, q_1 was taken as the closing price of Nikkei-225 on the day $t = 0$ (i.e., $q_1 = S(0)$), because $S(0)$ is identified as the price of the call option with strike 0. We chose the data according to the following rules to improve the data reliability:

1. Use prices of all Nikkei-225 call and put options which have more than 500 trade volume.

2. When both call and put options with the same strike and the same exercise date have more than 500 trade volumes, choose the one which has a larger trade volume. Then, if put option prices are chosen, determine the corresponding call option prices by applying the put-call parity (i.e., the relation between prices of a call option and a put option, see p.123 in [4]).

We confirmed that the call option prices obtained by the above rules mostly satisfy Condition A. An example is shown in Figure 3, which was computed from the data on March 11, 2004 (4 weeks before the exercise date April 8, 2004). In Figure 3, there is a large blank area between $k = 0$ and 8500, because we used not only prices of the call options with strikes $8500, 9000, \ldots, 13500$ but also the Nikkei-225 price $S(0) = 11297$ as the price of the call option with strike 0. For detailed data, refer to Tables 1 and 2 in Appendix. We note that Nikkei-225 options are

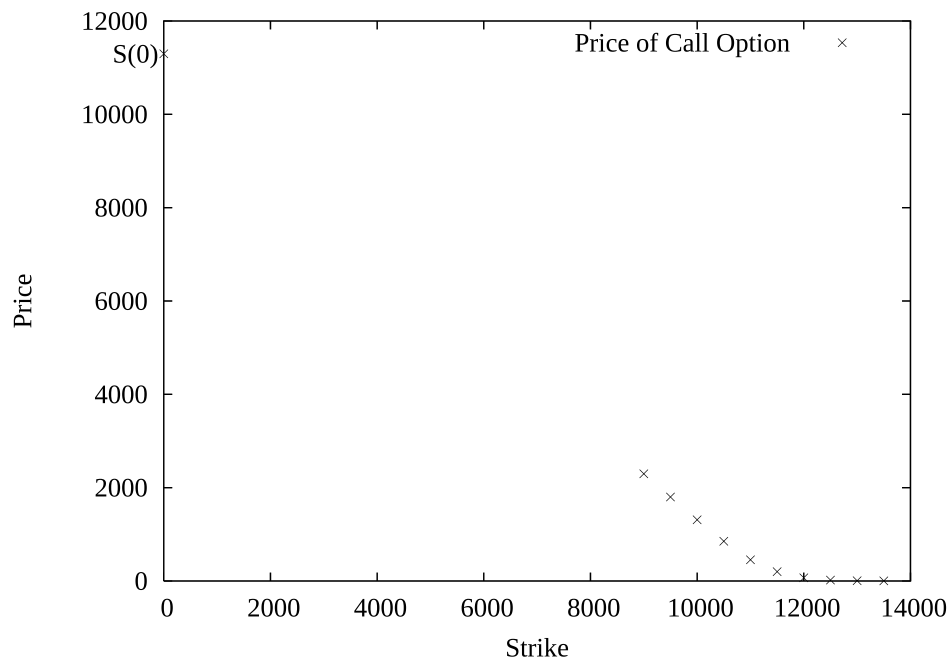

Figure 3: Prices of Nikkei-225 call options on March 11, 2004, with the exercise date of April 8, 2004.

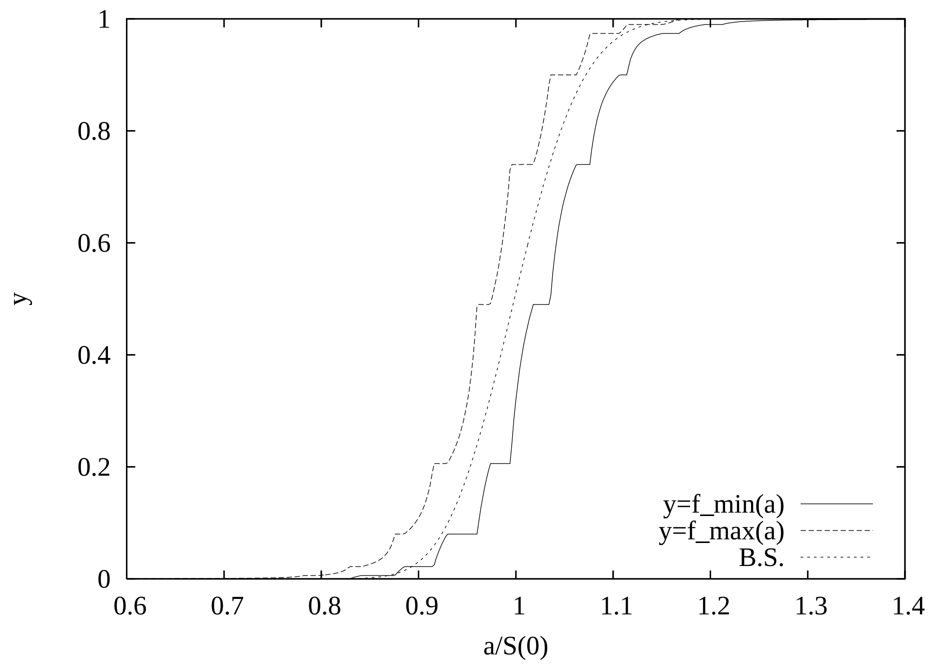

Figure 4: $f_{\max}(a)$ and $f_{\min}(a)$ on April 8, 2004.

