Convex Drawings of Graphs with Non-convex Boundary Constraints

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Abstract: In this paper, we study a new problem of convex drawing of planar graphs with *non-convex* boundary constraints. It is proved that every triconnected plane graph whose boundary is fixed with a star-shaped polygon admits a drawing in which every inner facial cycle is drawn as a convex polygon. We also prove that every four-connected plane graph whose boundary is fixed with a crown-shaped polygon admits such a drawing, called an *inner-convex drawing*. We present an algorithm to construct an inner-convex drawing in linear time.

Keywords: Graph Drawing, Convex Drawing, Crown-shaped Polygons, Star-shaped Polygons, Triconnected Planar Graphs, Four-connected Planar Graphs.

1 Introduction

Graph drawing has attracted much attention over the last ten years due to its wide range of applications, such as VLSI design, software engineering and bioinformatics. Two- or three-dimensional drawings of graphs with a variety of aesthetics and edge representations have been extensively studied (see [1]). One of the most popular drawing conventions is the *straight-line drawing*, where all the edges of a graph are drawn as straight-line segments. Every planar graph is known to have a planar straight-line drawing [7].

A straight-line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. Note that not all planar graphs admit a convex drawing. Tutte [17] gave a necessary and sufficient condition for a triconnected plane graph to admit a convex drawing. He also showed that every triconnected plane graph with a given boundary drawn as a convex polygon admits a convex drawing using the polygonal boundary. That is, when the vertices on the boundary are placed on a convex polygon, inner vertices can be placed on suitable positions so that each inner facial cycle forms a convex polygon. More specifically, he proposed a "barycentric mapping" method which computes a convex drawing of a triconnected plane graph with n vertices by solving a system of $O(n)$ linear equations. This requires $O(n^3)$ time and $O(n^2)$ space

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using the ordinary Gaussian elimination method, however it can be implemented in $O(n^{1.5})$ time and $O(n \log n)$ space using the sparse Gaussian elimination method [10].

Later, Thomassen [16] gave a necessary and sufficient condition for a biconnected plane graph to admit a convex drawing. Based on this result, Chiba et al. [6] presented a linear time algorithm for finding a convex drawing (if any) for a biconnected plane graph with a specified convex boundary.

In general, the convex drawing problem has been well investigated for the last ten years. For example, a problem of convex drawing of graphs with *grid* constraints has been well studied [3, 4, 5, 12]. A convex drawing is called a *convex grid drawing* if all the vertices are restricted to be placed on grid points. Every triconnected plane graph has a convex grid drawing on an $(n-2) \times (n-2)$ grid, and such a grid drawing can be found in linear time [5]. A linear time algorithm for finding a convex grid drawing of four-connected plane graphs with four or more vertices on the outer face was presented in [12]. Another variation of convex drawing with minimum outer apices was introduced in [11]. For constructing a strictly convex drawing of graphs, see [15].

However, not much attention has been paid to the problem of finding a convex drawing with a *non-convex* boundary. In this paper, we study a new problem of drawing planar graphs with *non-convex* boundary constraints. Our problem was originally inspired by a real world application, such as visualisation of sensor networks with given floor planning. More formally, a straight-line drawing is called an *inner-convex drawing* if every inner facial cycle is drawn as a convex polygon, and is simply called a *convex drawing* if no confusion arises.

One can easily observe that not every triconnected plane graph has a convex drawing if its boundary is fixed as a non-convex polygon. For example, Fig. 1 shows three examples of plane graphs which have no convex drawing; the inner facial cycle f_1 in Fig. 1(b) (respectively, one of the inner facial cycles f_1 and f_2 in Figs. 1(a) and (c)) cannot be drawn as a convex polygon.

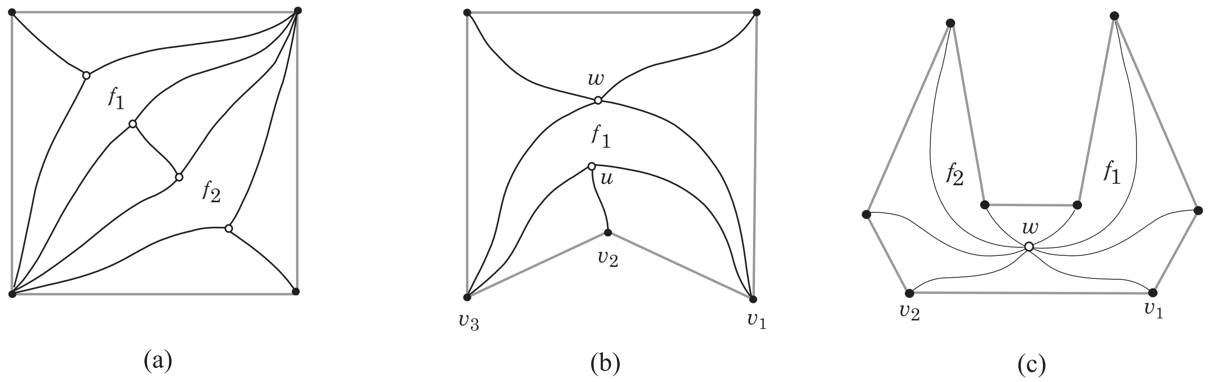

Figure 1: (a) A biconnected plane graph with a convex boundary; (b) An internally triconnected plane graph with a star-shaped boundary; (c) A triconnected plane graph whose boundary is crown-shaped but not star-shaped.

No characterisation is known for any class of plane graphs that have inner-convex drawings with non-convex boundaries. In this paper, we prove that every triconnected plane graph has an inner-convex drawing if its boundary is fixed with a star-shaped polygon P, i.e., a polygon P whose kernel (the set of all points from which all points in P are visible) is not empty. This is an extension of the classical result by Tutte [17] since any convex polygon is a star-shaped polygon. Our proof gives a linear time algorithm for computing an inner-convex drawing of a triconnected plane graph with a star-shaped boundary. We also prove that every four-connected plane graph whose boundary is fixed with a crown-shaped polygon admits an inner-convex drawing.

This paper is organized as follows. Section 2 reviews basic terminology and proves an important property of triconnected plane graphs and four-connected plane graphs. Section 3 proves that a triconnected plane graph has an archfree tree, a spanning tree with a special property. Section 4 reviews the necessary and sufficient condition for a biconnected plane graph with a boundary drawn as a convex polygon to admit a convex drawing. Section 5 presents a linear time algorithm to construct an inner-convex drawing of a triconnected plane graph with a star-shaped boundary constraints. Section 6 describes how to construct an inner-convex drawing of a four-connected plane graph with a crown-shaped boundary constraints. Section 7 concludes.

2 Preliminaries

Throughout the paper, a graph stands for a simple undirected graph. Let $G = (V, E)$ be a graph. The set of edges incident to a vertex $v \in V$ is denoted by $E(v)$. The degree of a vertex v in G is denoted by $d_G(v)$ (i.e., $d_G(v) = |E(v)|$). For a subset $X \subseteq E$ (respectively, $X \subseteq V$), $G-X$ denotes the graph obtained from G by removing the edges in X (respectively, the vertices in X together with the edges in $\cup_{v \in X} E(v)$).

