Robust Nash equilibria in N-person non-cooperative games: Uniqueness and reformulation*

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Abstract. In this paper we propose a general framework of distribution-free models for N-person non-cooperative games with uncertain information. In the model, we assume that each player's cost function and/or the opponents' strategies belong to some uncertainty sets, and each player chooses his/her strategy according to the robust optimization policy. Under such assumptions, we define the robust Nash equilibrium for N-person games by extending some existing definitions. We present sufficient conditions for existence and uniqueness of a robust Nash equilibrium. In order to compute robust Nash equilibria, we reformulate certain classes of robust Nash equilibrium problems to second-order cone complementarity problems. We finally show some numerical results to discuss the behavior of robust Nash equilibria.

1 Introduction

Game theory is a mathematical methodology to analyze various decisions in economics or societies [12, 22]. Nash [20, 21] proposed a concept of equilibrium, called Nash equilibrium, for non-cooperative games. To define the Nash equilibrium, we usually assume that each player has a complete knowledge about the game, that is, he[*1] can estimate the opponents' strategy and evaluate his own cost or profit exactly. This premise is called "complete information." However, in the real situation, it is not always satisfied, and hence, we need to define an alternative equilibrium concept.

There have been a large number of studies on games with uncertain data. Harsanyi [17, 18, 19] proposed a stochastic-based formulation for incomplete information games. He assumed that each player estimates the probability distribution for the uncertain information and maximizes his expected profit, or equivalently minimizes his expected cost. These assumptions

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*1 In this paper, we assume that all players are male.

are called "Bayesian hypothesis." Then, Harsanyi modeled incomplete information games as "Bayesian games" under some assumptions on probability distributions. Although Bayesian games are defined from incomplete information games, their elements (e.g., probability parameters) are actually the common knowledge, and hence, it is essentially a complete information game. Incomplete information games and Bayesian games can be considered to be equivalent from each player's strategical viewpoint.

On the other hand, distribution-free models based on the worst-case analysis attract much attention in recent years [1, 15]. In such models, each player makes a decision according to the idea of robust optimization [5, 6, 7, 9]. Originally, robust optimization is a technique for handling optimization problems with uncertain parameters, in which those uncertain parameters are assumed to belong to so-called *uncertainty sets*, and then the objective function is minimized (or maximized) by taking into account the worst possible case. An equilibrium resulting from the robust optimization by each player is called a robust Nash equilibrium, and the problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem. Aghassi and Bertsimas [1] considered the robust Nash equilibrium for N-person games in which each player solves a linear programming (LP) problem[*2]. Moreover, they proposed a method for solving the robust Nash equilibrium problem with convex polyhedral uncertain sets. Independently of their work, Hayashi, Yamashita, and Fukushima [15] defined the concept of robust Nash equilibria for bimatrix games. Under the assumption that uncertain sets are expressed by means of the Euclidean or the Frobenius norm, they showed that each player's problem reduces to a second-order cone program (SOCP) [2] and the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) [13, 14]. In addition, Hayashi et al. [15] studied robust Nash equilibrium problems in which uncertainty is contained in both opponents' strategies and each player's cost parameters, whereas Aghassi et al. [1] studied only the latter case.

In this paper, we extend the definition of robust Nash equilibria in [1] and [15] to N-person non-cooperative games with nonlinear cost functions. In particular, we show existence of robust Nash equilibria under the assumption that each player's cost function is *convex* with respect to his strategy, while [1] and [15] only considered the *linear* case. Moreover, we give some sufficient conditions for uniqueness of a robust Nash equilibrium. In order to solve certain classes of robust Nash equilibrium problems, we reformulate them to second-order cone complementarity problems.

This paper is organized as follows. In Section 2, we characterize the uncertainty in the incomplete information non-cooperative game, and define the robust Nash equilibrium. In Section 3, we give sufficient conditions under which the existence of Nash equilibria is guar-

[*2] In [1] a robust Nash equilibrium is called a robust-optimization equilibrium.

anteed. In Section 4, we discuss the uniqueness of a robust Nash equilibrium by way of the generalized variational inequality problem. In Section 5, we reformulate certain classes of robust Nash equilibrium problems as second-order cone complementarity problems, which can be solved by some modern algorithms. In Section 6, we show some numerical results and discuss the behavior of robust Nash equilibria.

Throughout the paper, we use the following notations. For a set X, $\mathcal{P}(X)$ denotes the set consisting of all the subsets of X. $\Re^n_+$ denotes the nonnegative orthant in $\Re^n$, that is, $\Re^n_+ := \{x \in \Re^n \mid x_i \geq 0 \ (i = 1, \ldots, n)\}$. For a vector $x \in \Re^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \Re^{n \times m}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^n \sum_{j=1}^m (M_{ij})^2)^{1/2}$.

2 Robust Nash equilibrium

In this paper, we consider an N-person non-cooperative game in which each player tries to minimize his own cost. Let $x^i \in \Re^{m_i}$, $S_i \subseteq \Re^{m_i}$, and $f_i : \Re^{m_1} \times \cdots \times \Re^{m_N} \to \Re$ be player i's strategy, strategy set, and cost function, respectively. Moreover, we denote

$$
\begin{aligned}
&\mathcal{I} := \{1, \ldots, N\}, \quad \mathcal{I}_{-i} := \mathcal{I} \setminus \{i\}, \quad m := \sum_{j \in \mathcal{I}} m_j, \quad m_{-i} := \sum_{j \in \mathcal{I}_{-i}} m_j, \\
&x := (x^j)_{j \in \mathcal{I}} \in \Re^m, \quad x^{-i} := (x^j)_{j \in \mathcal{I}_{-i}} \in \Re^{m_{-i}}, \\
&S := \prod_{j \in \mathcal{I}} S_j \subseteq \Re^m, \quad S_{-i} := \prod_{j \in \mathcal{I}_{-i}} S_j \subseteq \Re^{m_{-i}}.
\end{aligned}
$$

When the complete information is assumed, each player i decides his own strategy by solving the following optimization problem with the opponents' strategy x^{-i} fixed:

$$
\begin{array}{ll}
\underset{x^i}{\text{minimize}} & f_i(x^i, x^{-i}) \\
\text{subject to} & x^i \in S_i.
\end{array} \tag{2.1}
$$

A tuple $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$ satisfying $\overline{x}^i \in \operatorname{argmin}_{x^i \in S_i} f_i(x^i, \overline{x}^{-i})$ for each player $i = 1, \ldots, N$ is called a Nash equilibrium. In other words, if each player i chooses the strategy $\overline{x}^i$, then no player has an incentive to change his own strategy. The Nash equilibrium is well-defined only when each player can estimate his opponents' strategies and evaluate his own cost exactly. In the real situation, however, any information may contain uncertainty such as observation errors or estimation errors. Thus, in this paper, we focus on games with uncertainty.

To deal with such uncertainty, we introduce uncertainty sets U_i and $X_i(x^{-i})$, and assume the following statements for each player $i \in \mathcal{I}$:

(A) Player i's cost function involves a parameter $\hat{u}^i \in \Re^{\nu_i}$, i.e., it can be expressed as $f_i^{\hat{u}^i} : \Re^{m_i} \times \Re^{m_{-i}} \to \Re$. Although player i do not know the exact value of $\hat{u}^i$ itself, he can estimate that it belongs to a given nonempty set $U_i \subseteq \Re^{\nu_i}$.

(B) Although player i knows his opponents' strategies x^{-i}, his actual cost is evaluated with x^{-i} replaced by $\hat{x}^{-i} = x^{-i} + \delta x^{-i}$, where δx^{-i} is a certain error or noise. Player i cannot know the exact value of $\hat{x}^{-i}$. However, he can estimate that $\hat{x}^{-i}$ belongs to a certain nonempty set $X_i(x^{-i})$.

Then, each player is required to address the following family of problems involving uncertain parameters $\hat{u}^i$ and $\hat{x}^{-i}$:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \\ \text{subject to} & x^i \in S_i, \end{array} \tag{2.2}$$

where $\hat{u}^i \in U_i$ and $\hat{x}^{-i} \in X_i(x^{-i})$. We further assume that each player chooses his strategy according to the following criterion:

(C) Player i tries to minimize his worst cost under assumptions (A) and (B).

From assumption (C), each player considers the worst cost function $\tilde{f}_i : \Re^{m_i} \times \Re^{m_{-i}} \to (-\infty, +\infty]$ defined by

$$\tilde{f}_i(x^i, x^{-i}) := \sup\{f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \mid \hat{u}_i \in U_i, \hat{x}^{-i} \in X_i(x^{-i})\}, \tag{2.3}$$

and solves the following worst cost minimization problem:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \tilde{f}_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in S_i. \end{array} \tag{2.4}$$

Note that (2.4) is regarded as a complete information game with cost functions $\tilde{f}_i$. Based on the above discussions, we define the robust Nash equilibrium.

Definition 2.1. *Let $\tilde{f}_i$ be defined by* (2.3) *for $i = 1, \ldots, N$. A tuple $(\overline{x}^i)_{i \in \mathcal{I}}$ is called a robust Nash equilibrium of game* (2.2)*, if $\overline{x}^i \in \operatorname{argmin}_{x^i \in S_i} \tilde{f}_i(x^i, \overline{x}^{-i})$ for all i, i.e., a Nash equilibrium of game* (2.4)*. The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.*

3 Existence of robust Nash equilibria

In this section, we give sufficient conditions for the existence of a robust Nash equilibria. Note that $X_i(x^{-i})$ given in (B) can be regarded as a set-valued mapping $X_i(\cdot)$ with variable x^{-i}.