usually traded with 14 strikes, which are set at every 500 Japanese Yen around the present Nikkei-225 price.

Then, we compute $f_{\max}(a)$ and $f_{\min}(a)$ by Proposition 2 from the data in Figure 3, and illustrate them in Figure 4, where the scale of the x-axis is normalized by the present Nikkei-225 price $S(0)$. For comparison, we also show the risk-neutral measure obtained from the Black-Scholes model [5] with volatility $\sigma = 0.2$ (see B.S. in Figure 4); i.e., $\Phi((1/(\sigma\sqrt{T}))(\log(a/S(0)) + \sigma^2 T/2))$, where $\Phi(y) = (1/\sqrt{2\pi})\int_{-\infty}^{y} \mathrm{e}^{-x^2/2}\mathrm{d}x$ denotes the standard normal cumulative distribution. Since the risk-neutral measure in the B. S. model does not exactly satisfy the constraints in (5), the B.S. curve in Figure 4 slightly violates the boundaries of $f_{\max}(a)$ and $f_{\min}(a)$.

The results show that the difference between the upper and lower bounds is large in the region close to the present Nikkei-225 price $S(0)$ (i.e., $a/S(0) \approx 1$) but it is small in the region far from $S(0)$. We also computed the bounds for 32 different exercise dates from January, 2002 to August, 2004, and confirmed that a similar trend always held for all exercise dates (for example, see Figure 5 in Appendix showing the results for 3 different exercise dates in 2004).

In closing this section we suggest a few potential applications of our results. A first application of Proposition 2 is of course to use the upper and lower bounds on μ for the purpose of estimating the price of European options via (1).

Another use may be to utilize the above trend of the gap between the upper and lower bounds. It tells that adding extra strikes in the region close to $S(0)$ will reduce the difference between the upper and lower bounds more efficiently than adding them in far regions. Smaller the difference, the easier it becomes to hedge other European options with the same exercise date. As an extreme case, let us assume that call options with all nonnegative strikes are actually traded. In this case, the gap between the upper and lower bounds obtained from the observed prices becomes 0, and therefore all European options with the same exercise date can be replicated by a buy-and-hold portfolio consisting of several call options, meaning that the market is *complete* (for details see [14]). Since one of the important roles of the option market is to change the market closer to being *complete*, it is more meaningful to set the strikes, not equally spaced but less spaced in the region near the present Nikkei-225 price $S(0)$. In this way, we could use the bounds of Proposition 2 to set the strikes with which the call options are traded. This suggestion will also be supported by the observation that trade volume of the options became smaller for the strikes set farther from the present Nikkei-225 price $S(0)$ (e.g., see Table 1).

In general, the risk-neutral cumulative distribution function μ tells us how investors view the uncertain risk of $S(T)$. If the μ implied by the computed bounds is similar to the cumulative distribution function obtained from the historical data of the underlying asset price, we can expect that investors in the market are risk-neutral. This kind of observation will help us when we make investment in the market.

In analyzing Nikkei-225 data, we observed that the $f_{\max}$ and $f_{\min}$ computed by Proposition 2 showed different behaviors depending on whether $S(T)$ has actually become smaller or larger than the $S(0)$ of 4 weeks ago. This may suggest the possibility of using $f_{\max}$ and $f_{\min}$ to forecast the future price of an asset, which would be one of our future topics.

5 Conclusion

This paper investigated the problem of deriving the upper and lower bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call options, based only on the no-arbitrage assumption. The main contribution of this paper is to provide the bounds in closed forms, without solving the corresponding LPs. The bounds are easy to compute. Based on the bounds computed from the real data of the Nikkei-225 options, we made several observations and discussed possible applications, which could be used by investors. Finding more applications of the computed bounds remains as an important and interesting issue.

Acknowledgments

The authors would like to thank Professor Masao Fukushima of Kyoto University and Professor Hiroshi Nagamochi of Kyoto University for helpful suggestions.

References

[1] D. Bertsimas and N. Bushueva. Option pricing without price dynamics: a geometric approach. Working Paper, Massachusetts Institute of Technology, 2004.