A vertex in a connected graph is called a *cut vertex* if its removal from G results in a disconnected graph. A connected graph is called *biconnected* if it is simple and has no cut vertex. Similarly, a pair of vertices in a connected graph is called a *cut pair* (or *separation pair*) if its removal from G results in a disconnected graph. A connected graph is called *triconnected* if it is simple and has no cut pair. We say that a cut pair $\{u, v\}$ *separates* two vertices s and t if s and t belong to different components in $G - \{u, v\}$. In general, a graph G with more than k vertices is called *k-connected* if $G - X$ remains connected for any subset X of $k - 1$ vertices.

A graph $G = (V, E)$ is called *planar* if its vertices and edges are drawn as points and curves in the plane so that no two curves intersect except for their endpoints, where no two vertices are drawn at the same point. In such a drawing, the plane is divided into several connected regions, each of which is called a *face*. A face is characterized by the cycle of G that surrounds the region. Such a cycle is called a *facial cycle*. A set F of facial cycles in a drawing is called an *embedding* of a planar graph G. A *plane* graph $G = (V, E, F)$ is a planar graph $G = (V, E)$ with a fixed embedding F of G, where we always denote the outer facial cycle in F by f_o.

A vertex (respectively, an edge) in f_o is called an *outer vertex* (respectively, an *outer edge*), while a vertex (respectively, an edge) not in f_o is called an *inner vertex* (respectively, an *inner edge*). A path Q between two vertices s and t in G is called *inner* if every vertex in $V(Q)-\{s, t\}$ is an inner vertex. The region enclosed by a facial cycle $f \in F$ may be denoted by f for simplicity. The set of vertices, set of edges and set of facial cycles of a plane graph G may be denoted by

$V(G)$, $E(G)$ and $F(G)$, respectively.

A biconnected plane graph G is called *internally triconnected* if, for any cut pair $\{u, v\}$, u and v are outer vertices and each component in $G - \{u, v\}$ contains an outer vertex. For example, the graph in Fig. 1(a) is biconnected but not internally triconnected, the graph in Fig. 1(b) is internally triconnected but not triconnected, and the graph in Fig. 1(c) is triconnected. Note that every inner vertex in an internally triconnected plane graph must be of degree at least 3.

For a cut pair $\{u, v\}$ of an internally triconnected plane graph $G = (V, E, F)$, if u and v are not adjacent and there is an inner facial cycle $f \in F$ such that $\{u, v\} \in V(f)$, we say that f *separates* two vertices s and t if the cut pair $\{u, v\}$ separates them.

We then observe the following:

Lemma 1 *Let G be an internally triconnected plane graph. Then G has an inner path connecting two outer vertices s and t if and only if no facial cycle separates s and t.* ∎

We now show some key properties of a triconnected plane graph.

Lemma 2 *Every triconnected plane graph $G = (V, E, F)$ has a spanning tree T such that each vertex $v \in V(f_o)$ is a leaf of T. Such a tree can be found in linear time.*

Proof: Since G has no cut pair, there is an inner path between any two vertices in $V(f_o)$ by Lemma 1. Let $V(f_o) = \{v_1, v_2, \ldots, v_p\}$, where vertices $v_1, v_2, \ldots, v_p$ appear in this order when we traverse f_o in the clockwise order. For each $v \in V(f_o)$, let $e_v \in E(v)$ be the edge that appears after edge $(v = v_i, v_{i+1})$ when we visit the edges in $E(v)$ around $v = v_i$ in the clockwise order. To prove the lemma, it suffices to show that $G^* = G - \cup_{v \in V(f_o)}(E(v) - \{e_v\})$ remains connected, since any spanning tree T of the graph satisfies the condition of the lemma, and it is immediate to see that such a tree can be computed in linear time.

To show the connectedness of G^*, we define an inner path Q_v from $v = v_i$ to v_{i-1} as follows. Let $E(v) = \{(v, u_1 = v_{i+1}), (v, u_2), (v, u_3), \ldots, (v, u_{h-1}), (v, u_h = v_{i-1})\}$, where $(v, u_1), \ldots, (v, u_h)$ appear in this order when we visit the edges in $E(v)$ in the clockwise order around v, and $f_j \in F$, $j = 1, 2, \ldots, h$ be the facial cycle that contains edges (v, u_j) and (v, u_{j+1}), where $V(f_j) \cap V(f_o) = \{v_i\}$ and $f_j \neq f_{j'}$ for $j \neq j'$ by the triconnectivity of G. Then there is an inner path Q_v from $v = v_i$ to v_{i-1} which consists of subpaths $f_j - v$, $j = 2, 3, \ldots, h-1$. That is, path Q_v and edge $(v = v_i, u_h = v_{i-1})$ surround the union of faces f_j, $j = 2, 3, \ldots, h-1$.

Note that $G - (E(v) - \{e_v\})$ contains Q_v. Hence if G has an inner path Q from $v \in V(f_o)$ to a vertex $w \in V$ that does not use edge e_v, then Q must use an edge $(v, u_j) \in E(v) - \{e_v\}$ and the subpath from v to u_j along Q_v and the subpath from u_j to w along Q give rise to an inner path from v to w without using any edge in $E(v) - \{e_v\}$. By noting that $E(v) \cap E(Q_{v'}) = \emptyset$ for any two $v, v' \in V(f_o)$, this implies that any two vertices in $V(f_o)$ are connected by an inner path in $G^* = G - \cup_{v \in V(f_o)}(E(v) - \{e_v\})$ and that G^* contains a tree T' that connects all vertices in $V(f_o)$ (note that each vertex $v \in V(f_o)$ is a leaf in T' since the degree of v is 1 in $G - \cup_{v \in V(f_o)}(E(v) - \{e_v\})$).

To complete the proof for the connectedness of G^*, we show that an arbitrary vertex $u \in V - V(T')$ is connected to a vertex in $V(T')$ in G^*. There are adjacent vertices $w_1, w_2 \in V(f_o)$

such that u is located in the region enclosed by edge (w_1, w_2) and the path T'_{w_1,w_2} between w_1 and w_2 along T'. No vertex in this region is incident to $E(v)$ with $v \notin \{w_1, w_2\}$. Hence, if u is not connected to any vertex in T' in G^*, then $\{w_1, w_2\}$ would be a cut pair which separates u and a vertex not in the region, contradicting the triconnectivity of G.

Therefore, $G^* = G - \cup_{v \in V(f_o)}(E(v) - \{e_v\})$ remains connected, as required. ∎

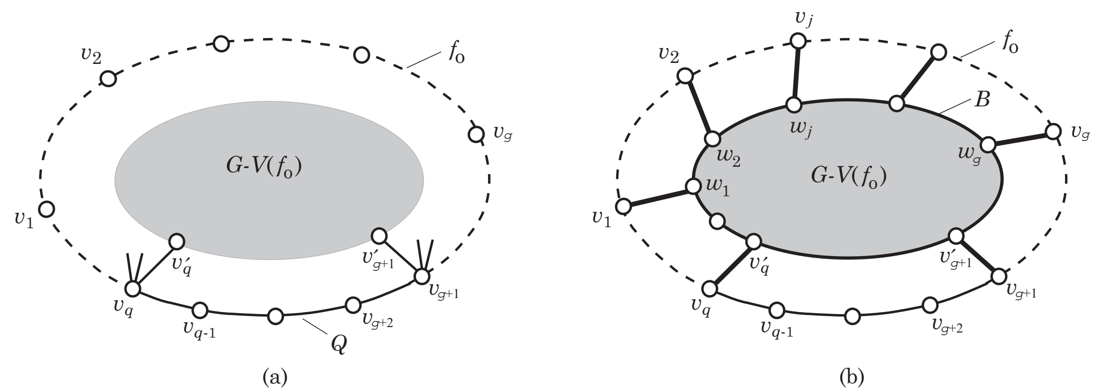

Figure 2: (a) Subgraph $G - V(f_o)$ and subpath Q in f_o; (b) Matching M joining outer vertices and inner vertices, where matching edges are displayed by thick lines.

