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A Gap Function Approach to the Generalized Nash Equilibrium Problem[1]

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Abstract. We consider an optimization reformulation approach for the generalized Nash equilibrium problem (GNEP) that uses the regularized gap function of a quasi-variational inequality problem (QVIP). The regularized gap function for QVIP is in general not differentiable, but only directionally differentiable. Moreover, a simple condition has yet to be established, under which any stationary point of the regularized gap function solves the QVIP. We tackle these issues for the GNEP in which the shared constraints are given by linear equalities, while the individual constraints are given by convex inequalities. First, we formulate the minimization problem involving the regularized gap function, and show the equivalence to GNEP. Next, we establish the differentiability of the regularized gap function and show that any stationary point of the minimization problem solves the original GNEP under some suitable assumptions. Then, by using a barrier technique, we propose an algorithm that sequentially solves minimization problems obtained from GNEPs with the shared equality constraints only. Further, we discuss the case of shared inequality constraints and present an algorithm that utilizes the transformation of the inequality constraints to equality constraints by means of slack variables. We present some results of numerical experiments to illustrate the proposed approach.

Keywords. Generalized Nash equilibrium problem, quasi-variational inequality, regularized gap function.

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1 Introduction

A multi-player non-cooperative game is called the Nash equilibrium problem (NEP), if the goal is to find a solution in which no player has any incentive to change his/her own strategy unilaterally. The generalized Nash equilibrium problem (GNEP) is a generalization of the NEP, in which each player's strategy set depends on the other players' strategies as well. A solution of the GNEP is called a generalized Nash equilibrium (GNE). The GNEP has many applications such as electric power market models [16, 21] and river basin pollution games [13, 18, 19].

Recently, an increasing effort has been made to develop algorithms for computing GNEs [4]. Some of them are based on the well-known fact that a NEP can be reformulated as a variational inequality problem (VIP) if each player's problem is a convex programming problem [5, 10, 12]. Pang and Fukushima [21] proposed an approach for GNEP that solves a sequence of VIPs corresponding to NEPs, which are obtained by approximating the original GNEP by means of a penalty technique. Along a similar line, Facchinei and Pang [6] proposed to use an exact penalty function. More recently, Fukushima [9] proposed a controlled penalty method to find a particular GNE called a restricted GNE that contains a normalized equilibrium of Rosen [22] as a special case.

Besides penalty methods, several algorithms have been proposed for GNEPs. Facchinei, Fischer and Piccialli [3] studied Newton-type methods for finding a normalized equilibrium by way of a VI reformulation of the GNEP with shared constraints, while von Heusinger, Kanzow and Fukushima [15] proposed a generalized Newton method applied to a fixed point problem derived from the original GNEP. Nabetani, Tseng and Fukushima [20] proposed parametrized VI approaches to GNEP, with particular emphasis on finding GNEs as many as possible. Krawczyk and Uryasev [19] and von Heusinger and Kanzow [14], among others, proposed Nikaido-Isoda function-type approaches to compute GNEs.

Yet another approach is based on the link between a GNEP and a quasi-variational inequality problem (QVIP). It is known that a GNEP can be reformulated as a QVIP under some assumptions [11, 21]. The relationship between the GNEP and QVIP in Hilbert space was studied by Bensoussan [2]. Harker [11] obtained some results for problems in a finite-dimensional Euclidean space. However, compared with the VIP, the study of the QVIP is still in its infancy, and only a few algorithms have been proposed to solve QVIPs numerically. Fukushima [8] defined the regularized gap function for a QVIP, which is an extension of the one for a VIP [7], and showed that the QVIP can be solved by minimizing the regularized gap function. However, there still remain some difficulties with this approach. Unlike the case of VIP, the regularized gap function for QVIP is in general not differentiable, but only directionally differentiable [8]. Moreover, for VIP, under some monotonicity assumption, it is proved that any stationary point of the regularized gap function solves the VIP [8]. However, such a simple condition for the QVIP has yet to be established.

In this paper, we focus on the GNEP in which the shared constraints are given by linear equalities, while the individual constraints are given by convex inequalities. First, we formulate the minimization problem with the regularized gap function, and show the equivalence between this minimization problem and the GNEP. Next, we establish the differentiability of the regularized gap function and show that any stationary point of the minimization problem solves the original GNEP under suitable assumptions. Then, by using a barrier technique, we propose an algorithm that sequentially solves minimization problems obtained from GNEPs with the shared equality constraints only. Further, we discuss the case of shared inequality constraints and present an algorithm that utilizes the transformation of inequality constraints to equality constraints by means of slack variables. Finally, we present some results of numerical experiments to illustrate the proposed approach.

We use the following notation throughout the paper. For vectors $x, y \in \mathbb{R}^n$, the inner product is denoted by $\langle x, y\rangle := x^\top y$, where $^\top$ denotes transposition. For a vector $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $\|x\| := \sqrt{\langle x, x\rangle}$. For a transposed vector comprised of several subvectors, we use a simplified notation $(x^1, x^2, \ldots, x^N)^\top$ instead of $((x^1)^\top, (x^2)^\top, \ldots, (x^N)^\top)^\top$.

2 Generalized Nash Equilibrium Problem

Consider an N-person non-cooperative game in which each player's strategy set depends on the other players' strategies. Specifically, let each player ν solve the following optimization problem for x^ν with $x^{-\nu}$ treated as exogenous:

$$P^\nu(x^{-\nu}) : \quad \begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & x^\nu \in S^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}, \end{array}$$

where

$$x := (x^\nu)_{\nu=1}^N \in \mathbb{R}^n, \quad x^{-\nu} := (x^{\nu'})_{\nu'=1,\nu'\neq\nu}^N \in \mathbb{R}^{n_{-\nu}}, \quad n := \sum_{\nu=1}^N n_\nu, \quad n_{-\nu} := n - n_\nu.$$

Here, $x^\nu \in \mathbb{R}^{n_\nu}$ denotes the strategy of player ν, and $x^{-\nu} \in \mathbb{R}^{n_{-\nu}}$ denotes the vector formed by the strategies of all players except player ν. The objective function $\theta^\nu \colon \mathbb{R}^{n_\nu} \times \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ of player ν is assumed to be a differentiable convex function for any fixed $x^{-\nu}$. Player ν's strategy set $S^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ is a convex set, and depends on the other player's strategies. Thus, each player's problem is a convex programming problem.

A GNE is then defined to be a tuple $x^* := (x^{*,\nu})_{\nu=1}^N$ such that $x^{*,\nu}$ is an optimal solution of the following optimization problem for each $\nu = 1, \ldots, N$:

$$P^\nu(x^{*,-\nu}) : \quad \begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^\nu, x^{*,-\nu}) \\ \text{subject to} & x^\nu \in S^\nu(x^{*,-\nu}). \end{array} \tag{1}$$

This means that, when each player ν chooses the strategy $x^{*,\nu}$, no player has any incentive to change his/her strategy unilaterally.

