Convex Drawings of Hierarchical Planar Graphs and Clustered Planar Graphs

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Abstract: *Hierarchical graphs* are graphs with layering structures; *clustered graphs* are graphs with recursive clustering structures. Both have applications in VLSI design, CASE tools, software visualisation and visualisation of social networks and biological networks. *Straight-line drawing* algorithms for hierarchical graphs and clustered graphs have been presented in [P. Eades, Q. Feng, X. Lin and H. Nagamochi, Straight-line drawing algorithms for hierarchical graphs and clustered graphs, Algorithmica, 44, pp. 1-32, 2006].

A straight-line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. In this paper, it is proved that every internally triconnected hierarchical plane graph with the outer facial cycle drawn as a convex polygon admits a convex drawing. We present an algorithm which constructs such a drawing.

We then extend our results to convex representations of clustered planar graphs. It is proved that every internally triconnected clustered plane graph with completely connected clustering structure admits a convex drawing. We present an algorithm to construct a convex drawing of clustered planar graphs.

Keywords: Graph Drawing, Convex Drawing, Hierarchical Graphs, Clustered Graphs, Straight-line Drawing, Triconnected Planar Graphs.

1 Introduction

Graph drawing has attracted much attention for the last ten years due to its wide range of applications such as VLSI design, software engineering and bioinformatics. Two or three dimensional drawings of graphs with a variety of aesthetics and edge representations have been extensively studied (see [1]).

One of the most popular drawing conventions is the *straight-line drawing*, where all the edges of a graph are drawn as straight-line segments. Every planar graph is known to have a planar straight-line drawing [10].

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A straight-line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. Note that not all planar graphs admit a convex drawing. Tutte [22] gave a necessary and sufficient condition for a triconnected plane graph to admit a convex drawing. He also showed that every triconnected plane graph with a given boundary drawn as a convex polygon admits a convex drawing using the polygonal boundary. That is, when the vertices on the boundary are placed on a convex polygon, inner vertices can be placed on suitable positions so that each inner facial cycle forms a convex polygon. More specifically, he proposed a "barycentric mapping" method which computes a convex drawing of a triconnected plane graph with n vertices by solving a system of $O(n)$ linear equations. This requires $O(n^3)$ time and $O(n^2)$ space using the ordinary Gaussian elimination method, however it can be implemented in $O(n^{1.5})$ time and $O(n \log n)$ space using the sparse Gaussian elimination method [14].

Later, Thomassen [21] gave a necessary and sufficient condition for a biconnected plane graph to admit a convex drawing. Based on this result, Chiba et al. [5] presented a linear time algorithm for finding a convex drawing (if any) for a biconnected plane graph with a specified convex boundary.

In general, the convex drawing problem has been well investigated for the last ten years by the Graph Drawing community. For example, a problem of convex drawing of graphs with *grid* constraints has been well studied [2, 3, 4, 17]. A convex drawing is called a *convex grid drawing* if all the vertices are restricted to be placed on grid points. Every triconnected plane graph has a convex grid drawing on an $(n-2) \times (n-2)$ grid, and such a grid drawing can be found in linear time [4]. A linear time algorithm for finding a convex grid drawing of four-connected plane graphs with four or more vertices on the outer face was presented in [17]. Another variation of convex drawing with minimum outer apices was introduced in [15]. For constructing a strictly convex drawing of graphs, see [20].

Recently, in our companion paper [11], we introduced a new problem of drawing planar graphs with *non-convex* boundary constraints. A straight-line drawing is called an *inner-convex drawing* if every inner facial cycle is drawn as a convex polygon. It is proved that every triconnected plane graph admits an *inner-convex* drawing if its boundary is fixed with a star-shaped polygon P, i.e., a polygon P whose kernel (the set of all points from which all points in P are visible) is not empty [11]. Note that this is an extension of the classical result by Tutte [22] since any convex polygon is a star-shaped polygon. We also presented a linear time algorithm for computing an inner-convex drawing of a triconnected plane graph with a star-shaped boundary [11].

In this paper, we present results on convex drawings of *hierarchical graphs* and *clustered graphs*. Hierarchical graphs and clustered graphs are useful graph models with structured relational information. Hierarchical graphs are graphs with layering structures; clustered graphs are graphs with recursive clustering structures. Both have applications in VLSI design, CASE tools, software visualisation and visualisation of social networks and biological networks [7].

Hierarchical graphs (sometimes called *level* graphs) are directed graphs with vertices assigned into *layers* (or levels). Hierarchical graphs are drawn with vertices of a layer on the same horizontal line, and edges as curves monotonic in y direction. A hierarchical graph is *hierarchical planar* (*h-planar*) (*or level-planar*) if it admits a drawing without edge crossings.

The hierarchical structure in hierarchical graphs imposes constraints on the y-coordinate.

In this paper, it is proved that every internally triconnected hierarchical plane graph with the outer facial cycle drawn as a convex polygon admits a *convex* drawing. We present an algorithm which constructs such a drawing. Note that this extends the previous known result that every hierarchical planar graph admits a straight-line drawing [7].

A *clustered graph* $C = (G, T)$ consists of an undirected graph (called the *underlying graph*) $G = (V, E)$ and a rooted tree (called the *inclusion tree* of C) $T = (\mathcal{V}, \mathcal{A})$, such that the leaves of T are exactly the vertices of G [7]. Each node ν of T represents a *cluster* $V(\nu)$, a subset of the vertices of G that are leaves of the subtree rooted at ν. A clustered graph $C = (G, T)$ is a *connected clustered graph* if each cluster $V(\nu)$ induces a connected subgraph $G(\nu)$ of G [7]. A clustered graph $C = (G, T)$ is *completely connected* if, for every non-root node ν of T, both subgraphs $G(\nu)$ and $G[V - V(\nu)]$ are connected [6].

In a *drawing* of a clustered graph $C = (G, T)$, graph G is drawn as points and curves as usual. For each node ν of T, the cluster is drawn as a simple closed region $R(\nu)$ enclosed by a simple closed curve such that the drawing of $G(\nu)$ is completely contained in the interior of $R(\nu)$, the regions for all sub-clusters of ν are completely contained in the interior of $R(\nu)$, and the regions for all other clusters are completely contained in the exterior of $R(\nu)$. A clustered graph is *compound planar* (*c-planar*) if it admits a c-planar drawing without edge crossings or *edge-region crossings* (i.e. the drawing of e crosses the boundary of region R more than once).

In this paper, it is proved that every connected clustered plane graph with internally triconnected underlying graph and completely connected clustering structure admits a convex drawing. We present an algorithm to construct a convex drawing of clustered planar graphs. Note that this extends the previous known results on straight-line drawings of connected clustered planar graphs [7].