We next show some key properties of a four-connected plane graph.

Lemma 3 *Let $G = (V, E, F)$ be a 4-connected plane graph, and denote the vertices in f_o by $v_1, v_2, \ldots, v_q$ such that they appear in this order when we walk along f_o in the clockwise order (see Fig. 2(a)). Let Q be a subpath of the outer facial cycle f_o with $|V(Q)| \geq 2$, where the vertices $V(Q)$ are denoted by $v_{g+1}, v_{g+2}, \ldots, v_q$. Let v'_{g+1} denote the first inner vertex when we visit all neighbours of v_{g+1} in the clockwise order starting with v_{g+2}, and v'_q be the first inner vertex when we visit all neighbours of v_q in the anticlockwise order starting with v_{q-1}. Then*

(i) *$G - V(f_o)$ and $G' = G - (V(f_o) - V(Q))$ are both internally triconnected.*

(ii) *Let B be the set of vertices in the boundary of G'. Then, there is a matching $M = \{(v_j, w_j) \in E \mid j = 1, \ldots, g\} \cup \{(v_{g+1}, v'_{g+1}), (v_q, v'_q)\}$ such that $w_j \in B$, $j = 1, \ldots, g$ (see Fig. 2(b)). Such a matching M can be obtained in linear time.*

PROOF: **(i)** We see that $|V - V(f_o)| \geq 2$, since otherwise G is not four-connected. We now show that G' is internally triconnected (internal triconnectivity of $G - V(f_o)$ can be treated analogously). Hence G' contains at least four vertices. We first show that G' is biconnected.

Let s, t be arbitrary vertices in G'. By Menger's theorem [2], G has four internally disjoint paths $P_1, P_2, \ldots, P_4$ between s and t, which divided the plane into four regions $R_1, R_2, \ldots, R_4$, where we assume without loss of generality R_1 contains the outer region of G surrounded by P_3 and P_4. Then paths P_1 and P_2 do not touch R_1 except at s and t, which means that these paths are also contained in G'. Since $|V(G')| \geq 3$ and any two vertices in G' are connected by two internally disjoint paths, G' is biconnected.

Assume that G' is not internally triconnected, i.e., for some cut pair $\{u, v\}$ of G', $G' - \{u, v\}$ has a component which has no outer vertex in the boundary of G'. This, however, implies that $\{u, v\}$ remains a cut pair in G, contradicting the four-connectivity. Therefore G' is internally triconnected.

(ii) We choose $w_j \in B$ by traversing the boundary of $G - V(f_o)$ in the clockwise order starting with v'_q. After starting from v'_q, we choose the first vertex $w \in B'$ that is adjacent to v_1 as w_1. Note that such a vertex $w \in B' - \{v'_g\}$ exists, since otherwise v_1 would have only one inner adjacent vertex, contradicting the four-connectivity of G. We can repeatedly choose w_{j+1} as the neighbour $w \in B'$ of v_{j+1} that appears after visiting w_j until a desired set M of matching edges is obtained.

From the above construction, it is easy to see that a desired matching M can be obtained in linear time. ∎

3 Archfree Paths and Archfree Trees

We say that a facial cycle f *arches* a path Q in a plane graph if there are two distinct vertices $a, b \in V(Q) \cap V(f)$ such that the subpath $Q_{a,b}$ of Q between a and b is not a subpath of f. A path Q is called *archfree* if no inner facial cycle f arches Q. Note that any subpath of a facial cycle in a triconnected plane graph is an archfree path.

Let Q be an inner path that is contained in an inner path Q' between two outer vertices s' and t' in a plane graph $G = (V, E, F)$, and let s and t be the end vertices of Q, where Q and Q' are viewed as directed paths from s' to t', as shown in Fig. 3. The outer facial cycle f_o consists of subpath f'_o from s' to t' and subpath f''_o from t' to s' when we walk along f_o in the clockwise order.

We say that an inner facial cycle $f \in F$ is *on the left side* if f is surrounded by f'_o and Q', and that f *arches* Q *on the left side* if f is on the left side of Q. The case of the right side is defined symmetrically. For example, facial cycles f, f_1 and f_2 in Fig. 3 arch path Q on the left side, where Q is displayed as thick lines.

Now we modify Q into a path $L(Q)$ from s to t such that no inner facial cycle arches $L(Q)$ on the left side. Let F_Q be the set of all inner facial cycles $f \in F$ that arch Q on the left side, but are not contained in the region enclosed by Q and any other $f' \in F$. For example, facial cycle f_1 in Fig. 3 is enclosed by Q and f, and thereby $f_1 \notin F_Q$.

The *left-aligned path* $L(Q)$ of Q is defined as an inner path from s to t obtained by replacing subpaths of Q with subpaths of cycles in F_Q as follows. For each $f \in F_Q$, let a_f and b_f be the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t, and f_Q be the subpath from a_f to b_f obtained by traversing f in the anticlockwise order. Let $L(Q)$ be the path obtained by replacing the subpath from a_f to b_f along Q with f_Q for all $f \in F_Q$ (see Fig. 3 for an example of $L(Q)$).

The following is then observed:

Lemma 4 *Given an inner path Q, the left-aligned path $L(Q)$ of Q can be constructed in $O(|E_Q| + |L(Q)|)$ time, where E_Q is the set of all edges incident to a vertex in Q.* ∎

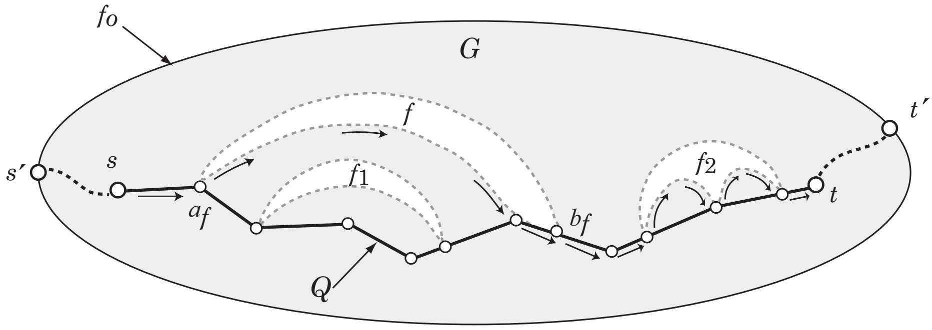

Figure 3: Construction of the left-aligned path $L(Q)$ from an inner path Q between s and t, where thick lines show Q and the path following the arrows shows $L(Q)$.