In particular, if each player's strategy set does not depend on the other players' strategies, then a GNE reduces to the classical Nash equilibrium.

3 Reformulation of GNEP as QVIP

Define the vector-valued function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and the point-to-set mapping $S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\begin{aligned} F(x) &:= (F^\nu(x))_{\nu=1}^N := \left(\nabla_{x^\nu}\theta^\nu(x^\nu, x^{-\nu})\right)_{\nu=1}^N \in \mathbb{R}^n, \\ S(x) &:= \prod_{\nu=1}^N S^\nu(x^{-\nu}) \subseteq \mathbb{R}^n. \end{aligned} \tag{2}$$

By assumption, problem (1) is a convex programming problem for each ν. Therefore, $x^{*,\nu}$ is an optimal solution of (1) if and only if $x^{*,\nu}$ is a stationary point of the function $\theta^\nu(\cdot, x^{*,-\nu})$ on the set $S(x^{*,-\nu})$, that is, $x^{*,\nu}$ satisfies

$$x^{*,\nu} \in S^\nu(x^{*,-\nu})$$

and

$$\langle \nabla_{x^\nu}\theta^\nu(x^{*,\nu}, x^{*,-\nu}), x^\nu - x^{*,\nu}\rangle \geq 0, \qquad \forall x^\nu \in S(x^{*,-\nu}).$$

Thus, the GNEP defined in Section 2 is equivalent to finding a vector $x^* \in \mathbb{R}^n$ such that $x^* \in S(x^*)$ and

$$\langle F(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in S(x^*). \tag{3}$$

This type of problem is called a quasi-variational inequality problem (QVIP). In particular, if $S(x) = \hat{S}$ for all x, where $\hat{S}$ is a nonempty closed convex set, then QVIP (3) reduces to a variational inequality problem (VIP).

4 A Merit Function for QVIP

Generally, a merit function of an equilibrium problem refers to a nonnegative-valued function f such that x is a solution of the problem if and only if $f(x) = 0$ and x satisfies the constraints of the problem. The equilibrium problem can be reformulated as an equivalent optimization problem by means of a merit function.

For VIPs, there have been several proposals of merit functions, such as the gap function [1] and the regularized gap function [7], and the properties of those functions have been studied extensively. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\hat{S} \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Consider the VIP of finding a vector $x^* \in \hat{S}$ such that

$$\langle F(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in \hat{S}. \tag{4}$$

The regularized gap function $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}$ for VIP (4) is defined by

$$\hat{f}(x) := -\inf_y \left\{ \langle F(x), y - x\rangle + \frac{1}{2}\langle y - x, H(y - x)\rangle \;\middle|\; y \in \hat{S} \right\}, \tag{5}$$

where H is an $n \times n$ symmetric positive definite matrix. The minimization problem on the right-hand side of (5) is a convex programming problem, and it has a unique optimal solution for any given x. Denote this optimal solution by $\hat{y}(x)$. The regularized gap function $\hat{f}$ has the following properties [7].

Theorem 4.1. *For each $x \in \hat{S}$, we have $\hat{f}(x) \geq 0$. Moreover, x solves VIP* (4) *if and only if $\hat{f}(x) = 0$ and $x \in \hat{S}$.*

Therefore, VIP (4) can be reformulated as the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \hat{f}(x) \\ \text{subject to} & x \in \hat{S}. \end{array} \tag{6}$$

Moreover, the function $\hat{f}$ possesses some favorable properties as shown in the next theorem [7].

Theorem 4.2. *Suppose that the function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then the regularized gap function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by* (5) *is continuous. Moreover, if F is continuously differentiable, then $\hat{f}$ is also continuously differentiable, and the gradient of $\hat{f}$ at x is given by*

$$\nabla \hat{f}(x) = F(x) - (\nabla F(x) - H)(\hat{y}(x) - x).$$

In particular, when $\nabla F(x)$ is positive definite for all x, any stationary point of the minimization problem (6)*, i.e. any point that satisfies the first-order optimality condition, solves VIP* (4)*.*

For QVIP (3), an extension of the regularized gap function is proposed by Fukushima [8]. This function, also called the regularized gap function for the QVIP, is defined by

$$f(x) := -\inf_{y} \left\{ \langle F(x), y - x \rangle + \frac{1}{2}\langle y - x, H(y - x) \rangle \;\middle|\; y \in S(x) \right\}, \tag{7}$$

where H is a symmetric positive definite matrix. Let the set $X \subseteq \mathbb{R}^n$ be defined by

$$X := \{x \in \mathbb{R}^n \mid x \in S(x)\},$$

which is called the feasible set of QVIP (3). Similarly to VIP, for any $x \in X$, the minimization problem on the right-hand side of (7) is a convex programming problem, and it has a unique optimal solution for any x. We denote this optimal solution by $y(x)$. Then the regularized gap function f is written as

$$f(x) = -\langle F(x), y(x) - x \rangle - \frac{1}{2}\langle y(x) - x, H(y(x) - x) \rangle.$$

The following result holds [8].

Theorem 4.3. *For each $x \in X$, we have $f(x) \geq 0$. Moreover, x solves QVIP* (3) *if and only if $f(x) = 0$ and $x \in X$.*

This theorem indicates that QVIP (3) can be reformulated as the following optimization problem:

$$Q : \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X. \end{array}$$

That is, the function f is a merit function of QVIP (3). Unfortunately, unlike the case of VIP, the regularized gap function f for QVIP is in general not differentiable, but only directionally differentiable when the function F is differentiable. Moreover, even if $\nabla F(x)$ is positive definite at any stationary point x of problem Q, it does not imply that x solves the original QVIP. In the next section, we will explore the possibility of avoiding these difficulties with the gap function approach for a class of GNEPs.

5 GNEP with Shared Equality Constraints and Barrier Method

Consider the GNEP with player ν's problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta^\nu(x^\nu, x^{-\nu}) \\ \text{subject to} \quad & \langle a_i^\nu, x^\nu \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle, \quad i = 1, \dots, m, \\ & h_j^\nu(x^\nu) \leq 0, \quad j = 1, \dots, l_\nu. \end{aligned} \tag{8}$$

Notice that the shared constraints are given by equalities, while the individual constraints are given by inequalities. We denote this GNEP as P.

In the remainder of this paper, we make the following assumption:

Assumption 5.1. $\theta^\nu(\,\cdot\,, x^{-\nu})\colon \mathbb{R}^{n_\nu} \to \mathbb{R}$ *is a twice continuously differentiable convex function for any fixed* $x^{-\nu} \in \mathbb{R}^{n_{-\nu}}$*, and* $h_j^\nu\colon \mathbb{R}^{n_\nu} \to \mathbb{R},\ j = 1, \dots, l_\nu$ *are twice continuously differentiable convex functions.*

We apply a barrier technique to the individual inequality constraints, thereby reformulating the GNEP into another GNEP with the shared equality constraints only. We then develop an optimization approach using the regularized gap function for the QVIP derived from the latter GNEP. Note that the proposed barrier method incorporates each player's individual constraints in the objective function by using the barrier function. This is different from the common approach where the penalty technique is applied to the shared constraints [6, 9, 21].

