A Simple Extreme Subsets Algorithm for Submodular and Posi-modular Set Functions

Hiroshi Nagamochi*

Dept. of Applied Mathematics and Physics
Graduate School of Informatics
Kyoto University
nag@amp.i.kyoto-u.ac.jp

Abstract: Let f be a set function on a finite set V. A pair $\{u, v\} \subseteq V$ is called *flat* in f if $f(X) \geq \min_{x \in X} f(\{x\})$ holds for all subsets $X \subseteq V$ with $|X \cap \{u, v\}| = 1$. A subset $X \subseteq V$ is called an extreme subset of f if $f(Y) > f(X)$ holds for all nonempty and proper subsets Y of X. In this paper, we first define a minimum degree ordering (MD ordering) of V, and show that the last two elements in an MD ordering give a flat pair of f if f is symmetric and crossing submodular. Based on this, we then give a simple $O(|V|^3)$-oracle time algorithm for computing all extreme subsets of a set function g if g is symmetric and crossing submodular or intersecting submodular and posi-modular.

Keywords: Edge-connectivity; Extreme subsets; Graph algorithms; Maximum adjacency orderings; Minimum cuts; Minimum degree orderings; Submodular functions; Posi-modular functions

1 Introduction

Let V be a finite set, where we denote $n = |V|$. A singleton set $\{v\}$ is called *trivial* and may be written as v. The union of a set X and a singleton $\{v\}$ may be written as $X + v$. A subset $X \subseteq V$ *separates* two elements $u, v \in V$ if $|X \cap \{u, v\}| = 1$. For two subsets $X, Y \subseteq V$, we say that X and Y *intersect* each other if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$ and $Y - X \neq \emptyset$ hold, and say that X and Y *cross* each other if, in addition, $V - (X \cup Y) \neq \emptyset$ holds. A family $\mathcal{X} \subseteq 2^V$ of subsets of V is called *laminar* if no two subsets in $\mathcal{X}$ intersect each other.

Let $\Re$ and $\Re_+$ denote the sets of reals and nonnegative reals, respectively. A *set function* f on V is a function $f : 2^V \to \Re$. A set function f is called *fully* (resp., *intersecting, crossing*) *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \tag{1}$$

holds for every (resp., intersecting, crossing) pair of sets $X, Y \subseteq V$ [1, 3]. A set function f is called *fully* (resp., *intersecting, crossing*) *posi-modular* if

$$f(X) + f(Y) \geq f(X - Y) + f(Y - X) \tag{2}$$

*Technical report 2007-006, February 6, 2007.

holds for every (resp., intersecting, crossing) pair of sets $X, Y \subseteq V$ [8]. A set function f is called *symmetric* if

$$f(X) = f(V-X) \text{ for all } X \subseteq V. \tag{3}$$

Every symmetric and fully (resp., intersecting, crossing) submodular set function is fully (resp., intersecting, crossing) posi-modular.

An important example of symmetric and fully submodular functions is the cut functions of hypergraphs. The cut function $d : 2^V \mapsto \Re^+$ of a hypergraph $G = (V, E)$ ($E \subseteq 2^V$) with an edge weight function $w : E \to \Re_+$ is defined by

$$d(X; G) = \sum\{w(e) \mid e \in E,\ e \cap X \neq \emptyset,\ e-X \neq \emptyset\} \tag{4}$$

where we let $d(\emptyset; G) = d(V; G) = 0$. We see that d is symmetric and fully submodular.

A nonempty proper subset X of V is called an *extreme subset* of f if

$$f(Y) > f(X) \text{ for all nonempty proper subsets } Y \text{ of } X.$$

We denote by $\mathcal{X}(f)$ the family of all extreme subsets of a set function f. Any trivial set $\{v\}$, $v \in V$ is an extreme subset. By definition, any nonempty subset X contains an extreme subset X' with $f(X') \leq f(X)$. In particular, $\mathcal{X}(f)$ contains a *minimizer* of f, i.e., a subset X with $f(X) = \min_{Y \in 2^V - \{\emptyset, V\}} f(Y)$.