The *right-aligned path* $R(Q)$ of Q is defined symmetrically to the left-aligned path.

Lemma 5 *Let $G = (V, E, F)$ be an internally triconnected plane graph, and Q be an inner path from a vertex s to a vertex t. Then the left-aligned path $L(Q)$ is an inner path from s to t, and no inner facial cycle arches $L(Q)$ on the left side. Moreover, if no inner facial cycle arches Q on the right side, then $L(Q)$ is an archfree path.*

PROOF: Since G is internally triconnected, we can extend Q to an inner path between two outer vertices s' and t'. Let F_Q be the set of all inner facial cycles $f \in F$ that arch Q on the left side, but are not enclosed by Q and any other $f' \in F$. It is clear that $L(Q)$ is an inner path, since a subpath of Q is replaced with a subpath of $f \in F_Q$ which is not adjacent to any outer vertex.

Assume that an inner facial cycle $f^* \in F - F_Q - \{f_o\}$ arches $L(Q)$ on the left side. Let a_{f^*} and b_{f^*} be the first and last vertices in $V(f^*)$ along path $L(Q)$. Note that neither a_{f^*} or b_{f^*} is contained in subpath f_Q for any $f \in F_Q$ since face f contains no edge. Then both a_{f^*} and b_{f^*} are in $V(Q)$. This, however, implies that f^* arches Q on the left side in G, contradicting the choice of F_Q.

We consider the case where Q has no inner facial cycle that arches Q on the right side. Assume that an inner facial cycle $\hat{f} \in F - F_Q - \{f_o\}$ arches $L(Q)$ on the right side. Let $a_{\hat{f}}$ and $b_{\hat{f}}$ be the first and last vertices in $V(\hat{f})$ along path $L(Q)$. Note that both $a_{\hat{f}}$ and $b_{\hat{f}}$ belong to Q or a subpath f_Q with $f \in F_Q$, since otherwise Q would intersect the interior of face $\hat{f}$. Since no inner facial cycle arches Q on the right side by the assumption on Q, both $a_{\hat{f}}$ and $b_{\hat{f}}$ must belong to subpath f_Q for some $f \in F_Q$. This implies that $a_{\hat{f}}$ and $b_{\hat{f}}$ is a cut pair. Since G is internally triconnected, it must hold $a_{\hat{f}}, b_{\hat{f}} \in V(f_o)$. However, it is clear that $G - \{a_{\hat{f}}, b_{\hat{f}}\}$ has at least three components, contradicting the internal triconnectivity of G. ∎

Corollary 6 *For any inner path Q from s to t in an internally triconnected plane graph G, the right-aligned path $R(L(Q))$ of the left-aligned path $L(Q)$ is an archfree path.* ∎

A path in a tree T is called a *base path* if it is a maximal induced path in T, i.e., end vertices v and internal vertices u (if any) in the path satisfy $d_T(v) \neq 2$ and $d_T(u) = 2$, respectively. A tree T in a plane graph G is called *archfree* if every base path is an archfree path in G.

Lemma 7 *For a triconnected plane graph $G = (V, E, F)$, let S be a subset of $V(f_o)$ with $|S| \geq 2$. Then G contains an archfree tree T such that the set of leaves of T is equal to S. Such a tree T can be obtained in linear time.*

PROOF: By Lemma 2, G contains a spanning tree T_1 such that each vertex $v \in V(f_o)$ is a leaf of T_1. Let T_2 be the tree obtained from T_1 by removing all vertices that are not in the path between any two vertices in S. Note that T_2 is a tree such that the set of leaves of T_2 is S and each base path of T_2 is an inner path of G. Choose a vertex $s \in S$ as the root of T_2 with each base path as a directed path from the root to leaves. Let T_3 be the tree obtained from T_2 by replacing each base path Q of T_2 with its left-aligned path $L(Q)$. Then T_4 be the tree obtained from T_3 by replacing each base path Q of T_3 with its right-aligned path $R(Q)$. From Corollary 6, T_4 is an archfree tree whose leaf set is S.

By Lemma 2, spanning tree T_1 can be computed in linear time. It is clear that T_2 can be constructed from T_1 in linear time. By Lemma 4, T_3 can be computed in $O(\sum\{|E_Q| + |L(Q)| \mid \text{base paths } Q \text{ in } T_2\}) = O(\sum_{f\in F} |E(f)|) = O(|V|)$ time. Analogously T_4 can be obtained in $O(|V|)$ time. ∎

4 Convex Drawing with a Convex Boundary

For three points a_1, a_2, and a_3 in the plane, the line segment whose end points are a_i and a_j is denoted by (a_i, a_j), and the angle (a_1, a_2, a_3) formed by line segments (a_1, a_2) and (a_2, a_3) is defined by the central angle of a circle with center a_2 when we traverse the circumference from a_1 to a_3 in the clockwise order (note that $(a_1, a_2, a_3) + (a_3, a_2, a_1) = \pi$).

A polygon P is given by a sequence $a_1, a_2, \ldots, a_p$ $(p \geq 3)$ of points, called *apices*, and edges (a_i, a_{i+1}), $i = 1, 2, \ldots, p$ (where $a_{p+1} = a_1$) such that no two line segments (a_i, a_{i+1}) and (a_j, a_{j+1}), $i \neq j$ intersect each other except at apices. Let $A(P)$ denote such a sequence of apices of a polygon P, where $A(P)$ may be used to denote the set of the apices in $A(P)$. The *inner angle* $\theta(a_i)$ of an apex a_i is the angle (a_{i+1}, a_i, a_{i-1}) formed by line segments (a_i, a_{i+1}) and (a_{i-1}, a_i), and an apex a_i is called *convex* (respectively, *concave* and *flat*) if $\theta(a_i) < \pi$ (respectively, $\theta(a_i) > \pi$ and $\theta(a_i) = \pi$). A polygon P has no apex a with $\theta(a) = 0$ since no two adjacent edges on the boundary intersect each other. Thus, the interior of a polygon has a positive area.

A polygon P is called *convex* if it has no concave apex. A k-gon is a polygon with exactly k apices, some of which may be flat or concave. A *side* of a polygon is a maximal line segment in its boundary, i.e., a sequence of edges $(a_i, a_{i+1}), (a_{i+1}, a_{i+2}), \ldots, (a_{i+h-1}, a_{i+h})$ such that a_i and a_{i+h} are non-flat apices and the other apices between them are flat. A k-gon has at most k sides.

A *straight-line drawing* of a graph $G = (V, E)$ in the plane is an embedding of G in the two dimensional space $\Re^2$ such that each vertex $v \in V$ is drawn as a point $\psi(v) \in \Re^2$ and each edge $(u, v) \in E$ is drawn as a straight-line segment $(\psi(u), \psi(v))$, where $\Re$ is the set of reals. Hence, a straight-line drawing of a graph $G = (V, E)$ is defined by a function $\psi : V \to \Re^2$. A straight-line drawing ψ of a plane graph $G = (V, E, F)$ is called an *inner-convex drawing* (or simply a convex drawing) if every inner facial cycle is drawn as a convex polygon. A convex drawing ψ of a plane graph $G = (V, E, F)$ is called a *strictly convex drawing* if it has no flat apex $\psi(v)$ for any vertex $v \in V$ with $d_G(v) \geq 3$. We say that a drawing ψ of a graph G is *extended* from a drawing ψ' of a subgraph G' of G if $\psi(v) = \psi'(v)$ for all $v \in V(G')$.