This paper is organized as follows: Section 2 reviews basic terminology. In Section 3 we introduce the concept of *archfree* paths and *archfree* trees, which play an important role in our convex drawing algorithm. Section 4 reviews the necessary and sufficient conditions for a convex drawing of a biconnected plane graph. Section 5 proves our main theorem on convex drawings of hierarchical planar graphs and presents an algorithm for constructing such a drawing. In Section 6, we prove our second theorem on convex drawings of clustered planar graphs and present an algorithm for constructing such a drawing. Section 7 concludes.

2 Preliminaries

Throughout the paper, a graph stands for a simple undirected graph unless stated otherwise. Let $G = (V, E)$ be a graph. The set of edges incident to a vertex $v \in V$ is denoted by $E(v)$. The degree of a vertex v in G is denoted by $d_G(v)$ (i.e., $d_G(v) = |E(v)|$). For a subset $X \subseteq V$, $G[X]$ denotes the subgraph induced by X (i.e., graph $(V, E - \cup_{v \in V - X} E(v))$), and $G - X$ denotes subgraph $G[V - X]$. For a subset $E' \subseteq E$, $G - E'$ denotes the graph obtained from G by removing the edges in E'.

A vertex in a connected graph is called a *cut vertex* if its removal from G results in a disconnected graph. A connected graph is called *biconnected* if it is simple and has no cut vertex. Similarly, a pair of vertices in a connected graph is called a *cut pair* (or *separation pair*) if its removal from G results in a disconnected graph. A connected graph is called *triconnected* if it is simple and has no cut pair. We say that a cut pair $\{u, v\}$ *separates* two vertices s and t if s and t belong to different components in $G - \{u, v\}$.

A graph $G = (V, E)$ is called *planar* if its vertices and edges are drawn as points and curves in the plane so that no two curves intersect except at their endpoints, where no two vertices are drawn at the same point. In such a drawing, the plane is divided into several connected regions, each of which is called a *face*. A face is characterized by the cycle of G that surrounds the region. Such a cycle is called a *facial cycle*. A set F of facial cycles in a drawing is called an *embedding* of a planar graph G. A *plane* graph $G = (V, E, F)$ is a planar graph $G = (V, E)$ with a fixed embedding F of G, where we always denote the outer facial cycle in F by $f_o \in F$.

A vertex (respectively, an edge) in f_o is called an *outer vertex* (respectively, an *outer edge*), while a vertex (respectively, an edge) not in f_o is called an *inner vertex* (respectively, an *inner edge*). A path Q between two vertices s and t in G is called *inner* if every vertex in $V(Q) - \{s, t\}$ is an inner vertex. The region enclosed by a facial cycle $f \in F$ may be denoted by f for simplicity. The set of vertices, set of edges and set of facial cycles of a plane graph G may be denoted by $V(G)$, $E(G)$ and $F(G)$, respectively.

A biconnected plane graph G is called *internally triconnected* if, for any cut pair $\{u, v\}$, u and v are outer vertices and each component in $G - \{u, v\}$ contains an outer vertex. Note that every inner vertex in an internally triconnected plane graph must be of degree at least 3.

For a cut pair $\{u, v\}$ of an internally triconnected plane graph $G = (V, E, F)$, if u and v are not adjacent and there is an inner facial cycle $f \in F$ such that $\{u, v\} \in V(f)$, we say that f *separates* two vertices s and t if the cut pair $\{u, v\}$ separates them.

3 Archfree Paths and Archfree Trees

In this section, we review definitions of *archfree paths* and *archfree trees*, which were used to construct inner convex drawings of triconnected plane graphs [11].

We say that a facial cycle f *arches* a path Q in a plane graph if there are two distinct vertices $a, b \in V(Q) \cap V(f)$ such that the subpath $Q_{a,b}$ of Q between a and b is not a subpath of f. A path Q is called *archfree* if no inner facial cycle f arches Q.

We easily observe the next property.

Lemma 1 *For an internally triconnected plane graph, a subpath of any inner facial cycle is an archfree path.* ∎

Let Q be an inner path that is contained in an inner path Q' between two outer vertices s' and t' in a plane graph $G = (V, E, F)$, and let s and t be the end vertices of Q, where Q and Q' are viewed as directed paths from s' to t', as shown in Fig. 1. The outer facial cycle f_o consists of a subpath f_o' from s' to t' and a subpath f_o'' from t' to s' when we walk along f_o in a clockwise direction.

We say that an inner facial cycle $f \in F$ is *on the left side* if f is surrounded by f_o' and Q', and that f *arches* Q *on the left side* if f is on the left side of Q. For example, facial cycles f, f_1

and f_2 in Fig. 1 arch path Q on the left side, where Q is displayed as thick lines. The case of the *right side* is defined symmetrically.

Now we modify Q into a path $L(Q)$ from s to t such that no inner facial cycle arches $L(Q)$ on the left side.

Let F_Q be the set of all inner facial cycles $f \in F$ that arch Q on the left side, but are not contained in the region enclosed by Q and any other $f' \in F$. For example, facial cycle f_1 in Fig. 1 is enclosed by Q and f, and thereby $f_1 \notin F_Q$.

The *left-aligned path* $L(Q)$ of Q is defined as an inner path from s to t obtained by replacing subpaths of Q with subpaths of cycles in F_Q as follows. For each $f \in F_Q$, let a_f and b_f be the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t, and f_Q be the subpath from a_f to b_f obtained by traversing f in an anticlockwise direction. Let $L(Q)$ be the path obtained by replacing the subpath from a_f to b_f along Q with f_Q for all $f \in F_Q$ (see Fig. 1 for an example of $L(Q)$).

It is not difficult to observe the next lemma.

Lemma 2 *Given an inner path Q, the left-aligned path $L(Q)$ of Q can be constructed in $O(|E_Q| + |L(Q)|)$ time, where E_Q is the set of all edges incident to a vertex in Q.* ▮

The *right-aligned path* $R(Q)$ of Q is defined symmetrically to the left-aligned path.

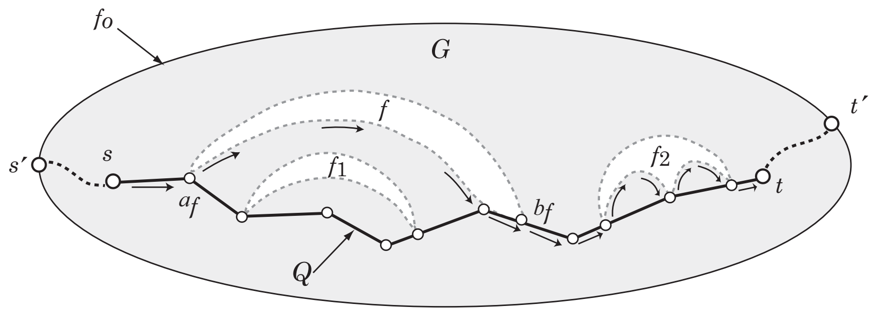


Figure 1: Construction of the left-aligned path $L(Q)$ from an inner path Q between s and t, where thick lines show Q and the path following the arrows show $L(Q)$.

The following results have been previously shown [11].