In what follows, we suppose that $X_i(\cdot)$, U_i, f^{u^i} and S_i in (A) and (B) satisfy the following assumption.

Assumption 1. *For every $i \in \mathcal{I}$, the following statements hold.*

(a) *The function $G_i : \Re^{m_i} \times \Re^{m_{-i}} \times \Re^{\nu_i} \to \Re$ defined by $G_i(x^i, x^{-i}, u^i) := f_i^{u^i}(x^i, x^{-i})$ is continuous.*

(b) *The set-valued mapping $X_i : \Re^{m_{-i}} \to \mathcal{P}(\Re^{m_{-i}})$ is continuous, and $X_i(x^{-i})$ is nonempty and compact for any $x^{-i} \in S_{-i}$.*

(c) *The set $U_i \subseteq \Re^{\nu_i}$ is nonempty and compact.*

(d) *The set S_i is nonempty, compact and convex, and function $f_i^{u^i}(\cdot, x^{-i}) : \Re^{m_i} \to \Re$ is convex on S_i for any fixed x^{-i} and u^i.*

Under Assumption 1, the function $\tilde{f}_i(x^i, x^{-i})$ defined by (2.3) has the following properties:

- $\tilde{f}_i(x^i, x^{-i})$ is continuous and finite at any $(x^i, x^{-i}) \in S_i \times S_{-i}$.
- For any fixed $x^{-i} \in S_{-i}$, function $\tilde{f}_i(\cdot, x^{-i}) : \Re^{m_i} \to \Re$ is convex on S_i.

The continuity and finiteness of $\tilde{f}_i$ can be verified from [4, Theorem 1.4.16], while the convexity of $\tilde{f}_i(\cdot, x^{-i})$ follows from [8, Proposition 1.2.4(c)].

The following lemma is a well-known result for N-person non-cooperative games.

Lemma 3.1. [3, Theorem 9.1.1] *Suppose that, for every player $i \in \mathcal{I}$,* (i) *the strategy set S_i is nonempty, convex and compact,* (ii) *the cost function $f_i : \Re^{m_i} \times \Re^{m_{-i}} \to \Re$ is continuous, and* (iii) *$f_i(\cdot, x^{-i})$ is convex for any $x^{-i} \in S_{-i}$. Then, game* (2.1) *has at least one Nash equilibrium.*

By this lemma, we obtain the following theorem for the existence of a robust Nash equilibrium in game (2.2).

Theorem 3.2. *Suppose that Assumption 1 holds. Then, game* (2.2) *has at least one robust Nash equilibrium.*

Proof. Let i be chosen from $\mathcal{I}$ arbitrarily. From Assumption 1, $\tilde{f}_i(x^i, x^{-i})$ is continuous and finite at any $(x^i, x^{-i}) \in S_i \times S_{-i}$. Moreover, function $\tilde{f}_i(\cdot, x^{-i})$ is convex on S_i for any $x^{-i} \in S_{-i}$. Therefore, from Lemma 3.1, game (2.4) has a Nash equilibrium, that is, game (2.2) has a robust Nash equilibrium. □

4 Uniqueness of the robust Nash equilibrium

In the previous section, we have studied sufficient conditions for existence of robust Nash equilibria. Under such conditions, there exist a number of robust Nash equilibria in general, and it is difficult to find them all. In this section, we therefore study conditions for uniqueness of a robust Nash equilibrium.

For complete information games, Rosen [23] gave some conditions for the uniqueness of a Nash equilibrium. Those conditions are essentially equivalent to the strict monotonicity of the vector-valued function involved in the equivalent variational inequality problem (VIP) [10]. Moreover, such a vector-valued function is defined by using the derivatives of all players' cost functions. However, since the worst cost function $\tilde{f}_i$ defined by (2.3) is in general nondifferentiable, the VIP reformulation approach cannot be applied directly. This fact prompts us to consider the generalized VIP (GVIP), which is defined by means of a set-valued mapping. Then, by using the uniqueness results for GVIP, we establish sufficient conditions for the uniqueness of a robust Nash equilibrium.

For a given set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ and a nonempty closed convex set Ω, GVIP$(\mathcal{F}, \Omega)$ is to find a vector $x \in \Omega$ such that

$$\mathrm{GVIP}(\mathcal{F}, \Omega) : \qquad \exists \xi \in \mathcal{F}(x), \quad \langle \xi, y - x \rangle \geq 0 \quad \forall y \in \Omega. \tag{4.1}$$

If the set-valued mapping $\mathcal{F}$ is given by $\mathcal{F}(x) = \{F(x)\}$ for a vector-valued function $F : \Re^n \to \Re^n$, then the GVIP reduces to the following VIP:

$$\mathrm{VIP}(F, \Omega) : \qquad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \Omega. \tag{4.2}$$

It is well known that if the function F is strictly monotone, then VIP (4.2) has at most one solution [10]. In fact, a similar result holds for GVIP [11]. Recall that the set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is said to be monotone (strictly monotone) on a nonempty convex set $\Omega \subseteq \Re^n$ if

$$\langle x - y, \xi - \eta \rangle \geq (>)\ 0$$

for all $x, y \in \Omega$ $(x \neq y)$ and $\xi \in \mathcal{F}(x), \eta \in \mathcal{F}(y)$.

Proposition 4.1. *Suppose that the set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is strictly monotone on Ω. Then, GVIP*(4.1) *has at most one solution.*

Next, we reformulate a robust Nash equilibrium problem as a GVIP. Specifically, the robust Nash equilibrium problem (2.4) is equivalent to GVIP$(\tilde{\mathcal{F}}, \Omega)$ with $\tilde{\mathcal{F}} : \Re^m \to \mathcal{P}(\Re^m)$ and Ω defined by

$$\tilde{\mathcal{F}}(x) := \left(\partial_i \tilde{f}_i(x^i, x^{-i}) \right)_{i \in \mathcal{I}} \tag{4.3}$$

and

$$\Omega := S = S_1 \times \cdots \times S_N,$$

respectively. Here, $\partial_i \tilde{f}_i$ denotes the subdifferential of $\tilde{f}_i$ with respect to player i's strategy x^i.

If Assumption 1 holds, then there exists at least one robust Nash equilibrium from Theorem 3.2. Moreover, by Proposition 4.1, if the set-valued mapping $\tilde{\mathcal{F}}$ defined by (4.3) is strictly monotone, then game (2.2) has a unique robust Nash equilibrium.

Next, we give sufficient conditions for $\tilde{\mathcal{F}}$ to be strictly monotone. To this end, we introduce the following assumption:

Assumption 2. *For each $i \in \mathcal{I}$, the following conditions hold:*

(a) *The set $X_i(x^{-i})$ is given by $X_i(x^{-i}) = x^{-i} + D_i$ for a nonempty compact set $D_i \subseteq \Re^{m_{-i}}$.*

(b) *Function $f_i^{u^i}$ is expressed as $f_i^{u^i}(x^i, x^{-i}) := g_i^{u^i}(x^i) + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top A_{ij} x^j$ with a convex function $g_i^{u^i} : \Re^{m_i} \to \Re$ and matrices $A_{ij} \in \Re^{m_i \times m_j}$ $(j \in \mathcal{I}_{-i})$.*

(c) *Either of the following statements holds:*

  (c-i) *For any $u^i \in U_i$ and $i \in \mathcal{I}$, the function $g_i^{u^i}$ is strongly convex with modulus $\gamma > -\lambda_{\min}(\overline{A}_0)$, where $\lambda_{\min}(\overline{A}_0)$ denotes the minimum eigenvalue of $\overline{A}_0 := (A_0 + A_0^\top)/2$ with*

$$A_0 := \begin{bmatrix} 0 & A_{12} & \cdots & A_{1N} \\ A_{21} & 0 & & A_{2N} \\ \vdots & & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & 0 \end{bmatrix}.$$

  (c-ii) *U_i is a singleton, i.e., $U_i = \{u^i\}$, and the set-valued mapping $\mathcal{F} : \Re^m \to \mathcal{P}(\Re^m)$ defined by*

$$\mathcal{F}(x) := (\partial_i f_i^{u^i}(x^i, x^{-i}))_{i \in \mathcal{I}} \tag{4.4}$$

  is strictly monotone.

Under the above assumption, we have the following lemma.

Lemma 4.2. *Suppose that Assumption 2 holds. Then, the set-valued mapping $\tilde{\mathcal{F}}$ defined by (4.3) is strictly monotone.*

Proof. For simplicity, we denote $A_{-i} := [A_{i1} \cdots A_{i\,i-1}\ A_{i\,i+1} \cdots A_{iN}] \in \Re^{m_i \times m_{-i}}$. Then, from Assumption 2(a)(b), we have

$$\begin{aligned} \tilde{f}_i(x) &= \max\left\{ g_i^{u^i}(x^i) + (x^i)^\top A_{-i}(x^{-i} + \delta x^{-i}) \;\middle|\; u^i \in U_i, \delta x^{-i} \in D_i \right\} \\ &= \tilde{g}_i(x^i) + (x^i)^\top A_{-i} x^{-i} + \psi_i(x^i), \end{aligned}$$

where $\tilde{g}_i(x^i) := \max_{u^i \in U_i} g_i^{u^i}(x^i)$ and $\psi_i(x^i) := \max_{\delta x^{-i} \in D_i} (x^i)^\top A_{-i} \delta x^{-i}$. Hence, we obtain

$$\begin{aligned} \tilde{\mathcal{F}}(x) &= \left(\partial_i \tilde{f}_i(x^i, x^{-i})\right)_{i \in \mathcal{I}} \\ &= \left(\partial \tilde{g}_i(x^i) + A_{-i} x^{-i} + \partial \psi_i(x^i)\right)_{i \in \mathcal{I}} \\ &= \left(\partial \tilde{g}_i(x^i)\right)_{i \in \mathcal{I}} + A_0 x + \left(\partial \psi_i(x^i)\right)_{i \in \mathcal{I}}, \end{aligned}$$

where the second equality holds from [8, Proposition 4.2.4].

