New Restricted NCP Functions and Their Applications to Stochastic NCP and Stochastic MPEC *

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Abstract. We focus on studying stochastic nonlinear complementarity problems (SNCP) and stochastic mathematical programs with equilibrium constraints (SMPEC). Instead of the NCP functions employed in the literature, we use the restricted NCP functions to define expected residual minimization formulations for SNCP and SMPEC. We then discuss level set conditions and error bounds of the new formulation. Numerical examples show that the new formulations have some desirable properties which the existing ones do not have.

Key words. Stochastic complementarity problem, stochastic mathematical program with equilibrium constraints, NCP function, restricted NCP function, level set, error bound.

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1 Introduction

The nonlinear complementarity problem (NCP) and mathematical program with equilibrium constraints (MPEC) are two important problems in optimization. Their applications can be found in many fields, see the monographs [6] and [12] for details. Since in many practical problems, some elements may involve uncertain data, stochastic versions of NCP and MPEC (called SNCP and SMPEC below) have been receiving more and more attention in the recent literature [1, 4, 5, 8–11, 15, 16, 19]. In this paper, we focus on dealing with the following stochastic nonlinear complementarity system:

$$x \geq 0,\ F(x,\omega) \geq 0,\ x^T F(x,\omega) = 0, \quad \omega \in \Omega \text{ a.s.,} \tag{1.1}$$

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where Ω is the underlying sample space, $F : \Re^n \times \Omega \to \Re^n$ is a continuous mapping, and "a.s." is the abbreviation for "almost surely" under the given probability measure.

Note that problem (1.1) may not have a solution in general. There have been proposed three ways to deal with (1.1): One way given in [8] is to find a vector x such that

$$x \geq 0, \; \mathbb{E}[F(x, \omega)] \geq 0, \; x^T \mathbb{E}[F(x, \omega)] = 0,$$

where $\mathbb{E}$ means expectation with respect to the random variable ω. Another way is presented by Chen and Fukushima [4], who make use of the so-called NCP functions to present the expected residual minimization (ERM) formulation for (1.1). The last is suggested by Lin and Fukushima [10], who formulate (1.1) as an SMPEC with recourse. The main contributions of the paper can be summarized as follows:

- We consider the case where F is nonlinear with respect to x. This is different from the works about the ERM formulation [4,5] in which F is assumed to be linear. We use the restricted NCP functions, which were introduced in [20] and contain the NCP functions as a subclass, to present a new ERM formulation for (1.1). We give some new restricted NCP functions and investigate their properties, including level set conditions and error bounds. We further give some examples to show that the new functions indeed have some better properties than the min function and the Fischer-Burmeister function in dealing with SNCP.

- We apply the restricted NCP functions to SMPEC and present an ERM formulation. This is different from the models given in the literature [9], because the new model is no longer an MPEC. In particular, when the upper-level decision variables does not exist, the new formulation reduces to the ERM formulation for SNCP [4].

2 New Restricted NCP Functions

The restricted NCP functions, which can be used to reformulate nonlinear complementarity problems as constrained optimization problems, were first introduced by Yamashita [20]. The definition is as follows.

Definition 2.1 *Let $\phi : \Re^2 \to \Re$. We call ϕ a restricted NCP function if the relation*

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad b \geq 0, ab = 0$$

holds for any $a \geq 0$.

In this paper, we mainly consider the following functions:

- $\phi_1(a,b) := \max(ab,0) + \max(-b,0)$, which can be written as $\phi_1(a,b) = \max(ab,-b)$ when $a \geq 0$.
- $\phi_2(a,b) := \max^2(ab,0) + \max^2(-b,0)$, which is a smoothed modification of ϕ_1.
- $\phi_3(a,b) := a^2b^2 + \max^2(-b,0)$, which may also be regarded as a smoothed modification of the function ϕ_1.