By adding the barrier term associated with the individual constraints to the objective function, problem (8) is approximated by the following problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta^\nu(x^\nu, x^{-\nu}) - \rho \sum_{j=1}^{l_\nu} \log(-h_j^\nu(x^\nu)) \\ \text{subject to} \quad & \langle a_i^\nu, x^\nu \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle, \quad i = 1, \dots, m, \end{aligned} \tag{9}$$

where $\rho > 0$ is a barrier parameter. Let P_ρ denote the GNEP with each player's problem given by (9). Since problem (9) is a convex programming problem, GNEP P_ρ can be reformulated as the following QVIP: Find a vector $x \in S_0(x) \cap \Sigma_0$ such that

$$\langle F(x) - \rho E(x), y - x \rangle \geq 0, \quad \forall y \in S_0(x), \tag{10}$$

where the function F is defined by (2), and $\Sigma_0 \subseteq \mathbb{R}^n$, $E\colon \Sigma_0 \to \mathbb{R}^n$ and $S_0\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are defined by

$$
\begin{aligned}
\Sigma_0 &:= \prod_{\nu=1}^{N} \left\{ x^\nu \mid h_j^\nu(x^\nu) < 0,\ j = 1, \ldots, l_\nu \right\}, \\
E(x) &:= \left(\sum_{j=1}^{l_\nu} \frac{\nabla h_j^\nu(x^\nu)}{h_j^\nu(x^\nu)} \right)_{\nu=1}^{N}, \\
S_0(x) &:= \prod_{\nu=1}^{N} \left\{ y^\nu \;\middle|\; \langle a_i^\nu, y^\nu \rangle = b_i - \sum_{\substack{\nu'=1 \\ \nu' \neq \nu}}^{N} \langle a_i^{\nu'}, x^{\nu'} \rangle,\ i = 1, \ldots, m \right\},
\end{aligned}
\tag{11}
$$

respectively. Then, the regularized gap function for this problem is defined by

$$
f_\rho(x) := -\inf \left\{ \langle F(x) - \rho E(x), y - x \rangle + \frac{1}{2} \langle y - x, H(y - x) \rangle \;\middle|\; y \in S_0(x) \right\}. \tag{12}
$$

Note that the function f_ρ is defined only on the open set Σ_0. By letting $f_\rho(x) = +\infty\ \forall x \notin \Sigma_0$, GNEP P_ρ is reformulated as the minimization problem

$$
Q_\rho : \quad \begin{array}{ll} \text{minimize} & f_\rho(x) \\ \text{subject to} & x \in X_0, \end{array}
$$

where the set $X_0 \subseteq \mathbb{R}^n$ is defined by

$$
X_0 := \left\{ x \;\middle|\; b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle = 0,\ i = 1, \ldots, m \right\}.
$$

This fact is formally stated as follows.

Theorem 5.1. *For each $x \in X_0$, we have $f_\rho(x) \geq 0$. Moreover, x solves QVIP* (10) *if and only if $f_\rho(x) = 0$ and $x \in X_0$.*

Now we consider the differentiability of the function f_ρ. Let $y(x)$ denote the unique solution of the optimization problem on the right-hand side of (12).

Lemma 5.1. *If $x \in X_0$, then $y(x) \in X_0$.*

Proof. Since $y(x) \in S_0(x)$, for each $\nu = 1, \ldots, N$, $y^\nu(x)$ satisfies

$$
\langle a_i^\nu, y^\nu(x) \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle, \quad i = 1, \ldots, m. \tag{13}
$$

Since $x \in X_0$ by assumption, we have

$$
b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle = 0, \quad i = 1, \ldots, m. \tag{14}
$$

Hence, by (13) and (14), for each $\nu = 1, \ldots, N$, we obtain

$$
\langle a_i^\nu, x^\nu \rangle = \langle a_i^\nu, y^\nu(x) \rangle, \quad i = 1, \ldots, m. \tag{15}
$$

Therefore, by (14) and (15), we have

$$0 = b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle = b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, y^\nu(x) \rangle, \quad i = 1, \dots, m,$$

which implies $y(x) \in X_0$. ■

The following theorem shows that the function f_ρ is directionally differentiable in general; moreover it is differentiable under suitable assumptions.

Theorem 5.2. *The function f_ρ defined by* (12) *is directionally differentiable at every $x \in \Sigma_0$ along any direction $d \in \mathbb{R}^n$, and the directional derivative is given by*

$$f'_\rho(x; d) = \min_{\mu \in M(x)} \left\{ \langle (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x), d \rangle \right.$$
$$\left. - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu \langle (a_i^1, \dots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \dots, a_i^N)^\top, d \rangle \right\},$$

where $M(x) \subseteq \mathbb{R}^{Nm}$ consists of all vectors $\mu := ((\mu_i^\nu)_{i=1}^m)_{\nu=1}^N \in \mathbb{R}^{Nm}$ satisfying

$$F(x) - \rho E(x) + H(y(x) - x) + \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu (0, \dots, 0, a_i^\nu, 0, \dots, 0)^\top = 0. \tag{16}$$

In particular, if $M(x)$ is a singleton, i.e.,

$$M(x) = \{\mu(x)\},$$

then f_ρ is differentiable at x and the gradient of f_ρ at x is given by

$$\nabla f_\rho(x) = (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x)$$
$$- \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu(x) (a_i^1, \dots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \dots, a_i^N)^\top.$$

Proof. The regularized gap function f_ρ is defined by substituting $F(x) - \rho E(x)$ for $F(x)$ in the definition (7) of the regularized gap function f_ρ. Since $F(x) - \rho E(x)$ is differentiable, the assertion of this theorem immediately follows from [8, Theorem 3]. ■

The next assumption ensures that the set $M(x)$ is a singleton for any x.

Assumption 5.2. *For each $\nu = 1, \dots, N$, the vectors a_i^ν, $i = 1, \dots, m$ are linearly independent.*

Theorem 5.3. *Let Assumption 5.2 hold. Then the function f_ρ is differentiable at any point $x \in \Sigma_0$, and its gradient is given by*

$$\nabla f_\rho(x) = -(\nabla F(x) - \rho \nabla E(x))(y(x) - x) - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu(x) a_i,$$

where $a_i := (a_i^1, \dots, a_i^N)^\top \in \mathbb{R}^n$.

