Evaluation of Firm's Loss Due to Incomplete Information in Real Investment Decision *

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Abstract

We investigate the effect of incomplete information in a model where a start-up with a unique idea and technology pioneers a new market but will eventually be expelled from the market by a large firm's subsequent entry. We evaluate the start-up's loss due to incomplete information about the large firm's behavior. We clarify conditions under which the start-up needs more information about the large firm, and reveal the risk of incomplete information about the competitor. The proposed method of evaluating the loss due to incomplete information could also be applied to other real options models involving several firms.

Keywords: Investment analysis; Real options; Incomplete information; Optimal stopping; Leader-follower game

1 Introduction

In recent years, the real options approach to investment has become the mainstream in corporate finance. In the real options approach, a firm that faces an irreversible investment generating uncertain profit in future is considered to have an option to make the investment. Then, in order to maximize the expected cash flow, the firm must invest at the timing when the NPV (net present value) of the investment becomes greater than

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the opportunity cost of investing (i.e., the value of the option to delay the investment). When the value of the option to delay the investment is taken into account, the firm's optimal investment timing becomes much later than the first date on which the NPV of the investment exceeds zero.

The real options approach to investment has provided a new insight into a firm's real investment decision which tended to rely on managerial experiences and intuitions since the proposal by Dixit [3] and McDonald and Siegel [11] in the late 1980's. Many results obtained in the early studies are summarized in [4], and the real options approach has gradually come to be applied to investment in the real world.

Although the early literature treated an investment decision of a monopolist, more recent studies have investigated how a firm's investment decision is affected by its rival firms' behaviors. This is natural, because a chance to make real investment (e.g., entering a new market), unlike financial options, can usually be shared by several firms in the same industry. One of the earliest results in the strategic real options approach has been obtained by Grenadier [6] who provided the symmetric equilibrium strategy for firms by using a continuous time Cournot-Nash framework. Weeds [15] has derived equilibrium strategies in two players who attempt to preempt a single patent from the other, and Huisman and Kort [9] have investigated a two player real options game in the context of the adoption of new technology. In [8] and [14], the strategic real options approach has been incorporated with the equilibrium concept in a timing game studied in [5]. The strategic real options approach can also be used by practitioners for making their real investment decisions (e.g., see [13]).

While the above studies assume complete information concerning the qualities (e.g., the investment cost and the profit flow) of the competitors, Lambrecht and Perraudin [10] consider incomplete information about the competitors' investment costs. They have shown that the optimal investment timing under incomplete information lies between the zero-NPV timing and the optimal timing for the monopolist. A patent race between an incumbent and a potential entrant has been studied in [7], where asymmetric and incomplete information is introduced in real options.

Bernardo and Chowdhry [1] and Décamps et al. [2] have also incorporated incomplete information in real investment problems from another perspective. By using the filtering theory, they have investigated models in which a firm has incomplete information about parameters of its own profit flow rather than the competitors' behavior.

The effect of incomplete information is practically significant, since how accurately a firm can estimate the behaviors of rival firms has a crucial effect on whether or not its real investment succeeds. Previous studies such as [7] and [10] dealing with incomplete information pay primary attention to how incomplete information affects the equilibrium strategy. However, there remains the following natural question: How great loss does a firm suffer with incomplete information compared with that in the case of complete information? Answers to this question will unveil a risk of a firm using the real options approach in the real world and also suggest how a firm should act under incomplete information.

In this paper, we answer the above question in a model with a start-up who pioneers a new market by a unique idea and technology and a large firm that will eventually take over the market from the start-up. We evaluate the start-up's loss due to incomplete information about the large firm that will make subsequent investment and drive out the start-up from the market eventually. Then, we clarify conditions under which the start-up needs more information about the large firm. In particular, we show that in some cases the real options strategy under incomplete information gives less expected payoff to the start-up than the zero-NPV strategy (i.e., investing at the timing when the NPV of the investment becomes positive) under the same incomplete information.

Our results imply that in some cases a firm using the real options approach to investment has a risk of incorrect conjectures about the behaviors of its competitors and therefore must be cautious of applying the approach in the real world. Although we consider the simple model involving two firms for the purpose of concentrating our attention on the loss due to incomplete information, the proposed method of evaluating the loss due to incomplete information could also be applied to other real options models involving several firms.

This paper is organized as follows. After the model is introduced in Section 2, Section 3 gives the start-up's value function and optimal strategy under complete information. Section 4 describes our main theoretical results, which show the start-up's strategy under incomplete information, its expected payoff, and the loss due to incomplete information. Section 5 gives numerical examples, and Section 6 concludes the paper.

2 Model

This section introduces the model treated in this paper. We consider the start-up's problem of determining the timing of entering the new market which may be taken over by the large firm eventually. In this problem, we will discuss how incomplete information about the large firm affects the expected payoff of the start-up. Throughout this paper, we assume that both stochastic process and random variable are defined on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. The model is described as follows:

Profit flows and investment costs of the two firms: The start-up can receive a profit flow $D_1(1,0)Y(t)$ in the new market by paying an indivisible investment cost I_1, but the flow will be reduced to $D_1(1,1)Y(t)$ after the date on which the large firm enters the new market. Here, $(1,0)$ and $(1,1)$ denote the situation in which the start-up is active alone in the market, and the situation in which both of the firms are active in the market, respectively. Quantities $I_1, D_1(1,1)$ and $D_1(1,0)$ are constants such that $I_1 > 0$ and $0 \leq D_1(1,1) < D_1(1,0)$, and $Y(t)$ is the state of the market satisfying the following geometric Brownian motion:

$$\begin{aligned} \mathrm{d}Y(t) &= \mu Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}B(t) \quad (t > 0), \\ Y(0) &= y, \end{aligned} \tag{1}$$

where $\mu\ (\geq 0), \sigma\ (> 0)$ and $y\ (> 0)$ are given constants, and $B(t)$ denotes one-dimensional $\mathcal{F}_t$ standard Brownian motion. In contrast, the large firm does not notice the existence of the potential market until the start-up's investment. The large firm can obtain a profit flow $D_1(1,0)Y(t)$ in the market by paying an indivisible investment cost I_2 after the start-up's investment. Here, I_2 and $D_2(1,1)$ are positive constants.

The large firm's investment decision: The large firm does not notice the opportunity to preempt the market until the date τ_1 on which the start-up invests in the market. Then, with discount rate $\rho\ (> \mu)$, the large firm optimizes its investment timing τ_2 by solving the following optimal stopping problem:

$$\sup_{\tau_2 \geq \tau_1} E[\int_{\tau_2}^{\infty} \mathrm{e}^{-\rho t} D_2(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_2} I_2], \tag{2}$$

where τ_2 is any $\mathcal{F}_t$ stopping time that satisfies $\tau_2 \geq \tau_1$. Let us call $Q_i = D_i(1,1)/I_i$ $(i = 1,2)$ the qualities of the start-up and the large firm, respectively, and let τ_2^q denote an optimal stopping time of problem (2) in which $Q_2 = D_2(1,1)/I_2$ is replaced with a general constant q (>0).

The start-up's investment decision: Since the start-up does not have complete information about the quality of the large firm, the start-up determines its investment timing τ_1 assuming that the quality of the large firm obeys a random variable X independent of filtration $\{\mathcal{F}_t\}$. Then, the start-up believes that the expected payoff of investing at τ_1 is equal to

$$E\left[\int_{\tau_1}^{\tau_2^X} \mathrm{e}^{-\rho t} D_1(1,0) Y(t) \mathrm{d}t + \int_{\tau_2^X}^{+\infty} \mathrm{e}^{-\rho t} D_1(1,1) Y(t) \mathrm{d}t - \mathrm{e}^{-\rho \tau_1} I_1\right], \tag{3}$$

where τ_2^X represents a random variable which takes a value $\tau_2^{X(\omega)}(\omega)$ for $\omega \in \Omega$ (note that τ_2^X also depends on τ_1). The start-up finds its investment timing τ_1 by solving the following optimal stopping problem:

$$\sup_{\tau_1} E\left[\int_{\tau_1}^{\tau_2^X} \mathrm{e}^{-\rho t} D_1(1,0) Y(t) \mathrm{d}t + \int_{\tau_2^X}^{+\infty} \mathrm{e}^{-\rho t} D_1(1,1) Y(t) \mathrm{d}t - \mathrm{e}^{-\rho \tau_1} I_1\right], \tag{4}$$

where τ_1 is any $\mathcal{F}_t$ stopping time. Let $V(y)$ and τ_1^* denote the value function and an optimal stopping time of problem (4), respectively (recall $y = Y(0)$). An optimal stopping time τ_1^* is expressed in a form independent of the starting point y, as will be shown in Sections 3 and 4. Let $V(y;q)$ and τ_1^q be the value function and an optimal stopping time of a special problem (4) in which the random variable X is replaced with a constant q (>0), respectively. We note that if the start-up knows the real value Q_2 of the quality of the large firm (i.e, in the case of complete information), the start-up invests at $\tau_1^{Q_2}$ and its expected payoff becomes equal to $V(y;Q_2)$.