It is not difficult to verify that they are restricted NCP functions. We will study their properties and compare these new functions with the following ones employed in [4]:

$$\begin{aligned}
\phi_{\min}(a,b) &:= |\min(a,b)|^2, \\
\phi_{\mathrm{FB}}(a,b) &:= (\sqrt{a^2+b^2} - a - b)^2.
\end{aligned}$$

3 ERM Formulation for SNCP

Let ϕ be a given nonnegative valued restricted NCP function. The ERM formulation, defined by ϕ, for the original stochastic complementarity problem (1.1) can be written as follows:

$$\min_{x\geq 0}\ \theta(x) := \sum_{i=1}^{n} \mathbb{E}\Big[\phi(x_i, F_i(x,\omega))\Big], \tag{3.1}$$

which is a standard stochastic programming problem. To solve this problem, sampling methods such as Monte Carlo methods [14] and quasi-Monte Carlo methods [13] have been proposed. Here, we mainly focus on studying properties such as level set conditions and error bounds of problem (3.1). Throughout this section, we denote

$$f(x) := \mathbb{E}[F(x,\omega)]$$

and, for a given scalar $c \geq 0$, we denote the level set of the objective function of (3.1) on $\Re^n_+ := \{x \in \Re^n \mid x_i \geq 0, i = 1, \cdots, n\}$ by

$$L^+_\theta(c) := \Big\{x \in \Re^n_+ \mid \theta(x) \leq c\Big\}.$$

Moreover, the following definitions will be used later on.

Definition 3.1 [3] *Let X be a nonempty subset of $\Re^n$.*

(1) *We say f is monotone on X if*

$$(x-y)^T(f(x)-f(y)) \geq 0, \qquad \forall x, y \in X.$$

(2) *We say f is strongly monotone with modulus $\mu > 0$ on X if*

$$(x-y)^T(f(x)-f(y)) \geq \mu\|x-y\|^2, \qquad \forall x, y \in X.$$

(3) *We say f is an R_0-function on X if for any sequence $\{x^k\} \subseteq X$ satisfying*

$$\begin{aligned} &\lim_{k\to\infty} \|x^k\| = +\infty, \\ &\liminf_{k\to\infty} \frac{\min\{x_1^k, \cdots, x_n^k\}}{\|x^k\|} \geq 0, \\ &\liminf_{k\to\infty} \frac{\min\{f_1(x^k), \cdots, f_n(x^k)\}}{\|x^k\|} \geq 0, \end{aligned} \tag{3.2}$$

there exists an index j such that $x_j^k \to +\infty$ and $f_j(x^k) \to +\infty$ as $k \to +\infty$.

Recall that the expected value (EV) formulation for the SNCP (1.1) is defined as

$$f(x) \geq 0, \quad x \geq 0, \quad x^T f(x) = 0.$$

This problem is denoted by NCP(f) in the following.

3.1 Level Boundedness

In this subsection, we study conditions for the boundedness of the level sets of the objective function of problem (3.1) defined by the new restricted NCP functions.

3.1.1 Boundedness of level sets

We first consider the function ϕ_1. Let

$$\theta_1(x) := \sum_{i=1}^n \mathbb{E}\Big[\phi_1(x_i, F_i(x,\omega))\Big].$$

Then, we have the following results.

Theorem 3.1 *Assume that the mapping f is monotone on $\Re^n_+$ and the problem NCP(f) has a nonempty and bounded solution set. Then the level set $L^+_{\theta_1}(c)$ is bounded for any nonnegative scalar c.*

