Semidefinite complementarity reformulation for robust Nash equilibrium problems with Euclidean uncertainty sets*

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Abstract. Consider the N-person non-cooperative game in which each player's cost function and the opponents' strategies are uncertain. For such an incomplete information game, the new solution concept called a robust Nash equilibrium has attracted much attention over the past several years. The robust Nash equilibrium results from each player's decision-making based on the robust optimization policy. In this paper, we focus on the robust Nash equilibrium problem in which each player's cost function is quadratic, and the uncertainty sets for the opponents' strategies and the cost matrices are represented by means of Euclidean and Frobenius norms, respectively. Then, we show that the robust Nash equilibrium problem can be reformulated as a semidefinite complementarity problem (SDCP), by utilizing the semidefinite programming (SDP) reformulation technique in robust optimization. We also give some numerical example to illustrate the behavior of robust Nash equilibria.

Keyword. non-cooperative games, robust Nash equilibrium, semidefinite programming, semidefinite complementarity problems

1 Introduction

In the field of game theory, there have been plenty of studies on games with uncertain data. Particularly, Harsanyi's stochastic-based model [11, 12, 13] for incomplete information games is one of the most popular contributions, which assumes so-called Bayesian hypothesis and formulates the Bayesian game that can be treated as a complete information game essentially. On the other hand, the *robust Nash equilibrium* [1, 15, 16, 17, 18] is a new solution concept based on the worst-case analysis of a distribution-free model. In such a model, each player decides his or her strategy according to the criterion of robust optimization [5, 6, 7]. Therefore, a robust Nash equilibrium is also called a robust optimization equilibrium.

Originally, the notion of a robust Nash equilibrium was proposed by Hayashi, Yamashita and Fukushima [15] and Aghassi and Bertsimas [1] independently. Aghassi and Bertsimas [1] considered a robust Nash equilibrium for N-person games in which each player solves a linear programming (LP) problem. Moreover, they proposed a method for solving the robust Nash equilibrium problem with polyhedral uncertainty sets. Hayashi et al. [15] defined the concept of robust Nash equilibria for bimatrix games in which the uncertainty is contained in the opponents' strategies as well as the players' cost matrices, while Aghassi and Bertsimas [1] considered the uncertainty in the cost matrices only. Hayashi et al. [15] also showed that the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) [10, 14] under the assumption that either the opponents' strategies or the players' cost matrices belong to Euclidean uncertainty sets.*1 Recently, Luo, An and Xia [16] and Luo and Li [17] extended the SOCCP reformulation technique in [15] to problems with different types of

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*1 When the uncertainty sets for cost matrices are represented by means of Frobenius norm, we also call them Euclidean uncertainty sets.

uncertainty sets. Nishimura, Hayashi and Fukushima [18] extended the definition of robust Nash equilibria in [1] and [15] to N-person non-cooperative games with *nonlinear* cost functions, and provided an SOCCP reformulation for the case where each player's cost function is quadratic.

In the existing results for the SOCCP reformulation, it was always assumed that either the opponents' strategies or the players' cost matrices could be observed certainly. Therefore, it has been an open problem whether or not the robust Nash equilibrium problem can be reformulated as a tractable problem such as an SOCCP, when *both* the opponents' strategies and the players' cost matrices are contained in Euclidean uncertainty sets. In this paper, we show that such a robust Nash equilibrium problem reduces to a *semidefinite complementarity problem* (SDCP) [9, 20] rather than an SOCCP. To this end, we utilize the semidefinite programming (SDP) reformulation technique [19] for a class of robust optimization problems, which is based on the duality theory in nonconvex quadratic programming [4].

This paper is organized as follows. In Section 2, we formally state the robust Nash equilibrium problem and describe the SDP reformulation technique for robust linear programming (LP) problems. Especially, the latter technique plays an important role in reformulating the robust Nash equilibrium problem as an SDCP. In Section 3, we show that the robust Nash equilibrium problem reduces to an SDCP under the Euclidean uncertainty assumption. In Section 4, we give a numerical example to observe the behavior of robust Nash equilibria. In Section 5, we conclude the paper with some remarks.