Remark 2.1 *The adapted process $Y(t)$ means the observable state of the market at time t, and it causes an exogenous change in the firms' profit flows in the market. In contrast, $D_i(\cdot,\cdot)$ $(i=1,2)$ represent endogenous changes due to the firms' entrance in the market.*

Remark 2.2 *For simplicity, this paper treats the two player leader-follower game as mentioned above, but similar results can be obtained in a more practical setting that permits several followers, by assuming that the followers make joint investment. There is a possibility that the followers make joint investment even if they are non-cooperative. In a strategic real options model involving several firms, a joint investment type equilibrium occurs under some condition, and in such a case a Pareto optimal equilibrium timing is of the same form as τ_2^q of problem (2). For details, see [8].*

References [4] and [8] have investigated a preemption model in which both two firms attempt to become a leader assuming complete information and $D_1(\cdot,\cdot) = D_2(\cdot,\cdot)$. Unlike their model, the model studied in this paper is a leader-follower game. In the remainder of this paper, we assume $0 = D_1(1,1) < D_2(1,1)$ to concentrate on the effect of incomplete information. Our model describes the following tendency which has been observed in practice frequently: A start-up with a unique idea and technology has an advantage of pioneering a new market, but it has the weakness of being taken over by a large firm when both firms compete in the market. For simplicity, we will denote $D_1 = D_1(1,0)$ and $D_2 = D_2(1,1)$ unless they cause confusion.

3 Case of complete information

This section derives the value function $V(y;q)$ and an optimal stopping time τ_1^q of the start-up who believes that the quality of the large firm is a constant q (>0) (i.e., $X \equiv q$ in problem (4)). They can be derived in the same fashion as in the case of complete information. When information is complete, the start-up's value function and optimal stopping time can be derived in the usual manner (e.g., see p. 314 in [4]). In order to solve an optimal stopping problem with discount rate ρ and state process $Y(t)$ that follows the geometric Brownian motion (1), we need a general solution of the following differential equation (see [4]):

$$\sigma y^2 \frac{\mathrm{d}^2 F}{\mathrm{d}y^2} + \left(\mu + \frac{\sigma^2}{2}\right) y \frac{\mathrm{d}F}{\mathrm{d}y} - \rho F = 0.$$

A general solution of this differential equation is of the form $F(y) = a_1 y^{\beta_1} + a_2 y^{\beta_2}$, where a_i $(i = 1,2)$ are any constants and β_i $(i = 1,2)$ are zeros of the quadratic function

$$\frac{\sigma^2 \beta(\beta - 1)}{2} + \mu\beta - \rho,$$

that is,

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1, \tag{5}$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < 0. \tag{6}$$

We can easily check the inequalities in (5) and (6) by taking into consideration that the above quadratic function takes negative values at $\beta = 0$ and $\beta = 1$ by $0 \leq \mu < \rho$. Furthermore, we introduce the following notation:

$$y_M(q) = \frac{\beta_1(\rho-\mu)}{(\beta_1-1)q} \qquad (q>0), \tag{7}$$

$$p(\beta, q_1, q_2) = \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} - \frac{q_2}{q_1} \qquad (\beta > 1, q_1, q_2 > 0). \tag{8}$$

Note that $y_M(q)$ denotes the optimal threshold of a monopolist with quality q, that is, an optimal stopping time for the monopolist with quality q is given by $\inf\{t \geq 0 \mid Y(t) \geq y_M(q)\}$ (e.g., see [4]).

Under the assumption that the start-up has already entered in the market at time τ_1, the large firm's problem becomes a problem for a monopolist. Therefore, an optimal stopping time τ_2^q of problem (2) in which Q_2 is replaced with a constant q is expressed as follows:

$$\tau_2^q = \inf\{t \geq \tau_1 \mid Y(t) \geq y_M(q)\}. \tag{9}$$

Notice that τ_2^q is a hitting time into the interval $[y_M(q), +\infty)$ since the start-up's investment timing τ_1. Then, we can derive the start-up's value function $V(y;q)$ and investment timing τ_1 in the case of $X \equiv q$ and $Y(0) = y$ in problem (4).

Proposition 3.1 *The start-up's value function $V(y;q)$ and optimal stopping time τ_1^q are given as follows: If*

$$p(\beta_1, Q_1, q) > 0, \tag{10}$$

then

$$V(y;q) = \begin{cases} A(q)y^{\beta_1} & (0 < y < y_M(Q_1)) \\ \dfrac{D_1 y}{\rho-\mu} - I_1 - \dfrac{D_1 y_M(q)^{-\beta_1+1} y^{\beta_1}}{\rho-\mu} & (y_M(Q_1) \leq y < y_U(q)) \\ B(q)y^{\beta_2} & (y \geq y_U(q)), \end{cases} \tag{11}$$

and τ_1^q is expressed as

$$\tau_1^q = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(q)]\}$$

(i.e., a hitting time into the interval $[y_M(Q_1), y_U(q)]$) regardless of the initial point $Y(0) = y$. Here, $y_M(\cdot)$ is defined by (7), and $A(q)$ is defined by

$$A(q) = y_M(Q_1)^{-\beta_1}\left(\frac{D_1 y_M(Q_1)}{\rho-\mu} - I_1 - \frac{D_1 y_M(q)^{-\beta_1+1} y_M(Q_1)^{\beta_1}}{\rho-\mu}\right) \qquad (q>0). \tag{12}$$

Moreover, $y_U(q)$ denotes the unique solution of the equation

$$\frac{(\beta_1-\beta_2)Q_1 y_M(q)^{-\beta_1+1}}{\rho-\mu}y^{\beta_1}+\frac{(\beta_2-1)Q_1}{\rho-\mu}y-\beta_2=0 \quad (y_M(Q_1)<y<y_M(q)), \tag{13}$$

and for $q>0$ satisfying (10), $B(q)$ is defined by

$$B(q)=y_U(q)^{-\beta_2}\left(\frac{D_1 y_U(q)}{\rho-\mu}-I_1-\frac{D_1 y_M(q)^{-\beta_1+1}y_U(q)^{\beta_1}}{\rho-\mu}\right). \tag{14}$$

If (10) does not hold, then $V(y;q)=0$ for all $y>0$ and $\tau_1^q=+\infty$.

(Proof) See Appendix A.

Remark 3.1 *Until the quality of the large firm q exceeds the solution of $p(\beta_1,Q_1,q)=0$, inequality (10) holds, and $y_U(q)$ and $V(y;q)$ monotonically decrease with q.*

Remark 3.2 *By taking $q=Q_2$ in Proposition 3.1, we can obtain the expected payoff $V(y;Q_2)$ of the start-up who has complete information about the quality of the large firm.*

Remark 3.3 *From Proposition 3.1, we have $y_U(q)\to+\infty$ and $\tau_1^q\to\inf\{t\geq 0\mid Y(t)\geq y_M(Q_1)\}$ as $q\to+0$; this means that the stopping time τ_1^q tends to an optimal stopping time in the case of the monopoly.*

The value function $V(y;q)$ is illustrated for some instances in Figures 1 and 2 shown in later sections, which will help understand Proposition 3.1. In Proposition 3.1, $y_U(q)$ means a threshold at which the start-up's expected payoff until the entry of the large firm with quality q exceeds the value of the start-up's option to delay its investment. Proposition 3.1 suggests that the start-up should delay its investment until the date on which the state of the market $Y(t)$ drops to the threshold $y_U(q)$, when the present state of the market $Y(0)=y$ is larger than $y_U(q)$. It is possible that $Y(t)$ decreases from the initial point y to the threshold $y_U(q)$ even with a positive drift μ in (1), as $Y(t)$ has a positive volatility σ in (1). Even if the start-up makes immediate investment in the case of $y>y_U(q)$, the large firm is quite likely to enter the market before the start-up gains enough income.

Inequality (10) can be interpreted as a prerequisite condition for the start-up's investment for the reason mentioned below. In [4] and [8], since it is assumed that $D_1(1,1)>0$, the expected payoff of the start-up's immediate investment for a very large y becomes always positive, even if the large firm enters the market as soon as the start-up invests. On the other hand, in our model, since it is assumed that $D_1(1,1) = 0$, the start-up's expected payoff never becomes positive for any time t and any value of $Y(t)$, unless (10) holds (for details see Appendix A).

Let us examine how the prerequisite condition (10) is affected by the values of parameters μ, ρ and σ. First observe from (5) that $\partial\beta_1/\partial\sigma < 0, \lim_{\sigma\to\infty}\beta_1 = 1, \lim_{\sigma\to+0}\beta_1 = \rho/\mu > 1, \partial\beta_1/\partial\mu < 0$ and $\partial\beta_1/\partial\rho > 0$ (also see [4]). Since $p(\beta, Q_1, q)$ is monotonically increasing for $\beta > 1$ by (8), the prerequisite condition (10) becomes more restrictive as the expected return μ and the volatility σ (resp. the discount rate ρ) of the market increase (resp. decrease). Moreover, we note that

$$\begin{aligned} \lim_{\beta\downarrow 1} p(\beta, Q_1, q) &= \frac{1}{\mathrm{e}} - \frac{q}{Q_1}, \\ \lim_{\beta\uparrow+\infty} p(\beta, Q_1, q) &= 1 - \frac{q}{Q_1}. \end{aligned}$$

Thus, when $q/Q_1 < 1/\mathrm{e}$, the prerequisite condition (10) always holds regardless of μ, ρ and σ, but when $q/Q_1 \geq 1$, the prerequisite condition never holds.