Proof. Suppose that there is a nonnegative number $\bar{c}$ such that the set $L^+_{\theta_1}(\bar{c})$ is unbounded. This implies that there exists a sequence $\{x^k\} \subseteq L^+_{\theta_1}(\bar{c})$ such that $\lim_{k\to\infty} \|x^k\| = +\infty$. We first have

$$\begin{aligned} (x^k)^T f(x^k) &\leq \max((x^k)^T f(x^k), 0) \\ &\leq \mathbb{E}[\max((x^k)^T F(x^k, \omega), 0)] \\ &\leq \sum_{i=1}^{n} \mathbb{E}\Big[\max(x_i^k F_i(x^k, \omega), 0) + \max(-F_i(x^k, \omega), 0)\Big] \\ &= \theta_1(x^k) \\ &\leq \bar{c}, \quad \forall k, \end{aligned} \tag{3.3}$$

where the second inequality follows from Jensen's inequality. In a similar manner, we obtain that, for any i and any k,

$$\max(-f_i(x^k), 0) \leq \mathbb{E}[\max(-F_i(x^k, \omega), 0)] \leq \theta_1(x^k) \leq \bar{c},$$

which implies that

$$f_i(x^k) \geq -\bar{c}, \quad \forall i,\ \forall k. \tag{3.4}$$

Since f is monotone and NCP(f) has a nonempty and bounded solution set, it has an interior feasible point, that is, there exists a vector $\bar{x} \in \Re^n_+$ such that $f(\bar{x}) > 0$ [2]. In addition, it follows from the monotonicity of the function f that

$$(x^k - \bar{x})^T (f(x^k) - f(\bar{x})) \geq 0, \quad \forall k. \tag{3.5}$$

Noting that

$$\bar{x} \geq 0,\ f(\bar{x}) > 0,\ x^k \geq 0,\ \lim_{k\to\infty} \|x^k\| = +\infty,$$

we have from (3.3)-(3.5) that

$$\begin{aligned} \bar{c} &\geq (x^k)^T f(x^k) \\ &\geq \bar{x}^T f(x^k) + (x^k)^T f(\bar{x}) - \bar{x}^T f(\bar{x}) \\ &\xrightarrow{k\to\infty} +\infty. \end{aligned}$$

This is a contradiction and hence $L^+_{\theta_1}(c)$ is bounded for any number $c \geq 0$. ■

Theorem 3.2 *Assume that the mapping f is an R_0-function on $\Re^n_+$. Then, for any nonnegative scalar c, the level set $L^+_{\theta_1}(c)$ is bounded.*

Proof. Similarly to the proof of Theorem 3.1, we suppose by contradiction that there exist a nonnegative constant $\bar{c}$ and a sequence $\{x^k\} \subseteq L^+_{\theta_1}(\bar{c})$ such that $\lim_{k\to\infty} \|x^k\| = +\infty$. Notice that (3.4) remains valid and hence condition (3.2) is true. Then, since f is an R_0-function over $\Re^n_+$, there exists an index j such that

$$\lim_{k\to\infty} x^k_j = +\infty, \quad \lim_{k\to\infty} f_j(x^k) = +\infty. \tag{3.6}$$

On the other hand, we have

$$x^k_j f_j(x^k) \le \max(x^k_j f_j(x^k), 0) \le \mathbb{E}[\max(x^k_j F_j(x^k, \omega), 0)] \le \theta_1(x^k) \le \bar{c}$$

for each k. This obviously contradicts (3.6) and hence the proof is complete. ■

We next consider the functions ϕ_2 and ϕ_3. Let

$$\begin{aligned} \theta_2(x) &:= \sum_{i=1}^n \mathbb{E}\Big[\phi_2(x_i, F_i(x,\omega))\Big], \\ \theta_3(x) &:= \sum_{i=1}^n \mathbb{E}\Big[\phi_3(x_i, F_i(x,\omega))\Big]. \end{aligned}$$

Then, we have the following results.

Theorem 3.3 *The level set $L^+_{\theta_1}(c)$ is bounded for any nonnegative scalar c if and only if the level set $L^+_{\theta_2}(c)$ is bounded for any nonnegative scalar c.*