Proof. By Assumption 5.2, the vectors $a_i^\nu,\ i = 1, \ldots, m$ are linearly independent for each $\nu = 1, \ldots, N$. Therefore, the vectors $(0, \ldots, 0, a_i^\nu, 0, \ldots, 0)^\top \in \mathbb{R}^n,\ i = 1, \ldots, m,\ \nu = 1, \ldots, N$ are linearly independent, and $M(x)$ has only one element $\mu(x)$. Hence, by Theorem 5.2, the function f_ρ is differentiable at any point $x \in \Sigma_0$.

Moreover, by Theorem 5.2, the gradient of f_ρ at x is given by

$$\begin{aligned}\nabla f_\rho(x) &= (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x) \\ &\quad - \sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x)(a_i^1, \ldots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \ldots, a_i^N)^\top. \end{aligned} \tag{17}$$

The last term on the right-hand side of (17) is rewritten as

$$\begin{aligned} &\sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x)(a_i^1, \ldots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \ldots, a_i^N)^\top \\ &= \sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x)(a_i^1, \ldots, a_i^{\nu-1}, a_i^\nu, a_i^{\nu+1}, \ldots, a_i^N)^\top - \sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x)(0, \ldots, 0, a_i^\nu, 0, \ldots, 0)^\top \\ &= \sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x) a_i + F(x) - \rho E(x) + H(y(x) - x), \end{aligned} \tag{18}$$

where the last equality follows from (16). By using (18), the formula (17) can be rewritten as

$$\nabla f_\rho(x) = -(\nabla F(x) - \rho \nabla E(x))(y(x) - x) - \sum_{i=1}^m \sum_{\nu=1}^N \mu_i^\nu(x) a_i.$$

■

The following theorem gives a condition under which any point that satisfies the first-order optimality condition for the optimization problem Q_ρ is a solution of GNEP P_ρ.

Theorem 5.4. *Suppose Assumption 5.2 holds. Let $x \in X_0$ be a stationary point of problem Q_ρ, and $\nabla F(x) - \rho \nabla E(x)$ be positive definite. Then x is a solution of QVIP* (10)*, i.e., x is a solution of GNEP P_ρ.*

Proof. First, note that $x \in \Sigma_0$. By Theorem 5.2, the function f_ρ is differentiable at the point x under the given assumptions. Thus, when x is a stationary point of problem Q_ρ, by making use of the fact that the feasible set X_0 is an affine set, we have

$$\langle \nabla f_\rho(x), y - x \rangle = 0, \quad \forall y \in X_0. \tag{19}$$

Note that $\langle a_i, x \rangle - b_i = 0$ holds by $x \in X_0$. Moreover, we have $y(x) \in X_0$ by Lemma 5.1, i.e.,

$$b_i - \langle a_i, y(x) \rangle = 0, \quad i = 1, \ldots, m.$$

Hence, we have

$$\langle a_i, y(x) - x \rangle = 0, \quad i = 1, \ldots, m. \tag{20}$$

Thus, it follows from Theorem 5.3 together with (19) and (20) that

$$\begin{aligned}
0 &= \langle \nabla f_\rho(x), y(x) - x\rangle \\
&= -\langle (\nabla F(x) - \rho\nabla E(x))(y(x) - x) + \sum_{i=1}^{m}\sum_{\nu=1}^{N} \mu_i^\nu(x) a_i, y(x) - x\rangle \\
&= -\langle (\nabla F(x) - \rho\nabla E(x))(y(x) - x), y(x) - x\rangle - \sum_{i=1}^{m}\sum_{\nu=1}^{N} \mu_i^\nu(x)\langle a_i, y(x) - x\rangle \\
&= -\langle (\nabla F(x) - \rho\nabla E(x))(y(x) - x), y(x) - x\rangle. \qquad (21)
\end{aligned}$$

Moreover, since $\nabla F(x) - \rho\nabla E(x)$ is positive definite by assumption, we must have $y(x) = x$ from (21). Then, the definition (12) of f_ρ yields $f_\rho(x) = 0$, and x is a solution of QVIP (10) according to Theorem 5.1. ■

Corollary 5.1. *Suppose Assumption 5.2 holds and $\nabla F(x)$ is positive definite at any point $x \in X_0 \cap \Sigma_0$. Then, for any $\rho > 0$, a stationary point x of problem Q_ρ is a solution of GNEP P_ρ.*

Proof. By direct calculation, it follows from the definition (11) of $E(x)$ that

$$\nabla E(x) = \mathrm{Diag}\left[\sum_{j=1}^{l_\nu}\left(\frac{\nabla^2 h_j^\nu(x^\nu)}{h_j^\nu(x^\nu)} - \frac{\nabla h_j^\nu(x^\nu)\nabla h_j^\nu(x^\nu)^\top}{h_j^\nu(x^\nu)^2}\right)\right]_{\nu=1}^{N},$$

where $\mathrm{Diag}[B_\nu]_{\nu=1}^N$ denotes the block diagonal matrix whose block diagonal elements are B_ν, $\nu = 1, \dots, N$. Notice that each $\nabla^2 h_j^\nu(x^\nu)$ is positive semidefinite since h_j^ν is convex. This implies that $\nabla E(x)$ is negative semidefinite for any $x \in \Sigma_0$, since $h_j^\nu(x^\nu) < 0$.

Hence, $\nabla F(x) - \rho\nabla E(x)$ is positive definite at any $x \in \Sigma_0$, whenever so is $\nabla F(x)$. Therefore, Theorem 5.4 ensures that, for any $\rho > 0$, a stationary point of Q_ρ is a solution of GNEP P_ρ. ■

6 Convergence of the Barrier Method

In the previous section, we have shown that, for every fixed $\rho > 0$, a solution of GNEP P_ρ can be obtained by solving the minimization problem Q_ρ. Here we present an algorithm for solving GNEP P by solving problems Q_ρ sequentially by letting the parameter ρ tend to zero.

Algorithm 6.1. *Choose a positive sequence $\{\rho_k\} \subset \mathbb{R}$ tending to zero. For each k, find a stationary point x_k of the minimization problem*

$$Q_{\rho_k}: \quad \begin{array}{ll} \text{minimize} & f_{\rho_k}(x) \\ \text{subject to} & x \in X_0. \end{array}$$

By imposing appropriate conditions, we can show that the sequence $\{x_k\}$ generated by Algorithm 6.1 converges to a solution of GNEP P.

Assumption 6.1. *$\nabla F(x)$ is positive definite at any point $x \in X_0 \cap \Sigma_0$.*

Theorem 6.1. *Suppose that Assumptions 5.2 and 6.1 hold, and the set $\Sigma^\nu := \{x^\nu \in \mathbb{R}^{n_\nu} \mid h^\nu(x^\nu) \le 0\}$ is bounded for each $\nu = 1, \dots, N$. Let x_∞ be any accumulating point of the sequence $\{x_k\}$ generated by Algorithm 6.1. Suppose the following Mangasarian-Fromovotz constraint qualification (MFCQ) holds for each $\nu = 1, \dots, N$:*

$$\left.\begin{array}{l} \displaystyle\sum_{j \in \gamma_\infty^\nu} \lambda_j^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^m \mu_i^\nu a_i^\nu = 0 \\ \qquad \lambda_j^\nu \ge 0, \quad j \in \gamma_\infty^\nu \\ \qquad \mu_i^\nu \in \mathbb{R}, \quad i = 1, \dots, m \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \lambda_j^\nu = 0, & j \in \gamma_\infty^\nu \\ \mu_i^\nu = 0, & i = 1, \dots, m, \end{array}\right.$$

where $\gamma_\infty^\nu := \{j \mid h_j^\nu(x_\infty^\nu) = 0\} \subseteq \{1, 2, \dots, l_\nu\}$. Then x_∞ is a solution of GNEP P.