We first consider the case where (c-i) holds. Since $g_i^{u^i}$ is strongly convex with modular γ, so is $\tilde{g}_i$, and hence $\partial\tilde{g}_i$ is strongly monotone with modulus γ [3]. Then, for any $x, y \in \Re^m$ with $x \neq y$, we have

$$\begin{aligned}
&\min\left\{ (x-y)^\top(\xi-\eta) \;\middle|\; \xi \in \tilde{\mathcal{F}}(x), \eta \in \tilde{\mathcal{F}}(y) \right\} \\
&= \min\left\{ \sum_{i\in\mathcal{I}} (x^i-y^i)^\top(\xi^i-\eta^i) \;\middle|\; \begin{array}{l} \xi^i \in \partial\tilde{g}_i(x^i) + A_{-i}x^{-i} + \partial\psi_i(x^i),\ i \in \mathcal{I} \\ \eta^i \in \partial\tilde{g}_i(y^i) + A_{-i}y^{-i} + \partial\psi_i(y^i),\ i \in \mathcal{I} \end{array} \right\} \\
&= (x-y)^\top A_0(x-y) + \sum_{i\in\mathcal{I}} \min\left\{ (x^i-y^i)^\top(\xi^i_\alpha - \eta^i_\alpha) \;\middle|\; \xi^i_\alpha \in \partial\tilde{g}_i(x^i), \eta^i_\alpha \in \partial\tilde{g}_i(y^i) \right\} \\
&\quad + \sum_{i\in\mathcal{I}} \min\left\{ (x^i-y^i)^\top(\xi^i_\beta - \eta^i_\beta) \;\middle|\; \xi^i_\beta \in \partial\psi_i(x^i), \eta^i_\beta \in \partial\psi_i(y^i) \right\} \\
&\geq (x-y)^\top \overline{A}_0(x-y) + \sum_{i\in\mathcal{I}} \gamma \|x^i-y^i\|^2 > 0,
\end{aligned}$$

where the first inequality follows from the strong monotonicity of $\partial\tilde{g}_i$ and the monotonicity of $\partial\psi_i$, and the last inequality is due to $\gamma > -\lambda_{\min}(\overline{A}_0)$ and $x \neq y$. Thus, the set-valued mapping $\tilde{\mathcal{F}}$ is strictly monotone.

We next consider the case where (c-ii) holds. Then, we can rewrite $\tilde{\mathcal{F}}(x)$ as

$$\begin{aligned}
\tilde{\mathcal{F}}(x) &= \left(\partial_i f_i^{u^i}(x^i, x^{-i}) + \partial\psi_i(x^i)\right)_{i\in\mathcal{I}} \\
&= \mathcal{F}(x) + (\partial\psi_i(x^i))_{i\in\mathcal{I}}.
\end{aligned}$$

From the strict monotonicity of $\mathcal{F}$ and the monotonicity of $\partial\psi_i$, the set-valued mapping $\tilde{\mathcal{F}}$ is strictly monotone. □

By the above lemmas, we obtain the following theorem on the uniqueness of a robust Nash equilibrium.

Theorem 4.3. *Suppose that Assumptions* 1 *and* 2 *hold. Then, game* (2.2) *has a unique robust Nash equilibrium.*

Proof. By Assumption 1 and Theorem 3.2, game (2.2) has at least one robust Nash equilibrium. On the other hand, by Proposition 4.1 and Lemma 4.2, game (2.2) cannot have multiple robust Nash equilibria. Hence, game (2.2) has a unique robust Nash equilibrium. □

5 SOCCP formulation of robust Nash equilibrium problem

In this section, we focus on the game in which each player takes a mixed strategy and minimizes a convex quadratic cost function with respect to his own strategy. We show that the robust Nash equilibrium problem then reduces to an SOCCP. We also discuss the existence and uniqueness properties by using the results obtained heretofore.

Recall that SOCCP [13, 14] is a problem to find a triple $(\xi, \eta, \zeta) \in \Re^l \times \Re^l \times \Re^\nu$ such that

$$\mathcal{K} \ni \xi \perp \eta \in \mathcal{K}, \quad G(\xi, \eta, \zeta) = 0, \tag{5.1}$$

where $G : \Re^l \times \Re^l \times \Re^\nu \to \Re^l \times \Re^\nu$ is a given function, $\xi \perp \eta$ means $\xi^\top \eta = 0$, $\mathcal{K}$ is a closed convex cone defined by $\mathcal{K} = \mathcal{K}^{l_1} \times \mathcal{K}^{l_2} \times \cdots \times \mathcal{K}^{l_m}$ with l_j-dimensional second-order cones $\mathcal{K}^{l_j} := \{(\zeta_1, \zeta_2) \in \Re \times \Re^{l_j - 1} \mid \|\zeta_2\| \leq \zeta_1\}$, $j = 1, \dots, m$, and $l = \sum_{j=1}^m l_j$. SOCCP can be solved by some existing algorithms such as a smoothing and regularization method [14]. Here, we consider an SOCCP of the form

$$\mathcal{K} \ni M\zeta + q \perp N\zeta + r \in \mathcal{K}, \quad C\zeta = d \tag{5.2}$$

with variable $\zeta \in \Re^{l+\tau}$ and constants $M, N \in \Re^{l \times (l+\tau)}$, $q, r \in \Re^l, C \in \Re^{\tau \times (l+\tau)}$ and $d \in \Re^\tau$. Note that, by introducing auxiliary variables $\xi, \eta \in \Re^l$, SOCCP (5.2) reduces to SOCCP (5.1) with $G : \Re^{3l+\tau} \to \Re^{2l+\tau}$ defined by

$$G(\xi, \eta, \zeta) := \begin{bmatrix} \xi - M\zeta - q \\ \eta - N\zeta - r \\ C\zeta - d \end{bmatrix}.$$

Throughout this section, the cost functions and the strategy sets are given as follows.

(i) Player i's cost function $f_i^{\hat{u}^i}$ is given by

$$f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) = \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + (x^i)^\top \Bigg(\sum_{j \in \mathcal{I}_{-i}} \hat{A}_{ij} \hat{x}^j + \hat{c}^i \Bigg), \tag{5.3}$$

where $\hat{A}_{ij} \in \Re^{m_i \times m_j}$ $(j \in \mathcal{I})$ and $\hat{c}^i \in \Re^{m_i}$ are given constants involving uncertainties.

(ii) Player i takes a mixed strategy, i.e.,

$$S_i = \{x^i \mid x^i \geq 0, \ e_{m_i}^\top x^i = 1\}, \tag{5.4}$$

where e_{m_i} denotes the vector $(1, 1, \dots, 1)^\top \in \Re^{m_i}$.

We call $\hat{A}_{ij}$ and $\hat{c}^i$ a cost matrix and a cost vector, respectively. Note that these constants correspond to the cost function parameter $\hat{u}^i$, i.e.,

$$\hat{u}^i = \mathrm{vec}\, [\hat{A}_{i1} \ \cdots \ \hat{A}_{iN} \ \hat{c}^i] \in \Re^{m_i(m+1)} \tag{5.5}$$

where vec denotes the vectorization operator that creates an nm-dimensional vector $[(p_1^c)^\top \cdots (p_m^c)^\top]^\top$ from a matrix $P \in \Re^{n \times m}$ with column vectors $p_1^c, \dots, p_m^c$.

5.1 Uncertainty in the opponents' strategy

In this subsection, we consider the case where each player knows the cost matrices and vectors exactly but the opponents' strategies uncertainly. More specifically, we suppose the following assumption holds.

Assumption 3. *For each $i \in \mathcal{I}$, uncertainty sets $X_i(\cdot)$ and U_i $(i \in \mathcal{I})$ are given as follows.*

(a) $X_i(x^{-i}) = \prod_{j \in \mathcal{I}_{-i}} X_{ij}(x^j)$*, where* $X_{ij}(x^j) := \{x^j + \delta x^{ij} \mid \|\delta x^{ij}\| \leq \rho_{ij},\ e_{m_j}^\top \delta x^{ij} = 0\}$ *with a given constant* $\rho_{ij} \geq 0$*.*

(b) $U_i := \{u^i\} = \left\{\mathrm{vec}\,[A_{i1} \cdots A_{iN}\ c^i]\right\}$*. Moreover,* A_{ii} *is symmetric and positive semidefinite.*

In Assumption 3(a), the condition $e_{m_j}^\top \delta x^{ij} = 0$ is provided so that $e_{m_j}^\top (x^j + \delta x^{ij}) = 1$ holds for $x^j \in S_j$. Under this assumption, the worst cost function $\tilde{f}_i$ can be expressed explicitly as follows:

$$
\begin{aligned}
&\tilde{f}_i(x^i, x^{-i}) \\
&\quad = \max\left\{ \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j \in \mathcal{I}_{-i}} A_{ij}(x^j + \delta x^{ij}) + (c^i)^\top x^i \;\middle|\; \|\delta x^{ij}\| \leq \rho_{ij}, e_{m_j}^\top \delta x^{ij} = 0 \ (j \in \mathcal{I}_{-i}) \right\} \\
&\quad = \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j \in \mathcal{I}_{-i}} A_{ij} x^j + (c^i)^\top x^i + \sum_{j \in \mathcal{I}_{-i}} \max\left\{ (x^i)^\top A_{ij} \delta x^{ij} \;\middle|\; \|\delta x^{ij}\| \leq \rho_{ij}, e_{m_j}^\top \delta x^{ij} = 0 \right\} \\
&\quad = \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j \in \mathcal{I}_{-i}} A_{ij} x^j + (c^i)^\top x^i + \sum_{j \in \mathcal{I}_{-i}} \rho_{ij} \|\tilde{A}_{ij}^\top x^i\|, \qquad (5.6)
\end{aligned}
$$

where $\tilde{A}_{ij} := A_{ij}(I_{m_j} - m_j^{-1} e_{m_j} e_{m_j}^\top)$, and the last equality follows since $\tilde{A}_{ij}^\top x^i$ is the projection of $A_{ij}^\top x^i$ onto the hyperplane $\pi_j := \{x^j \mid e_{m_j}^\top x^j = 0\}$ and hence the maximum is attained when $\delta x^{ij} = \rho_{ij}(\tilde{A}_{ij}^\top x^i)/\|\tilde{A}_{ij}^\top x^i\|$.