The next section describes our main results, which evaluate the start-up's loss due to incomplete information about the quality of the large firm.

4 Loss due to incomplete information

This section evaluates the start-up's loss due to incomplete information about the quality of the large firm by the following procedure:

1. Derive the start-up's value function $V(y)$ and its optimal stopping time τ_1^* of problem (4) which the start-up believes.

2. Derive the expected payoff $\tilde{V}(y)$ which can be obtained by the start-up who invests at the timing τ_1^* derived in Step 1.

3. Compute $W(y) = V(y; Q_2) - \tilde{V}(y)$, which is the difference between the expected payoff $V(y; Q_2)$ of the start-up who has complete information about the quality of the large firm and the expected payoff $\tilde{V}(y)$ of the start-up who invests at the timing τ_1^* under incomplete information.

The quantity $W(y)$ computed in Step 3 is regarded as the loss due to incomplete information. In this paper, we use the above method to evaluate the loss which the firm suffers with incomplete information compared with the case of complete information. The proposed method may also be applied to other real options models involving several firms. The loss due to incomplete information is identified as the value of information about the rival firm, and hence it tells us whether the firm should conduct a further survey on the rival firm or not. Subsection 4.1, Subsection 4.2 and Subsection 4.3 describe Step 1, Step 2 and Step 3, respectively.

4.1 Start-up's strategy under incomplete information

With incomplete information, the start-up determines its investment timing, believing that the quality of the large firm obeys a random variable X independent of $\{\mathcal{F}_t\}$. Here we assume that $X > 0$ and $E[X^{\beta_1 - 1}] < +\infty$. Given the initial state $Y(0) = y$, we can compute the expected payoff $g(y)$ which the start-up believes that its immediate investment (i.e., $t = 0$) generates, that is, the expectation of (3) in the case of $\tau_1 = 0$, as follows:

$$g(y) = \frac{D_1 y}{\rho - \mu} - \frac{D_1 E\left[(y_M(X) \vee y)^{-\beta_1 + 1}\right] y^{\beta_1}}{\rho - \mu} - I_1 \qquad (y > 0), \tag{15}$$

where $a \vee b$ means $\max\{a, b\}$. The equation (15) can be derived by using the independence between X and $\{\mathcal{F}_t\}$ (for details, see the proof of Proposition 4.1 in Appendix B). Generally, it is hard to derive an explicit form of the value function $V(y)$ and an optimal stopping time of problem (4). However, we can show that problem (4) is reduced to the problem with $X \equiv q$ (i.e., the problem discussed in Section 3) for some constant q, provided that the following condition holds:

Condition (a): The inequality $g(y) \leq V(y; \tilde{Q}_2)$ holds for all $y > 0$, where $\tilde{Q}_2 = E[X^{\beta_1 - 1}]^{1/(\beta_1 - 1)}$.

In relation to (15), we define

$$h(y) = \frac{D_1 y}{\rho - \mu} - \frac{D_1 E\left[y_M(X)^{-\beta_1 + 1}\right] y^{\beta_1}}{\rho - \mu} - I_1 \qquad (y > 0). \tag{16}$$

From the definitions of $g(y), h(y)$ and $\tilde{Q}_2$, it immediately follows that

$$h(y) \leq g(y) \qquad (y > 0) \tag{17}$$

and

$$\begin{aligned} h(y) &= \frac{D_1 y}{\rho-\mu} - \frac{D_1 y_M(\tilde{Q}_2)^{-\beta_1+1} y^{\beta_1}}{\rho-\mu} - I_1 \\ &= V(y;\tilde{Q}_2) \qquad (y_M(Q_1) < y < y_U(\tilde{Q}_2)), \end{aligned} \tag{18}$$

when $p(\beta_1, Q_1, \tilde{Q}_2) > 0$. By using this formula, we can show the following proposition, which is the key result to evaluating the loss due to incomplete information.

Proposition 4.1 *Assume that Condition* (a) *holds. Then, the value function $V(y)$ and an optimal stopping time of problem* (4) *which the start-up believes are given as $V(y) = V(y;\tilde{Q}_2)$ and $\tau_1^* = \tau_1^{\tilde{Q}_2}$ for all $y > 0$, respectively, where, $V(y;\tilde{Q}_2)$ and τ_1^q are given in Proposition 3.1.*

(Proof) See Appendix B.

Remark 4.1 *In Section 5, we will observe that Condition* (a) *is likely to hold when the support of X is not wide. In particular, we can easily show that Condition* (a) *certainly holds whenever X is a constant.*

Remark 4.2 *Figure 1 illustrates the function $V(y) = V(y;\tilde{Q})$ together with the functions $g(y)$ and $h(y)$. In particular, it shows that $V(y) = V(y;\tilde{Q}_2) = g(y) = h(y)$ holds for $y \in [y_M(Q_1), y_U(\tilde{Q}_2)]$.*

By Proposition 4.2, the start-up who regards the quality of the large firm as the random variable X invests at the same timing as the start-up who regards the quality of the large firm as the constant $\tilde{Q}_2$, provided Condition (a) holds. However, this is not always true unless Condition (a) holds. In the general case, the start-up's strategy (i.e., an optimal stopping time of problem (4)) need not be expressed as a hitting time, like $\tau_1^{\tilde{Q}_2}$, that has two thresholds. In the rest of the paper, we will restrict our attention to the case where Condition (a) is satisfied.

4.2 The expected payoff of the start-up

We derive the expected payoff $\tilde{V}(y)$ of the start-up who enters the new market at time τ_1^* given in Proposition 4.1 (Step 2). Since the large firm actually has quality Q_2, its real investment timing is equal to $\tau_2^{Q_2}$, where

$$\tau_2^{Q_2} = \inf\{t \geq \tau_1^* \mid Y(t) \geq y_M(Q_2)\} \tag{19}$$

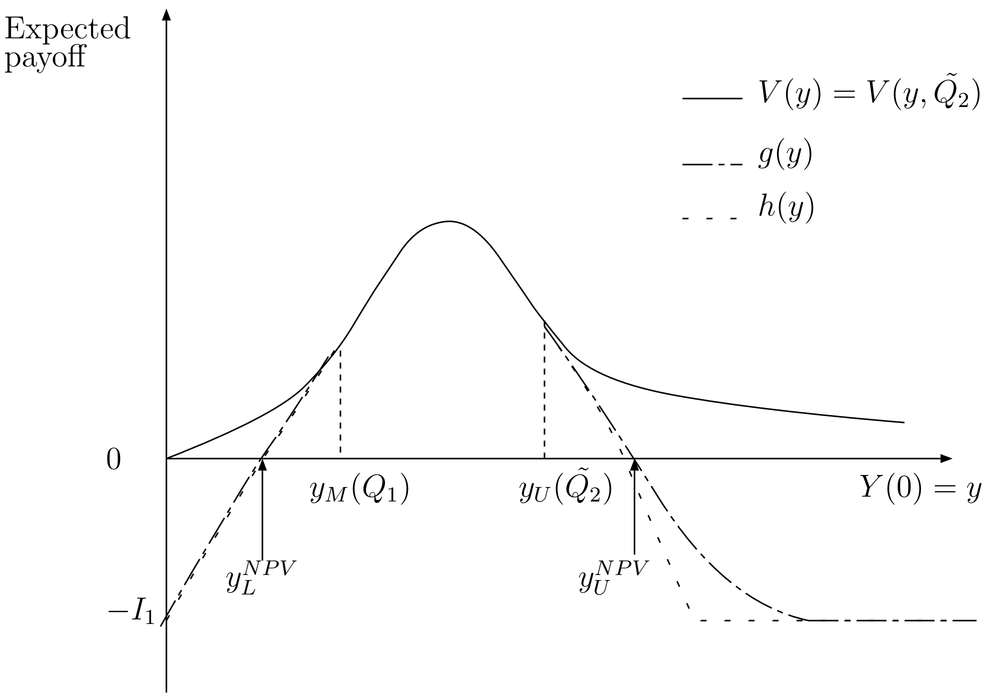

Figure 1: $g(y), h(y)$ and $V(y) = V(y; \tilde{Q}_2)$.

by (9). Then, the expected payoff $\tilde{V}(y)$ becomes

$$\tilde{V}(y) = E\left[\int_{\tau_1^*}^{\tau_2^{Q_2}} \mathrm{e}^{-\rho t} D_1 Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_1^*} I_1\right]. \tag{20}$$

We can show the following proposition by computing the expectation of (20).