Lemma 1 *Let f be a set function on a finite set V. If f is intersecting posi-modular or symmetric and crossing submodular, then the family $\mathcal{X}(f)$ of extreme subsets of f is laminar.*

Proof: Recall that a symmetric and crossing submodular set function is crossing posi-modular. Let $X, Y \in \mathcal{X}(f)$ be two intersecting extreme subsets. By definition we would have $f(X) + f(Y) < f(X-Y) + f(Y-X)$, which is a contradiction if f is intersecting posi-modular or X and Y cross each other. Hence we only have to consider the case where f is symmetric and X and Y do not cross each other. In this case, we have $f(X-Y) = f(V-Y) = f(Y)$ and $f(Y-X) = f(V-X) = f(X)$, which again contradicts the above inequality. ∎

Extreme subsets are originally introduced by Watanabe and Nakamura [12] for the cut functions of graphs to solve the edge-connectivity augmentation problem, and extreme subsets of graphs are currently an important tool to solve many graph connectivity problems such as the source location problem. It is known [6] that all extreme subsets in a graph G with n vertices and m edges can be computed in $O(mn + n^2 \log n)$ time. Figure 1 illustrates all extreme subsets in $\mathcal{X}(d)$ for the cut function d of an edge-weighted graph $G = (V, E)$.

For a set function f on a set V, let T_f denote the time to evaluate the function value $f(X)$ of a given subset $X \subseteq V$. In this paper, we prove the next result, which also solves the minimization of f since $\mathcal{X}(f)$ contains a minimizer of f.

Theorem 2 *Let f be a set function on V with $n = |V| \geq 2$. If f is symmetric and crossing submodular or intersecting submodular and posi-modular, then the family $\mathcal{X}(f)$ of extreme subsets of f can be found in $O(n^3 T_f)$ time.* ∎

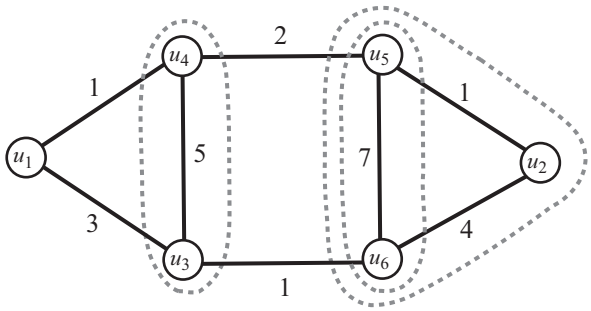

Figure 1: Extreme subsets in $\mathcal{X}(d)$ for the cut function d of an edge-weighted graph $G = (V, E)$, where the numbers beside edges indicate their weights and each nontrivial extreme subset $X \in \mathcal{X}(d)$ is depicted by a dotted closed curve.

2 Minimum Degree Orderings

To show Theorem 2, this section will introduce a new ordering of V for set functions f. Before showing this, we first review a related ordering, called a maximum adjacency ordering. A pair of elements $u, v \in V$ is called a *pendent pair* of f if

$$f(X) \geq \min\{f(u), f(v)\}$$

holds for all subsets X that separate u and v. Given a set function f on V with $n = |V| \geq 2$, an ordering $\sigma = (v_1, v_2, \ldots, v_n)$ is a *maximum adjacency ordering* (MA ordering, for short) of V if it satisfies

$$f(v_i) - f(V_{i-1} + v_i) \geq f(v_j) - f(V_{i-1} + v_j), \quad 1 \leq i \leq j \leq n, \tag{5}$$

where $V_0 = \emptyset$ and $V_i = \{v_1, v_2, \ldots, v_i\}$ $(1 \leq i \leq n-1)$.

By extending the results on the minimum cut algorithms in graphs due to Nagamochi and Ibaraki [9] and Stoer and Wagner [11], Queyranne [10] obtained the following result.