5.1.1 Reformulation as SOCCP

We first show that the robust Nash equilibrium problem reduces to the SOCCP (5.2). By using the explicit expression (5.6) of $\tilde{f}_i$ and auxiliary variables $y_{ij} \in \Re$ $(j \in \mathcal{I}_{-i})$, player i's worst cost minimization problem (2.4) can be reformulated as the following SOCP:

$$
\begin{aligned}
\underset{x^i, y_{ij}}{\text{minimize}} \quad & \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j \in \mathcal{I}_{-i}} A_{ij} x^j + (c^i)^\top x^i + \sum_{j \in \mathcal{I}_{-i}} \rho_{ij} y_{ij} \\
\text{subject to} \quad & \|\tilde{A}_{ij}^\top x^i\| \leq y_{ij} \ \ (j \in \mathcal{I}_{-i}), \quad x^i \geq 0, \quad e_{m_i}^\top x^i = 1.
\end{aligned}
$$

Moreover, the Karush-Kuhn-Tucker (KKT) conditions of this problem can be written as the following SOCCP:

$$
\begin{aligned}
\mathcal{K}^{m_j+1} \ni \begin{bmatrix} \mu_{ij} \\ \lambda^{ij} \end{bmatrix} \perp \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_{ij}^\top \end{bmatrix} \begin{bmatrix} y_{ij} \\ x^i \end{bmatrix} \in \mathcal{K}^{m_j+1} \ (j \in \mathcal{I}_{-i}) \\
\Re_+^{m_i} \ni x^i \perp A_{ii}x^i + \sum_{j \in \mathcal{I}_{-i}} \left(A_{ij}x^j - \tilde{A}_{ij}\lambda^{ij} \right) + c^i + e_{m_i}s_i \in \Re_+^{m_i}, \quad e_{m_i}^\top x^i = 1, \\
\mu_{ij} = \rho_{ij} \ (j \in \mathcal{I}_{-i}),
\end{aligned}
$$

where $\lambda^{ij} \in \Re^{m_j}$ and $s_i \in \Re$ are Lagrange multipliers, and $\mu_{ij} \in \Re$ are auxiliary variables. Noticing that the above KKT conditions hold for all players simultaneously, the robust Nash equilibrium problem can be reformulated as the SOCCP (5.2) with

$$
l = N(m+N-1), \ \tau = N(N+1), \ \mathcal{K} = \prod_{i \in \mathcal{I}} \left(\prod_{j \in \mathcal{I}_{-i}} \mathcal{K}^{m_j+1} \right) \times \prod_{i \in \mathcal{I}} \Re_+^{m_i},
$$

$$
\begin{aligned}
\zeta = \left[\begin{array}{ccccc|ccc|ccc} y_1 & (x^1)^\top & \cdots & y_N & (x^N)^\top & (\Lambda^1)^\top & \cdots & (\Lambda_N)^\top & s_1 & \cdots & s_N \end{array} \right]^\top, \\
M = \begin{bmatrix} 0 & M_{12} & 0 \\ M_{21} & 0 & 0 \end{bmatrix}, \ q = 0, \quad N = \begin{bmatrix} N_{11} & 0 & 0 \\ N_{21} & N_{22} & N_{23} \end{bmatrix}, \ r = \begin{bmatrix} 0 \\ r_2 \end{bmatrix}, \\
C = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \end{bmatrix}, \ d = \left[\begin{array}{ccc|ccc} 1 & \cdots & 1 & \rho_1 & \cdots & \rho_N \end{array} \right]^\top,
\end{aligned}
$$

where

$$
\begin{aligned}
y_i = (y_{ij})_{j \in \mathcal{I}_{-i}} \in \Re^{N-1}, \quad \Lambda_i = \begin{bmatrix} \mu_{ij} \\ \lambda_{ij} \end{bmatrix}_{j \in \mathcal{I}_{-i}} \in \Re^{m_{-i}+N-1}, \\
\rho_i = (\rho_{ij})_{j \in \mathcal{I}_{-i}} \in \Re^{N-1}, \quad r_2 = (c^i)_{i \in \mathcal{I}} \in \Re^m,
\end{aligned}
$$

for $i \in \mathcal{I}$. Moreover, N_{21} is a block matrix whose (i,j)-block elements are

$$
(N_{21})_{ij} = \begin{bmatrix} 0 & A_{ij} \end{bmatrix} \in \Re^{m_i \times (m_j+N-1)} \quad (i, j \in \mathcal{I}),
$$

and $M_{12}, M_{21}, N_{11}, N_{22}, N_{23}, C_{11}$ and C_{22} are block diagonal matrices whose block diagonal elements are

$$
(M_{12})_{ii} = I_{m_{-i}+N-1}, \ (M_{21})_{ii} = \begin{bmatrix} 0 & I_{m_i} \end{bmatrix} \in \Re^{m_i \times (m_i+N-1)},
$$

$$
(N_{11})_{ii} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \tilde{A}_{i1}^\top \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \tilde{A}_{i2}^\top \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & \tilde{A}_{iN}^\top
\end{bmatrix} \in \Re^{(m_{-i}+N-1) \times (m_i+N-1)},
$$

$$(N_{22})_{ii} = \begin{bmatrix} 0 & -\tilde{A}_{i1} & \cdots & 0 & -\tilde{A}_{iN} \end{bmatrix} \in \Re^{m_i \times (m_{-i}+N-1)}, \quad (N_{23})_{ii} = e_{m_i} \in \Re^{m_i},$$
$$(C_{11})_{ii} = \begin{bmatrix} 0 & e_{m_i}^\top \end{bmatrix} \in \Re^{1 \times (m_i+1)}, \quad (C_{22})_{ii} = \Gamma_i \in \Re^{(N-1)\times(m_{-i}+1)}$$

with Γ_i being the block diagonal matrix whose block diagonal elements are

$$(\Gamma_i)_{jj} = \begin{bmatrix} 1 & 0 \end{bmatrix} \in \Re^{1\times(m_j+1)} \ (j \in \mathcal{I}_{-i}).$$

5.1.2 Existence and uniqueness of robust Nash equilibrium

Next, we study existence and uniqueness of the robust Nash equilibrium under Assumption 3. In the following analyses, we make use of the results from Theorems 3.2 and 4.3.

Theorem 5.1. *Suppose that the cost functions and the strategy sets are given by* (5.3) *and* (5.4)*, respectively. Suppose further that Assumption* 3 *holds. Then, there exists at least one robust Nash equilibrium.*

Proof. From Theorem 3.2, it suffices to show Assumption 1 holds. Assumption 1(a) holds since $G_i(x^i, x^{-i}, u^i) = G_i(x^i, x^{-i}, (A_{ij})_{j\in\mathcal{I}}, c^i) = \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top (\sum_{j\in\mathcal{I}_{-i}} A_{ij} x^j + c^i)$. It is easily seen that Assumption 3 implies Assumption 1(b)(c). Moreover, Assumption 1(d) holds since $A_{ii} \succeq 0$ and each player takes a mixed strategy. This completes the proof. □

Theorem 5.2. *Suppose that the cost functions and the strategy sets are given by* (5.3) *and* (5.4)*, respectively. Suppose further that Assumption* 3 *holds. Then there exists a unique robust Nash equilibrium, provided that*

$$A := \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{N1} & \cdots & \cdots & A_{NN} \end{bmatrix} \succ 0. \tag{5.7}$$

Proof. From Theorem 4.3, it suffices to show that Assumptions 1 and 2 hold. We can see that Assumption 1 holds in a way analogous to the proof of Theorem 5.1. Assumption 2(a)(b) readily follows from (5.3) and Assumption 3(a) with $D_i = \{\delta x^{-i} = (\delta x^{ij})_{j\in\mathcal{I}_{-i}} \mid \|\delta x^{ij}\| \leq \rho_{ij},\ e_{m_j}^\top \delta x^{ij} = 0,\ j \in \mathcal{I}_{-i}\}$. Assumption 2(c-ii) also holds from (5.7) and $\nabla\mathcal{F}(x)^\top = A$. □

5.2 Uncertainty in the cost matrices and vectors

In this subsection, we consider the case where each player can estimate the opponents' strategies exactly, but estimates his cost matrices and vectors uncertainly. We first make the following assumption.

Assumption 4. *For each $i \in \mathcal{I}$, uncertainty sets $X_i(\cdot)$ and U_i $(i \in \mathcal{I})$ are given as follows.*

(a) $X_i(x^{-i}) := \{x^{-i}\}$.