Proof. From Corollary 5.1, x_k is a solution of the following QVIP: Find $x \in S_0(x)$ such that

$$\langle F(x) - \rho_k E(x), y - x \rangle \ge 0, \quad \forall y \in S_0(x). \tag{22}$$

Let $\{x_k\}_{k \in \kappa}$ be a convergent subsequence whose limit is x_∞. By Assumption 5.2, the linear independence constraint qualification holds for problem P_{ρ_k}. Thus, it follows from the Karush-Kuhn-Tucker (KKT) condition for problem (22) that for any k there exist Lagrange multipliers $\left(\mu_{k,i}^\nu\right)_{i=1}^m$ such that

$$F^\nu(x_k) - \rho_k \sum_{j=1}^{l_\nu} \frac{1}{h_j^\nu(x_k^\nu)} \nabla h_j^\nu(x_k^\nu) + \sum_{i=1}^m \mu_{k,i}^\nu a_i^\nu = 0, \qquad \nu = 1, \dots, N. \tag{23}$$

Put $\lambda_{k,j}^\nu := -\rho_k / h_j^\nu(x_k^\nu) \ge 0,\ j = 1, \dots, l_\nu$, and define the vectors

$$\phi_k^\nu := \begin{pmatrix} \lambda_k^\nu \\ \mu_k^\nu \end{pmatrix},$$

where $\lambda_k^\nu := \left(\lambda_{k,j}^\nu\right)_{j=1}^{l_\nu}$ and $\mu_k^\nu := \left(\mu_{k,i}^\nu\right)_{i=1}^m$. Let us show that the sequence $\{\phi_k^\nu\}_{k \in \kappa}$ is bounded for each ν. In fact, if $\{\phi_k^\nu\}_{k \in \kappa}$ is unbounded, then there exists a further subsequence $\{\phi_k^\nu\}_{k \in \kappa'}$ such that

$$\lim_{\kappa' \ni k \to \infty} \|\phi_k^\nu\| = \infty.$$

By dividing both sides of (23) by $\|\phi_k^\nu\|$, we have

$$\frac{1}{\|\phi_k^\nu\|} F^\nu(x_k) + \sum_{j=1}^{l_\nu} \frac{\lambda_{k,j}^\nu}{\|\phi_k^\nu\|} \nabla h_j^\nu(x_k^\nu) + \sum_{i=1}^m \frac{\mu_{k,i}^\nu}{\|\phi_k^\nu\|} a_i^\nu = 0, \qquad \nu = 1, \dots, N.$$

Since $\{\lambda_{k,j}^\nu / \|\phi_k^\nu\|\}_{k \in \kappa'}$ and $\{\mu_{k,i}^\nu / \|\phi_k^\nu\|\}_{k \in \kappa'}$ are bounded, these sequences have accumulation points $\bar{\lambda}_j^\nu$ and $\bar{\mu}_i^\nu$, respectively. Therefore, we have

$$\sum_{j=1}^{l_\nu} \bar{\lambda}_j^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^m \bar{\mu}_i^\nu a_i^\nu = 0, \qquad \nu = 1, \dots, N. \tag{24}$$

Now notice that $\bar{\lambda}_j^\nu \geq 0$ for all j. In particular, since

$$\limsup_{k\to\infty} h_j^\nu(x_k^\nu) < 0, \qquad \forall j \notin \gamma_\infty^\nu,$$

we have $\lambda_{k,j}^\nu = 0$, for all $k \in \kappa'$ sufficient large, implying $\bar{\lambda}_j^\nu = 0$ for all $j \notin \gamma_\infty^\nu$. Thus, it follows from (24) that

$$\sum_{j\in\gamma_\infty^\nu} \bar{\lambda}_j^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^m \bar{\mu}_i^\nu a_i^\nu = 0, \qquad \nu = 1,\ldots,N.$$

However, this along with the fact that

$$\bar{\lambda}_j^\nu \geq 0, \quad \forall j \in \gamma_\infty^\nu, \quad \text{and} \quad \left\| \begin{pmatrix} \bar{\lambda}^\nu \\ \bar{\mu}^\nu \end{pmatrix} \right\| = 1$$

contradicts the assumed MFCQ. This implies that $\{\phi_k^\nu\}$ is bounded, and that $\{\lambda_k^\nu\}$ and $\{\mu_k^\nu\}$ have accumulation points λ_∞^ν and μ_∞^ν, respectively. Therefore, x_∞ satisfies

$$\left.\begin{array}{l} F^\nu(x_\infty) + \displaystyle\sum_{j=1}^{l_\nu} \lambda_{\infty,j}^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^m \mu_i^\nu a_i^\nu = 0 \\ h_j^\nu(x_\infty^\nu) \leq 0, \quad \lambda_{\infty,j}^\nu \geq 0, \quad \lambda_{\infty,j}^\nu h_j^\nu(x_\infty^\nu) = 0, \quad j = 1,\ldots,l_\nu \\ \langle a_i^\nu, x_\infty^\nu\rangle + \displaystyle\sum_{\nu'\neq\nu} \langle a_i^{\nu'}, x_\infty^{\nu'}\rangle - b_i = 0, \quad i = 1,\ldots,m \end{array}\right\}, \quad \nu = 1,\ldots,N.$$

This is nothing but the KKT condition for problem (3) with $S(x)$ defined by the constraints in problems (8). Consequently, x_∞ is a solution of GNEP P. ∎

7 Extension to GNEP with Shared Inequality Constraints

The GNEP considered in the previous section assumes that each player's shared constraints are defined by equalities only. In practice, however, the shared constraints often contain inequalities. In this section, we discuss the case of shared linear inequality constraints and present an approach that relies on the transformation to the equality constraints by means of slack variables.