Theorem 3 [10] *For a given symmetric and crossing submodular function f on V with $n = |V| \geq 2$, let $\sigma = (v_1, v_2, \ldots, v_n)$, be an MA ordering of V. Then the last two elements v_{n-1} and v_n give a pendent pair.* ∎

Based on this, the following results are known.

Theorem 4 [10] *For a given symmetric and crossing submodular set function f on V with $n = |V| \geq 2$, a set $X \in 2^V - \{\emptyset, V\}$ that minimizes f can be found in $O(n^3 T_f)$ time.* ∎

Theorem 5 [8] *For a given intersecting submodular and posi-modular set function f on V with $n = |V| \geq 2$, a set $X \in 2^V - \{\emptyset, V\}$ that minimizes f can be found in $O(n^3 T_f)$ time.* ∎

In this paper, we introduce a new pair and a new ordering of V for set functions f. We call a pair of elements $u, v \in V$ a *flat pair* of f if $f(X) \geq \min_{x \in X} f(x)$ holds for all subsets X that separate u and v.

Given a set function f on V with $n = |V| \geq 2$, we call an ordering $\pi = (v_1, v_2, \ldots, v_n)$ a *minimum degree ordering* (MD ordering, for short) of V if it satisfies

$$f(v_i) + f(V_{i-1} + v_i) \leq f(v_j) + f(V_{i-1} + v_j), \quad 1 \leq i \leq j \leq n, \tag{6}$$

where $V_0 = \emptyset$ and $V_i = \{v_1, v_2, \ldots, v_i\}$ $(1 \leq i \leq n-1)$. It is not difficult to see that, after choosing V_{i-1}, the next element v_i can be chosen from $V - V_i$ by evaluating $f(v) + f(V_{i-1} + v)$ for all $v \in V - V_{i-1}$ and that an MD ordering can be found in $O(n^2 T_f)$ time.

We here consider the time complexity for computing an MD ordering of the cut function d in (4). Let $G = (V, E)$ be an edge-weighted hypergraph with an edge weight function $w : E \to \Re_+$. For a subset $X \subseteq V$, the graph $G\langle X \rangle$ induced by X is defined to be an edge-weighted hypergraph $(X, E_X \cup E_{\overline{X}})$ with an edge weight function $w_X : E \to \Re_+$ such that

$$\begin{aligned} E_X &= \{e \in E \mid e - X \neq \emptyset, |e \cap X| \geq 2\}, \\ E_{\overline{X}} &= \{e \in E \mid e \cap X = \emptyset\}, \\ w_X(e) &= \begin{cases} w(e)/2 & \text{if } e \in E_X \\ w(e) & \text{if } e \in E_{\overline{X}}. \end{cases} \end{aligned}$$

Note that if G contains only graph edges e (i.e., $|e| = 1$) then $E_X = \emptyset$.

Lemma 6 *For an edge-weighted hypergraph $G = (V, E)$, an ordering $\pi = (v_1, v_2, \ldots, v_n)$ such that*

$$d(v_i; G\langle V - V_{i-1} \rangle) = \min\{d(v; G\langle V - V_{i-1} \rangle) \mid v \in V - V_{i-1}\}, \ i = 0, 1, \ldots, n \tag{7}$$

is an MD ordering of the cut function d. An MD ordering π of d can be found in $O(m + n \log n)$ time and $O(m + n)$ space, where $n = |V|$ and $m = \sum_{e \in E} |e|$.