Let $G = (V, E, F)$ be a plane graph with an outer facial cycle f_o, and P be a $|V(f_o)|$-gon. A drawing ϕ of f_o on P is a bijection $\phi : V(f_o) \rightarrow A(P)$ such that the vertices in $V(f_o)$ appear along f_o in the same order as the corresponding apices in sequence $A(P)$.

Lemma 8 [6, 16] *Let $G = (V, E, F)$ be a biconnected plane graph. Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if and only if the following conditions* (i)-(iii) *hold:*

(i) *For each inner vertex v with $d_G(v) \geq 3$, there exist three paths disjoint except v, each connecting v and an outer vertex;*

(ii) *Every cycle of G which has no outer edge has at least three vertices v with $d_G(v) \geq 3$; and*

(iii) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. The graph $G - V(f_o)$ has no component H such that all the outer vertices adjacent to vertices in H are contained in a single path Q_i, and there is no inner edge (u, v) whose end vertices are contained in a single path Q_i.* ∎

Since every inner vertex of degree 2 must be drawn as a point sub-dividing a line segment in any convex drawing, we can assume without loss of generality that a given biconnected plane graph has no inner vertex of degree 2. Then Lemma 8 can be restated as follows.

Lemma 9 *Let $G = (V, E, F)$ be a biconnected plane graph which has no inner vertex with degree 2. Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if and only if the following conditions* (a) *and* (b) *hold:*

(a) *G is internally triconnected.*

(b) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. Each Q_i is an archfree path in G.* ∎

5 Convex Drawing with Star-Shaped Boundary Constraints

A kernel $K(P)$ of a polygon P is the set of all points from which all points in P are visible. The boundary of a kernel, if any, is a convex polygon. A polygon P is called *star-shaped* if $K(P) \neq \emptyset$. Throughout the paper, we assume that for a given star-shaped polygon, its kernel has a positive area.

Let ϕ be a drawing of the outer facial cycle f_o of a plane graph G on a star-shaped polygon P, and let $S = \{v_1, v_2, \ldots, v_p\}$ be a subset of $V(f_o)$, where the vertices $v_1, v_2, \ldots, v_p$ in S appear in this order when we traverse f_o in the clockwise order (where $v_{p+1} = v_1$). A subset S is *valid* if S contains all vertices $v \in V(f_o)$ such that $\phi(v)$ is a concave apex of P and for any point $a \in K(P)$, the angle $(\phi(v_i), a, \phi(v_{i+1}))$ formed by line segments $(\phi(v_i), a)$ and $(a, \phi(v_{i+1}))$ is less than π. Obviously $S = V(f_o)$ is valid if P is a star-shaped polygon.

For a tree T of G whose leaf set is $S = \{v_1, v_2, \ldots, v_p\}$, we denote by $H = T + f_o$ the plane subgraph of G obtained by joining T and f_o, i.e., $V(H) = V(T) \cup V(f_o)$, $E(H) = E(T) \cup E(f_o)$ and $F(H) = \{f_1, f_2, \ldots, f_p\}$, where f_i is the cycle consisting of the path between v_i and v_{i+1} along T and the subpath from v_i to v_{i+1} in f_o. We now describe an important lemma.

Lemma 10 *Let $G = (V, E, F)$ be a triconnected plane graph, and ϕ be a drawing of the outer facial cycle f_o on a star-shaped polygon P. For drawing ϕ, let S be a valid subset of $V(f_o)$, and T be an archfree tree whose leaf set is S. Then ϕ can be extended to a strictly convex drawing ψ of $H = T + f_o$. Such a drawing ψ can be obtained in linear time.*

Proof: Let K be a circle contained in $K(P)$, where the center of K is denoted by c. It suffices to show that the lemma holds only for the case where T contains no vertex v with $d_T(v) = 2$, since two edges in T that are adjacent at such a vertex v can be replaced with a single edge and the eliminated vertex v can be re-inserted in a line segment in a straight-line drawing. See Fig. 4(a), which illustrates $H = T + f_o$ with no vertex v with $d_T(v) = 2$ and K centered at a point c.

Now T has at least three leaves (since S is valid) but no vertex v with $d_T(v) = 2$. We call a non-leaf vertex in T a *fringe vertex* if it has no more than one neighbor u with $d_T(u) \geq 2$. If T has exactly one fringe vertex, i.e., T is a star centered at a vertex v^*, then ψ with $\psi(u) = \phi(u)$, $u \in S$ and $\psi(v^*) = c$ is a strictly convex drawing of H since each facial cycle f_i is drawn as a triangle whose sides do not intersect with any side of P by the validity of S.

We now consider the case where T has at least two fringe vertices. Each fringe vertex u has at least two adjacent leaves and all adjacent leaves appear consecutively along f_o, when we visit the leaves of T along P in the clockwise order. We denote its first leaf (respectively, last leaf) by a_u (respectively, b_u). For example, vertex v in Fig. 4(a) has $a_v = v_1$ and $b_v = v_h$.

A fringe vertex u is called *wide* in a drawing ϕ if the angle $(\phi(a_u), c, \phi(b_u))$ formed by line segments $(c, \phi(a_u))$ and $(c, \phi(b_u))$ is no less than π (for example, vertex u in Fig. 4(a) is a wide fringe vertex). Note that T has at most one wide fringe vertex.

We prove that a drawing ϕ of f_o on P can be extended to a strictly convex drawing ψ of $H = T + f_o$ such that ψ satisfies the following two conditions:

$$\text{all non-leaf vertices of } T \text{ are drawn strictly inside } K, \tag{1}$$

$$\begin{array}{l}\text{the angle } (\psi(a_u), u, \psi(b_u)) < \pi \text{ for all fringe vertices } u \\ \text{except for a wide fringe vertex.}\end{array} \tag{2}$$

We prove the lemma by induction on the number of vertices in a tree T. In the base case, T has exactly two fringe vertices u_1 and u_2, where the wide fringe vertex (if any) is denoted by u_1. Draw vertex u_1 as the center c of K, and vertex u_2 as a point at the intersection of K and triangle u_1, a_{u_2}, b_{u_2}. By the choice of u_1, the resulting drawing ψ of $H = T + f_o$ is strictly convex satisfying (1) and (2).

We now assume that the lemma holds for any tree T with at most k vertices. Let T be a tree with $k + 1$ vertices. We prove that the lemma holds for T. Let $v^* \in V(T)$ be a fringe vertex which is not wide, and $v_1, v_2, \ldots, v_h$ be the leaves adjacent to v^*, which appear in this order when we traverse f_o in the clockwise order, as shown in Fig. 4(a). We distinguish two cases, $h \geq 3$ and $h = 2$.