(b) $U_i := (\prod_{j\in\mathcal{I}} D_{A_{ij}}) \times D_{c^i}$ *with* $D_{A_{ij}} := \{A_{ij} + \delta A_{ij} \mid \|\delta A_{ij}\|_F \leq \rho_{ij}\} \subseteq \Re^{m_i\times m_j}$ *and* $D_{c^i} := \{c^i + \delta c^i \mid \|\delta c^i\| \leq \gamma_i\} \subseteq \Re^{m_i}$ *for some nonnegative scalars* ρ_{ij} *and* γ_i. *Moreover,* $A_{ii} + \rho_{ii} I$ *is symmetric and positive semidefinite.*

Under this assumption, the worst cost function $\tilde{f}_i$ in (2.4) can be rewritten as follows:

$$\begin{aligned}\tilde{f}_i(x^i, x^{-i}) &= \max\left\{\frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij} x^j + (\hat{c}^i)^\top x^i \;\middle|\; \hat{A}_{ij} \in D_{A_{ij}}, \hat{c}^i \in D_{c^i}\ (j \in \mathcal{I})\right\}\\ &= \frac{1}{2}(x^i)^\top A_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} (x^i)^\top A_{ij} x^j + (c^i)^\top x^i\\ &\quad + \max_{\substack{\|\delta A_{ij}\|_F \leq \rho_{ij}\\ j\in\mathcal{I}}} \left\{\frac{1}{2}(x^i)^\top \delta A_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} (x^i)^\top \delta A_{ij} x^j\right\} + \max_{\|\delta c^i\|\leq\gamma_i} \{(\delta c^i)^\top x^i\}\\ &= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + (c^i)^\top x^i + \sum_{j\in\mathcal{I}_{-i}} \left((x^i)^\top A_{ij} x^j + \rho_{ij}\|x^i\|\|x^j\|\right) + \gamma_i \|x^i\|.\end{aligned} \tag{5.8}$$

The last equality follows from

$$\max_{\|M\|_F\leq\rho} y^\top M z = \max_{\|M\|_F\leq\rho} (z\otimes y)^\top \mathrm{vec}\,(M) = \|z \otimes y\|\rho = \rho\|y\|\|z\|,$$

for any $y \in \Re^n$, $z \in \Re^m$ and $\rho \geq 0$, where $\otimes$ denotes the Kronecker product [16, Sections 4.2 and 4.3].

5.2.1 Reformulation as SOCCP

We first reformulate the robust Nash equilibrium problem as SOCCP (5.2) under Assumption 4. By using (5.8) and an auxiliary variable $y_i \in \Re$, the minimization problem (2.4) can be rewritten as the following SOCP:

$$\begin{array}{ll}\underset{x^i, y_i}{\text{minimize}} & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j\in\mathcal{I}_{-i}} \left((x^i)^\top A_{ij} x^j + \rho_{ij}\|x^j\| y_i\right) + \gamma_i y_i\\ \text{subject to} & \|x^i\| \leq y_i, \quad x^i \geq 0, \quad e_{m_i}^\top x^i = 1,\end{array} \tag{5.9}$$

and its KKT conditions are given by

$$\begin{gathered}\mathcal{K}^{m_i+1} \ni \begin{bmatrix} y_i \\ x^i \end{bmatrix} \perp \begin{bmatrix} \sum_{j\in\mathcal{I}_{-i}} \rho_{ij}\|x^j\| + \gamma_i \\ (A_{ii} + \rho_{ii} I) x^i + \sum_{j\in\mathcal{I}_{-i}} A_{ij} x^j + e_{m_i} s_i - \lambda^i + c^i \end{bmatrix} \in \mathcal{K}^{m_i+1}\\ \Re_+^{m_i} \ni \lambda^i \perp x^i \in \Re_+^{m_i}, \quad e_{m_i}^\top x^i = 1,\end{gathered} \tag{5.10}$$

where $\lambda^i \in \Re^{m_i}$ and $s_i \in \Re$ are Lagrange multipliers. It is not straightforward to reformulate the robust Nash equilibrium problem as SOCCP (5.2), since the KKT conditions (5.10) contains the nonlinear term $\|x^j\|$. However, by introducing auxiliary variables $z_j \in \Re, u^j \in \Re^{m_j}$, we can rewrite (5.10) as follows:

$$\mathcal{K}^{m_i+1} \ni \begin{bmatrix} y_i \\ x^i \end{bmatrix} \perp \begin{bmatrix} \sum_{j\in\mathcal{I}_{-i}} \rho_{ij} z_j + \gamma_i \\ (A_{ii} + \rho_{ii} I)x^i + \sum_{j\in\mathcal{I}_{-i}} A_{ij}x^j + e_{m_i}s_i - \lambda^i + c^i \end{bmatrix} \in \mathcal{K}^{m_i+1}, \quad e_{m_i}^\top x^i = 1,$$
$$\Re_+^{m_i} \ni \lambda^i \perp x^i \in \Re_+^{m_i}, \quad \mathcal{K}^{m_j+1} \ni \begin{bmatrix} z_j \\ x^j \end{bmatrix} \perp \begin{bmatrix} y_j \\ u^j \end{bmatrix} \in \mathcal{K}^{m_j+1} \ (j \in \mathcal{I}_{-i}). \tag{5.11}$$

In fact, the equivalence between (5.10) and (5.11) can be verified as follows. If SOCCP (5.10) holds, then we readily obtain (5.10) by letting

$$z_j := \|x^j\|, \quad u^j := -\frac{x^j y_j}{\|x^j\|}.$$

Conversely, suppose that (5.11) holds for any $i \in \mathcal{I}$. Then, by the complementarity condition

$$\mathcal{K}^{m_j+1} \ni \begin{bmatrix} z_j \\ x^j \end{bmatrix} \perp \begin{bmatrix} y_j \\ u^j \end{bmatrix} \in \mathcal{K}^{m_j+1} \tag{5.12}$$

in (5.11), we have

$$0 = z_j y_j + (x^j)^\top u^j \geq z_j y_j - \|x^j\|\|u^j\| \geq z_j y_j - \|x^j\| y_j, \tag{5.13}$$

where the inequalities follow from the Cauchy-Schwarz inequality and $\left[\begin{smallmatrix} y_j \\ u^j \end{smallmatrix}\right] \in \mathcal{K}^{m_j+1}$. Moreover, we must have $y_j > 0$ since $e_{m_j}^\top x^j = 1$ and $\left[\begin{smallmatrix} y_j \\ x^j \end{smallmatrix}\right] \in \mathcal{K}^{m_j+1}$ from (5.11) with i replaced by j. Dividing (5.13) by $y_j > 0$, we obtain $\|x^j\| \geq z_j$. However, since $\|x^j\| \leq z_j$ from (5.12), we obtain $\|x^j\| = z_j$. This implies (5.10) holds.

Now, we can reformulate the robust Nash equilibrium problem as SOCCP (5.2) with

$$l = 3m + 2N, \ \tau = N, \ \mathcal{K} = \prod_{i\in\mathcal{I}} \mathcal{K}^{m_i+1} \times \prod_{i\in\mathcal{I}} \Re_+^{m_i} \times \prod_{i\in\mathcal{I}} \mathcal{K}^{m_i+1},$$

$$\zeta = \begin{bmatrix} y_1 \ (x^1)^\top \ z_1 \ (u^1)^\top \ \cdots \ y_N \ (x^N)^\top \ z_N \ (u^N)^\top \ (\lambda^1)^\top \ \cdots \ (\lambda^N)^\top \ s_1 \ \cdots \ s_N \end{bmatrix}^\top,$$
$$M = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ M_{31} & 0 & 0 \end{bmatrix}, q = 0, \quad N = \left[\begin{array}{c|c|c} N_{11} & N_{12} & N_{13} \\ N_{21} & 0 & 0 \\ N_{31} & 0 & 0 \end{array}\right], \quad r = \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & e_{m_1}^\top & 0 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & e_{m_N}^\top & 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

where N_{11} is a block matrix whose (i,j)-block elements are given by

$$(N_{11})_{ij} = \begin{cases} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{ii} + \rho_{ii} I & 0 & 0 \end{bmatrix} (i = j) \\ \\ \begin{bmatrix} 0 & 0 & \rho_{ij} & 0 \\ 0 & A_{ij} & 0 & 0 \end{bmatrix} (i \neq j) \end{cases} \qquad (i, j \in \mathcal{I}),$$

$M_{11}, M_{22}, M_{31}, N_{12}, N_{13}, N_{21}$ and N_{31} are block diagonal matrices whose block diagonal elements are

$$(M_{11})_{ii} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & I_{m_i} & 0 & 0 \end{bmatrix} \in \Re^{(m_i+1)\times 2(m_i+1)}, \quad (M_{22})_{ii} = I_{m_i},$$
$$(M_{31})_{ii} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & I_{m_i} & 0 & 0 \end{bmatrix} \in \Re^{(m_i+1)\times 2(m_i+1)},$$
$$(N_{12})_{ii} = \begin{bmatrix} 0 \\ -I_{m_i} \end{bmatrix}, \quad (N_{13})_{ii} = \begin{bmatrix} 0 \\ e_{m_i} \end{bmatrix},$$
$$(N_{21})_{ii} = \begin{bmatrix} 0 & I_{m_i} & 0 & 0 \end{bmatrix}, \quad (N_{31})_{ii} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m_i} \end{bmatrix},$$

and $r_1 = \begin{bmatrix} \gamma_i \\ c^i \end{bmatrix}_{i \in \mathcal{I}} \in \Re^{m+N}$.