PROOF: Note that, for each $x \in V - V_{i-1}$, it holds

$$\begin{aligned} & d(x; G) + d(V_{i-1} + x; G) - d(V_{i-1}; G) \\ &= 2\sum\{w(e) \mid x \in e \in E, e \subseteq V - V_{i-1}\} \\ &\quad + \sum\{w(e) \mid x \in e \in E, e \cap V_{i-1} \neq \emptyset \neq e \cap (V - V_{i-1} - x)\} \\ &= 2\sum\{w(e) \mid e \in x \in e \in E, e \cap (V - V_{i-1}) = \emptyset\} \\ &\quad + \sum\{w(e) \mid x \in e \in E, e \cap V_{i-1} \neq \emptyset, |e \cap (V - V_{i-1})| \geq 2\} \\ &= 2d(x; G\langle V - V_{i-1} \rangle). \end{aligned}$$

Hence any ordering $\pi = (v_1, v_2, \ldots, v_n)$ in the statement of the lemma is an MD ordering of d. We consider the time complexity. A hypergraph $G = (V, E)$ can be stored as a set of adjacency lists $Adj(u)$, $u \in V$ in $O(m + n)$ space, by which we can find the set $\Gamma_G(u)$ of all vertices adjacent to u in $O(|\Gamma_G(u)|)$ time. To obtain an MD ordering of d, we repeat choosing a vertex u with the minimum degree $d(v; G\langle V - V_{i-1} \rangle)$ in the current graph $G\langle V - V_{i-1} \rangle$ and removing the vertex $v_i = u$ from the graph to obtain $G\langle V - V_i \rangle$. We maintain a data structure Q that contains all vertices $u \in V - V_i$ along with the current degree $d(u; G\langle V - V_{i-1} \rangle)$, $u \in V - V_i$.

After choosing v_i, we update the degree $d(u; G\langle V - V_{i-1}\rangle)$ of each vertex $u \in \Gamma_G(v_i) - V_i$ by decreasing by

$$\sum\{\frac{w(e)}{2} \mid u \in e \in E, e \cap V_i = \{v_i\}\} + \sum\{\frac{w(e)}{2} \mid v_i \in e \in E, e - V_i = \{u\}\}.$$

We then choose a vertex u^* with the minimum degree $d(u^*; G\langle V - V_i\rangle)$ from Q, and then delete u^* from Q. It is not difficult to see that the total number of updating degrees is $O(m)$, and the number of extracting a vertex with the minimum degree from Q is n. By using data structure of Fibonacci heap [2] to maintain elements in Q, the above algorithm can be implemented to run in $O(m + n \log n)$ time and $O(m + n)$ space. ∎

In Fig 1, the vertices $u_1, u_2, \ldots, u_6$ in the graph are indexed by an MD ordering. Orderings of (7) in a special case where G is an unweighted simple graph are known as δ-slicings [4] or smallest-last orderings [5] which are introduced to study the structure of induced subgraphs.

As an analogous result to Theorem 3, we show that a flat pair in a symmetric and crossing submodular function f can be found by an MD ordering of f.

Theorem 7 *For a symmetric and crossing submodular set function f on V with $n = |V| \geq 2$, let $\pi = (v_1, v_2, \ldots, v_n)$ be an MD ordering of V. Then the last two vertices v_{n-1} and v_n give a flat pair.* ∎

We prove Theorem 7 after showing a lemma.

Lemma 8 *Let f be a symmetric and crossing submodular set function on V. For a subset $Z \subset V$, let g be a set function on $V - Z$ such that $g(X) = f(X) + f(Z \cup X)$, $X \subseteq V - Z$. Then g is symmetric and crossing submodular.*

Proof: Let X be an arbitrary subset of Z. We show $g(X) = g((V-Z)-X))$. By definition of g, we have $g((V-Z)-X)) = f((V-Z)-X)) + f(Z \cup ((V-Z)-X)) = f(V-(Z \cup X)) + f(V-X)$, which is $f(Z \cup X) + f(X) = g(X)$ by symmetry of f. Hence g is symmetric. For two crossing subsets $X, Y \subseteq V - Z$, we have by the submodularity of f

$$\begin{aligned} g(X) + g(Y) &= f(X) + f(Y) + f(Z \cup X) + f(Z \cup Y) \\ &\geq f(X \cap Y) + f(X \cup Y) + f(Z \cup (X \cap Y)) + f(Z \cup X \cup Y) \\ &\geq g(X \cap Y) + g(X \cup Y). \end{aligned}$$