Case-1. $h \geq 3$. Let T' be the tree obtained from T by removing leaves $v_2, v_3, \ldots, v_{h-1}$, $S' = S - \{v_2, v_3, \ldots, v_{h-1}\}$, f'_o be the cycle obtained from f_o by replacing its subpath from v_1 to v_h with an edge (v_1, v_h). Let P' be the polygon obtained from P by replacing the edges between v_1 and v_h with a single edge, and ϕ' be the resulting drawing of f'_o on P' (see Fig. 4(b)). Note that v^* remains a non-wide fringe vertex in T' and hence, S' is valid in P'.

By the inductive hypothesis, ϕ' can be extended to a strictly convex drawing ψ' of $H' = T' + f'_o$ that satisfies (1) and (2) (see Fig. 4(b)). We show that the original drawing ϕ of f_o on P can be extended to a strictly convex drawing ψ of $H = T + f_o$ satisfying (1) and (2).

Such a drawing ψ is obtained from ϕ and ψ' by setting $\psi(u) = \psi'(u)$, $u \in V(H')$ and $\psi(v_i) = \phi(v_i)$, $i = 2, 3, \ldots, h-1$. In the resulting drawing ψ, each deleted edge (v^*, v_i), $i = 2, 3, \ldots, h-1$ is drawn as a line segment $(\psi'(v^*), \phi(v_i))$ (see Fig. 4(c)). We see that each of new facial cycle $\{v^*, v_{i-1}, v_i\}$, $i = 2, 3, \ldots, h-1$ is drawn as a triangle, which does not intersect with P since $\psi'(v^*)$ is in K and v remains as a non-wide fringe vertex in T by (2). Therefore, ϕ is a strictly convex drawing of H which satisfies (1) and (2).

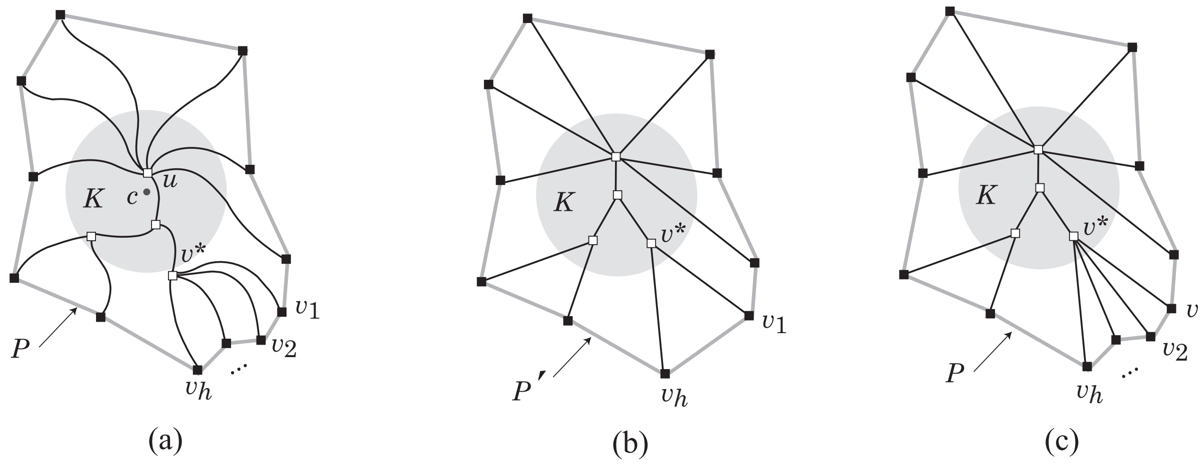

Figure 4: (a) Reducing the number of leaves adjacent to v^* in tree T; (b) Re-inserting the deleted leaves in a convex drawing of $H' = T' + f'_o$ with boundary P'; (c) A convex drawing of $H = T + f_o$ with boundary P.

Case-2. $h = 2$. Let w be the unique neighbour of v^* with $d_T(w) \geq 3$, and $e_a = (w, u_a)$ (respectively, $e_b = (w, u_b)$) be the edge incident to w that appears immediately before edge (w, v^*) (respectively, immediately after edge (w, v^*)) when we walk around w in the clockwise order. Since $d_T(w) \geq 3$, we have $e_a \neq e_b$ (see Fig. 5(a)).

Let T' be the tree obtained from T by contracting edge (v^*, w) into a vertex w. Obviously the set of leaves in T' is S, which remains valid in P. By the inductive hypothesis, a drawing ϕ of f_o on P can be extended to a strictly convex drawing ψ' of $H' = T' + f_o$ that satisfies (1) and (2) (see Fig. 5(b)).

We show that the original drawing ϕ of f_o on P can be extended to a strictly convex drawing

ψ of $H = T + f_o$ satisfying (1) and (2). Such a drawing ψ is obtained from ϕ and ψ' by setting $\psi(u) = \psi'(u)$, $u \in V(H') - \{v^*\}$ and choosing $\psi(v^*)$ of v as follows. Let f_a (respectively, f_b) be the facial cycle containing edges e_a and (w, v_1) (respectively, e_b and (w, v_h)), and L_a (respectively, L_b) be the half line which starts at $\phi'(w)$ in the direction from $\phi'(u_a)$ to $\phi'(w)$ (respectively, from $\phi'(u_b)$ to $\phi'(w)$). See Fig. 5(b).

To keep the faces f_a and f_b strictly convex after re-inserting edge (v^*, w), the position $\psi(v^*)$ of v^* must be in the region R enclosed by the boundary of K and the two lines L_a and L_b. Since $e_a \neq e_b$, we can choose a position $\psi(v^*)$ of v strictly inside K so that the resulting drawing ψ is strictly convex that satisfies (1) and (2). This completes the proof for the existence of a desired convex drawing ψ of $H = T + f_o$.

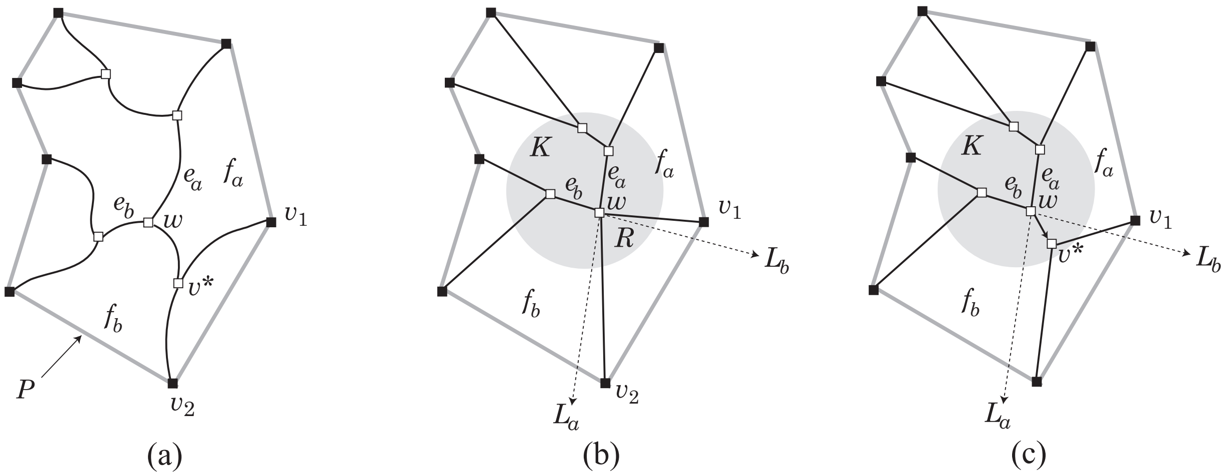

Figure 5: (a) Contracting edge (v^*, w) in tree T; (b) Re-inserting a contracted edge (v^*, w) in a convex drawing ψ' of $H' = T' + f_o$; (c) A convex drawing of $H = T + f_o$ with boundary P.