Proposition 4.2 *Assume that Condition* (a) *holds. Then the expected payoff $\tilde{V}(y)$ of the start-up who invests at τ_1^* is given as follows. If $p(\beta_1, Q_1, \tilde{Q}_2) > 0$, then*

$$\tilde{V}(y) = \begin{cases} \tilde{A}(Q_2)y^{\beta_1} & (0 < y < y_M(Q_1)) \\ \dfrac{D_1 y}{\rho-\mu} - \dfrac{D_1\left(y \vee y_M(Q_2)\right)^{-\beta_1+1} y^{\beta_1}}{\rho-\mu} - I_1 & (y_M(Q_1) \le y < y_U(\tilde{Q}_2)) \\ \tilde{B}(\tilde{Q}_2)y^{\beta_2} & (y \ge y_U(\tilde{Q}_2)), \end{cases}$$

where $y_M(\cdot)$ is defined by (7)*, $y_U(\cdot)$ is the unique solution of equation* (13)*, and $\tilde{A}(\cdot)$ and $\tilde{B}(\cdot)$ are given by*

$$\tilde{A}(q) = y_M(Q_1)^{-\beta_1}\left(\frac{D_1 y_M(Q_1)}{\rho-\mu} - \frac{D_1\left(y_M(Q_1) \vee y_M(q)\right)^{-\beta_1+1} y_M(Q_1)^{\beta_1}}{\rho-\mu} - I_1\right), \tag{21}$$

$$\tilde{B}(q) = y_U(q)^{-\beta_2}\left(\frac{D_1 y_U(q)}{\rho-\mu} - \frac{D_1(y_U(q) \vee y_M(Q_2))^{-\beta_1+1} y_U(q)^{\beta_1}}{\rho-\mu} - I_1\right). \tag{22}$$

If $p(\beta_1, Q_1, \tilde{Q}_2) \le 0$, then $\tilde{V}(y) = 0$ for all $y > 0$.

(Proof) See Appendix C.

Remark 4.3 *Propositions 3.1, 4.1 and 4.2 ensure that, under Condition* (a)*, we have $\tilde{V}(y) = V(y; Q_2) = V(y; \tilde{Q}_2) = V(y)$ whenever $\tilde{Q}_2 = Q_2$.*

4.3 The start-up's loss due to incomplete information

Finally we evaluate the start-up's loss $W(y) = V(y; Q_2) - \tilde{V}(y)$ due to incomplete information about the quality of the large firm (Step 3). The loss $W(y)$ varies according to the relation between $\tilde{Q}_2$ and Q_2. Notice that $y_M(\cdot)$ is monotonically decreasing by (7).

Case 1: $\tilde{Q}_2 < Q_2$ The start-up underestimates the quality of the large firm, and the inequality $y_M(\tilde{Q}_2) > y_M(Q_2)$ holds with respect to the threshold of the large firm's investment

Case 2: $\tilde{Q}_2 = Q_2$ The start-up correctly estimates the quality of the large firm, and the equality $y_M(\tilde{Q}_2) = y_M(Q_2)$ holds with respect to the threshold of the large firm's investment.

Case 3: $\tilde{Q}_2 > Q_2$ The start-up overestimates the quality of the large firm, and the inequality $y_M(\tilde{Q}_2) < y_M(Q_2)$ holds with respect to the threshold of the large firm's investment.

The following proposition describes the start-up's loss $W(y)$ in each case.

Proposition 4.3 *Assume that Condition* (a) *holds. The start-up's loss* $W(y)$ *due to incomplete information is given as follows.*

Case 1: $\tilde{Q}_2 < Q_2$

Case 1.1: $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$

$$W(y) = 0 \qquad (y > 0).$$

Case 1.2: $p(\beta_1, Q_1, \tilde{Q}_2) > 0$ *and* $p(\beta_1, Q_1, Q_2) \leq 0$

$$W(y) = -\tilde{V}(y) \qquad (y > 0).$$

Case 1.3: $p(\beta_1, Q_1, Q_2) > 0$

$$W(y) = \begin{cases} 0 & (0 < y < y_U(Q_2)) \\ B(Q_2)y^{\beta_2} - \dfrac{D_1 y}{\rho - \mu} + \dfrac{D_1 (y \vee y_M(Q_2))^{-\beta_1+1} y^{\beta_1}}{\rho - \mu} + I_1 & (y_U(Q_2) \leq y < y_U(\tilde{Q}_2)) \\ \left(B(Q_2) - \tilde{B}(\tilde{Q}_2)\right) y^{\beta_2} & (y \geq y_U(\tilde{Q}_2)). \end{cases}$$

Case 2: $\tilde{Q}_2 = Q_2$

$$W(y) = 0 \qquad (y > 0).$$

Case 3: $\tilde{Q}_2 > Q_2$

Case 3.1: $p(\beta_1, Q_1, Q_2) \leq 0$

$$W(y) = 0 \qquad (y > 0).$$

Case 3.2: $p(\beta_1, Q_1, Q_2) > 0$ *and* $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$

$$W(y) = V(y; Q_2) \qquad (y > 0).$$

Case 3.3: $p(\beta_1, Q_1, \tilde{Q}_2) > 0$

$$W(y) = \begin{cases} 0 & (0 < y < y_U(\tilde{Q}_2)) \\ \dfrac{D_1 y}{\rho - \mu} - I_1 - \dfrac{D_1 y_M(Q_2)^{-\beta_1+1} y^{\beta_1}}{\rho - \mu} - \tilde{B}(\tilde{Q}_2) & (y_U(\tilde{Q}_2) \leq y < y_U(Q_2)) \\ \left(B(Q_2) - \tilde{B}(\tilde{Q}_2)\right) y^{\beta_2} & (y \geq y_U(Q_2)). \end{cases}$$

Here, $y_U(\cdot)$ *is the unique solution of equation* (13)*, and* $B(\cdot)$ *and* $\tilde{B}(\cdot)$ *are defined by* (14) *and* (22)*, respectively.*

(Proof) In Case 1 (i.e., $\tilde{Q}_2 < Q_2$), we have $p(\beta_1, Q_1, Q_2) < p(\beta_1, Q_1, \tilde{Q}_2)$ by (8). Thus, we can easily compute $W(y) = V(y; Q_2) - \tilde{V}(y)$ from Propositions 3.1 and 4.2 for each of Cases 1.1, 1.2, and 1.3. In Case 2 (i.e., $\tilde{Q}_2 = Q_2$), we have $\tilde{V}(y) = V(y; Q_2)$ and hence $W(y) = V(y; Q_2) - \tilde{V}(y) = 0$. In Case 3 (i.e., $\tilde{Q}_2 > Q_2$), we have $p(\beta_1, Q_1, \tilde{Q}_2) < p(\beta_1, Q_1, Q_2)$. Then, we can compute $W(y)$ from Propositions 3.1 and 4.2 for each of Cases 3 3.1, 3.2, and 3.3. □

Let us explain the start-up's investment strategy in each case. Needless to say, in Case 2 the start-up's strategy becomes optimal as $\tau_1^* = \tau_1^{\tilde{Q}_2} = \tau_1^{Q_2}$, and hence the start-up suffers no loss $W(y)$ for any initial point $y > 0$. In Cases 1.1 and 3.1, the prerequisite condition for the start-up's investment does not actually hold (i.e., $p(\beta_1, Q_1, Q_2) \leq 0$), and the start-up never attempts to invest. As a result, in these cases the start-up's strategy becomes optimal, and the loss $W(y)$ never arises for any $y > 0$.

Case 1.2 and Case 3.2 correspond to the case where the start-up attempts to invest although the prerequisite condition does not actually hold, and the case where the start-up never attempts to invest although the prerequisite condition actually holds, respectively. Therefore, in both cases, the start-up suffers the loss $W(y)$ for all $y > 0$. We note that $\tilde{V}(y) < V(y; Q_2) = 0$ for all $y > 0$ in Case 1.2.

In Cases 1.3 and 3.3, the prerequisite condition actually holds, and also the start-up attempts to invest. The start-up however makes its investment at $\tau_1^{\tilde{Q}_2} = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\}$, though the optimal investment timing $\tau_1^{Q_2}$ is given as $\inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(Q_2)]\}$. In Case 1.3, since $y_U(\tilde{Q}_2) > y_U(Q_2)$, the start-up makes investment earlier than $\tau_1^{Q_2}$ and suffers the loss $W(y)$ when $y > y_U(Q_2)$; conversely, in Case 3.3, since $y_U(\tilde{Q}_2) < y_U(Q_2)$, the start-up makes investment later than $\tau_1^{Q_2}$ and suffers the loss $W(y)$ when $y > y_U(\tilde{Q}_2)$.

Corollary 4.1 *Assume that Condition* (a) *holds. Also assume that the random variable* X (*the start-up's prospect for the quality of the large firm*) *has a support* $(0, Q_U]$ *for some constant* Q_U, *and that* Q_2 (*the real quality of the large firm*) *satisfies* $Q_2 \in (0, Q_U]$. *If*

$$p(\beta_1, Q_1, Q_U) > 0 \tag{23}$$

and

$$y \le y_U(Q_U) \tag{24}$$

hold, then the start-up suffers no loss $W(y)$ *due to incomplete information. Here,* $y_U(\cdot)$ *is defined as the unique solution of equation* (13).