Therefore g is crossing submodular. ∎

Proof of Theorem 7 For each $i = 0, 1, \ldots, n-2$, we define set function f_i of $V - X_i$ by

$$f_i(X) = f(X) + f(V_i \cup X),\ X \subseteq V - V_i,$$

which is symmetric and crossing submodular on $V - V_i$ by Lemma 8. By induction on $i = n-2, n-3, \ldots, 1, 0$, we prove that

$$f_i(X) \geq \min_{x \in X} f_i(x) \text{ for all } X \subseteq V - V_i \text{ that separate } v_{n-1} \text{ and } v_n. \tag{8}$$

Since $f_0(X) = 2f(X)$, it suffices to show that (8) holds for $i = 0$. We easily see that (8) holds for $i = n - 2$. Now we assume that (8) holds for $i = j$. We prove that (8) holds for $i = j - 1$. Let X be an arbitrary subset of $V - V_i$ that separates v_{n-1} and v_n.

Case-1. $v_j \notin X$: Let $x^* = \mathrm{argmin}\{f_{j-1}(x) \mid x \in X\}$. For two crossing sets $V_{j-1} \cup X$ and $V_j + x^*$, we have by submodularity of f

$$f(V_{j-1} \cup X) + f(V_j + x^*) \geq f(V_j \cup X) + f(V_{j-1} + x^*).$$

From this and induction hypothesis $f_j(X) \geq f_j(x^*)$, we have

$$\begin{aligned} f_{j-1}(X) - f_{j-1}(x^*) &= f(X) + f(V_{j-1} \cup X) - f(V_{j-1} + x^*) - f(x^*) \\ &\geq f(X) + f(V_j \cup X) - f(V_j + x^*) - f(x^*) \\ &= f_j(X) - f_j(x^*) \geq 0. \end{aligned}$$

Hence $f_{j-1}(X) \geq f_{j-1}(x^*) \geq \min_{x \in X} f_{j-1}(x)$.

Case-2. $v_j \in X$: By the choice of v_j, $f_{j-1}(v_j) = \min_{x \in V - V_j} f_{j-1}(x)$. Consider subset $Y = (V - V_j) - X$, which separates v_{n-1} and v_n, and hence $f_{j-1}(Y) \geq \min_{y \in Y} f_{j-1}(y)$ holds by the argument in Case-1. Since $f_{j-1}(Y) = f_{j-1}(X)$ holds by symmetry, we have $f_{j-1}(X) = f_{j-1}(Y) \geq \min_{y \in Y} f_{j-1}(y) \geq f_{j-1}(v_j) = \min_{x \in X} f_{j-1}(x)$, as required.

Therefore, (8) holds for $i = j$. This proves that (8) holds for $i = 0$, i.e., Theorem 7. ∎

Corollary 9 *Let V be a finite set $n = |V| \geq 2$. Every symmetric and crossing submodular function f on V such that $f(v) = k$, $v \in V$ for some $k \in \Re$ has a pair $\{u, v\} \subseteq V$ that is pendent and flat.*

PROOF: We easily see that any MD ordering π of f is also an MA ordering if $f(v) = k$, $v \in V$. Hence the last two vertices in π is pendent and flat by Theorems 3 and 7. ∎

There is a symmetric and crossing submodular function f which has no pair that is pendent and flat. For example, the cut function d in Fig. 1 has no such pairs, since $\{u_3, u_4\}$ and $\{u_5, u_6\}$ are the flat pairs of d, but neither of them is pendent.