Throughout the paper, we use the following notations. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \ (i = 1, \dots, n)\}$. $\mathcal{S}^n$ denotes the set of $n \times n$ real symmetric matrices. $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \mathbb{R}^{m\times n}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^m \sum_{j=1}^n (M_{ij})^2)^{1/2}$, and $\ker M$ denotes the kernel of matrix M, i.e., $\ker M := \{x \in \mathbb{R}^n \mid Mx = 0\}$. For two matrices $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{k\times l}$, $M \otimes N \in \mathbb{R}^{mk\times nl}$ denotes the Kronecker product, and $\langle M, N\rangle := \mathrm{tr}(M^\top N)$ denotes the inner product when $m = k$ and $n = l$.

2 Preliminaries

2.1 Definition of robust Nash equilibrium

In this subsection, we recall the definition of a robust Nash equilibrium and some related properties. Consider an N-person non-cooperative game in which each player tries to minimize his own cost. Let $x^i \in \mathbb{R}^{m_i}$, $S_i \subseteq \mathbb{R}^{m_i}$, and $f_i : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N} \to \mathbb{R}$ be player i's strategy, strategy set, and cost function, respectively, where m_i are positive integers. Moreover, denote

$$\begin{aligned}
&\mathcal{I} := \{1, \dots, N\}, \quad \mathcal{I}_{-i} := \mathcal{I} \setminus \{i\}, \quad m := \sum_{j\in\mathcal{I}} m_j, \quad m_{-i} := \sum_{j\in\mathcal{I}_{-i}} m_j,\\
&x := (x^j)_{j\in\mathcal{I}} \in \mathbb{R}^m, \quad x^{-i} := (x^j)_{j\in\mathcal{I}_{-i}} \in \mathbb{R}^{m_{-i}},\\
&S := \prod_{j\in\mathcal{I}} S_j \subseteq \mathbb{R}^m, \quad S_{-i} := \prod_{j\in\mathcal{I}_{-i}} S_j \subseteq \mathbb{R}^{m_{-i}}.
\end{aligned}$$

When the complete information is assumed, each player $i \in \mathcal{I}$ decides his own strategy by solving the following optimization problem with the opponents' strategies $x^{-i} \in \mathbb{R}^{m_{-i}}$ fixed:

$$\begin{array}{ll}
\underset{x^i}{\text{minimize}} & f_i(x^i, x^{-i}) \\
\text{subject to} & x^i \in S_i.
\end{array} \tag{2.1}$$

A tuple $(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^N) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_N}$ such that $\overline{x}^i \in \mathrm{argmin}_{x^i\in S_i} f_i(x^i, \overline{x}^{-i})$ for each $i \in \mathcal{I}$ is called a Nash equilibrium. In other words, if each player i chooses the strategy $\overline{x}^i$, then no player has an incentive to change his own strategy unilaterally. The Nash equilibrium is well-defined only when each player can estimate his opponents' strategies and can evaluate his own cost exactly. In the real situation, however, any information may contain uncertainty such as observation errors or estimation errors.

To deal with such uncertainty, we introduce the uncertainty sets U_i and $X_i(x^{-i})$, and assume the following situations for each player $i \in \mathcal{I}$:

(A) Player i's cost function involves a parameter $\hat{u}^i \in \mathbb{R}^{s_i}$, where s_i is a positive integer, and is denoted as $f_i^{\hat{u}^i} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to \mathbb{R}$. Although player i does not know the exact value of $\hat{u}^i$ itself, he can estimate that it belongs to a given nonempty set $U_i \subseteq \mathbb{R}^{s_i}$.

(B) When player i perceives his opponents' strategies as x^{-i}, his actual cost is evaluated with x^{-i} replaced by $\hat{x}^{-i} := x^{-i} + \delta x^{-i}$, where δx^{-i} is a certain error or noise. Player i cannot know the exact value of $\hat{x}^{-i}$. However, he can estimate that $\hat{x}^{-i}$ belongs to a certain nonempty set $X_i(x^{-i}) \subseteq \mathbb{R}^{m_{-i}}$.