Lemma 3 [11] *Let $G = (V, E, F)$ be an internally triconnected plane graph, and Q be an inner path from a vertex s to a vertex t. Then the left-aligned path $L(Q)$ is an inner path from s to t, and no inner facial cycle arches $L(Q)$ on the left side. Moreover, if no inner facial cycle arches Q on the right side, then $L(Q)$ is an archfree path.* ▮

Corollary 4 [11] *For any inner path Q from s to t in an internally triconnected plane graph G, the right-aligned path $R(L(Q))$ of the left-aligned path $L(Q)$ is an archfree path.* ▮

A path in a tree T is called a *base path* if it is a maximal induced path in T, i.e., end vertices v and internal vertices u (if any) in the path satisfy $d_T(v) \neq 2$ and $d_T(u) = 2$, respectively. A tree T in a plane graph G is called *archfree* if every base path is an archfree path in G.

4 Convex Drawings of Planar Graphs

In this section, we review the previous results on convex drawings of planar graphs.

For three points a_1, a_2, and a_3 in the plane, the line segment whose end points are a_i and a_j is denoted by (a_i, a_j), and the angle (a_1, a_2, a_3) formed by line segments (a_1, a_2) and (a_2, a_3) is defined by the central angle of a circle with center a_2 when we traverse the circumference from a_1 to a_3 in the clockwise order (note that $(a_1, a_2, a_3) + (a_3, a_2, a_1) = \pi$).

A polygon P is given by a sequence $a_1, a_2, \ldots, a_p$ ($p \geq 3$) of points, called *apices*, and edges (a_i, a_{i+1}), $i = 1, 2, \ldots, p$ (where $a_{p+1} = a_1$) so that no two line segments (a_i, a_{i+1}) and (a_j, a_{j+1}), $i \neq j$, intersect each other except at apices. Let $A(P)$ denote such a sequence of apices of a polygon P, where $A(P)$ may be used to denote the set of the apices in $A(P)$.

The *inner angle* $\theta(a_i)$ of an apex a_i is the angle (a_{i+1}, a_i, a_{i-1}) formed by two line segments (a_i, a_{i+1}), (a_{i-1}, a_i), and an apex a_i is called *convex* (respectively, *concave* and *flat*) if $\theta(a_i) < \pi$ (respectively, $\theta(a_i) > \pi$ and $\theta(a_i) = \pi$). A polygon P has no apex a with $\theta(a) = 0$ since no two adjacent edges on the boundary intersect each other. A polygon P is called *convex* if it has no concave apex.

A k-gon is a polygon with exactly k apices, some of which may be flat or concave. A *side* of a polygon is a maximal line segment in its boundary, i.e., a sequence of edges (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), ..., (a_{i+h-1}, a_{i+h}) such that a_i and a_{i+h} are non-flat apices and the other apices between them are flat. A k-gon has at most k sides.

A *straight-line drawing* of a graph $G = (V, E)$ in the plane is an embedding of G in the two dimensional space $\Re^2$ so that each vertex $v \in V$ is drawn as a point $\psi(v) \in \Re^2$ and each edge $(u, v) \in E$ is drawn as a straight-line segment $(\psi(u), \psi(v))$, where $\Re$ is the set of real numbers. Hence, a straight-line drawing of a graph $G = (V, E)$ is defined by a function $\psi : V \to \Re^2$.

A straight-line drawing ψ of a plane graph $G = (V, E, F)$ is called an *inner-convex drawing* (or simply a convex drawing) if every inner facial cycle is drawn as a convex polygon.

A convex drawing ψ of a plane graph $G = (V, E, F)$ is called a *strictly convex drawing* if it has no flat apex $\psi(v)$ for any vertex $v \in V$ with $d_G(v) \geq 3$. We say that a drawing ψ of a graph G is *extended* from a drawing ψ' of a subgraph G' of G if $\psi(v) = \psi'(v)$ for all $v \in V(G')$.

Let $G = (V, E, F)$ be a plane graph with an outer facial cycle f_o, and P be a $|V(f_o)|$-gon. A drawing ϕ of f_o on P is a bijection $\phi : V(f_o) \to A(P)$ so that the vertices in $V(f_o)$ appear along f_o in the same order as the corresponding apices in sequence $A(P)$.

Lemma 5 [5, 21] *Let $G = (V, E, F)$ be a biconnected plane graph. Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if, and only if, the following conditions* (i)-(iii) *hold:*

(i) *For each inner vertex v with $d_G(v) \geq 3$, there exist three paths disjoint except v, each connecting v and an outer vertex;*

(ii) *Every cycle of G which has no outer edge has at least three vertices v with $d_G(v) \geq 3$; and*

(iii) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. The graph $G - V(f_o)$ has no component H such that all the outer vertices adjacent to vertices in H*

are contained in a single path Q_i, and there is no inner edge (u, v) whose end vertices are contained in a single path Q_i. ∎

Every inner vertex of degree 2 must be drawn as a point subdividing a line segment in any convex drawing. Hence we can assume without loss of generality that a given a plane has no inner vertex of degree 2, since any convex drawing of the plane graph obtained by replacing each maximal path containing inner vertices v with $deg_G(v) = 2$ with a single edge gives a convex drawing of G by subdividing the replaced edges.

Then Lemma 5 can be restated as follows:

Lemma 6 [11] *Let $G = (V, E, F)$ be a biconnected plane graph with no inner vertices of degree 2. Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if, and only if, the following conditions* (a) *and* (b) *hold:*

(a) *G is internally triconnected.*

(b) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. Each Q_i is an archfree path in G.* ∎

5 Convex Drawings of Hierarchical Planar Graphs

In this section, we now present the main result of this paper. More specifically, we prove that every internally triconnected hierarchical plane graph with the outer face fixed with a convex polygon admits a convex drawing. First, however, we review basic terminology related to hierarchical planar graphs.

An edge with a tail u and a head v is denoted by (u, v). A *hierarchical graph* $H = (V, A, \lambda, k)$ consists of a directed graph (V, A), a positive integer k, and, for each vertex u, an integer $\lambda(u) \in 1, 2, \ldots, k$, with the property that if $(u, v) \in A$, then $\lambda(u) < \lambda(v)$. For $1 \leq i \leq k$, the ith *layer* L_i of G is the set $\{u \mid \lambda(u) = i\}$. The *span* of an edge (u, v) is $\lambda(v) - \lambda(u)$. An edge of span greater than one is *long*, and a hierarchical graph with no long edges is *proper*.

For each vertex v in H, denote $\{u \in V \mid (v, u) \in A\}$ by $V_H^+(v)$ and $\{u \in V \mid (u, v) \in A\}$ by $V_H^-(v)$. A vertex v is called a *source* (respectively *sink*) if $V_H^-(v) = \emptyset$ (respectively $V_H^+(v) = \emptyset$). For a non-sink vertex v, a vertex $w \in V_H^+(v)$ is called an *up-neighbor* of v (see Fig. 2). Further, w is called the *highest up-neighbor* if $\lambda(w) = \max\{\lambda(u) \mid u \in V_H^+(v)\}$. Similarly, for a non-source vertex v, a vertex $w \in V_H^-(v)$ is called a *down-neighbor* of v, and w is called the *lowest down-neighbor* if $\lambda(w) = \min\{\lambda(u) \mid u \in V_H^-(v)\}$.