[2] D. Bertsimas and N. Bushueva. Option pricing without price dynamics: a probabilistic approach. Working Paper, Massachusetts Institute of Technology, 2004.

[3] D. Bertsimas and I. Popescu. On the relation between option and stock prices: a convex optimization approach. *Operations Research*, 50:358–374, 2002.

[4] T. Bjork. *Arbitrage Theory in Continuous Time*. Oxford University Press, Oxford, 2004.

[5] F. Black and M. Scholes. The pricing options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[6] P. Boyle and X. S. Lin. Bounds on contingent claims based on several assets. *Journal of Financial Economics*, 46:383–400, 1997.

[7] E. Derman and I. Kani. Riding on a smile. *Risk*, 7:32–39, 1994.

[8] B. Dupre. Pricing with a smile. *Risk*, 7:18–20, 1994.

[9] F. Fornari and A. Mele. Recovering the probability density function of asset prices using garch as diffusion. *Journal of Empirical Finance*, 8:83–110, 2001.

[10] B. Grundy. Option prices and the underlying asset's return distribution. *Journal of Finance*, 46:1045–1069, 1991.

[11] N. V. Hartvig, J. L. Jensen, and J. Pedersen. A class of risk neutral densities with heavy tails. *Finance and Stochastic*, 5:115–128, 2001.

[12] A. Lo. Semi-parametric upper bounds for option prices and expected payoffs. *Journal of Financial Economics*, 19:373–387, 1987.

[13] R. C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[14] M. Nishihara, M. Yagiura, and T. Ibaraki. Duality in option pricing based on prices of other derivatives. Technical Reports 013, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, 2005.

[15] M. Rubinstein. Implied binomial trees. *Journal of Finance*, 49:771–819, 1994.

Appendix

Table 1: Data of Nikkei-225 options on March 11, 2004, with the exercise date April 8, 2004.

Strike	Call Option Price	Trade Volume	Put Option Price	Trade Volume	Choose
7500	N/A	0	N/A	0	N/A
8000	N/A	0	N/A	0	N/A
8500	2780	30	N/A	0	N/A
9000	N/A	0	1	2840	Put
9500	N/A	0	4	3409	Put
10000	N/A	0	15	4128	Put
10500	810	32	55	3940	Put
11000	455	1188	180	866	Call
11500	200	1249	415	163	Call
12000	70	3044	775	94	Call
12500	20	1908	N/A	0	Call
13000	7	2328	N/A	0	Call
13500	2	1127	N/A	0	Call

Table 2: Prices of call options on March 11, 2004, with the exercise date April 8, 2004.

Strike k_i	Price q_i
0	11297
9000	2298
9500	1801
10000	1312
10500	852
11000	455
11500	200
12000	70
12500	20
13000	7
13500	2

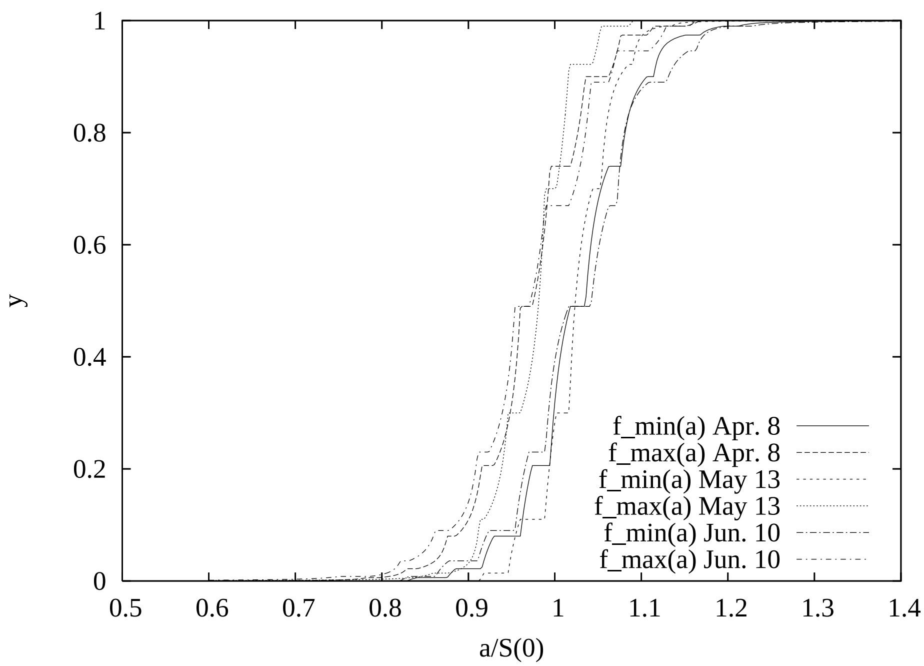

Figure 5: $\mathrm{f}_{\max}(a)$ and $\mathrm{f}_{\min}(a)$ on different dates in 2004.