Proof. Note that, for any $x \in \Re^n_+$,

$$\begin{aligned} \theta_1(x) &= \sum_{i=1}^n \mathbb{E}\Big[\max(x_i F_i(x,\omega), 0) + \max(-F_i(x,\omega), 0)\Big] \\ &\le \sqrt{2} \sum_{i=1}^n \mathbb{E}\Big[\sqrt{\max{}^2(x_i F_i(x,\omega), 0) + \max{}^2(-F_i(x,\omega), 0)}\Big] \\ &\le \sqrt{2n}\sqrt{\sum_{i=1}^n \Big(\mathbb{E}\Big[\sqrt{\max{}^2(x_i F_i(x,\omega), 0) + \max{}^2(-F_i(x,\omega), 0)}\Big]\Big)^2} \\ &\le \sqrt{2n \sum_{i=1}^n \mathbb{E}\Big[\max{}^2(x_i F_i(x,\omega), 0) + \max{}^2(-F_i(x,\omega), 0)\Big]} \\ &= \sqrt{2n\,\theta_2(x)}, \end{aligned}$$

where the first and second inequalities follow from the Cauchy-Schwarz inequality and the third inequality follows from Jensen's inequality. It then follows that

$$L^+_{\theta_2}(c) \subseteq L^+_{\theta_1}(\sqrt{2nc}), \quad \forall c \ge 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}
\theta_2(x) &= \sum_{i=1}^n \mathbb{E}\Big[\max{}^2(x_iF_i(x,\omega),0) + \max{}^2(-F_i(x,\omega),0)\Big] \\
&\le \sum_{i=1}^n \mathbb{E}\Big[\Big(\max(x_iF_i(x,\omega),0) + \max(-F_i(x,\omega),0)\Big)^2\Big] \\
&\le \Big(\sum_{i=1}^n \mathbb{E}\Big[\max(x_iF_i(x,\omega),0) + \max(-F_i(x,\omega),0)\Big]\Big)^2 \\
&= \theta_1^2(x),
\end{aligned}$$

which means

$$L^+_{\theta_1}(c) \subseteq L^+_{\theta_2}(c^2), \quad \forall c \ge 0. \tag{3.8}$$

The conclusion follows from (3.7) and (3.8) immediately. ■

Theorem 3.4 *Suppose that the level set $L^+_{\theta_1}(c)$ is bounded for any nonnegative scalar c. Then, the level set $L^+_{\theta_3}(c)$ is also bounded for any nonnegative scalar c.*

Proof. Since $\theta_2(x) \le \theta_3(x)$ for any $x \in \Re^n_+$, there holds $L^+_{\theta_3}(c) \subseteq L^+_{\theta_2}(c)$ for any $c \ge 0$. The conclusion follows from Theorem 3.3 immediately. ■

3.1.2 Comparison with the functions $\phi_{\min}$ and ϕ_{FB}

Now we consider the functions $\phi_{\min}$ and ϕ_{FB}. Let

$$\begin{aligned}
\theta_{\min}(x) &:= \sum_{i=1}^n \mathbb{E}\Big[\phi_{\min}(x_i, F_i(x,\omega))\Big], \\
\theta_{\mathrm{FB}}(x) &:= \sum_{i=1}^n \mathbb{E}\Big[\phi_{\mathrm{FB}}(x_i, F_i(x,\omega))\Big].
\end{aligned}$$

In what follows, we give two examples to show that the properties stated in Theorems 3.1 and 3.2 do not hold for $\phi_{\min}$ and ϕ_{FB}. Since

$$\left(\frac{2}{\sqrt{2}+2}\right)^2 \phi_{\min}(a,b) \;\le\; \phi_{\mathrm{FB}}(a,b) \;\le\; (\sqrt{2}+2)^2\phi_{\min}(a,b)$$

for any scalars a and b [18], it is sufficient to consider the function $\phi_{\min}$ only.

Example 3.1 Suppose that the random variable ω is uniformly distributed on the sample space $\Omega := [-1, 1]$. Consider the function given by $F(x, \omega) = 1$ for any $x \in \Re$ and $\omega \in \Omega$. Then, the expectation function $f(x) = \mathbb{E}[F(x, \omega)] \equiv 1$, which is a monotone function on $\Re_+$. Moreover, NCP(f) has a unique solution $x^* = 0$. However, since $\theta_{\min}(x) = \min^2(x, 1)$, the level set is $L^+_{\theta_{\min}}(c) \equiv \Re_+$ for any $c \geq 1$. This indicates that the property stated in Theorem 3.1 does not hold for this example.