Corollary 10 *Let f be a set function f on V with $n = |V| \geq 2$. If f is symmetric and crossing submodular or intersecting submodular and posi-modular, then a flat pair of f can be found in $O(n^2 T_f)$ time.*

PROOF: If f is symmetric and crossing submodular, then we compute an MD ordering π of f in $O(n^2 T_f)$ time and choose a pair of the last two elements in π, which is flat by Theorem 7. Consider the case where f is intersecting submodular and posi-modular, where we assume $f(\emptyset) = f(V) = -\infty$, which does not lose the intersecting submodularity and posi-modularity of f. In this case, we work with the following set function $g : 2^{V+s} \mapsto \Re \cup \{-\infty\}$ (where s is a new element): For each $X \subseteq V + s$, let

$$g(X) = \begin{cases} f(X) & \text{if } s \notin X \\ f(V - (X - s)) & \text{if } s \in X. \end{cases} \tag{9}$$

It is known [8] that, for an intersecting submodular and posi-modular set function f on V, the above set function g is symmetric and crossing submodular on $V + s$. Let π_g be an MD ordering of g, where the first element in π_g must be s since $g(s) = f(V) = +\infty$. Then the last two elements $u, v \in V$ in π_g give a flat pair of g. We see that $\{u, v\}$ is also flat in f, since any subset $X \subseteq V$ with $f(X) < \min_{x \in X} f(x)$ would imply $g(X) < \min_{x \in X} g(x)$, contradicting that $\{u, v\}$ is flat in g. Therefore, we can find a flat pair in $O(n^2 T_f)$ time even if f is intersecting submodular and posi-modular. ∎

3 Computing Extreme Subsets

This section presents an algorithm for computing all extreme subsets in a set function by using flat pairs. For any nonempty subset $Y \subseteq V$, there is an extreme subset $Y^* \in \mathcal{X}(f)$ with $f(Y) \geq f(Y^*)$. Hence we see that $f(Y) > f(X)$ for all nonempty $Y \subset X$ if and only if $f(Z) > f(X)$ for all $Z \subset X$ with $Z \in \mathcal{X}(f)$. From this observation and the fact that no nontrivial extreme subset $X \in \mathcal{X}(f)$ separates any flat pair, we obtain the following algorithm for computing all extreme subsets in a set function.

After intialzing by setting $\mathcal{X} := \{\{v\} \mid v \in V\}$ $(\subseteq \mathcal{X}(f))$, we repeat a procedure of contracting a flat pair $n - 2$ times. Let V^i, $i = n, n-1, \ldots, 2$, be the set of elements obtained after contracting the first $n - i$ flat pairs, where $|V^i| = i$ holds. For each element $x \in V^i$, let $V[x] \subseteq V$ denote the set of all elements that have been contracted into x. We maintain the property that

$$\mathcal{X} \text{ consists of all extreme subsets } X \in \mathcal{X}(f) \text{ with } X \subseteq V[x] \text{ and } x \in V^i. \tag{10}$$

After contracting a flat pair $u^i, v^i \in V^i$ into a single element z^i, we add $V[z^i]$ to $\mathcal{X}$ if $V[z^i] \in \mathcal{X}(f)$, so that (10) holds in the resulting set $V^{i-1} = (V^i - \{u^i, v^i\}) \cup \{z^i\}$ of elements. We can test whether $V[z^i] \in \mathcal{X}(f)$ or not by checking if $f(V[z^i]) < \min_{Y \in \mathcal{X}: Y \subset V[z^i]} f(Y)$. To facilitate this test, we also maintain $\mu(x) = \min_{Y \in \mathcal{X}: Y \subset V[x]} f(Y)$, $x \in V^i$ for each $i = n, n-1, \ldots, 2$. The entire algorithm is described as follows.