5.2.2 Existence and uniqueness of robust Nash equilibrium

Next, we study existence and uniqueness of the robust Nash equilibrium under Assumption 4. Unlike the analyses in Subsection 5.1.2, we do not use the results from Theorems 3.2 and 4.3. Instead of them, we exploit the concrete structure (5.8) of the worst cost function $\tilde{f}_i$.

Theorem 5.3. *Suppose that the cost functions and the strategy sets are given by* (5.3) *and* (5.4)*, respectively. Suppose further that Assumption* 4 *holds. Then, there exists at least one robust Nash equilibrium.*

Proof. From (5.8), for arbitrarily fixed $x^{-i} \in S_{-i}$, the function $\tilde{f}_i$ can be expressed as $\tilde{f}_i(x^i, x^{-i}) = \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \alpha \|x^i\| + \xi^\top x^i$ with some $\alpha \in \Re$ and $\xi \in \Re^{m_i}$ not depending on x^i. Since $A_{ii} + \rho_{ii} I \succeq 0$ and $\alpha \geq 0$, $\tilde{f}_i(\cdot, x^{-i})$ is convex for any fixed $x^{-i} \in S_{-i}$. Hence, letting $\theta_i(x^i, x^{-i}) := \tilde{f}_i(x^i, x^{-i})$ in Lemma 3.1 yields the desired result. □

Note that Theorem 3.2 cannot be applied to the proof, since Assumption 4 does not imply Assumption 1(d). In fact, $A_{ii} + \delta A_{ii}$ is not necessarily positive semidefinite even if A_{ii} is positive semidefinite, that is, $f_i^{u^i}(\cdot, x^{-i})$ may be nonconvex for some δA_{ii} and $x^{-i} \in S_{-i}$.

We next give sufficient conditions for the uniqueness of a robust Nash equilibrium. To simplify the notations, we define the following vector and matrices:

$$
\begin{gathered}
A := (A_{ij})_{i\in\mathcal{I},\, j\in\mathcal{I}}, \;\; P := (\rho_{ij})_{i\in\mathcal{I},\, j\in\mathcal{I}} \\
Q(x) := \operatorname{diag}\left[\left(\frac{1}{\|x^i\|}\sum_{j=1}^{N}\rho_{ij}\|x^j\|\right)\left(I - v^i(v^i)^\top\right)\right], \\
V(x) := \operatorname{diag}\left(v^1, \dots, v^N\right), \;\; \text{where } v^i := x^i/\|x^i\|.
\end{gathered}
$$

Then, we have the following lemma.

Lemma 5.4. *For each $i \in \mathcal{I}$, let $\tilde{f}_i : \Re^{m_i} \to \Re$ and $S_i \subset \Re^m$ be given by* (5.8) *and* (5.4)*, respectively. Then, for any $x \in S$, the set-valued mapping $\tilde{\mathcal{F}}$ given by* (4.3) *satisfies $\tilde{\mathcal{F}}(x) = \{\tilde{F}(x)\}$ with $\tilde{F}(x) := \left(\nabla_i \tilde{f}_i(x^i, x^{-i})\right)_{i\in\mathcal{I}}$. Moreover, the following statements hold.*

(a) *Function $\tilde{F}$ is differentiable at any $x \in S$ with the Jacobian*

$$
\nabla\tilde{F}(x)^\top = A + V(x)PV(x)^\top + Q(x).
$$

(b) *$Q(x) \succeq 0$ for any $x \in S$.*

(c) *If $P \succ 0$, then $V(x)PV(x)^\top + Q(x) \succ 0$ for any $x \in S$.*

Proof. In what follows, we write $V = V(x)$ and $Q = Q(x)$ for convenience.

First, we show (a). Since $0 \notin S_i$ for all $i \in \mathcal{I}$, the derivative of $\tilde{f}_i$ with respect to x^i is given by

$$
\nabla_i \tilde{f}_i(x^i, x^{-i}) = (A_{ii} + \rho_{ii} I)x^i + \sum_{j\in\mathcal{I}_{-i}}\left(A_{ij}x^j + \rho_{ij}\|x^j\|\frac{x^i}{\|x^i\|}\right) + c^i.
$$

Moreover, for each $i \in \mathcal{I}$, the partial derivative of $\nabla_i \tilde{f}_i(x^i, x^{-i})$ with respect to x^k is given by

$$
\nabla_{ki}\tilde{f}_i(x^i, x^{-i}) = \begin{cases} A_{ii} + \rho_{ii}I + \dfrac{1}{\|x^i\|}\left(\displaystyle\sum_{j\in\mathcal{I}_{-i}}\rho_{ij}\|x^j\|\right)(I - v^i(v^i)^\top) & (k = i) \\ A_{ik}^\top + \rho_{ik}v^k(v^i)^\top & (k \neq i). \end{cases} \tag{5.14}
$$

Arraying (5.14) for $(k,i) = (1,1), (1,2), \dots, (N,N)$, we obtain (a).

Next, we show (b). Let $w = (w^1, \dots, w^N) \in \Re^{m_1} \times \cdots \times \Re^{m_N}$ be an arbitrary vector. Then, we have

$$
w^\top Q w = \sum_{i=1}^{N}\left(\frac{1}{\|x^i\|}\sum_{j=1}^{N}\rho_{ij}\|x^j\|\right)\left(\|w^i\|^2 - ((w^i)^\top v^i)^2\right) \geq 0, \tag{5.15}
$$

where the inequality is due to $\|v^i\| = 1$ and the Cauchy-Schwarz inequality. Hence, Q is positive semidefinite.

Finally, we show (c). Let $w = (w^1, \dots, w^N) \in \Re^{m_1} \times \cdots \times \Re^{m_N}$ be an arbitrary nonzero vector. Since $VPV^\top \succeq 0$ from $P \succ 0$, we have $w^\top(VPV^\top)w \geq 0$, where the equality holds only when $w \in \ker V^\top$, i.e., w lies in the orthogonal complement of the subspace generated by $v^1, \ldots, v^N$. In addition, from (5.15) we have $w^\top Q w \geq 0$, where the equality holds only when $w^i = \lambda_i v^i$ for some $\lambda_i \in \Re$ $(i \in \mathcal{I})$. Therefore, we have $w^\top(VPV^\top + Q)w \geq 0$, and the equality holds only if $w \in \ker V^\top$ and $w^i = \lambda_i v^i$ $(i \in \mathcal{I})$. However, there is no vector satisfying these two conditions except zero. Hence, we have $w^\top(VPV^\top + Q)w > 0$. □

We now obtain the following theorem.

Theorem 5.5. *Suppose that the cost functions and the strategy sets are given by* (5.3) *and* (5.4)*, respectively. Suppose further that Assumption* 4 *holds. Then, there exists a unique robust Nash equilibrium, if either* (i) $A \succ 0$ *and* $P \succeq 0$ *or* (ii) $A \succeq 0$ *and* $P \succ 0$ *holds.*

Proof. If (i) holds, then we have $A + VPV^\top + Q \succ 0$, since $VPV^\top \succeq 0$ and $Q \succeq 0$ from Lemma 5.4(b). If (ii) holds, then we also have $A + VPV^\top + Q \succ 0$, since $VPV^\top + Q \succ 0$ from Lemma 5.4(c). Hence, by Lemma 5.4(a), we have $\nabla \tilde{F}(x) \succ 0$ for any $x \in S$, i.e., $\mathcal{F}$ is strictly monotone on S. Thus, from Proposition 4.1 and Theorem 5.3, the game has a unique robust Nash equilibrium. □

6 Numerical experiments

In this section, we solve some robust Nash equilibrium problems with various sizes of uncertainty sets, by using the SOCCP reformulation approaches discussed in the previous section. Then, we observe some properties of obtained equilibria and values of the cost functions. For solving the reformulated SOCCPs, we apply the Newton-type method combined with a smoothing regularization technique [14]. All programs are coded in MATLAB 7 and run on a computer with 3.06GHz CPU and 1GB memories.

6.1 Relationship between actual costs and size of uncertain sets

In the first experiment, we consider a three-person game where the cost functions are given by (5.3) with cost matrices and vectors:

$$A_{ii} = \begin{bmatrix} 8 & 2 & -4 \\ 2 & 7 & -2 \\ -4 & -2 & 13 \end{bmatrix}, \ A_{ij} = \begin{bmatrix} 2 & -1 & 0 \\ -4 & 0 & -2 \\ -3 & 1 & 2 \end{bmatrix}, \ c^i = \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix} \quad (j \in \mathcal{I}_{-i})$$

for each $i = 1, 2, 3$. We note that each player has the same cost function.

We first consider the case where Assumption 3 holds and each player $i \in \mathcal{I}$ chooses parameters ρ_{ij} as

$$(\rho_{ij}) = \begin{bmatrix} * & 0.0001 & 0.0001 \\ 0.02 & * & 0.02 \\ 0.05 & 0.05 & * \end{bmatrix}. \tag{6.1}$$

This implies that player 1 hardly takes the uncertainty into consideration, whereas player 3 is more careful in choosing his strategy. Under such assumptions, the game has a unique robust Nash equilibrium $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ given by

$$\tilde{x}^1 = (0.310, 0.318, 0.372),\ \tilde{x}^2 = (0.353, 0.284, 0.363),\ \tilde{x}^3 = (0.410, 0.240, 0.350).$$

As assumed in (B) of Section 2, each player's actual cost is evaluated with $\tilde{x}^{-i}$ replaced by $\tilde{x}^{-i} + \delta x^{-i}$ with a certain noise vector $\delta x^{-i} \in \Re^6$. In our experiment, we generate $\delta x^{-i} := (\delta x^{ij})_{j \in \mathcal{I}_{-i}} \in \Re^6$ as follows: we first generate random vectors $\delta y^{ij} \in \Re^2$ for each $j \in \mathcal{I}_{-i}$ so that each component follows the normal distribution $N(0, 0.01)$, and then, map them onto the hyperplane $\{x \in \Re^3 \mid x_1 + x_2 + x_3 = 0\}$ by using an appropriate orthonormal transformation.