Example 3.2 Suppose that the random variable ω is uniformly distributed on the sample space $\Omega := [-2, 2]$. Consider the function $F : \Re_+ \times \Omega \to \Re$ defined by

$$F(x, \omega) := \begin{cases} 2 + \omega, & \omega \in [-2, 0] \\ 2 - \omega, & \omega \in (0, 2] \end{cases}$$

for $x \in [0, 1]$ and

$$F(x, \omega) := \begin{cases} 2x + x^3\omega, & \omega \in [-\frac{2}{x^2}, -\frac{1}{x^2}] \\ x + x^3 + x^5\omega, & \omega \in (-\frac{1}{x^2}, 0] \\ x + x^3 - x^5\omega, & \omega \in (0, \frac{1}{x^2}] \\ 2x - x^3\omega, & \omega \in (\frac{1}{x^2}, \frac{2}{x^2}] \\ 0, & \omega \in [-2, -\frac{2}{x^2}) \cup (\frac{2}{x^2}, 2] \end{cases}$$

for $x \in (1, +\infty)$. The function F is obviously continuous on $\Re_+ \times \Omega$. By straightforward calculation, we have

$$f(x) = \mathbb{E}[F(x, \omega)] = \frac{1}{4}\max(x, 1) + \frac{3}{4\max(x, 1)},$$

which is an R_0-function on $\Re_+$. We next calculate the function $\theta_{\min}$.

(1) Suppose $x \in [0, 1]$. Then

$$\begin{aligned} \theta_{\min}(x) &= \frac{1}{4}\int_\Omega \min{}^2(x, F(x, \omega))\, d\omega \\ &= \frac{1}{4}\int_{-2}^{-2+x} (2 + \omega)^2\, d\omega + \frac{1}{4}\int_{-2+x}^{2-x} x^2\, d\omega + \frac{1}{4}\int_{2-x}^{2} (2 - \omega)^2\, d\omega \\ &= x^2 - \frac{1}{3}x^3. \end{aligned}$$

(2) Suppose $x \in (1, +\infty)$. Then

$$\begin{aligned} \theta_{\min}(x) &= \frac{1}{4}\int_\Omega \min{}^2(x, F(x, \omega))\, d\omega \\ &= \frac{1}{4}\int_{-2/x^2}^{-1/x^2} (2x + x^3\omega)^2\, d\omega + \frac{1}{4}\int_{-1/x^2}^{1/x^2} x^2\, d\omega + \frac{1}{4}\int_{1/x^2}^{2/x^2} (2x - x^3\omega)^2\, d\omega \\ &= \frac{2}{3}. \end{aligned}$$

Therefore, for any $c \geq \frac{2}{3}$, the level set is $L^+_{\theta_{\min}}(c) \equiv \Re_+$. This indicates that the property stated in Theorem 3.2 does not hold for this example.

3.2 Error Bounds

In this subsection, we study the conditions for error bounds.

Theorem 3.5 *Suppose that the mapping f is strongly monotone with modulus $\mu > 0$ on $\Re^n_+$. Let x^* be the unique solution of NCP(f). Then, for any $x \in \Re^n_+$, we have*

$$\|x - x^*\| \leq \sqrt{\lambda\, \theta_1(x)}, \tag{3.9}$$

where $\lambda := \frac{1}{\mu}\max\{x_1^*, \cdots, x_n^*, 1\}$.