A hierarchical graph is conventionally drawn with layer L_i on the horizontal line $y = i$, and edges as curves monotonic in y direction. If no pair of non-incident edges intersect in the drawing, then we say it is a *hierarchical planar* (*h-planar*) drawing. Note that a nonproper hierarchical graph can be transformed into a proper hierarchical graph by adding dummy vertices on long edges. It is easily shown that a nonproper hierarchical graph is h-planar if, and only if, the corresponding proper hierarchical graph is h-planar.

A *hierarchical planar embedding* of a proper hierarchical graph is defined by the ordering of vertices on each layer of the graph. Note that every such embedding has a unique external

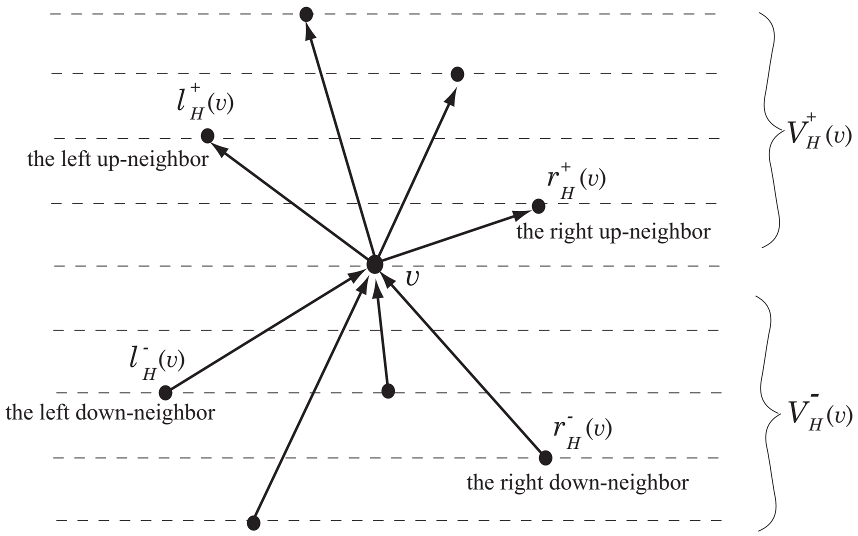


Figure 2: Definition of left-right relations in $V_H^-(v)$ and $V_H^+(v)$.

face. Also note that every proper h-planar graph admits a *straight-line hierarchical drawing*; that is, a drawing where edges are drawn as straight-line segments. However, for nonproper hierarchical graphs, the problem is not trivial, since no bends are allowed on long edges.

We call a plane embedded hierarchical graph a *hierarchical plane graph.* If a hierarchical plane graph has only one source s and one sink t, then we call it a *hierarchical-st plane graph.* Observe that a hierarchical-st plane graph is a connected graph, and its source s and sink t must lie on the bottom layer and the top layer, respectively. Figure 3(a) shows a hierarchical-st plane graph H.

The embedding of a hierarchical plane graph H determines, for every vertex v, a left-right relation among up-neighbors of v (see Fig. 2). The head w of the rightmost (respectively leftmost) edge outgoing from v is called the *right up-neighbor* (respectively the *left up-neighbor*) of v, and is denoted by $r_H^+(v)$ (respectively $\ell_H^+(v)$). The *right down-neighbor* $r_H^-(v)$ and the *left down-neighbor* $\ell_H^-(v)$ of v are defined analogously.

Hierarchical graphs are directed graphs and thus we can borrow much of the standard terminology of graph theory. The terms "path", "cycle", and "biconnectivity", when applied to a directed graph in this paper, refer to the underlying undirected graph.

To denote a cycle of a plane graph, we use the sequence of vertices on the cycle in clockwise direction. For a cycle or path $\mathcal{P} = (v_1, v_2, \dots, v_k)$, an edge between two nonconsecutive vertices in $\mathcal{P}$ is called a *chord* of $\mathcal{P}$. A cycle or path is called *chordless* if it has no chord. In hierarchical graphs, edges are directed from a lower layer to a higher layer. A path is called *monotonic* if the directions of the edges do not change along the path. In other words, a path is monotonic if the layer increases (or decreases) as we go along the path.

Note that from a vertex v, a monotonic and chordless path from v to a sink can be obtained by traversing the highest up-neighbors one after another. Similarly, a monotonic and chordless path from a source to v can be found by tracing the lowest down-neighbors from v.

For straight-line drawings of hierarchical-st plane graphs, the next result is shown.

Theorem 7 [7] *Let H be a triangulated hierarchical-st plane graph, and its outer facial cycle f_o be drawn as a convex polygon P such that, for each chord (u, z) of f_o, on each of the two paths of cycle $\mathcal{C}$ between u and z, there exists a vertex v which is drawn as a convex apex of polygon P. Then there exists a planar straight-line hierarchical drawing of H with external face P, and such a drawing can be constructed in linear time.* ▮

The theorem implies that every hierarchical-st plane graph H admits a straight-line hierarchical planar drawing, because we can easily augment a given biconnected hierarchical-st plane graph into a triangulated hierarchical-st plane graph H' by triangulating each non-triangle inner face. However, the drawing of H obtained by deleting added edges from the drawing of H' may not be a convex drawing.

In this paper, we prove the following result.

Theorem 8 *For every hierarchical-st plane graph H which is internally triconnected, any convex polygon P for the outer facial cycle f_o can be extended to a convex drawing of H. Such a drawing can be computed in $O(n^2)$ time.* ▮

We first observe a key lemma to derive the theorem.

Lemma 9 *Let H be a hierarchical-st plane graph that satisfies conditions* (a) *and* (b) *in Lemma 6. For any monotonic inner path Q from a vertex u to a vertex v, $R(L(Q))$ is a monotonic archfree path.*

Proof: By Corollary 4, $R(L(Q))$ is an archfree path. It suffices to show that operation L of constructing path $L(Q)$ from a monotonic inner path Q preserves the monotonicity of Q (operation R can be treated symmetrically). For this, we consider an inner facial cycle f on the left side of Q, where a_f and b_f are the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t (see Fig. 1), and prove that the subpath P_f from a_f to b_f obtained by traversing f in an anticlockwise direction is monotonic.