It is clear that the above inductive proof gives an algorithm for computing a desired drawing ϕ of H. We now show that it runs in linear time.

The kernel $K(P)$ can be computed in linear time [14], and a circle K in $K(P)$ can be chosen in linear time. The operation in Case-1 can be executed in $O(1)$ time by placing v on a point in R sufficiently closed to $\phi'(w)$. The operation in Case-2 can be executed in $O(1)$ time per edge to be deleted. Following the above construction, a desired convex drawing ϕ of $H = T + f_o$ can be constructed in linear time.

This completes the proof. ∎

We are now ready to prove Theorem 11.

Theorem 11 *Every drawing ϕ of the outer facial cycle f_o of a triconnected plane graph $G = (V, E, F)$ on a star-shaped polygon can be extended to a convex drawing ψ_G of G. Such a drawing ψ_G can be computed in linear time.*

Proof: Choose a valid subset $S = \{v_1, v_2, \ldots, v_p\}$ of $V(f_o)$ (for example $S = V(f_o)$), and compute an archfree tree T of G whose leaf set is S, which can be computed in linear time by Lemma 7 (see Fig. 6(a)). By Lemma 10, we can compute a strictly convex drawing ψ of $H = T + f_o$ as an extension of drawing ϕ of f_o on P in linear time (see Fig. 6(b)).

Now each facial cycle f_i of H is drawn as a strictly convex polygon P_i. Let G_i be the subgraph of G that consists of vertices and edges in the face f_i (including those on the boundary of f_i). Since each side of P_i is an archfree path and ψ is a strictly convex drawing, G_i with the boundary P_i satisfies condition (b) of Lemma 9.

It is clear that G_i with P_i also satisfies condition (a) of Lemma 9, since otherwise G with P would violate (a), contradicting the triconnectivity of G. Hence, each plane graph G_i with boundary P_i has a convex drawing ψ_i, and such a drawing ψ_i can be computed in $O(|V(G_i)| + |E(G_i)|)$ time by using the algorithm in [6].

Therefore, after computing convex drawings ψ_i for all G_i and placing them in the corresponding faces in ψ, we obtain a convex drawing ψ_G of G which is an extension of drawing ϕ of f_o on P (see Fig. 6(c)).

The above algorithm runs in linear time. ∎

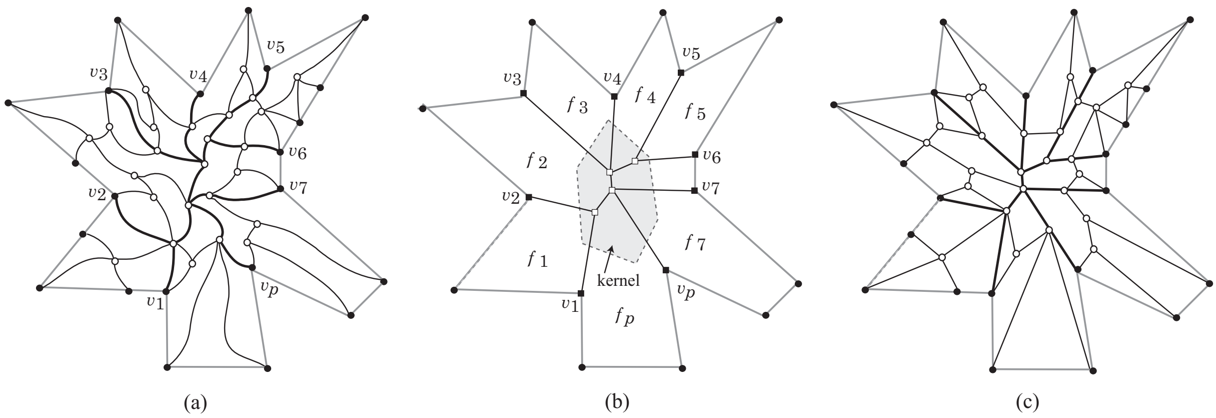

Figure 6: (a) A triconnected plane graph $G = (V, E, F)$ with a star-shaped boundary P, where an archfree tree T is denoted by thick black lines; (b) A convex drawing of $H = T + f_o$ with boundary P; (c) A convex drawing of G with boundary P.

6 Convex Drawing with Crown-Shaped Boundary Constraints

A polygon P is called *crown-shaped* if it has a side Q of P and a point p^* outside P such that, for any point p inside P, the line segment (p, p^*) does not intersect with any other side of P than Q (i.e., all points in P are visible from a point p^* if Q is removed from P). We call such a side Q and a point p^* the *base side* and the *pivot* of a crown-shaped polygon. Note that any two points in the boundary of P and p^* are not collinear.

For example, Fig. 1(c) and Fig. 7(a) show plane graphs whose boundaries are drawn as crown-shaped polygons. We easily see that if we divide a star-shaped polygon into two or more polygons by a single straight line, then each of the resulting polygon is either star-shaped or crown-shaped.

As shown in Fig. 1(c), there is a triconnected plane graph with a crown-shaped boundary that cannot be extended to a convex drawing. We show that four-connectivity suffices for a plane graph with a crown-shaped boundary to admit a convex drawing.

Theorem 12 *Every drawing ϕ of the outer facial cycle f_o of a four-connected plane graph $G = (V, E, F)$ on a crown-shaped polygon can be extended to a convex drawing ψ_G of G. Such a drawing ψ_G can be computed in linear time.* ∎

Let P be the boundary of G which is a crown-shaped polygon, and Q and p^* the base side and pivot of P. We denote the vertices in f_o and Q as $v_1, v_2, \dots, v_q$ and $v_{g+1}, v_{g+2}, \dots, v_q$, as in Lemma 3. Let V^* be the set of vertices $v \in V(f_o)$ such that $\phi(v)$ is a concave apex of P. If $V^* = \emptyset$, i.e., P is convex, then theorem follows from the result by Chiba et al. [6]. We now assume that $V^* = \{v_{j_1}, v_{j_2} \dots, v_{j_h}\}$ $(h \geq 1)$.

Consider graph $G' = G - (V(f_o) - V(Q))$, which is biconnected by Lemma 3(i), and v'_{g+1} and v'_q be the neighbours of v_{g+1} and v_q defined in Lemma 3. Fig. 7(a) shows an example of a four-connected plane graph G.