Proof. First note that, since f is strongly monotone on $\Re^n_+$, the problem NCP(f) has a unique solution. Let $x \in \Re^n_+$. It follows that $x^T f(x^*) \geq 0$ and $(x^*)^T f(x^*) = 0$. Therefore, we have

$$\begin{aligned}
\mu\|x - x^*\|^2 &\leq (x - x^*)^T(f(x) - f(x^*)) \\
&= x^T f(x) - (x^*)^T f(x) - x^T f(x^*) + (x^*)^T f(x^*) \\
&\leq x^T f(x) - (x^*)^T f(x) \\
&\leq \max(x^T f(x), 0) + (x^*)^T \max(-f(x), 0) \\
&\leq \mathbb{E}\Big[\max(x^T F(x,\omega), 0) + (x^*)^T \max(-F(x,\omega), 0)\Big] \\
&\leq \max\{x_1^*, \cdots, x_n^*, 1\} \sum_{i=1}^n \mathbb{E}\Big[\max(x_i F_i(x,\omega), 0) + \max(-F_i(x,\omega), 0)\Big] \\
&= \max\{x_1^*, \cdots, x_n^*, 1\}\, \theta_1(x),
\end{aligned} \tag{3.10}$$

where the fourth inequality follows from Jensen's inequality. Thus, (3.9) follows from (3.10) immediately. ■

We further have the following result from Theorems 3.3 and 3.4.

Corollary 3.1 *Suppose that the mapping f is strongly monotone with modulus $\mu > 0$ on $\Re^n_+$ and x^* is the unique solution of NCP(f). Then, for any $x \in \Re^n_+$, we have*

$$\|x - x^*\| \leq \sqrt[4]{2n\lambda^2\theta_2(x)} \leq \sqrt[4]{2n\lambda^2\theta_3(x)},$$

where λ is the same as in Theorem 3.5.

We next consider the functions $\phi_{\min}$ and ϕ_{FB}. We will show that, under the assumptions in Theorem 3.5, there do not exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|x - x^*\| \le \alpha[\theta_{\min}(x)]^\beta, \qquad \forall x \in \Re^n_+ \tag{3.11}$$

or

$$\|x - x^*\| \le \alpha[\theta_{FB}(x)]^\beta, \qquad \forall x \in \Re^n_+.$$

As mentioned in Section 4.2, it is sufficient to consider the function $\phi_{\min}$ only.

Example 3.3 Let $\beta > 0$ be an arbitrary constant. We show that there exists an NCP(f) such that (3.11) fails to hold for any $\alpha > 0$. Suppose that the random variable ω is uniformly distributed on the sample space $\Omega := [0, 1]$. Let the function $F : \Re^2 \times \Omega \to \Re^2$ be defined by

$$F(x, \omega) := \begin{pmatrix} x_1 - \frac{1}{2\beta+1} x_2^{2\beta+1} \\ \\ x_2 + x_2^{4\beta+1} \omega^{4\beta} \end{pmatrix}.$$

Then we have

$$f(x) = \mathbb{E}[F(x, \omega)] = \begin{pmatrix} x_1 - \frac{1}{2\beta+1} x_2^{2\beta+1} \\ \\ x_2 + \frac{1}{4\beta+1} x_2^{4\beta+1} \end{pmatrix}.$$

(1) We first show that f is strongly monotone on $\Re^2$. This is equivalent to showing that the Jacobian of f, denote by

$$J_f(x) := \begin{pmatrix} 1 & -x_2^{2\beta} \\ \\ 0 & 1 + x_2^{4\beta} \end{pmatrix},$$

is uniformly positive definite on $\Re^2$ in the sense that there exists a constant $\nu > 0$ satisfying

$$(y - z)^T J_f(x)(y - z) \ge \nu \|y - z\|_2^2$$

for any y and z in $\Re^2$. In fact, it is not difficult to see that the eigenvalues of the symmetric matrix

$$M_f(x) := \frac{1}{2}\Big(J_f(x) + [J_f(x)]^T\Big) = \begin{pmatrix} 1 & -\frac{1}{2}x_2^{2\beta} \\ \\ -\frac{1}{2}x_2^{2\beta} & 1 + x_2^{4\beta} \end{pmatrix}$$

are always larger than $\frac{3}{4}$ for any $x \in \Re^2$. Therefore, for any y and z in $\Re^2$,

$$(y - z)^T J_f(x)(y - z) = (y - z)^T M_f(x)(y - z) \ge \frac{3}{4}\|y - z\|_2^2.$$

In consequence, the function f is strongly monotone on $\Re^2$. It is easy to see that the solution of NCP(f) is $x^* = (0, 0)$.