Condition (23) means that it is certain that the quality of the start-up Q_1 is sufficiently better than the quality of the large firm Q_2. Condition (24) means that the initial state of the new market $Y(0) = y$ cannot generate great profit immediately, and ensures that it will take some time for the large firm to enter the market. Thus, by Proposition 4.3, more detailed information about the large firm is of little value when the quality of the start-up (such as the technology) is much better than that of the large firm in the new market that cannot generate great profit immediately.

In the case of incomplete information, the expected payoff $\tilde{V}(y)$ obtained by the real options strategy τ_1^* may generate less profit than the expected payoff $\tilde{V}_{NPV}(y)$ obtained by the zero-NPV strategy (which means to invest when the NPV of the investment becomes positive) with the same prospect X. To see this, consider the function $g(y)$ defined by (15) and assume that the equation $g(y) = 0$ $(y > 0)$ has exactly two solutions denoted $0 < y_L^{NPV} < y_U^{NPV}$ (see Figure 1), which is expected to hold in many cases. Then, the start-up who employs the zero-NPV strategy invests at time $\tau_1^{NPV} = \inf\{t \ge 0 \mid Y(t) \in [y_L^{NPV}, y_U^{NPV}]\}$, although the start-up who uses the real options strategy invests at $\tau_1^* = \inf\{t \ge 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\}$. Since $y_L^{NPV} < y_M(Q_1) < y_U(\tilde{Q}_2) < y_U^{NPV}$ (see Figure 1), the zero-NPV timing τ_1^{NPV} is not later than the real options timing τ_1^*. We define Q_{NPV} as the unique solution of $y_U(q) = y_U^{NPV}$. Taking into consideration that the zero-NPV timing is expressed as $\inf\{t \ge 0 \mid Y(t) \in [y_L^{NPV}, y_U(Q_{NPV})]\}$, we can show the following corollary.

Corollary 4.2 *Assume that Condition* (a) *holds. Also assume that the equation* $g(y) = 0$ $(y > 0)$ *has exactly two solutions. Then,* $\tilde{V}_{NPV}(y) > \tilde{V}(y)$ *holds if and only if one of the following three conditions is satisfied in Case 3.3. (i.e.,* $\tilde{Q}_2 > Q_2$ *and* $p(\beta_1, Q_1, \tilde{Q}_2) > 0$*):*

- $Q_{NPV} < Q_2$ *and* $y > y_U(\tilde{Q}_2)$
- $Q_2 \leq Q_{NPV}, \tilde{B}(\tilde{Q}_2) < \tilde{B}(Q_{NPV})$ *and* $y > y_U(\tilde{Q}_2)$
- $Q_2 \leq Q_{NPV}, \tilde{B}(Q_{NPV}) \leq \tilde{B}(\tilde{Q}_2)$ *and* $y_U(\tilde{Q}_2) < y < y_C$

Here, $y_U(\cdot)$ *is the unique solution of equation* (13) *and* $\tilde{B}(\cdot)$ *is defined by* (22). *Moreover,* y_C *is the unique solution of the equation*

$$\frac{D_1}{\rho - \mu} (y \vee y_M(Q_2))^{-\beta_1+1} y^{\beta_1} + \tilde{B}(\tilde{Q}_2) y^{\beta_2} - \frac{D_1}{\rho - \mu} y + I_1 = 0 \quad (y_U(Q_2) < y \leq y_U(Q_{NPV})),$$

which is obtained as the intersection of the graphs of two functions $\tilde{V}_{NPV}(y) = D_1 y/(\rho - \mu) - I_1 - D_1 (y \vee y_M(Q_2))^{-\beta_1+1} y^{\beta_1}/(\rho - \mu)$ *and* $\tilde{V}(y) = \tilde{B}(\tilde{Q}_2) y^{\beta_2}$.

To conclude this section, let us examine the special case where the mean of the start-up's prospect X is equal to the real quality of the large firm Q_2. In this case, contrary to intuition, the start-up's strategy is generally different from the optimal one.

Proposition 4.4 *Assume that Condition* (a) *holds and* $E[X] = Q_2$. *Then,*

$$\tilde{Q}_2 \begin{cases} \leq & Q_2 \quad (1 < \beta_1 < 2) \\ = & Q_2 \quad (\beta_1 = 2) \\ \geq & Q_2 \quad (\beta_1 > 2). \end{cases}$$

(Proof) Note that

$$\begin{aligned} {\tilde{Q}_2}^{\beta_1 - 1} &= E[X^{\beta_1 - 1}] \\ &\begin{cases} \leq E[X]^{\beta_1 - 1} = {Q_2}^{\beta_1 - 1} & (1 < \beta_1 < 2) \\ = Q_2 & (\beta_1 = 2) \\ \geq E[X]^{\beta_1 - 1} = {Q_2}^{\beta_1 - 1} & (\beta_1 > 2), \end{cases} \end{aligned} \tag{25}$$

where (25) follows from the Jensen inequality (e.g., see [12]), since the function $x^{\beta_1 - 1}$ $(x > 0)$ is concave when $1 < \beta_1 < 2$ and it is convex when $\beta_1 > 2$. We can deduce the conclusion, because $x^{\beta_1 - 1}$ $(x > 0)$ is continuous and monotonically increasing. $\square$

Remark 4.4 *The difference between Q_2 and $\tilde{Q}_2$ is usually quite small when $E[X] = \tilde{Q}_2$, compared with the case where $E[X]$ is far different from Q_2, as we will see in Section 5.*

Recall that, from (5), $\partial\beta_1/\partial\mu < 0, \partial\beta_1/\partial\sigma < 0$ and $\partial\beta_1/\partial\rho > 0$. When the expected return μ and the volatility σ of the market are high and the discount rate ρ is low, β_1 satisfies $1 < \beta_1 < 2$, and hence the start-up takes the strategy in the case of underestimating the large firm's quality (i.e., $\tilde{Q}_2 \leq Q_2$) by Proposition 4.4. On the other hand, when the expected return and the volatility of the market are low and the discount rate is high, the start-up takes the strategy in the case of overestimating the large firm's quality by Proposition 4.4. In other words, the start-up in the market with high expected return, high volatility and low discount rate can take a better strategy (i.e., $\tilde{Q}_2$ becomes closer to Q_2) when the mean $E[X]$ of the prospect slightly exceeds Q_2. In the market with low expected return, low volatility and high discount rate, the contrary holds.

5 Numerical Examples

This section presents some examples in which the start-up's loss $W(y)$ due to incomplete information is numerically computed. We set the parameters related to the state of the new market $Y(t)$ and the start-up as in [10], i.e.,

$$\mu = 0,\ \rho = 0.07,\ \sigma = 0.1,\ D_1 = 1,\ I_1 = 4.$$

Then, by definition, we have the start-up's quality $Q_1 = D_1/I_1 = 0.25$, and by (5), (6) and (8) we can compute

$$\beta_1 = 4.2749,\ \beta_2 = -3.2749,\ y_M(Q_1) = 0.3655.$$

Thus, the prerequisite condition (8) for the start-up's investment is given by

$$p(\beta_1, q_1, q_2) = 0.6417 - \frac{q_2}{q_1} > 0.$$

We set the quality of the large firm as $Q_2 = 1/2Q_1$, that is, $Q_2 = 0.125$. Notice that in this case, $p(\beta_1, Q_1, Q_2) = 0.1417 > 0$ holds, and hence the prerequisite condition actually holds.

First, we computed the value function $V(y; Q_2)$ of the start-up who has complete information about the quality of the large firm (see Figure 2). Moreover, in order to examine when Condition (a) (i.e., $g(y) \le V(y; \tilde{Q}_2)$) is satisfied, we computed $g(y)$ together with $V(y; Q_2)$, for various uniform distributions of the random variable X that satisfy $\tilde{Q}_2 = Q_2$ but have different support widths $0.005, 0.01$ and 0.015. From Figure 2, we can observe that Condition (a) does not hold when the support of X is too wide. Since the random variable X means the start-up's prospect for the large firm's quality, this corresponds to the situation where the start-up's prospect about the quality of the large firm is obscure. On the other hand, it is expected that Condition (a) holds when the start-up's prospect about the quality of the large firm is decisive.

Next, we computed the start-up's expected payoff $\tilde{V}(y)$ and loss $W(y)$ in the case of incomplete information. Figures 3 and 4 illustrate $\tilde{V}(y)$ and $W(y)$ of the start-up who believes the quality of the large firm follows uniform distributions with various supports. In comparison with the first experiment, here, we employed uniform distributions that have the same support width 0.05 but different values of $\tilde{Q}_2$, in order to calculate the loss in various cases. For instance, $[0.06, 0.11]$ in Figure 3 shows $\tilde{V}(y)$ in the case where the start-up's prospect X follows the uniform distribution with support $[0.06, 0.11]$.