Suppose that, for each ν, player ν's problem is given as

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & \langle a_i^\nu, x^\nu\rangle \leq b_i - \displaystyle\sum_{\nu'\neq\nu} \langle a_i^{\nu'}, x^{\nu'}\rangle, \quad i = 1,\ldots,m, \\ & h_j^\nu(x^\nu) \leq 0, \quad j = 1,\ldots,l_\nu. \end{array} \tag{25}$$

Denote this GNEP as $\hat{P}$. Introducing slack variables $s^\nu := (s_1^\nu,\ldots,s_m^\nu)$ as supplementary variables for each player ν, problem (25) is rewritten as

$$\begin{array}{ll} \underset{x^\nu, s^\nu}{\text{minimize}} & \theta^\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & \langle a_i^\nu, x^\nu\rangle + s_i^\nu = b_i - \displaystyle\sum_{\nu'\neq\nu} \left(\langle a_i^{\nu'}, x^{\nu'}\rangle + s_i^{\nu'}\right), \quad i = 1,\ldots,m, \\ & h_j^\nu(x^\nu) \leq 0, \quad j = 1,\ldots,l_\nu, \\ & s_i^\nu \geq 0, \quad i = 1,\ldots,m. \end{array} \tag{26}$$

Denote this GNEP as $\check{P}$. The vector consisting of all slack variables is denoted by $s := (s^\nu)_{\nu=1}^N \in \mathbb{R}^{Nm}$. The next result shows that, under some conditions, a solution of GNEP $\check{P}$ is also a solution of GNEP $\hat{P}$.

Theorem 7.1. *Let (x, s) be a solution of GNEP $\check{P}$. If the relation*

$$s_i^\nu = 0 \ \textit{for some}\ \nu \quad \Longrightarrow \quad s_i^\nu = 0 \ \textit{for all}\ \nu \tag{27}$$

holds for all $i = 1, \ldots, m$, then x is a solution of GNEP $\hat{P}$.

Proof. Define the Lagrangian for problem (26) by

$$\mathcal{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) := \theta^\nu(x^\nu, x^{-\nu}) + \sum_{i=1}^m \mu_i^\nu \left(\langle a_i^\nu, x^\nu \rangle + s_i^\nu + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle + \sum_{\nu' \neq \nu} s_i^{\nu'} - b_i \right)$$
$$+ \sum_{j=1}^{l_\nu} \lambda_j^\nu h_j^\nu(x^\nu) - \sum_{i=1}^m \eta_i^\nu s_i^\nu.$$

A solution (x, s) of GNEP $\check{P}$ satisfies the following KKT conditions for all ν:

$$\nabla_{x^\nu} \mathcal{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) = \nabla_{x^\nu} \theta^\nu(x^\nu, x^{-\nu}) + \sum_{i=1}^m \mu_i^\nu a_i^\nu + \sum_{j=1}^{l_\nu} \lambda_j^\nu \nabla h_j^\nu(x^\nu) = 0, \tag{28a}$$
$$\nabla_{s^\nu} \mathcal{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) = \mu^\nu - \eta^\nu = 0, \tag{28b}$$
$$\langle a_i^\nu, x^\nu \rangle + s_i^\nu + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle + \sum_{\nu' \neq \nu} s_i^{\nu'} - b_i = 0, \quad i = 1, \ldots, m, \tag{28c}$$
$$\lambda_j^\nu \geq 0, \quad \lambda_j^\nu h_j^\nu(x^\nu) = 0, \quad h_j^\nu(x^\nu) \leq 0, \quad j = 1, \ldots, l_\nu, \tag{28d}$$
$$\eta_i^\nu \geq 0, \quad \eta_i^\nu s_i^\nu = 0, \quad s_i^\nu \geq 0, \quad i = 1, \ldots, m. \tag{28e}$$

By the relation (27), we have for each i either (i) $s_i^\nu = 0$ for all ν, or (ii) $s_i^\nu > 0$ for all ν. Let us consider these two cases separately.

(i) Suppose $s_i^\nu = 0$ for all ν. By (28e), we have

$$\eta_i^\nu \geq 0.$$

Then it follows from (28b) and (28c) that

$$\mu_i^\nu \geq 0$$

and

$$\langle a_i^\nu, x^\nu \rangle + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle - b_i = 0$$

for all ν.

(ii) Suppose $s_i^\nu > 0$ for all ν. Then, by (28e), we have

$$\eta_i^\nu = 0.$$

Therefore, from (28b) and (28c), we obtain

$$\mu_i^\nu = 0$$

and

$$\langle a_i^\nu, x^\nu \rangle + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle - b_i < 0$$

for all ν.

Hence, the following complementarity conditions hold for all i:

$$\mu_i^\nu \geq 0, \quad \mu_i^\nu \left(\sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \right) = 0, \quad \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \leq 0. \tag{29}$$

Combining (28a), (28d) and (29), we have for all ν

$$\begin{aligned}
&\nabla_{x^\nu} \theta^\nu(x^\nu, x^{-\nu}) + \sum_{i=1}^{m} \mu_i^\nu a_i^\nu + \sum_{j=1}^{l_\nu} \lambda_j^\nu \nabla h_j^\nu(x^\nu) = 0, \\
&\mu_i^\nu \geq 0, \quad \mu_i^\nu \left(\sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \right) = 0, \quad \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \leq 0, \quad i = 1, \ldots, m, \\
&\lambda_j^\nu \geq 0, \quad \lambda_j^\nu h_j^\nu(x^\nu) = 0, \quad h_j^\nu(x^\nu) \leq 0, \quad j = 1, \ldots, l_\nu.
\end{aligned}$$

This implies that for each ν, x^ν satisfies the KKT condition for problem (25) with given $x^{-\nu}$. Thus, $x = (x^\nu)_{\nu=1}^N$ is a solution of GNEP $\hat{P}$. ■

By adding the barrier term associated with the individual constraints to the objective function, player ν's problem (26) is approximated by the following problem:

$$\begin{aligned}
&\underset{x^\nu, s^\nu}{\text{minimize}} \quad \theta^\nu(x^\nu, x^{-\nu}) - \rho \left(\sum_{j=1}^{l_\nu} \log(-h_j^\nu(x^\nu)) + \sum_{i=1}^{m} \log s_i^\nu \right) \\
&\text{subject to} \quad \langle a_i^\nu, x^\nu \rangle + s_i^\nu = b_i - \sum_{\nu' \neq \nu} \left(\langle a_i^{\nu'}, x^{\nu'} \rangle + s_i^{\nu'} \right), \quad i = 1, \ldots, m.
\end{aligned} \tag{30}$$

Denote this GNEP as $\check{P}_\rho$. Let the function $\check{F} \colon \mathbb{R}^{n+Nm} \to \mathbb{R}^{n+Nm}$ be defined by

$$\check{F}(x, s) := \begin{pmatrix} F(x) \\ 0 \end{pmatrix} \in \mathbb{R}^{n+Nm},$$

where $F(x)$ is given by (2). Define the function $G \colon \mathbb{R}^{Nm} \to \mathbb{R}^{Nm}$ by

$$G(s) := \left((1/s_i^\nu)_{i=1}^m\right)_{\nu=1}^N.$$

Moreover, let the function $\check{E} \colon \mathbb{R}^{n+Nm} \to \mathbb{R}^{n+Nm}$ be defined by

$$\check{E}(x, s) := \begin{pmatrix} E(x) \\ G(s) \end{pmatrix} \in \mathbb{R}^{n+Nm},$$

where $E(x)$ is given by (11).