(2) We next show that (3.11) does not hold for any $\alpha > 0$. To this end, we choose

$$x^k := \Big(\frac{1}{2\beta+1} k^{2\beta+1}, k\Big), \qquad k = 1, 2, \cdots .$$

By straightforward calculation, we have

$$\lim_{k\to\infty} \frac{\|x^k - x^*\|_2}{[\theta_{\min}(x^k)]^\beta} = \lim_{k\to\infty} \frac{\sqrt{(2\beta+1)^{-2}\, k^{4\beta+2} + k^2}}{k^{2\beta}} = +\infty,$$

which means that (3.11) does not hold for any constant $\alpha > 0$.

4 ERM Formulation for SMPEC

Consider the following SMPEC:

$$\begin{aligned} \min \quad & \mathbb{E}[u(x, y, \omega)] \\ \text{s.t.} \quad & x \in X, \\ & 0 \le y \perp G(x, y, \omega) \ge 0, \quad \omega \in \Omega \ \text{ a.s.}, \end{aligned} \tag{4.1}$$

where X is a nonempty subset of $\Re^n$, $x \in \Re^n$ is an upper-level decision variable, $y \in \Re^m$ is a lower-level decision variable, $u : \Re^n \times \Re^m \times \Omega \to \Re$ and $G : \Re^n \times \Re^m \times \Omega \to \Re^m$ are continuous mappings. Moreover, both decisions x and y have to be made before ω is observed. This problem is called a "here-and-now" model in the literature [9].

In general, for any fixed x, there does not exist a decision y meeting all random situations. To obtain a proper deterministic formulation of (4.1), a recourse variable is introduced in [9]. Here, we deal with (4.1) in a different manner. Let ϕ be a given nonnegative-valued restricted NCP function. Then problem (4.1) becomes

$$\begin{aligned} \min \quad & \mathbb{E}[u(x, y, \omega)] \\ \text{s.t.} \quad & x \in X, \ y \ge 0, \\ & \phi(y_i, G_i(x, y, \omega)) = 0, \ \omega \in \Omega \text{ a.s.}, \ i = 1, \cdots, m. \end{aligned}$$

Making use of a penalty technique, we can reformulate this problem as follows:

$$\begin{aligned} \min \quad & \mathbb{E}\Big[u(x, y, \omega) + \rho \sum_{i=1}^{m} \phi(y_i, G_i(x, y, \omega))\Big] \\ \text{s.t.} \quad & x \in X, \ y \ge 0. \end{aligned} \tag{4.2}$$

where $\rho > 0$ is a penalty parameter.

Unlike the models given in [9], problem (4.2) is no longer an MPEC and hence it may be relatively easy to deal with. Algorithms based on sample average approximations can be used to solve (4.2). Moreover, when X is a singleton and $u(x, y, \omega) \equiv 0$, problem (4.2) reduces to the ERM formulation (3.1) for SNCP. In a similar way to Section 3, we may obtain some results about level set conditions and error bounds. As an example, we state one result as follows.

Theorem 4.1 *Let $\zeta_1(x, y) := \mathbb{E}\Big[u(x, y, \omega) + \rho \sum_{i=1}^m \phi_1(y_i, G_i(x, y, \omega))\Big]$. Suppose that*

(1) *X is bounded and closed;*

(2) *$\mathbb{E}[u(\cdot, \cdot, \omega)]$ is bounded below on $X \times \Re^m_+$, i.e., there is a constant τ such that $\mathbb{E}[u(x, y, \omega)] \geq \tau$ for all $x \in X$ and $y \geq 0$;*

(3) *for each $x \in X$, $g(x, \cdot) := \mathbb{E}[G(x, \cdot, \omega)]$ is monotone on $\Re^m_+$ and NCP($g(x, \cdot)$) has a nonempty and bounded solution set.*

Then, for any scalar c, the level set $L_{\zeta_1}(c) := \Big\{x \in X, y \geq 0 \,\Big|\, \zeta_1(x, y) \leq c\Big\}$ is bounded.