Since these examples satisfy Condition (a), $\tilde{V}(y)$ and $W(y)$ can be computed by the formulas given in Propositions 4.2 and 4.3. Table 1 shows quantities $\tilde{Q}_2, y_M(\tilde{Q}_2), p(\beta_1, Q_1, \tilde{Q}_2)$ and $y_U(\tilde{Q}_2)$ for each uniform distribution of X, where the top row shows the values in the case of $\tilde{Q}_2 = Q_2 = 0.125$. From $\tilde{Q}_2$ and $p(\beta_1, Q_1, \tilde{Q}_2)$ in Table 1, we see that $[0.06, 0.11]$ and $[0.08, 0.13]$ belong to Case 1.3 (i.e., $\tilde{Q}_2 < Q_2$ and $p(\beta_1, Q_1, Q_2) > 0$), while $[0.1, 0.15]$ and $[0.12, 0.17]$ belong to Case 3.3 (i.e., $\tilde{Q}_2 > Q_2$ and $p(\beta_1, Q_1, \tilde{Q}_2) > 0$). Since $[0.14, 0.19]$ does not satisfy the prerequisite condition (i.e., $p(\beta_1, Q_1, \tilde{Q}_2) = -0.0239 < 0$), it corresponds to Case 3.2 (i.e., $\tilde{Q}_2 > Q_2$ and $p(\beta_1, Q_1, \tilde{Q}_2) \le 0$). Moreover, in the case of $[0.14, 0.19]$, we have $\tilde{V}(y) = 0$ $(y > 0)$ and $W(y)$ becomes equal to $V(y; Q_2)$ in Figure 2, and hence they are not shown in Figures 3 and 4. In the case of $[0.1, 0.15]$, the mean of the start-up's prospect X agrees with Q_2, but $\tilde{Q}_2 = 0.1269 > 0.125 = Q_2$ and therefore the start-up suffers the loss when $y > 0.5131$ (cf. Proposition 4.4). However, in this case, since the loss is quite small (the maximum loss is just $W(y) = 0.0082$ for $y = 0.523$), we do not depict $W(y)$ in Figure 4.

Since $\tilde{Q}_2 \neq Q_2$ holds in all cases, the start-up's investment timing $\tau_1^* = \tau_1^{\tilde{Q}_2}$ differs from the optimal one $\tau_1^{Q_2}$ that realizes the value function $V(y; Q_2)$. Thus, $\tilde{V}(y)$ is not the value function of the optimal stopping problem, and fails to be continuously differentiable at the threshold $y = y_U(\tilde{Q}_2)$ (see Figure 3), which is a necessary condition for the value function to satisfy (i.e., Smooth Pasting, see [4]).

In Figure 4, it is remarkable that the start-up's loss in the case of underestimating the quality of the large firm (i.e., Case 1) is greater than that of the overestimation case (i.e., Case 3). This is because, in this example, the value function $V(y; Q_2)$ is much smaller than the investment cost $I_1 = 4$ for all $y > 0$. We note that the maximum loss is $I_1 + V(y; Q_2)$ in Case 1 and is $V(y; Q_2)$ in Case 3. Therefore, we may not observe such a remarkable phenomenon when $V(y; Q_2)$ is larger than I_1 in a wide area of $y > 0$.

Finally, we examined how the start-up's loss $W(y)$ due to incomplete information varies with the volatility σ of the new market. Figure 5 illustrates the relative loss $W(y)/V(y; Q_2)$ for $\sigma = 0.1, 0.15, 0.2$ and 0.25, where the start-up's prospect X is assumed to follow the uniform distribution with support $[0.08, 0.13]$. We attempted to compute the loss for $\sigma = 0.05$ and 0.3, but for $\sigma = 0.05$ Condition (a) does not hold and for $\sigma = 0.3$ we have $p(\beta_1, \beta_2, Q_2) < 0$ and $V(y; Q_2) = 0$ (recall that the prerequisite condition is restrictive when the volatility σ is high as mentioned in Section 3). In these examples, the difference between $\tilde{Q}_2$ and the real value Q_2 is monotonically increasing with respect to σ. However we did not observe a remarkable relation between the volatility σ and the relative loss $W(y)/V(y; Q_2)$. A possible reason for this is that β_1 and β_2 also vary with σ.

Table 1: $\tilde{Q}_2, y_M(\tilde{Q}_2)$ and $y_U(\tilde{Q}_2)$ for various uniform distributions of X.

X	$\tilde{Q}_2$	$y_M(\tilde{Q}_2)$	$p(\beta_1, Q_1, \tilde{Q}_2)$	$y_U(\tilde{Q}_2)$
0.125	0.125	0.731	0.1417	0.523
[0.06,0.11]	0.087	0.1053	0.2937	0.8029
[0.08,0.13]	0.1072	0.8524	0.2129	0.6312
[0.1,0.15]	0.1269	0.7201	0.1341	0.5131
[0.12,0.17]	0.1466	0.6233	0.0553	0.4217
[0.14,0.19]	0.1664	0.5491	-0.0239	N/A

6 Conclusion

This paper has investigated the effect of incomplete information in the model in which a start-up with a unique idea and technology pioneers a new market that will be taken

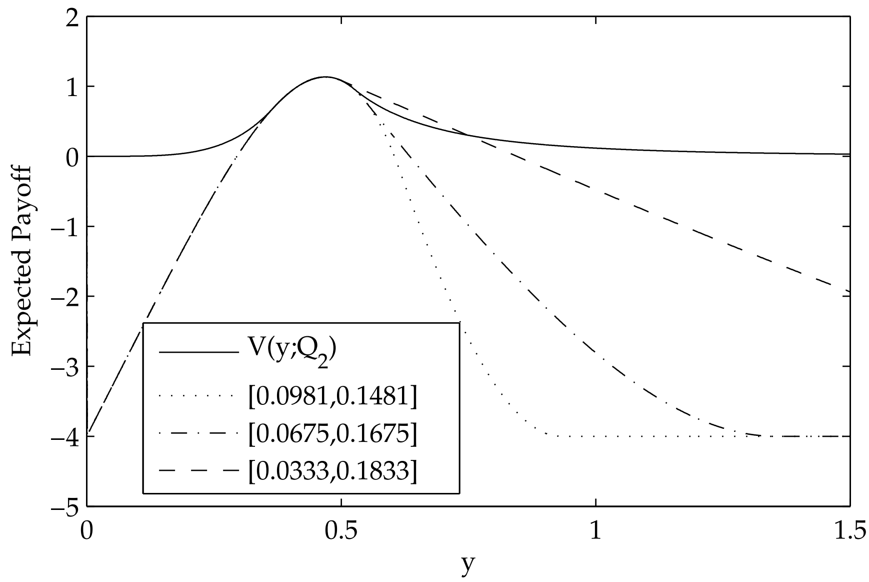

Figure 2: $V(y;Q_2)$ and $g(y)$.

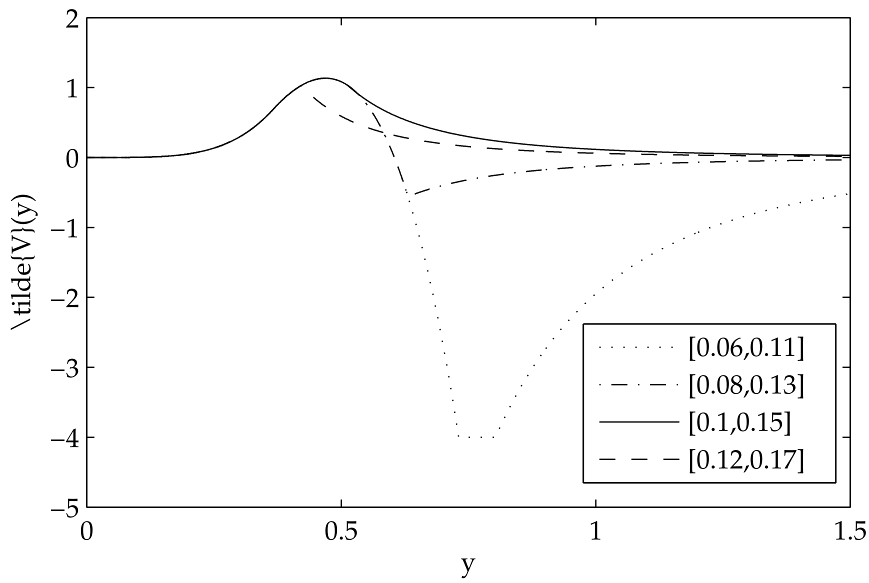

Figure 3: The start-up's expected payoff $\tilde{V}(y)$.