Since problem (30) is a convex programming problem, GNEP $\check{P}_\rho$ can be reformulated as the following QVIP: Find $(x,s) \in \check{S}_0(x,s) \cap \check{\Sigma}_0$ such that

$$\left\langle \check{F}(x,s) - \rho\check{E}(x,s), (y,t) - (x,s)\right\rangle \geq 0, \quad \forall (y,t) \in \check{S}_0(x,s), \tag{31}$$

where $\check{\Sigma}_0 \subseteq \mathbb{R}^{n+Nm}$ and $\check{S}_0 \colon \mathbb{R}^{n+Nm} \rightrightarrows \mathbb{R}^{n+Nm}$ are defined by

$$\check{\Sigma}_0 := \prod_{\nu=1}^{N} \left\{ (x^\nu, s^\nu) \mid h_j^\nu(x^\nu) < 0,\ j = 1, \dots, l_\nu, s_i^\nu > 0,\ i = 1, \dots, m \right\},$$

$$\check{S}_0(x,s) := \prod_{\nu=1}^{N} \left\{ (y^\nu, t^\nu) \,\middle|\, \langle a_i^\nu, y^\nu\rangle + t_i^\nu = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'}\rangle - \sum_{\nu' \neq \nu} s_i^{\nu'},\ i = 1, \dots, m \right\},$$

respectively. Then, the regularized gap function for this problem is given by

$$\begin{aligned} \check{f}_\rho(x,s) \quad := \quad -\inf \Bigg\{ & \langle \check{F}(x,s) - \rho\check{E}(x,s), (y,t) - (x,s)\rangle \\ & + \frac{1}{2}\langle (y,t) - (x,s), \check{H}((y,t) - (x,s))\rangle \,\Bigg|\, (y,t) \in \check{S}_0(x,s) \Bigg\}, \end{aligned}$$

where $\check{H}$ is a symmetric positive definite matrix. Note that the function $\check{f}_\rho$ is defined only on the open set $\check{\Sigma}_0$. By letting $\check{f}_\rho(x,s) := +\infty\ \forall (x,s) \notin \check{\Sigma}_0$, GNEP $\check{P}_\rho$ is reformulated as the minimization problem

$$\check{Q}_\rho : \quad \begin{array}{ll} \text{minimize} & \check{f}_\rho(x,s) \\ \text{subject to} & (x,s) \in \check{X}_0, \end{array}$$

where the set $\check{X}_0 \subseteq \mathbb{R}^{n+Nm}$ is defined by

$$\check{X}_0 := \left\{ (x,s) \,\middle|\, b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu\rangle - \sum_{\nu=1}^{N} s_i^\nu = 0,\ i = 1, \dots, m \right\}.$$

This fact implies that for each $\rho > 0$, the set of optimum solutions of problem $\check{Q}_\rho$ equals the set of solutions of GNEP $\check{P}_\rho$. We will then show that any stationary point for problem $\check{Q}_\rho$ is a solution of GNEP $\check{P}_\rho$ under some conditions.

Lemma 7.1. *Suppose $x \in \Sigma_0$, $s > 0$ and $\rho > 0$. Let $\nabla F(x) - \rho\nabla E(x)$ be positive definite. Then, $\nabla\check{F}(x,s) - \rho\nabla\check{E}(x,s)$ is also positive definite.*

Proof. By direct calculation, we have

$$\nabla G(s) = \operatorname{diag}\left(\left(-1/(s_i^\nu)^2 \right)_{i=1}^m \right)_{\nu=1}^N,$$

which is a negative definite matrix. Thus, by the given assumption, for all $\xi \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{Nm}$ such that $(\xi, \sigma) \neq (0,0)$, it follows that

$$\begin{aligned} (\xi,\sigma)^\top \left(\nabla\check{F}(x,s) - \rho\nabla\check{E}(x,s) \right)(\xi,\sigma) &= (\xi,\sigma)^\top \left(\begin{pmatrix} \nabla F(x) & 0 \\ 0 & 0 \end{pmatrix} - \rho \begin{pmatrix} \nabla E(x) & 0 \\ 0 & \nabla G(s) \end{pmatrix} \right)(\xi,\sigma) \\ &= \xi^\top (\nabla F(x) - \rho\nabla E(x))\xi + \sigma^\top \left(-\rho\nabla G(s) \right)\sigma \\ &> 0 \end{aligned}$$

for $\rho > 0$. This completes the proof. ■

Theorem 7.2. *Suppose Assumption 5.2 holds. Let $(x,s) \in \check{X}_0$ be a stationary point of problem $\check{Q}_\rho$, and $\nabla F(x) - \rho \nabla E(x)$ be positive definite for $\rho > 0$. Then the point (x,s) is a solution of QVIP* (31)*, i.e., (x,s) is a solution of GNEP $\check{P}_\rho$.*

Proof. By Assumption 5.2, the vectors a_i^ν, $i = 1,\dots,m$ are linearly independent for each $\nu = 1,\dots,N$. Therefore, the vectors $((0,\dots,0,a_i^\nu,0,\dots,0)^\top, e_i^\nu) \in \mathbb{R}^{n+Nm}$, $i = 1,\dots,m$, $\nu = 1,\dots,N$ are linearly independent, where $e_i^\nu \in \mathbb{R}^{Nm}$ denotes the unit vector whose element corresponding to s_i^ν is one and the others are zero. Thus, the following set is a singleton:

$$\begin{aligned}\check{M}(x,s) \;:=\; \Bigg\{ \mu \in \mathbb{R}^{Nm} \;\Bigg|\; & \check{F}(x,s) - \rho \check{E}(x,s) + \check{H}((y(x,s),t(x,s)) - (x,s)) \\ & + \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu ((0,\dots,0,a_i^\nu,0,\dots,0)^\top, e_i^\nu) = 0 \Bigg\}.\end{aligned}$$

That is, $\check{M}(x,s)$ has only one element $\mu(x,s)$. Hence, by Theorem 5.2, the function $\check{f}_\rho$ is differentiable at any point $(x,s) \in \check{\Sigma}_0$.

Since $\nabla F(x) - \rho \nabla E(x)$ is positive definite, we have that $\nabla \check{F}(x,s) - \rho \nabla \check{E}(x,s)$ is positive definite by Lemma 7.1. Therefore, Theorem 7.2 ensures that any stationary point (x,s) of problem $\check{Q}_\rho$ solves QVIP (31), which means (x,s) is a solution of GNEP $\check{P}_\rho$. ■

To solve the minimization problem $\check{Q}_\rho$, we apply a descent-type iterative method using the gradient information of the objective function $\check{f}_\rho$. We may expect that such a method generates a sequence converging to a stationary point of problem $\check{Q}_\rho$, which turns out to be a solution of GNEP $\check{P}_\rho$ under the conditions given in Theorem 7.2.