Algorithm ExtremeSubsets
Input: A set function f on a finite set V, where $n = |V| \geq 2$.
Output: A laminar family $\mathcal{X} \subseteq 2^V - \{\emptyset, V\}$.
1 $\mathcal{X} := \{\{v\} \mid v \in V\}$;
2 Let $\mu(v) := f(v)$ for all $v \in V$;
3 $V^n := V$; $f^n := f$;
4 **for** $i := n$ to 3 **do**
5 Find a flat pair $\{u^i, v^i\}$ of f^i;
6 $V^{i-1} := (V^i - \{u^i, v^i\}) \cup \{z^i\}$;
7 Let f^{i-1} be the set function on V^{i-1} obtained from f^i by contracting elements u^i and v^i into a single element z^i;
8 Let $V[z^i] \subseteq V$ be the set of all elements that have been contracted into z^i; /* $f(V[z^i]) = f^{i-1}(z^i)$ */

9 **if** $f^{i-1}(z^i) < \min\{\mu(u^i), \mu(v^i)\}$ **then**
10 $\mathcal{X} := \mathcal{X} \cup \{V[z^i]\}$; $\mu(z^i) := f^{i-1}(z^i)$
11 **else**
12 $\mu(z^i) := \min\{\mu(u^i), \mu(v^i)\}$
13 **end if**
14 **end for**

From the above argument, we see that algorithm ExtremeSubsets computes the set $\mathcal{X}(f)$ of all extreme sets of f correctly as long as we can always find a flat pair in line 5. If a given set function f is symmetric and crossing submodular or intersecting submodular and posi-modular, then it is not difficult to see that each set function f^i obtained from f by contracting elements remains symmetric and crossing submodular or intersecting submodular and posi-modular. Therefore, by Corollary 10, we can find a flat pair in $O(n^2 T_f)$ time. Then the running time of ExtremeSubsets is $O(n^3 T_f)$. This establishes Theorem 2.

By Lemma 6, we easily see that, for hypergraphs, algorithm ExtremeSubsets can be implemented to run in $O(n(m + n\log n))$ time and $O(m + n)$ space.

Corollary 11 *For an edge-weighted hypergraph $G = (V, E)$, the set $\mathcal{X}(d)$ of all extreme subsets can be found in $O(n(m + n\log n))$ time and $O(m + n)$ space, where $n = |V|$ and $m = \sum_{e\in E} |e|$.*

∎

Acknowledgement

This research was partially supported by the Scientific Grant-in-Aid from Ministry of Education, Science, Sports and Culture of Japan.

References

[1] A. Frank, Applications of submodular functions, in Surveys in Combinatorics, Keith Walker, ed., London Math. Soc. Lecture Notes, Ser. 187 (1993) 85–136.

[2] M. L. Fredman and R. E. Tarjan, Fibonacci heaps and their uses in improved network optimization algorithms, J. Assoc. Compt. Mach., 34 (1987) 596–615.

[3] S. Fujishige, Submodular Functions and Optimization, North-Holland, Amsterdam, (1991).

[4] D. W. Matula, k-Components, clusters, and slicings in graphs, SIAM J. Applied Mathematics, vol. 22, (1972) 459–480.

[5] D. W. Matula, and L. L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, J. ACM, vol. 30, (1983) 417–427.

[6] H. Nagamochi, Graph algorithms for network connectivity problems, J. Operations Research Society of Japan, 47 (2004) 199–223.

[7] H. Nagamochi and T. Ibaraki, Computing edge-connectivity of multigraphs and capacitated graphs, SIAM J. Discrete Mathematics, 5 (1992) 54–66.

[8] H. Nagamochi and T. Ibaraki, A note on minimizing submodular functions, Information Processing Letters, 67 (1998) 239–244.

[9] H. Nagamochi and T. Ibaraki, Augmenting edge-connectivity over the entire range in $\tilde{O}(nm)$ time, J. Algorithms, 30 (1999) 253–301.

[10] M. Queyranne, Minimizing symmetric submodular functions, Mathematical Programming, 82 (1998) 3–12.

[11] M. Stoer and F. Wagner, A simple min cut algorithm, J. ACM, 44 (1997) 585–591.

[12] T. Watanabe and A. Nakamura, Edge-connectivity augmentation problems, J. Comp. System Sci., 35 (1987) 96–144.