For each $i = 1, 2, 3$, we generate 10000 different samples of noise vector δx^{-i}, and observe the distribution of the actual cost $f_i(\tilde{x}^i, \tilde{x}^{-i} + \delta x^{-i})$. Moreover, we compare the actual cost $f_i(\tilde{x}^i, \tilde{x}^{-i} + \delta x^{-i})$ with the presumed worst cost $\tilde{f}_i(\tilde{x})$. The results are shown in Table 1 and Figures 1 − 3.

Table 1 Uncertainty in the opponents' strategy

	player 1	player 2	player 3
$\tilde{f}_i(\tilde{x}^i, \tilde{x}^{-i})$	−1.5105	−1.2782	−0.9680
$E(f_i(\tilde{x}^i, \tilde{x}^{-i} + \delta x^{-i}))$	−1.5087	−1.4316	−1.3346
Percentage of worse cases	50.98%	20.90%	0.63%

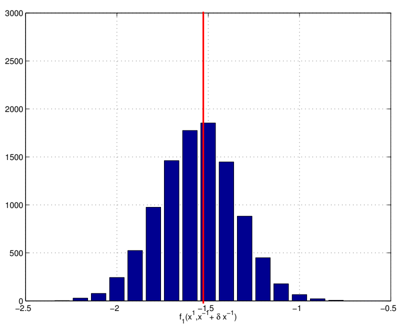

Figure 1 Player 1's cost

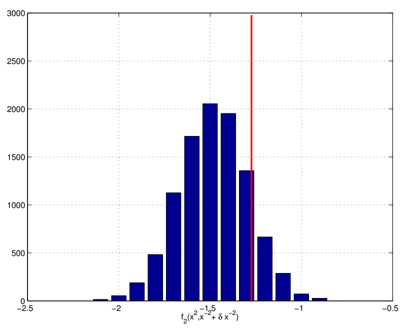

Figure 2 Player 2's cost

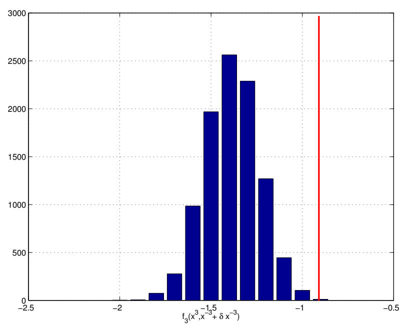

Figure 3 Player 3's cost

In each row of Table 1, we give the values of $\tilde{f}_i(\tilde{x})$, the mean values of 10000 samples of $f_i(\tilde{x}^i, \tilde{x}^{-i} + \delta x^{-i})$, and the percentage of which the value of $f_i(\tilde{x}^i, \tilde{x}^{-i} + \delta x^{-i})$ is greater than

$\tilde{f}_i(\tilde{x}^i, \tilde{x}^{-i})$ among 10000 samples. From Table 1, we can see that the mean value of player 1's cost is smaller than that of player 3, though player 3 take the uncertainty into account more than player 1. In fact, such a result does not always hold, and we can see an opposite result in another game. However, the last row of the table shows that, as a player considers the region of uncertainty larger, the possibility of avoiding the presumed worst case becomes higher. Figures 1−3 are histograms which show each player's actual costs for 10000 cases. The width of each bar is 0.1 and a vertical line represents the value of $\tilde{f}_i(\tilde{x})$. Indeed, the histograms show that player 1's actual cost exceeds the presumed worst cost for almost a half of 10000 samples, whereas player 3's actual cost seldom exceeds the presumed one. Moreover, even if it becomes worse, its difference is very small.

Next, we consider the case where Assumption 4 holds with the following parameters:

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 0.0001 & 0.0001 & 0.0001 \\ 0.50 & 0.50 & 0.50 \\ 1.50 & 1.50 & 1.50 \end{bmatrix}, \quad \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} 0.0001 \\ 0.50 \\ 1.50 \end{bmatrix} \tag{6.2}$$

Then the robust Nash equilibrium $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ is uniquely given by

$$\tilde{x}^1 = (0.364, 0.272, 0.365), \ \tilde{x}^2 = (0.344, 0.294, 0.362), \ \tilde{x}^3 = (0.334, 0.309, 0.358).$$

Similarly to the previous experiment, we generate 10000 samples of noise matrix $\delta A = (\delta A_{ij})_{(i,j)\in\mathcal{I}\times\mathcal{I}}$ and vector $\delta c = (\delta c^i)_{i\in\mathcal{I}}$ so that each element follows the standard normal distribution. The results are shown in Table 2 and Figures 4−6, where the width of each bar is 0.2.

Table 2 Uncertainty in the cost matrices and vectors

	player 1	player 2	player 3
$\tilde{f}_i(\tilde{x}^i, \tilde{x}^{-i})$	−1.3713	−0.9512	−0.0692
$E\big(f_i^{u^i+\delta u^i}(\tilde{x}^i, \tilde{x}^{-i})\big)$	−1.3690	−1.3838	−1.4038
Percentage of worse cases	50.50%	28.54%	3.65%

The table and figures show, like in the previous experiment, that the actual cost of player 3 is rarely worse than the presumed worst cost. However, the mean of player 3's actual cost is smaller than player 1's mean cost.

6.2 Relationship between size of uncertain sets and robust Nash equilibria

In this subsection, we change the size of uncertain sets variously, and see the trajectory of the robust Nash equilibria.

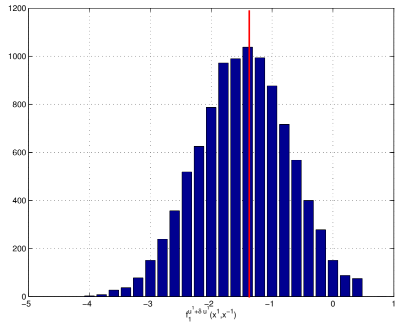

Figure 4　Player 1's cost

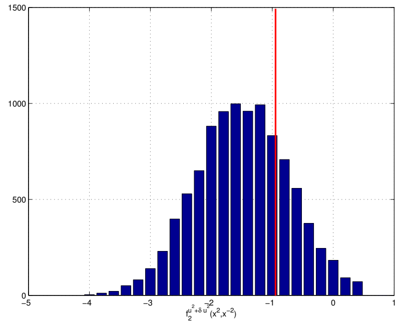

Figure 5　Player 2's cost

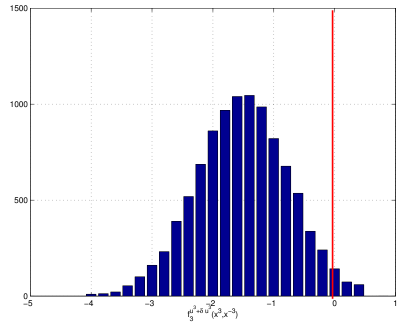

Figure 6　Player 3's cost

First, we consider the three-person game where the cost functions are defined by (5.3) with the following cost matrices and vectors:

$$A_{11} = \begin{bmatrix} 27 & -4 & 9 \\ -4 & 18 & 0 \\ 9 & 0 & 19 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 6 & 2 & 13 \\ -3 & -10 & 0 \\ -4 & -4 & 3 \end{bmatrix}, \ A_{13} = \begin{bmatrix} -10 & 6 & 10 \\ -19 & 0 & -7 \\ 12 & -10 & -1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 5 & -3 & -2 \\ 0 & -12 & -2 \\ 13 & 2 & 3 \end{bmatrix}, \ A_{22} = \begin{bmatrix} 18 & -7 & 2 \\ -7 & 41 & 0 \\ 2 & 0 & 18 \end{bmatrix}, \ A_{23} = \begin{bmatrix} -4 & -9 & 1 \\ 0 & 5 & 12 \\ 1 & 5 & -3 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} -7 & 17 & 10 \\ 7 & -4 & -13 \\ -10 & -10 & 0 \end{bmatrix}, \ A_{32} = \begin{bmatrix} -3 & 4 & 0 \\ -13 & 3 & 4 \\ 3 & 9 & 1 \end{bmatrix}, \ A_{33} = \begin{bmatrix} 24 & 9 & -17 \\ 9 & 28 & -5 \\ -17 & -5 & 31 \end{bmatrix}$$

$$c^1 = c^2 = c^3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top.$$

Then, the game has a unique Nash equilibrium $\overline{x} = (\overline{x}^1, \overline{x}^2, \overline{x}^3)$ given by

$$\overline{x}^1 = (0.0000, 0.4967, 0.5033),\ \overline{x}^2 = (0.7036, 0.0000, 0.2964),\ \overline{x}^3 = (0.0831, 0.4304, 0.4866).$$

We also consider the robust Nash equilibrium problem under Assumption 3 with $\rho_{ij} = \rho$ for all $i, j = 1, 2, 3 (j \neq i)$, where ρ is chosen as 0.05, 0.1 and 0.2. Table 3 and Figure 7 show the change of the robust Nash equilibria with the choice of ρ. In Figure 7, the horizontal and vertical axes denote the first and second components of each player's strategy, respectively[*3]. This figure intimates that the robust Nash equilibria move continuously as the sizes of uncertainty sets change continuously.