If P_f is not monotonic, then there are three vertices u_1, u_2 and u_3 which appear in this order on P_f and whose y-coordinates $y(u_1)$, $y(u_2)$ and $y(u_3)$ satisfy $y(u_1) > y(u_2) < y(u_3)$. This, however, implies that u_2 is another source, since H is a hierarchical-st plane graph, which contradicts the situation where H has no other source than s. This proves the lemma. ▮

We prove Theorem 8 by induction on the number of inner faces. It suffices to show the case where there is no inner vertex of degree 2. The theorem holds if H has only one inner face. Consider an internally triconnected hierarchical-st plane graph H, and assume that the theorem holds for any triconnected hierarchical-st plane graph which has a smaller number of inner faces than H. We distinguish two cases:

Case 1: There is an outer vertex v ($\neq s, t$) of degree 2. Let v' and v'' be the up- and down-neighbours of v. If v is a flat apex in P, then we see that H with P has a convex drawing

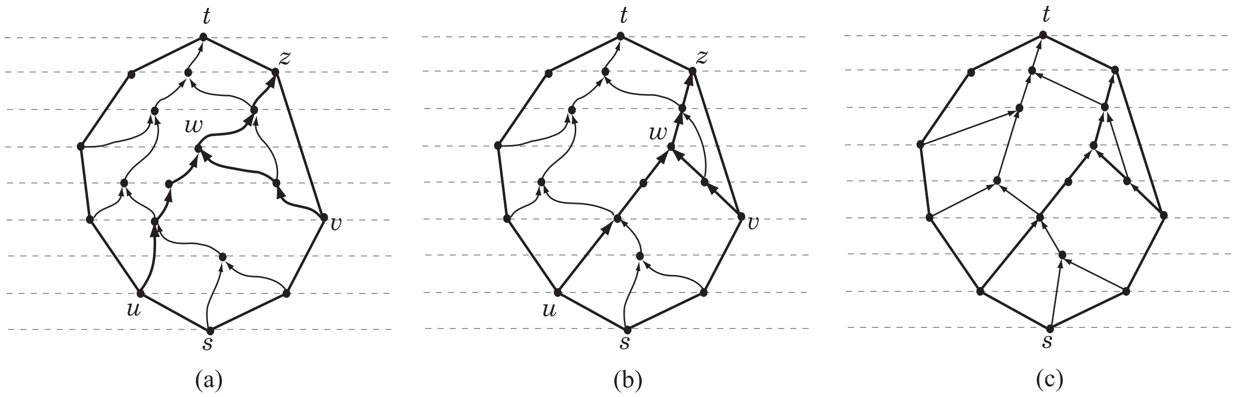


Figure 3: (a) A hierarchical-st plane graph H with a convex polygon P, where three monotonic paths with thick lines are archfree paths. (b) Three hierarchical-st plane graphs obtained by the archfree paths. (c) A convex drawing of H in (a).

D since replacing two edges (v'', v) and (v, v') with a new edge (v'', v') deleting vertex v results in a hierarchical-st plane graph H' with a new convex boundary P', which admits a convex drawing D' by the induction hypothesis (D can be obtained from D' by regaining point v on P').

We now assume that v is a convex apex in P. We construct a hierarchical-st plane graph H' from H by removing v and adding a new edge (v'', v') if $v'' \notin V_H^-(v')$. Let P' be the polygon obtained by replacing the segments (v'', v) and (v, v') in P with a new segment (v'', v'), which will form a new side of P'. We claim that H' with boundary P' satisfies conditions (a) and (b) in Lemma 6.

Consider the graph H'' obtained from H by adding a new edge (v'', v') if $v'' \notin V_H^-(v')$ ($H'' = H$ if $v'' \in V_H^-(v')$). We easily see that H'' with P still satisfies conditions in (a) and (b) in Lemma 6, and that $H'' - v$ remains to be internally triconnected due to the existence of edge (v'', v'). We show that path $Q = (v'', v')$, which corresponds to the new side of P', is an archfree path in H'.

Assume that Q is not an archfree path, i.e., there is a facial cycle f that arches Q. Thus, f contains two vertices v'' and v', but not edge (v'', v'). This, however, means that removal of v'' and v' from H results in at least three components, which contradicts to the internal triconnectivity of H. Hence, this proves the claim.

By the induction hypothesis, H' with P' admits a convex drawing D'. It is not difficult to see that D' can be modified into a convex drawing of H with P by adding segments (v'', v) and (v, v') and deleting segment (v'', v') if $v'' \notin V_H^-(v')$.

Case 2: Every outer vertex $v(\neq s, t)$ is of at least 3 degrees. There must be an outer vertex v $(\neq s, t)$ which is a convex apex in P; without loss of generality, v is on the rightmost path from s to t.

We consider the leftmost monotonic path Q_v from v (i.e., a path obtained by traversing the left up-neighbors), and let w be the first vertex in Q_v that has at least one down-neighbor, and

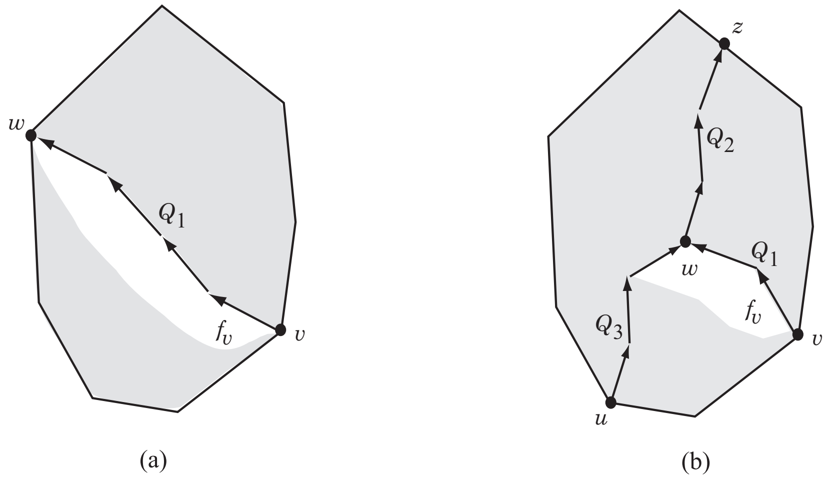


Figure 4: (a) A monotonic archfree path between two outer vertices v and w. (b) Constructing three monotonic archfree paths starting from v, w and u, respectively.

Q_1 be the subpath from v to w along Q_v. For example, see vertices v and w in Fig. 3(a) and (b). Hence, Q_1 is a subpath of an inner facial cycle f_v which contains v, and is an archfree path by Lemma 1.

Here we consider the two subcases: (i) w is an outer vertex (see Fig. 4(a)) and (ii) w is an inner vertex (see Fig. 4(b)).

Case 2(i): The outer facial cycle f_o together with path Q_1 splits the interior of P into two regions, say R_1 and R_2. We split H into two hierarchical-st plane graphs H_1 and H_2 such that H_1 and H_2 share Q_1, and each H_i, $i = 1, 2$, contains all faces in R_i. We draw Q_1 as a straight line between two points v and w, by which the x-coordinates of all points on Q_i are uniquely determined. Let P_1 and P_2 be the two convex polygons for the boundaries of H_1 and H_2, respectively. By choosing the vertex with the (respectively, lowest) highest position on P_i as s (respectively, t) of graph H_i, we can regard H_i as a hierarchical-st plane graph.