Proof. Suppose that there exists a number $\bar{c}$ such that $L_{\zeta_1}(\bar{c})$ is unbounded. This indicates that there is an unbounded sequence $\{x^k, y^k\}$ in $L_{\zeta_1}(\bar{c})$. Since X is bounded, it follows that $\{y^k\}$ is unbounded. Taking a subsequence if necessary, we assume that $\lim_{k \to \infty} \|y^k\| = +\infty$ and there exists a vector $\bar{x} \in X$ such that $x^k \to \bar{x}$ as $k \to +\infty$. As in the proof of Theorem 3.1, we can show that

$$\begin{aligned} (y^k)^T g(x^k, y^k) &\leq \mathbb{E}[\max((y^k)^T G(x^k, y^k, \omega), 0)] \\ &\leq \sum_{i=1}^n \mathbb{E}\Big[\max(y_i^k G_i(x^k, y^k, \omega), 0) + \max(-G_i(x^k, y^k, \omega), 0)\Big] \\ &= \frac{\zeta_1(x^k, y^k) - \mathbb{E}[u(x^k, y^k, \omega)]}{\rho} \\ &\leq \frac{\bar{c} - \tau}{\rho}, \qquad \forall k \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} g_i(x^k, y^k) &\geq -\max(-g_i(x^k, y^k), 0) \\ &\geq -\mathbb{E}[\max(-G_i(x^k, y^k, \omega), 0)] \\ &\geq -\frac{\zeta_1(x^k, y^k) - \mathbb{E}[u(x^k, y^k, \omega)]}{\rho} \\ &\geq -\frac{\bar{c} - \tau}{\rho}, \qquad \forall i, \forall k. \end{aligned} \tag{4.4}$$

On the other hand, since $g(\bar{x},\cdot)$ is monotone and NCP$(g(\bar{x},\cdot))$ has a nonempty and bounded solution set, there exists a vector $\bar{y} \in \Re^m_+$ such that $g(\bar{x},\bar{y}) > 0$ [2]. Note that $g(\cdot,\bar{y})$ is continuous on X and $\lim_{k\to\infty} x^k = \bar{x}$. Therefore, there is an integer k_0 such that

$$g(x^k,\bar{y}) \geq \frac{1}{2} g(\bar{x},\bar{y}) > 0, \qquad \forall k \geq k_0. \tag{4.5}$$

For each k, it follows from the monotonicity of $g(x^k,\cdot)$ that

$$(y^k - \bar{y})^T (g(x^k,y^k) - g(x^k,\bar{y})) \geq 0. \tag{4.6}$$

Noting that $y^k \geq 0$ for each k and $\lim_{k\to\infty} \|y^k\| = +\infty$, we have from (4.3)-(4.6) that

$$\begin{aligned} \frac{\bar{c}-\tau}{\rho} &\geq (y^k)^T g(x^k,y^k) \\ &\geq \bar{y}^T g(x^k,y^k) + (y^k)^T g(x^k,\bar{y}) - \bar{y}^T g(x^k,\bar{y}) \\ &\xrightarrow{k\to\infty} +\infty. \end{aligned}$$

This is a contradiction. In consequence, the level set $L_{\zeta_1}(c)$ is bounded for any number c. ■

5 Conclusions

By means of the restricted NCP functions, we have proposed some ERM formulations for SNCP and SMPEC. As shown in the paper, different functions may have different properties. In particular, we have shown that the new restricted NCP functions given in this paper have some desirable properties the min function and the Fischer-Burmeister function do not have, when employed in the expected residual minimization method for SNCP and SMPEC.

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