To find a solution of GNEP $\hat{P}$, we sequentially solve problems $\check{Q}_{\rho_k}$ with a positive sequence $\{\rho_k\}$ decreasing to zero. Then we may expect that a sequence of solutions to GNEPs $\check{Q}_{\rho_k}$ tends to a solution of GNEP $\check{P}$. From Theorem 7.1, however, we have to require condition (27) to hold in the limit, in order that a solution of GNEP $\check{P}$ yields a solution of GNEP $\hat{P}$. To this end, we introduce a correction phase that enforces the slack variables $s = ((s_i^\nu)_{i=1}^m)_{\nu=1}^N$ to satisfy condition (27) after problem $\check{Q}_\rho$ is solved for each ρ. Specifically, let $\varepsilon > 0$ be a small constant chosen a priori. Once a solution (x,s) of problem $\check{Q}_\rho$ is obtained, do the following for every i:

$$\text{If } s_i^\nu < \varepsilon \text{ for some } \nu, \text{ then put } s_i^\nu := \frac{1}{N} \sum_{\nu=1}^{N} s_i^\nu \text{ for all } \nu. \tag{32}$$

Note that this modification does not change the value of the sum $\sum_{\nu=1}^N s_i^\nu$ for each i, and hence all the equality constraints

$$b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - \sum_{\nu=1}^{N} s_i^\nu = 0, \quad i = 1,\dots,m$$

remain to be satisfied.

A sequential minimization algorithm for solving GNEP $\hat{P}$ is now stated as follows:

Algorithm 7.1. *Choose a positive sequence* $\{\rho_k\} \subseteq \mathbb{R}$ *tending to zero. For each* k*, solve the minimization problem* $\check{Q}_{\rho_k}$*, apply the correction* (32) *if necessary, and proceed to the next step by increasing* k *by one.*

8 Numerical Results

In this section, we report our numerical experience with Algorithm 7.1 for some examples. The matrix $\check{H}$ in the definition of the function $\check{f}_\rho$ is chosen to be the identity matrix. Algorithn 7.1 is terminated when the barrier parameter ρ_k becomes less than 10^{-15}. To solve minimization problems $\check{Q}_{\rho_k}$, we use the fmincon solver in MATLAB, based on the Sequential Quadratic Programming (SQP) method for constrained optimization problems.

Example 1. We consider Harker's example [11]. In this game, there are two players who solve the following problems:

$$P_1(x^2): \quad \begin{array}{ll} \underset{x^1}{\text{minimize}} & (x^1)^2 + (8/3)x^1x^2 - 34x^1 \\ \text{subject to} & 0 \leq x^1 \leq 10, \\ & \quad x^1 + x^2 \leq 15. \end{array}$$

$$P_2(x^1): \quad \begin{array}{ll} \underset{x^2}{\text{minimize}} & (x^2)^2 + (5/4)x^1x^2 - 24.25x^2 \\ \text{subject to} & 0 \leq x^2 \leq 10, \\ & \quad x^1 + x^2 \leq 15. \end{array}$$

The solution set of this GNEP is known to be

$$\left\{ \begin{pmatrix} 5 \\ 9 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} t \\ 15-t \end{pmatrix} \,\middle|\, 9 \leq t \leq 10 \right\}.$$

We have tried various starting points and barrier parameters in implementing Algorithm 7.1, and then observed that a generated sequence always converged to the particular GNE $x = (5,9)^\top$.

Example 2. We consider the three-person river basin pollution game [19]. In this game, the problem of each player $\nu \in \{1,2,3\}$ is given by

$$P^\nu(x^{-\nu}): \quad \begin{array}{ll} \text{minimize} & (c_{1\nu} + c_{2\nu}x^\nu)x^\nu - \left(d_1 - d_2(x^1+x^2+x^3)\right)x^\nu \\ \text{subject to} & \displaystyle\sum_{\nu=1}^{3} u_{\nu l} e_\nu x^\nu \leq K_l, \quad l = 1,\, 2 \\ & x^\nu \geq 0, \end{array}$$

where $d_1 = 3.0,\ d_2 = 0.01,\ K_l = 100,\ l = 1,2,$ and the other constants are shown in Table 1.

We implemented Algorithm 7.1 by using 10000 starting points randomly generated in the feasible set, and found many different GNEs as shown in Figure 1.

Table 1: Problem data for the river basin pollution game.

Player ν	$c_{1\nu}$	$c_{2\nu}$	e_ν	$u_{\nu 1}$	$u_{\nu 2}$
1	0.10	0.01	0.50	6.5	4.583
2	0.12	0.05	0.25	5.0	6.250
3	0.15	0.01	0.75	5.5	3.750

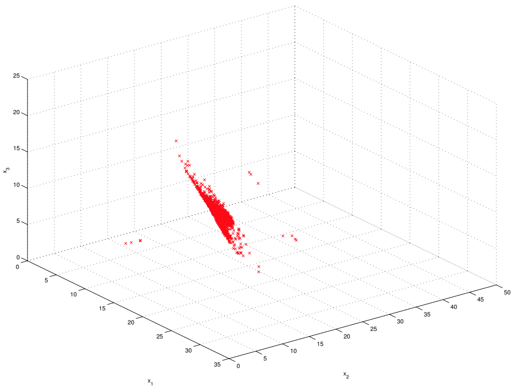

Figure 1: GNEs of the river basin pollution game found by Algorithm 7.1.

Example 3. We consider the internet switching model [17]. In this game, the problem of each player $\nu \in \{1, \dots, N\}$ is given by

$$
P^\nu(x^{-\nu}): \quad \begin{array}{ll} \text{minimize} & \theta^\nu(x^\nu, x^{-\nu}) = \dfrac{x^\nu}{B} - \dfrac{x^\nu}{\sum_{\nu=1}^N x^\nu} \\ \text{subject to} & \displaystyle\sum_{\nu=1}^N x^\nu \leq B, \\ & x^\nu \geq 0.01. \end{array}
$$

We set $N = 10$ and $B = 1$. Using 100 starting points randomly generated in the set $\Sigma_0 := \{x \mid \sum_{\nu=1}^N x^\nu < B,\ x^\nu > 0.01,\ \nu = 1, \dots, N\}$, we implemented Algorithm 7.1 and always obtained the point $x = (0.09, 0.09, \dots, 0.09)^\top$, which is the unique solution of the GNEP [17].

9 Conclusion

We have proposed a gap function approach to the GNEP in which the shared constraints are given by linear equalities, while the individual constraints are given by convex inequalities. We apply a barrier technique to individual inequality constraints and transform each player's problem into a problem involving the shared equality constraints only. Further, we have shown that the proposed approach can be extended to GNEPs with shared linear inequality constraints by means of slack variables. Finally, we have implemented the proposed sequential minimization method on some examples and confirmed that the method can find a solution of those problems.

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