We claim that each H_i with boundary P_i still satisfies the two conditions in Lemma 6. Since Q_1 is an archfree path and any side of P is an archfree path, H_i with P_i satisfies condition (b) in Lemma 5. It can be immediately seen that condition (a) in Lemma 6 still holds for each H_i with P_i, since it is a subgraph of H which contains all vertices and edges enclosed by a cycle. This proves the claim.

Therefore each H_i admits a convex drawing D_i by the induction hypothesis, implying that a convex drawing of H with P can be obtained by placing D_1 and D_2 in the corresponding region inside P.

Case 2(ii): We choose a monotonic path Q_z that starts from w and ends up with an outer vertex z. We also choose a monotonic path Q_u to w, from an outer vertex u. By Lemma 9, H has monotonic archfree paths Q_2 from w to z and Q_3 from u to w. By placing w at a point in the interior of P, we draw each path Q_i, $i = 1, 2, 3$ as a straight line between its end points, by which the x-coordinates of all points on Q_i are uniquely determined (see Fig. 3(b)). Note that v, z and w are not on a single straight line since v is a convex apex in P. Therefore, w can be placed at a point such that the polygons obtained from P by the straight lines are all convex. By choosing the vertex with the (respectively, lowest) highest position on P_i as s (respectively, t) of graph H_i, we can regard H_i as a hierarchical-st plane graph.

Since each Q_i is an archfree path and each of the resulting subgraphs H_i is internally triconnected, we see that H_i with P_i satisfies the condition in Lemma 6. We then have a convex drawing D_i of H_i with P_i by the inductive hypothesis. We see that a convex drawing of G with P can be obtained by combining D_1, D_2 and D_3 in the corresponding region inside P (see Fig. 3(c)).

This completes an induction, proving the existence of a desired convex drawing of H in Theorem 8.

It is not difficult to see that an $O(n^2)$ time algorithm for drawing a convex drawing of H can be obtained from the above proof. ∎

6 Convex Drawings of Clustered Plane Graphs

In this section, we present our main results on convex drawings of clustered planar graphs. First, we define terminology related to clustered planar graphs.

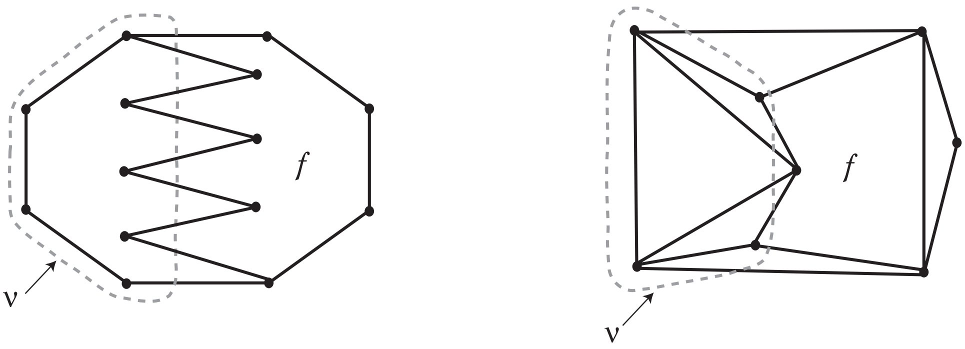
A *clustered graph* $C = (G, T)$ consists of an undirected graph $G = (V, E)$ and a rooted tree $T = (\mathcal{V}, \mathcal{A})$, such that the leaves of T are exactly the vertices of G. We call G the *underlying graph* and T the *inclusion tree* of C. To distinguish vertices in T from those in G, vertices in T are called *nodes*. Each node ν of T represents a *cluster* $V(\nu)$, a subset of the vertices of G that are leaves of the subtree rooted at ν. Then for the root ν of T, $V(\nu) = V$. A node ν in T (or its cluster $V(\nu)$) is called *nontrivial* if ν is neither the root or a leaf of T. Let $G(\nu)$ denote the subgraph $G[V(\nu)]$ of G induced by $V(\nu)$. Note that the tree T represents a laminar family of subsets of the vertices in G. If a node ν' is a descendant of a node ν in the tree T, then we say that the cluster of ν' is a *sub-cluster* of ν.

A clustered graph $C = (G, T)$ is a *connected clustered graph* if each cluster $V(\nu)$ induces a connected subgraph $G(\nu)$ of G. A clustered graph $C = (G, T)$ is *completely connected* if, for every non-root node ν of T, both subgraphs $G(\nu)$ and $G[V - V(\nu)]$ are connected [6]. Note that G is biconnected if $C = (G, T)$ is completely connected, since $G[V - \{v\}]$ is connected for every leaf cluster $\{v\}$ in T.

For example, among clustered graphs C_1 and C_2 in Fig. 5 and C_3 in Fig. 6(a), C_2 and C_3 are connected clustered graphs, and only C_3 is completely connected.

In a *drawing* of a clustered graph $C = (G, T)$, graph G is drawn as points and curves as usual. For each node ν of T, the cluster is drawn as a simple closed region $R(\nu)$ enclosed by a simple closed curve such that:

- the drawing of $G(\nu)$ is completely contained in the interior of $R(\nu)$;
- the regions for all sub-clusters of ν are completely contained in the interior of $R(\nu)$;
- the regions for all other clusters are completely contained in the exterior of $R(\nu)$.



(a) C_1 (b) C_2

Figure 5: (a) A c-planar clustered graph C_1, which is not a connected clustered graph. (b) A c-planar and connected clustered graph C_2, which is not completely connected, where only nontrivial clusters are enclosed by grey curves in (a) and (b).

We say that the drawing of edge e and region R have an *edge-region crossing* if the drawing of e crosses the boundary of R more than once. A drawing of a clustered graph is *compound planar* (*c-planar*, for short) if there are no edge crossings or edge-region crossings. If a clustered graph C has a c-planar drawing, then we say that it is *c-planar*. Figure 5(a) and (b) show examples of c-planar drawings of clustered graphs.

The characterization of c-planar clustered graphs is known only for connected clustered graphs.

Theorem 10 [8] *A connected clustered graph $C = (G, T)$ is c-planar if and only if graph G is planar and there exists a planar drawing of G, such that for each node ν of T, all the vertices and edges of $G[V - V(\nu)]$ are in the external face of the drawing of $G(\nu)$.*

It is shown that a completely connected clustered graph $C = (G, T)$ is c-planar if and only if the underlying graph G is planar [6, 13].

A c-planar drawing of a clustered graph is called *a planar straight-line convex cluster drawing* if edges are drawn as straight-line segments and clusters are drawn as convex polygons. The drawings in Fig. 5(a) and (b) are planar straight-line convex cluster drawing.