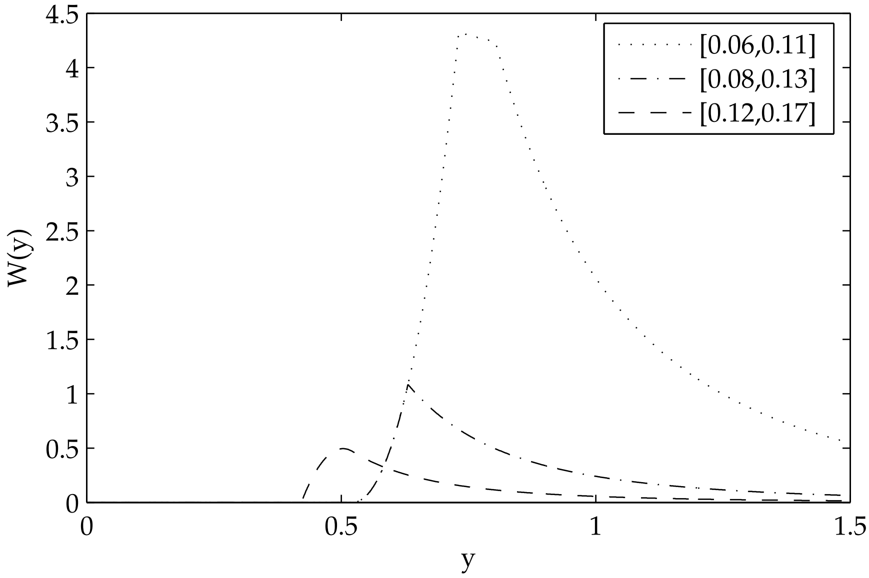

Figure 4: The start-up's loss due to incomplete information, i.e., $W(y)$.

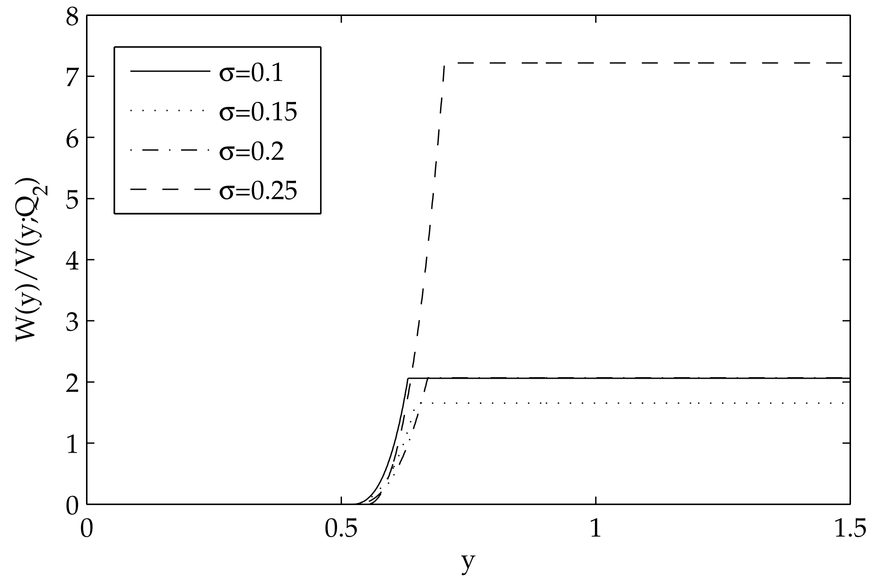

Figure 5: $W(y)/V(y;Q_2)$ for various parameters σ.

over by a large firm eventually. The main contribution of this paper is to evaluate the start-up's loss due to incomplete information about the large firm. The proposed method could be applied in other real options models involving several firms. The results obtained in this paper can be summarized as follows.

If the quality of the start-up is much better than that of the large firm and the current state of the market cannot generate great profit immediately, then the start-up requires no further survey on the quality of the large firm.

On the other hand, information about the quality of the large firm is valuable in the market that can readily generate great profit, even if the quality of the start-up is much better than that of the large firm. In this case, it is quite likely that the start-up's immediate investment does not produce much income for the start-up even before the large firm takes over the market from the start-up.

When it is doubtful that the quality of the start-up overwhelms that of the large firm, information about the quality of the large firm is always valuable regardless of the state of the market. The reason for this is that there is a possibility that the investment in the market is of no value (i.e., the prerequisite condition for the start-up's investment does not hold), in addition to the same risk as in the previous case, that is, the possibility that the start-up obtains little profit until the large firm's entry.

Furthermore, under incomplete information, the expected payoff of the start-up investing at the zero-NPV trigger could become greater than that of the start-up obeying the real options approach, and the start-up usually suffers the loss due to incomplete information even if the mean of the start-up's prospect for the quality of the large firm is equal to the real value.

In the real world, a start-up who has a unique idea and technology but is not competitive in the market may want to sell its idea and technology to a large firm, instead of investing in the new market by itself. Then, the value function which the start-up believes can be interpreted as a reward which the start-up demands for its idea and technology. As revealed in this paper, the value of the investment which the start-up believes under incomplete information is generally different from the real value of the investment which is regarded as the proper value of the start-up's idea and technology. Because of this gap, negotiations between the start-up and the large firm may not go smoothly. It remains as an interesting issue of future research to reveal the effect of incomplete information in such a negotiation problem of a firm having an option to sell its idea and technology to the rival firm.

Appendix A Proof of Proposition 3.1

Taking account of (9), we can compute (3) as follows:

$$\begin{aligned}
& E\left[\int_{\tau_1}^{\tau_2^q} \mathrm{e}^{-\rho t} D_1 Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_1} I_1\right] \\
= \; & E\left[\mathrm{e}^{\rho\tau_1}\left(D_1 E^{Y(\tau_1)}\left[\int_0^{\tau_2^q} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t\right] - I_1\right)\right] && (26) \\
= \; & E\left[\mathrm{e}^{\rho\tau_1}\left(D_1 E^{Y(\tau_1)}\left[\int_0^{+\infty} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t - \int_{\tau_2}^{+\infty} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t\right] - I_1\right)\right] \\
= \; & E\left[\mathrm{e}^{\rho\tau_1}\left(D_1 E^{Y(\tau_1)}\left[\int_0^{+\infty} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_2^q} E^{Y(\tau_2^q)}\left[\int_0^{+\infty} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t\right]\right] - I_1\right)\right] && (27) \\
= \; & E\left[\mathrm{e}^{\rho\tau_1}\left(D_1 E^{Y(\tau_1)}\left[\int_0^{+\infty} \mathrm{e}^{-\rho t} Y(t)\mathrm{d}t - \frac{\mathrm{e}^{-\rho\tau_2^q}(Y(0)\vee y_M(q))}{\rho-\mu}\right] - I_1\right)\right] \\
= \; & E\left[\mathrm{e}^{\rho\tau_1}\left(\frac{D_1 Y(\tau_1)}{\rho-\mu} - \frac{D_1\left(Y(\tau_1)\vee y_M(q)\right)^{-\beta_1+1} Y(\tau_1)^{\beta_1}}{\rho-\mu} - I_1\right)\right], && (28)
\end{aligned}$$

where we use the strong Markov property (e.g. see [12]) of the geometric Brownian motion $Y(t)$ to deduce (26) and (27), and use the formula of the expectation involving a hitting time (e.g. see [4]) to deduce (28). Here, for a random variable Z, $E^{Y(\tau_i)}[Z]$ denotes a random variable $G(Y(\tau_i))$, where for $y' > 0$, $G(y')$ is defined as an expectation $E[Z]$ in the case where $Y(t)$ starts at $Y(0) = y'$. Thus, problem (4) with X replaced by q is equivalent to

$$\sup_{\tau_1} E\left[\mathrm{e}^{-\rho\tau_1} f(Y(\tau_1); q)\right], \qquad (29)$$

where

$$f(y;q) = \frac{D_1 y}{\rho-\mu} - \frac{D_1\left(y \vee y_M(q)\right)^{-\beta_1+1} y^{\beta_1}}{\rho-\mu} - I_1. \qquad (30)$$

Consider the case where $f(y;q) \leq 0$ for all $y > 0$. In this case, the value function and an optimal stopping time are trivially given by $V(y;q) = 0$ and $\tau_1^q = +\infty$, respectively, for all $y > 0$. Now, let us derive a necessary and sufficient condition for $f(y;q) \leq 0$ to hold for all $y > 0$. Since $f(y;q)$ is concave for $y \in [0, y_M(q)]$ by $\beta_1 > 1$ and $f(y;q) = -I_1$ holds for $y = 0$ and $y \geq y_M(q)$, $f(y;q)$ $(y > 0)$ takes the maximum value at $y = \beta_1^{-1/(\beta_1-1)} y_M(q)$, which is the unique solution of $\partial f(y;q)/\partial y = 0$ $(0 \leq y \leq y_M(q))$. Since we have

$$\begin{aligned} f\left(\beta_1{}^{-\frac{1}{\beta_1-1}} y_M(q); q\right) &= \frac{D_1 \beta_1{}^{-\frac{1}{\beta_1-1}} \beta_1}{(\beta_1-1)q} - \frac{D_1 \beta_1{}^{-\frac{1}{\beta_1-1}}}{(\beta_1-1)q} - I_1 \\ &= \frac{D_1}{q}\left(\beta_1{}^{-\frac{1}{\beta_1-1}} - \frac{qI_1}{D_1}\right) \\ &= \frac{D_1}{q} p(\beta_1, Q_1, q) \end{aligned}$$

by (7), (8), (30), and $Q_1 = D_1/I_1$, we can deduce that $p(\beta_1, Q_1, q) \leq 0$ is a necessary and sufficient condition for $f(y;q) \leq 0$ to hold for all $y > 0$. Thus, if $p(\beta_1, Q_1, q) \leq 0$, we have $V(y;q) = 0$ and $\tau_1^q = +\infty$.