Under these assumptions, each player encounters the difficulty of addressing the following family of problems involving uncertain parameters $\hat{u}^i$ and $\hat{x}^{-i}$:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \\ \text{subject to} & x^i \in S_i, \end{array} \tag{2.2}$$

where $\hat{u}^i \in U_i$ and $\hat{x}^{-i} \in X_i(x^{-i})$. To overcome the difficulty, we further assume that each player chooses his strategy according to the following criterion of rationality:

(C) Player i tries to minimize his worst cost under assumptions (A) and (B).

From assumption (C), each player considers the worst cost function $\tilde{f}_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to (-\infty, +\infty]$ defined by

$$\tilde{f}_i(x^i, x^{-i}) := \sup\{f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \mid \hat{u}_i \in U_i, \hat{x}^{-i} \in X_i(x^{-i})\}, \tag{2.3}$$

and then solves the following worst cost minimization problem:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \tilde{f}_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in S_i. \end{array} \tag{2.4}$$

Note that, for fixed x^{-i}, problem (2.4) corresponds to the robust counterpart of the uncertain cost minimization problem (2.2). Also, (2.4) can be regarded as a complete information game with cost functions $\tilde{f}_i$ $(i \in \mathcal{I})$. Based on the above discussions, we define the robust Nash equilibrium.

Definition 2.1. Let $\tilde{f}_i$ be defined by (2.3) for $i \in \mathcal{I}$. Then a tuple $(\overline{x}^i)_{i \in \mathcal{I}}$ is called a robust Nash equilibrium of game (2.2) if $\overline{x}^i \in \operatorname{argmin}_{x^i \in S_i} \tilde{f}_i(x^i, \overline{x}^{-i})$ for all $i \in \mathcal{I}$, i.e., a Nash equilibrium of game (2.4). The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.

Finally, we give sufficient conditions for existence of a robust Nash equilibrium. Since the following theorem follows directly from the Nash equilibrium existence theorem [2, Theorem 9.1.1], we omit the proof.

Theorem 2.2. *Suppose that, for every player $i \in \mathcal{I}$,* (i) *the strategy set S_i is nonempty, convex and compact,* (ii) *the worst cost function $\tilde{f}_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to \mathbb{R}$ is continuous, and* (iii) *$\tilde{f}_i(\cdot, x^{-i})$ is convex for any $x^{-i} \in S_{-i}$. Then, game* (2.4) *has at least one Nash equilibrium, i.e., game* (2.2) *has at least one robust Nash equilibrium.*

2.2 SDP reformulation of a robust linear programming problem

In this subsection, we describe the SDP reformulation technique for a class of robust LPs discussed in [19]. This technique is based on Beck and Eldar's duality theory [4] in nonconvex quadratic programming, and will play an essential role in deriving the SDCP reformulation of the robust Nash equilibrium problem in the next section.

Consider the following uncertain LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \leq 0 \quad (i = 1, \ldots, K) \\ & x \in \Omega, \end{array} \tag{2.5}$$

where $\hat{\gamma}^i \in \mathbb{R}^{m_i}$ and $(\hat{A}^i, \hat{b}^i) \in \mathbb{R}^{m_i \times (n+1)}$ are uncertain vectors and matrices, respectively, and Ω is a given closed convex set with no uncertainty. Let $\mathcal{U}_i \subseteq \mathbb{R}^{m_i}$ and $\mathcal{V}_i \subseteq \mathbb{R}^{m_i \times (n+1)}$ be the uncertainty sets for $\hat{\gamma}^i$ and $(\hat{A}^i, \hat{b}^i)$, respectively, which satisfy the following assumption.

Assumption A. *For $i = 0, 1, \ldots, K$, the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are expressed as*

$$\mathcal{U}_i := \left\{ (\hat{A}^i, \hat{b}^i) \,\middle|\, (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{s_i} u^i_j (A^{ij}, b^{ij}),\ (u^i)^\top u^i \leq 1 \right\},$$
$$\mathcal{V}_i := \left\{ \hat{\gamma} \,\middle|\, \hat{\gamma} = \gamma^{i0} + \sum_{j=1}^{t_i} v^i_j \gamma^{ij},\ (v^i)^\top v^i \leq 1 \right\},$$

respectively, where $A^{ij} \in \mathbb{R}^{m_i \times n}, b^{ij} \in \mathbb{R}^{m_i}$ $(j = 0, 1, \ldots, s_i)$ and $\gamma^{ij} \in \mathbb{R}^{m_i}$ $(j = 1, \ldots, t_i)$ are given matrices and vectors, and s_i and t_i are given positive integers.