Theorem 11 [7] *Let $C = (G, T)$ be a c-planar clustered graph with n vertices. A planar straight-line convex cluster drawing of C in which each cluster is a convex hull of points in the cluster can be constructed in $O(n^2)$ time.* ∎

In this paper, we define a *fully convex drawing* as a planar straight-line convex cluster drawing such that facial cycles are also drawn as convex polygons. Among the four c-planar drawings in Fig. 5 and Fig. 6, only the drawing in Fig. 6(b) is fully convex. In what follows, we only consider c-planar and connected clustered graphs, and present a characterization of these clustered graphs that have fully convex drawings.

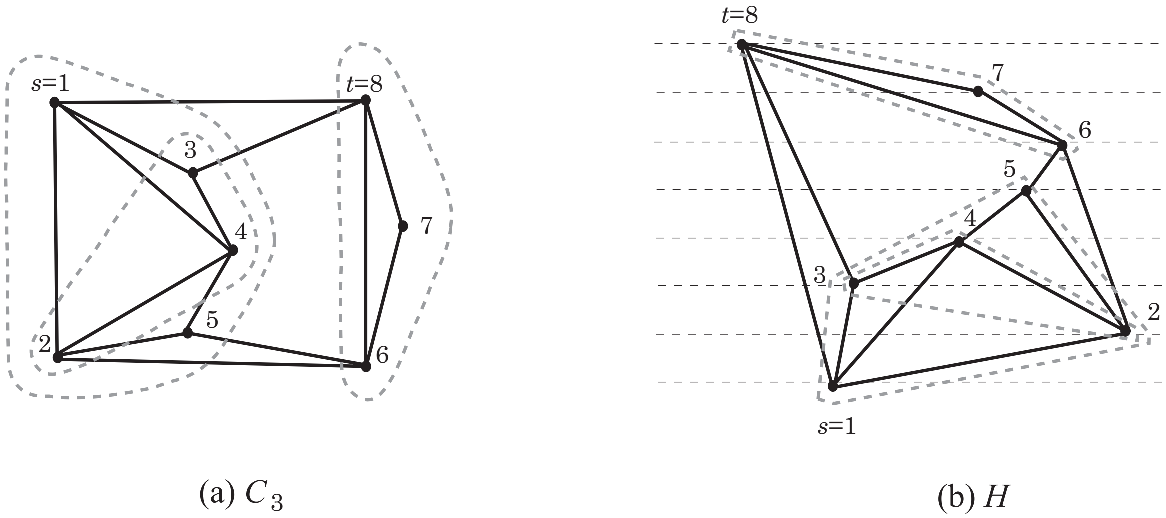


Figure 6: (a) A c-planar and completely connected clustered graph C_3 together with a *c-st* numbering of vertices, where non-leaf and non-root clusters are enclosed by grey curves. (b) A convex drawing of the hierarchical-st plane graph H obtained by the *c-st* numbering in (a).

We define a *region-face crossing* as follows. In a *c*-planar drawing of a clustered graph $C = (G, T)$, the region R of a cluster and a cycle f of G *cross* each other if the boundary of R cross the drawing of more than two edges in f (i.e., removing vertices in R from the facial cycle f leaves more than one subpath).

We now prove the following main result.

Theorem 12 *Let $C = (G, T)$ be a c-planar and connected clustered graph that has a c-planar drawing ψ such that the outer facial cycle does not cross any cluster region, and G be internally triconnected. Then*

(i) *There exists a fully convex drawing of C if and only if C is completely connected.*

(ii) *A fully convex drawing of C (if any) can be constructed from drawing ψ in $O(n^2)$ time, where each cluster is a convex hull of points in the cluster and n is the number of vertices in G.* ∎

Necessity of Theorem 12(i): We prove the necessity in Theorem 12(i) via several lemmas.

Lemma 13 *If a c-planar and connected clustered graph C admits a planar straight-line convex cluster drawing such that the drawing of G is inner-convex, then there is no region-face crossing between regions for clusters and inner facial cycles.*

Proof: We see that if there is a region-face crossing between the region $R(\nu)$ of a cluster ν and an inner facial cycle f in G, then it is impossible to draw both $R(\nu)$ and f as convex polygons in one drawing (note that $G(\nu)$ is connected). Therefore, if C admits a fully convex drawing, then there is no region-face crossing between regions for clusters and inner facial cycles. ∎

Lemma 14 *Suppose that a connected clustered graph $C = (G, T)$ has a c-planar drawing which has no region-face crossing between regions for clusters and inner facial cycles. Then for every non-root node ν of T, every component of the subgraph $G[V - V(\nu)]$ contains an outer vertex.*

PROOF: Let ψ be such a c-planar drawing of C. In ψ, G is a plane graph (V, E, F), and we denote the outer facial cycle of G by f_o. Let ν be a non-root node of T. To derive a contradiction, assume that $G[V - V(\nu)]$ has a component G_A that contains no vertex in f_o. In ψ, G_A is a plane graph, and we denote its boundary by B_A and the set of inner faces of G_A by F_A. Since $R(\nu)$ is a simple closed region, G_A is not completely surrounded by $R(\nu)$.

Consider a facial cycle $f \in F - F_A$ that shares a vertex with B_A. Since G_A is an inclusionwise maximal component in $G[V - V(\nu)]$, a vertex in f is contained in $R(\nu)$. By assumption, there is no region-face crossing in ψ, and hence f consists of two subpaths f' and f'', one is the intersection with B_1 and the other with $R(\nu)$.

We now consider all such facial cycle $f_1, f_2, \ldots, f_p \in F - F_A$ that share vertices with B_A. Then we see that the subpaths $f_1'', f_2'', \ldots, f_p''$ of them form a closed curve, which implies that G_A is enclosed by $R(\nu)$, a contradiction. This proves the lemma. ∎

Lemma 15 *Suppose that a connected clustered graph $C = (G, T)$ has a c-planar drawing which has no region-face crossing. Then C is completely connected.*

PROOF: Let ψ be such a c-planar drawing of C. In ψ, G is a plane graph (V, E, F), and we denote the outer facial cycle of G by f_o. Since C is a connected clustered graph, it suffices to show that, for every non-root node ν of T, subgraph $G[V - V(\nu)]$ is connected.

Let ν be a non-root node of T. To derive a contradiction, assume that $G[V - V(\nu)]$ is not connected, and G_A and G_B be two components in $G[V - V(\nu)]$. By Lemma 14, both G_A and G_B contain outer vertices in f_o. This, however, contradicts that region $R(\nu)$ and f_o has no region-face crossing. ∎

We are ready to prove the necessity in Theorem 12(i).

Let ψ be a fully convex drawing of C. Clearly ψ is a convex drawing on G, and G must be internally triconnected by Lemma 6(i). We show that C is completely connected. By Lemma 13, there is no region-face crossing between regions for clusters and inner facial cycles. By assumption on C, there is no region-face crossing between regions for clusters and the outer facial cycle, either. By Lemma 15, C is completely connected. This proves the necessity in Theorem 12(i).