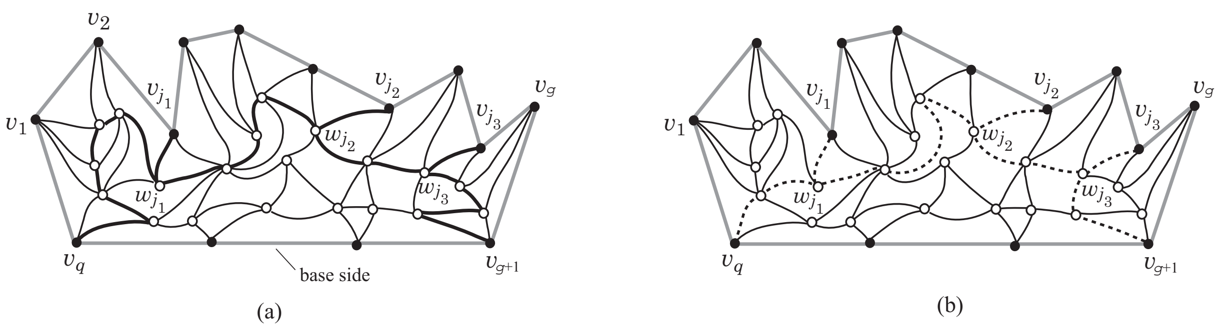

Figure 7: (a) A four-connected plane graph $G = (V, E, F)$ with a crown-shaped boundary P, where the path Q' from v_q to v_g of the boundary of G' and matching M are displayed as thick lines; (b) A archfree tree T^* obtained from tree T with $E(T) = E(Q') \cup M$, where edges in T are displayed as dashed lines.

By Lemma 3(ii), there is a matching $M = \{(v_{j_i}, w_{j_i}) \mid i = 1, 2, \dots, h\} \cup \{(v_{g+1}, v'_{g+1}), (v_q, v'_q)\}$, where each w_{j_i} belongs to the boundary of G'. Let Q' be the subpath from v_q to v_g of the boundary of G'. Note that the edges in $E(Q') \cup M$ form a tree T spanning vertices in $V^* \cup \{v_{g+1}, v_q\}$, where the degree of each w_{j_i} in T is 3 (see Fig. 7(a)).

By applying Lemma 5 to every base path of T, we construct an archfree tree T^* from T, which can be done in linear time as analysed in Lemma 7. Note that the degree of each w_{j_i} in T^* remains 3 (see Fig. 7(b)).

Lemma 13 *Let $G = (V, E, F)$ be a four-connected plane graph, and ϕ be a drawing of the outer facial cycle f_o on a crown-shaped polygon P. Let v_{g+1} and v_q denote the vertices drawn as the endvertices of the base side Q of P, and $V^* \neq \emptyset$ be the set of vertices $v \in V(f_o)$ such that $\phi(v)$ is a concave apex of P. Let T^* be an archfree tree such that its leaf set is $V^* \cup \{v_{g+1}, v_q\}$ and* *its edge set can be partitioned into a path between v_{g+1} and v_q and a matching covering V^*. Then ϕ can be extended to a strictly convex drawing ψ of $H = T^* + f_o$. Such a drawing ψ can be obtained in linear time.*

PROOF: As observed in the proof of Lemma 10, it suffices to show that the lemma holds only for the case where T^* contains no vertex v with $d_{T^*}(v) = 2$.

By definition, there is a pivot p^*. Place each internal vertex w_{j_i} of T^* on a point in line segment $(\phi(v_{j_i}), p^*)$ and strictly inside P (see Fig. 8(a)). Since all points inside P are visible from p^* if we ignore the base side Q, we easily see that the resulting drawing of T^* divides the interior of P into $h+2$ convex regions.

This construction takes linear time. ∎

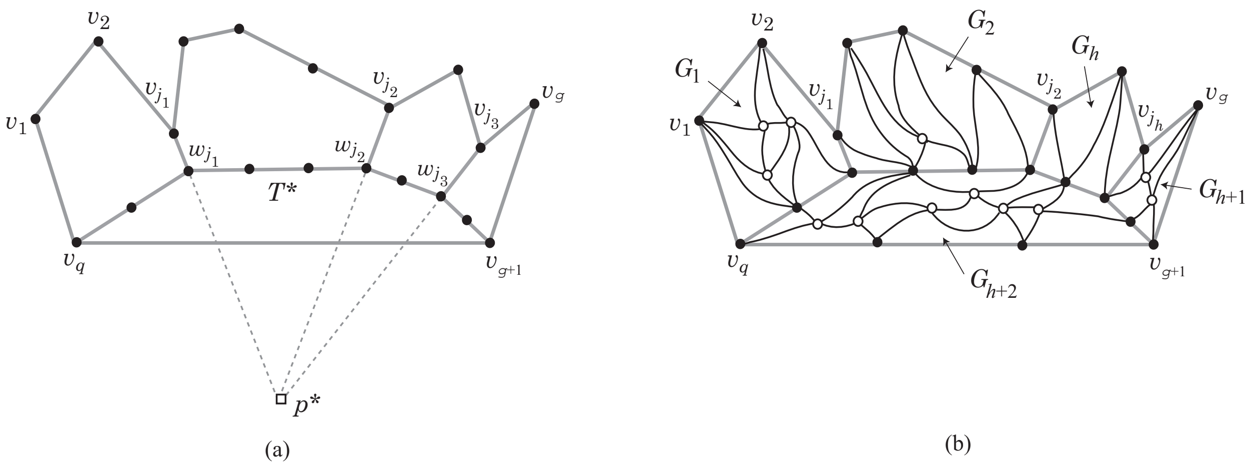

Figure 8: (a) A strictly convex drawing ψ of $H = T^* + f_o$; (b) Internally triconnected plane graphs G_i, $i = 1, 2, \ldots, h+2$, with convex boundaries.

We are ready to complete the proof of Theorem 12 in a similar manner of the proof of Theorem 11.

By Lemma 13, we can compute a strictly convex drawing ψ of $H = T^* + f_o$ as an extension of drawing ϕ of f_o on P in linear time (see Fig. 8(b)).

Now each facial cycle f_i of H is drawn as a strictly convex polygon P_i. Let G_i be the subgraph of G that consists of vertices and edges in the face f_i (including those on the boundary of f_i). Since each side of P_i is an archfree path and ψ is a strictly convex drawing, G_i with the boundary P_i satisfies condition (b) of Lemma 9.

It is clear that G_i with P_i also satisfies condition (a) of Lemma 9, since otherwise G with P would violate (a), contradicting the four-connectivity of G. Hence, each plane graph G_i with boundary P_i has a convex drawing ψ_i, and such a drawing ψ_i can be computed in $O(|V(G_i)| + |E(G_i)|)$ time by using the algorithm in [6].

Therefore, after computing convex drawings ψ_i for all G_i and placing them in the corresponding faces in ψ, we obtain a convex drawing ψ_G of G which is an extension of drawing ϕ of f_o on P.

The above algorithm runs in linear time.

7 Conclusion

In this paper, we initiate a new problem of convex drawings of planar graphs with non-convex boundary constraints.

It is proved that every triconnected plane graph with a star-shaped boundary admits an inner-convex drawing. We present a linear time algorithm for computing such an inner-convex drawing. Similar results hold for four-connected graphs with crown-shaped polygon.

Recently, we prove that every internally triconnected hierarchical-st plane graph with a convex boundary satisfying condition (b) in Lemma 9 admits a convex drawing [9].

It is left open to find a characterisation for a plane graph with a star-shaped boundary to admit an inner-convex drawing.

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