Next, we consider another game where the cost functions are defined by (5.3) with cost matrices and vectors:

$$A_{11} = \begin{bmatrix} 12.486 & 1.249 & 5.650 \\ 1.249 & 2.516 & 4.361 \\ 5.650 & 4.361 & 13.980 \end{bmatrix}, A_{12} = \begin{bmatrix} -5.095 & -7.403 & -4.152 \\ -1.459 & -8.215 & -2.511 \\ -6.228 & -3.783 & -5.306 \end{bmatrix}, A_{13} = \begin{bmatrix} -8.250 & -8.514 & -7.015 \\ -8.178 & -2.222 & -1.091 \\ -2.004 & -5.367 & -4.486 \end{bmatrix}$$

[*3] Since each player takes the mixed strategy, the third component is uniquely determined.

Table 3 Sizes of uncertainty sets and robust Nash equilibria

ρ	robust Nash equilibrium $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$
0.05	((0.0000,0.5230,0.4770), (0.6978,0.0283,0.2738), (0.0394,0.4938,0.4668))
0.10	((0.0000,0.5348,0.4652), (0.6659,0.0244,0.3097), (0.0677,0.4521,0.4802))
0.20	((0.0000,0.5396,0.4604), (0.6100,0.0228,0.3673), (0.1162,0.3812,0.5026))

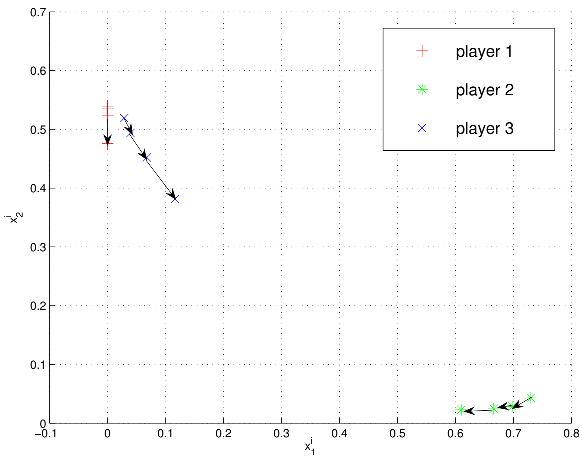

Figure 7 Trajectory of each player's strategy at the robust Nash equilibria

$$A_{21} = \begin{bmatrix} -7.236 & -2.175 & -5.223 \\ -1.980 & -7.579 & -3.141 \\ -3.180 & -4.678 & -1.155 \end{bmatrix}, A_{22} = \begin{bmatrix} 2.064 & 3.041 & 3.228 \\ 3.041 & 6.563 & 2.341 \\ 3.228 & 2.341 & 14.720 \end{bmatrix}, A_{23} = \begin{bmatrix} -5.420 & -1.153 & -1.514 \\ -4.874 & -6.610 & -3.609 \\ -7.741 & -7.763 & -5.577 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} -2.338 & -2.981 & -6.197 \\ -7.629 & -4.076 & -4.096 \\ -5.475 & -6.967 & -6.298 \end{bmatrix}, A_{32} = \begin{bmatrix} -3.912 & -3.988 & -1.043 \\ -4.867 & -1.407 & -1.981 \\ -4.844 & -7.212 & -3.992 \end{bmatrix}, A_{33} = \begin{bmatrix} 34.478 & -13.084 & -1.478 \\ -13.084 & 17.336 & -1.243 \\ -1.478 & -1.243 & 20.047 \end{bmatrix}$$

$$c^1 = c^2 = c^3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top .$$

This game has the following three Nash equilibria[*4]:

1: $(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \big((0.490, 0.510, 0.000),\ (0.000, 0.688, 0.312),\ (0.195, 0.360, 0.443)\big)$.

2: $(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \big((0.715, 0.011, 0.274),\ (1.000, 0.000, 0.000),\ (0.234, 0.501, 0.266)\big)$,

3: $(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \big((0.671, 0.304, 0.025),\ (0.596, 0.208, 0.196),\ (0.208, 0.456, 0.335)\big)$,

Moreover, we consider the robust Nash equilibrium problems under Assumption 4 with parameters

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{21} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 0.01 + k & 0.01 & 0.01 \\ 0.01 & 0.01 + k & 0.01 \\ 0.01 & 0.01 & 0.01 + k \end{bmatrix}, \quad \gamma_1 = \gamma_2 = \gamma_3 = 0,$$

[*4] We can find all Nash equilibria by using a branch and bound based approach.

where k is chosen as $k = 0.1$, 0.5, 1.0, 1.1485, 1.5. In order to obtain as many equilibria as possible, we solve the equivalent SOCCP with randomly generated 100 starting points[*5]. Table 4 shows the concrete values of obtained robust Nash equilibria. For $k = 0.1, 0.5, 1.0, 1.1485$, we obtain three robust Nash equilibria. However, for $k = 1.5$, we obtain only one robust Nash equilibrium. Figure 8 shows the trajectory of player 1's strategies at the robust Nash equilibria for each k[*6], in which the vertical and horizontal axes denote the first and second components of the robust Nash equilibria, respectively. Figure 8 indicates that two of the three equilibria are getting closer to each other as k increases, and they almost coincide at $k = 1.1485$. Furthermore, at $k = 1.5$, the two equilibria disappear and only one equilibrium is obtained.

Table 4 Sizes of uncertainty sets and obtained robust Nash equilibria

k		robust Nash equilibria
0.1	1:	((0.490, 0.510, 0.000), (0.000, 0.685, 0.315), (0.198, 0.360, 0.442))
	2:	((0.708, 0.020, 0.272), (1.000, 0.000, 0.000), (0.234, 0.499, 0.267))
	3:	((0.667, 0.294, 0.039), (0.608, 0.200, 0.193), (0.210, 0.457, 0.333))
0.5	1:	((0.492, 0.508, 0.000), (0.000, 0.676, 0.324), (0.199, 0.363, 0.439))
	2:	((0.684, 0.057, 0.259), (0.949, 0.000, 0.051), (0.232, 0.491, 0.277))
	3:	((0.657, 0.252, 0.091), (0.660, 0.161, 0.179), (0.216, 0.460, 0.325))
1.0	1:	((0.493, 0.507, 0.000), (0.000, 0.666, 0.334), (0.201, 0.363, 0.436))
	2:	((0.658, 0.094, 0.249), (0.895, 0.000, 0.105), (0.231, 0.483, 0.286))
	3:	((0.650, 0.155, 0.195), (0.800, 0.059, 0.141), (0.226, 0.473, 0.301))
1.1485	1:	((0.494, 0.506, 0.000), (0.000, 0.664, 0.336), (0.202, 0.364, 0.435))
	2:	((0.6507, 0.1026, 0.2467), (0.8810, 0.0000, 0.1190), (0.2312, 0.4807, 0.2881))
	3:	((0.6507, 0.1027, 0.2466), (0.8809, 0.0001, 0.1190), (0.2312, 0.4807, 0.2881))
1.5	1:	((0.507, 0.493, 0.000), (0.052, 0.619, 0.329), (0.204, 0.372, 0.425))

[*5] Since we employ an iterative method, we can choose an arbitrary starting point. Indeed, it is expected that a different starting point can lead to a different solution when the SOCCP has multiple solutions.

[*6] We omit the other players' trajectories since they are similar to player 1's.

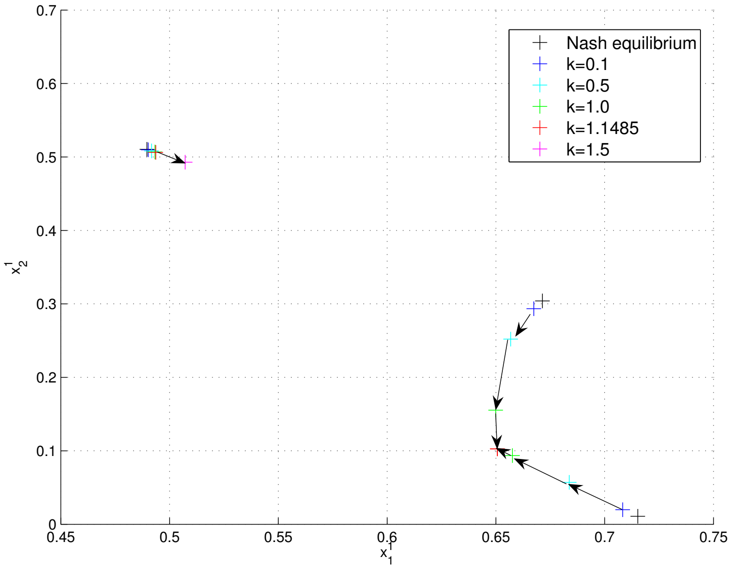

Figure 8 Trajectory of player 1's strategies at the robust Nash equilibria

7 Concluding remarks

In this paper, we have extended the concept of robust Nash equilibrium to N-person non-cooperative games with nonlinear cost functions, and derived sufficient conditions for existence and uniqueness of the robust Nash equilibria by means of the GVIP or VIP reformulation techniques. In addition, we have shown that the robust Nash equilibrium problems with quadratic cost functions and uncertainty sets can be reformulated as SOCCPs. We also solved some examples of the robust Nash equilibrium problem, and observed some numerical properties.

We still have some future issues to be addressed. One important issue is to weaken the sufficient conditions for uniqueness of the robust Nash equilibrium. In fact, the uniqueness conditions shown in the paper is rather restrictive, and there seems to remain much room for the improvement. Another issue is to consider the SOCCP reformulation for the robust Nash equilibrium problem in which both the cost function parameters and the opponents' strategies are uncertain. In this paper, we have only considered the case where either of them is uncertain. However, in the real situation, it would be natural to assume that both of them involve uncertainties.

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