Sufficiency of Theorem 12(i):

To prove the sufficiency of Theorem 12(i), we follow an approach due to Eades et al. [7] which was used to derive Theorem 11. They used the *c-st numbering* of vertices in G, which is an extension of *st* numberings. Suppose that (s, t) is an edge of a biconnected graph G with n vertices. In an *st* numbering, the vertices of G are numbered from 1 to n so that vertex s

receives number 1, vertex t receives number n, and any vertex except s and t is adjacent both to a lower-numbered vertex and a higher-numbered vertex.

A *c-st numbering* for a clustered graph $C = (G, T)$ is an *st* numbering of the vertices in G such that the vertices that belong to the same cluster are numbered consecutively. This numbering gives us a layer assignment of the vertices of G. Hence, a c-planar clustered graph C is transformed to a hierarchical-st plane graph H, and each cluster has consecutive layers.

Based on this property, Eades et al. [7] proved that a planar straight-line convex cluster drawing can be constructed from the straight-line hierarchical drawing. In this method, a given underlying graph is augmented to a triangulated graph to ensure the existence of *c-st* numberings in the clustered graph, and then all added edges are removed from a drawing of the triangulated graph to obtain a desired straight-line drawing of C.

However, the resulting drawing may not be convex. Since we aim to construct a convex drawing of G, we cannot use triangulation to find *c-st* numberings. Fortunately, complete connectedness ensures the existence of *c-st* numberings instead.

Lemma 16 *Suppose that $C = (G, T)$ is a c-planar and completely connected clustered graph. Then C has a* c-st *numbering such that s and t can be chosen as adjacent vertices in the outer facial cycle, and such a* c-st *numbering can be computed in linear time.*

PROOF: We give only a sketch. It is known [7] that the lemma holds if, in addition, G is triangulated, i.e., all facial cycles are triangles (note that every connected clustered graph C is completely connected if its underlying graph is triangulated). The correctness of the proof in [7] relies on only complete connectedness of $C = (G, T)$, not on triangulation.

Hence the argument can be carried over to the case of completely connected clustered graphs, and the lemma holds. ∎

We are ready to prove the sufficiency in Theorem 12(i) and Theorem 12(ii).

Let $C = (G, T)$ be a c-planar and completely connected clustered graph, and ψ be a c-planar drawing such that the outer facial cycle f_o does not cross any cluster region.

First, we compute a *c-st* numbering of $C = (G, T)$ such that s and t are chosen as adjacent vertices in the outer facial cycle f_o. This can be done in linear time. For example, Figure 6(a) illustrates a c-planar and completely connected clustered graph C_3 and its *c-st* numbering.

Next, we regard the plane graph G as a hierarchical-st plane graph by assigning the layer of each vertex with its *c-st* number. Figure 6(b) illustrates the hierarchical-st plane graph H obtained from clustered graph C_3 by its *c-st* numbering.

Then, we compute a convex drawing ψ^* of H. Such a drawing ψ^* can be obtained in $O(n^2)$ time by Theorem 8.

Finally, for each cluster $V(\nu)$, we let the convex hull of the vertices in $V(\nu)$ in ψ^* be the region $R(\nu)$ of the cluster. A convex hull of a given simple polygon with m apices can be constructed in $O(m)$ time [19]. Then the total time of computing all convex hulls in $C = (G, T)$ is $O(n^2)$.

Overall, the entire running time of the above algorithm is $O(n^2)$.

To prove the sufficiency in Theorem 12(i) and Theorem 12(ii), we only need to prove that the resulting drawing $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is a fully convex drawing.

By Theorem 8, ψ^* is a convex drawing. We show that $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is c-planar.

Since $R(\nu)$ is a convex hull of the vertices in $V(\nu)$ in ψ^* and edges are drawn as line-segments, it contains the drawing of $G(\nu)$ (and hence the regions of all sub-clusters of ν). The *c-st* numbers within a cluster are consecutive, thus if the convex hulls of two clusters overlap in y-coordinate, then one is a sub-cluster of the other. This keeps the disjoint clusters apart; For two clusters ν and ν' with $V(\nu) \cap V(\nu') = \emptyset$, the convex hulls of ν and ν' are disjoint.

We finally see that there are no edge crossings and no edge-region crossings. Since ψ^* is a plane drawing of G, it has no edge crossings. Assume that the drawing of an edge e intersects region $R(\nu)$ of a cluster ν twice (i.e., the endvertices of e are outside $R(\nu)$). This, however, contradicts that ψ^* is a plane drawing of G, since $G(\nu)$ is connected and the line-segment for e must create an edge crossing with some edge in $G(\nu)$. Therefore, $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is a fully convex drawing of C.

This completes the proof of Theorem 12.

From the argument for establishing Theorem 12, we easily derive the following corollary.

Corollary 17 *Let $C = (G, T)$ be a c-planar and connected clustered graph, and G be internally triconnected. Then*

(i) *There exists a planar straight-line convex cluster drawing of C such that the drawing of G is inner-convex if and only if, for every non-root node ν of T, every component of the subgraph $G[V - V(\nu)]$ contains an outer vertex.*

(ii) *Such a drawing of C in* (i) *(if any) can be constructed from drawing ψ in $O(n^2)$ time, where each cluster is a convex hull of points in the cluster and n is the number of vertices in G.*

PROOF: **(i):** By Lemmas 13 and 14, we see the necessity of this corollary. To show the sufficiency, we augment the clustered graph C as follows. Let V_o be the set of outer vertices in G. We create a new vertex s^* in the exterior of G, join G and s^* with new edges (s^*, v), $v \in V_o$ to obtain a new plane graph G^*, where its boundary f_o^* is a triangle, and add new clusters $\nu^* = V \cup \{s^*\}$, $\nu_v = \{v\}$, $v \in V_o$ to T.

We see that the resulting clustered graph $C^* = (G^*, T^*)$ remains c-planar and connected. Also we easily see that C^* has no region-face crossing between regions for clusters and facial cycles. Hence by applying Theorem 12 to C^*, there is a fully convex drawing ψ of C^*. After removing the drawing of s^* and edges (s^*, v), $v \in V_o$ from ψ, we obtain a desired drawing of C.

(ii): Immediate from (i) and Theorem 12(ii). ∎

7 Conclusion

In this paper, we extend the previous known results on straight-line drawings of hierarchical planar graphs and clustered planar graphs into convex drawings.

It is proved that every internally triconnected hierarchical plane graph with the outer facial cycle drawn as a convex polygon admits a convex drawing. We then extend our results to convex representations of clustered planar graphs. We have proved that every internally triconnected clustered plane graph with completely connected clustering structure admits a convex drawing. We also present an algorithm to construct convex drawings of hierarchical planar graphs and clustered planar graphs.

It would be interesting to apply the notion of archfree paths/trees to convex drawings or inner convex drawings of other types of graphs. Further, adding more constraints such as angles of vertices to convex representations may be an interesting research direction in the future.

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