Next, we consider the case where $p(\beta_1, Q_1, q) > 0$. In this case, if we can check that the right-hand side of (11), denoted $\phi(y)$, is a continuously differentiable function satisfying the following conditions:

$$\sigma y^2 \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}(y) + \left(\mu + \frac{\sigma^2}{2}\right) y \frac{\mathrm{d}\phi}{\mathrm{d}y}(y) - \rho\phi(y) \begin{cases} \leq 0 & \text{for all } y > 0, \\ = 0 & \text{for all } y \notin [y_M(Q_1), y_U(q)], \end{cases}$$

$$\phi(y) - f(y) \begin{cases} \geq 0 & \text{for all } y > 0, \\ = 0 & \text{for all } y \in [y_M(Q_1), y_U(q)], \end{cases}$$

$$\lim_{y \downarrow 0} \phi(y) = \lim_{y \uparrow +\infty} \phi(y) = 0,$$

and $\phi(y)$ is twice continuously differentiable at any $y > 0$ such that $y \neq y_M(Q_1)$ and $y \neq y_U(q)$, then we obtain the value function $V(y;q) = \phi(y)$ and an optimal stopping time $\tau_1^q = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(q)]\}$ via the relation between optimal stopping and variational inequalities (see [12] for details). Note that the thresholds $y_M(q)$ and $y_U(q)$ are defined so that $\phi(y)$ is continuously differentiable at the thresholds (i.e, value matching and smooth pasting, see also [4]). Since we can check all the conditions for $\phi(y)$ by direct calculation, we obtain the proposition. □

Appendix B Proof of Proposition 4.1

Note that

$$
\begin{aligned}
& E\left[\int_{\tau_1}^{\tau_2^X} \mathrm{e}^{-\rho t} D_1(1,0)Y(t)\mathrm{d}t + \int_{\tau_2^X}^{+\infty} \mathrm{e}^{-\rho t} D_1(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_1} I_1\right] \\
= \ & \int_0^{+\infty} E\left[\int_{\tau_1}^{\tau_2^X} \mathrm{e}^{-\rho t} D_1(1,0)Y(t)\mathrm{d}t + \int_{\tau_2^X}^{+\infty} \mathrm{e}^{-\rho t} D_1(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_1} I_1 \mid X = q\right] \mathrm{d}\Psi_X(q) \\
= \ & \int_0^{+\infty} E\left[\int_{\tau_1}^{\tau_2^q} \mathrm{e}^{-\rho t} D_1(1,0)Y(t)\mathrm{d}t + \int_{\tau_2^q}^{+\infty} \mathrm{e}^{-\rho t} D_1(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-\rho\tau_1} I_1\right] \mathrm{d}\Psi_X(q) \qquad (31) \\
= \ & \int_0^{+\infty} E\left[\mathrm{e}^{-\rho\tau_1} f(Y(\tau_1);q)\right] \mathrm{d}\Psi_X(q) \qquad (32) \\
= \ & E\left[\mathrm{e}^{-\rho\tau_1} g(Y(\tau_1))\right], \qquad (33)
\end{aligned}
$$

where $\Psi_X(q)$ denotes the distribution of X, and f and g are defined by (30) and (15), respectively. Here, (31) and (33) follow from the independence between X and $Y(t)$, and (32) follows from the strong Markov property as in Appendix A.

First, we consider the case where $g(y) \le 0$ for all $y > 0$. In this case, apparently, the value function and an optimal stopping time are given by $V(y) = 0$ and $\tau_1^* = +\infty$, respectively, for all $y > 0$. Since $h(y) \le 0$ holds for all $y > 0$ by (17), $V(y;\tilde{Q}_2) = 0$ and $\tau_1^{\tilde{Q}_2} = +\infty$ hold for all $y > 0$. This implies $V(y) = V(y;\tilde{Q}_2)$ and $\tau_1^* = \tau_1^{\tilde{Q}_2}$ for all $y > 0$.

Next, let us assume that there exists some $\hat{y} > 0$ such that $g(\hat{y}) > 0$. We have $V(\hat{y};\tilde{Q}_2) > 0$ by Condition (a) (i.e., $g(y) \le V(y;\tilde{Q}_2)$ for all $y > 0$). Then, we can deduce that $p(y, Q_1, \tilde{Q}_2) > 0$, taking into consideration that $V(y;\tilde{Q}_2) = 0$ holds for all $y > 0$ whenever $p(y, Q_1, \tilde{Q}_2) \le 0$ by Proposition 3.1. We can check the following conditions for $\phi(y) = V(y;\tilde{Q}_2)$ (i.e., the right-hand side of (11) with q replaced by $\tilde{Q}_2$):

$$
\sigma y^2 \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}(y) + \left(\mu + \frac{\sigma^2}{2}\right) y \frac{\mathrm{d}\phi}{\mathrm{d}y}(y) - \rho\phi(y) \begin{cases} \le 0 & \text{for all } y > 0, \\ = 0 & \text{for all } y \notin [y_M(Q_1), y_U(\tilde{Q}_2)], \end{cases}
$$

$$
\phi(y) - g(y) \begin{cases} \ge 0 & \text{for all } y > 0, \\ = 0 & \text{for all } y \in [y_M(Q_1), y_U(\tilde{Q}_2)], \end{cases}
$$

$$
\lim_{y\downarrow 0} \phi(y) = \lim_{y\uparrow+\infty} \phi(y) = 0,
$$

and $\phi(y)$ is twice continuously differentiable at any $y > 0$ such that $y \ne y_M(Q_1)$ and $y \ne y_U(\tilde{Q}_2)$. The above conditions except for the second can be checked directly as mentioned in Appendix A. Condition (a) ensures $\phi(y) - g(y) \ge 0$ for all $y > 0$. By (17) and (18), for all $y \in [y_M(Q_1), y_U(\tilde{Q}_2)]$, we have

$$\begin{aligned} \phi(y) - g(y) &= h(y) - g(y) \\ &\leq 0, \end{aligned}$$

where $h(y)$ is defined by (16). These imply the second condition. Therefore, we obtain $V(y) = \phi(y)$ and $\tau_1^* = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\}$ via the relation between optimal stopping and variational inequalities (e.g., see [12]). □

Appendix C Proof of Proposition 4.2

By (20), we have

$$\tilde{V}(y) = E\left[\int_{\tau_1^*}^{\tau_2^{Q_2}} \mathrm{e}^{-\rho t} D_1 Y(t) \mathrm{d}t - \mathrm{e}^{-\rho \tau_1^*} I_1\right]. \tag{34}$$

First, we assume $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$. In this case, we have $\tau_1^* = \tau_1^{\tilde{Q}_2} = +\infty$ by Propositions 3.1 and 4.1, and hence we have also $\tau_2^{Q_2} = +\infty$ by (19). Thus, $\tilde{V}(y) = 0$ holds for all $y > 0$. Next, let us assume $p(\beta_1, Q_1, \tilde{Q}_2) > 0$. In this case, we have

$$\begin{aligned} \tau_1^* &= \tau_1^{\tilde{Q}_2} \\ &= \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\} \end{aligned} \tag{35}$$

by Propositions 3.1 and 4.1. As in Appendix A, by the strong Markov property, (34) is equal to (28) with τ_1 and q replaced by τ_1^* and Q_2, respectively, that is, $\tilde{V}(y) = E\left[\mathrm{e}^{-\rho \tau_1^*} f(Y(\tau_1^*); Q_2)\right]$, where f is defined by (30). Since $Y(\tau_1^*)$ is a constant such that

$$Y(\tau_1^*) = \begin{cases} y_M(Q_1) & (0 < y < y_M(Q_1)) \\ y & (y_M(Q_1) \leq y < y_U(\tilde{Q}_2)) \\ y_U(\tilde{Q}_2) & (y \geq y_U(\tilde{Q}_2)) \end{cases} \tag{36}$$

by (35), we have

$$\tilde{V}(y) = f(Y(\tau_1^*); Q_2) E\left[\mathrm{e}^{-\rho \tau_1^*}\right]. \tag{37}$$

Thus, by (35), (36) and the formula of the expectation involving a hitting time (e.g. see

[4]), (37) can be computed as

$$\tilde{V}(y) = \begin{cases} f(y_M(Q_1); Q_2) y_M(Q_1)^{-\beta_1} y^{\beta_1} & (0 < y < y_M(Q_1)) \\ f(y; Q_2) & (y_M(Q_1) \le y < y_U(\tilde{Q}_2)) \\ f(y_U(\tilde{Q}_2); Q_2) y_U(\tilde{Q}_2)^{-\beta_2} y^{\beta_2} & (y \ge y_U(\tilde{Q}_2)). \end{cases} \tag{38}$$

By (30) and (38), we obtain the formula of $\tilde{V}(y)$ given in the proposition. □

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