Then, the robust counterpart (RC) for (2.5) can be written as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \displaystyle\sup_{(\hat{A}^0, \hat{b}^0) \in \mathcal{U}_0,\ \hat{\gamma}^0 \in \mathcal{V}_0} (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \leq 0 \quad \forall (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \forall \hat{\gamma}^i \in \mathcal{V}_i \quad (i = 1, \ldots, K) \\ & x \in \Omega. \end{array} \tag{2.6}$$

Now, by means of the reformulation technique in [19], we introduce the following SDP associated with RC (2.6):

$$\begin{array}{ll} \underset{x,\alpha,\beta,\lambda_0}{\text{minimize}} & -\lambda_0 \\ \text{subject to} & \begin{bmatrix} P^0_0(x) & q^0(x) \\ q^0(x)^\top & r^0(x) - \lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P^0_1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_0 \begin{bmatrix} P^0_2 & 0 \\ 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} P^i_0(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P^i_1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P^i_2 & 0 \\ 0 & 1 \end{bmatrix} \ (i = 1, \ldots, K), \\ & \alpha := (\alpha_0, \alpha_1, \ldots, \alpha_K) \in \mathbb{R}^{K+1}_+, \quad \beta := (\beta_0, \beta_1, \ldots, \beta_K) \in \mathbb{R}^{K+1}_+, \\ & \lambda_0 \in \mathbb{R}, \quad x \in \Omega, \end{array} \tag{2.7}$$

where the problem data are defined by

$$
\begin{aligned}
P_0^i(x) &:= -\frac{1}{2}\begin{bmatrix} 0 & (\Gamma_i^\top \Phi_i(x))^\top \\ \Gamma_i^\top \Phi_i(x) & 0 \end{bmatrix}, \quad q^i(x) := -\frac{1}{2}\begin{bmatrix} \Phi_i(x)^\top \gamma^i \\ \Gamma_i^\top (A^{i0}x + b^{i0}) \end{bmatrix}, \\
r^i(x) &:= -(\gamma^i)^\top (A^{i0}x + b^{i0}), \quad P_1^i := \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^i := \begin{bmatrix} 0 & 0 \\ 0 & -I_{t_i} \end{bmatrix}, \\
\Gamma_i &:= \begin{bmatrix} \gamma^{i1} & \cdots & \gamma^{it} \end{bmatrix}, \quad \Phi_i(x) := \begin{bmatrix} A^{i1}x + b^{i1} & \cdots & A^{is_i}x + b^{is_i} \end{bmatrix}.
\end{aligned} \tag{2.8}
$$

Then, we can show that the optimum of SDP (2.7) also solves RC (2.6) under a certain nonsingularity assumption.

Theorem 2.3. [19, Theorem 3.1] *Suppose that Assumption A holds and $z^* := (x^*, \alpha^*, \beta^*, \lambda_0^*)$ is an optimum of SDP (2.7). Then, x^* is feasible in RC (2.6), and $-\lambda_0^*$ serves as an upper bound of the optimal value of* (2.6)*. Moreover, if there exists an $\varepsilon > 0$ such that*

$$
\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \quad (i = 0, 1, \ldots, K) \tag{2.9}
$$

for all $(x, \alpha, \beta, \lambda_0^*) \in B(z^*, \varepsilon)$, then x^* solves RC (2.6)*.*

In general, it is difficult to judge whether or not the condition (2.9) holds, since we have to check all the functional values around the optimum z^*. However, if the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are spherical, then (2.9) holds automatically [19]. We thus make the following assumption.

Assumption B. *The uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are expressed as in Assumption A. Moreover, for each $i = 0, 1, \ldots, K$, matrices (A^{ij}, b^{ij}) $(j = 1, \ldots, m_i(n+1))$ and vectors γ^{ij} $(j = 1, \ldots, t_i)$ $(t_i \geq 2)$ satisfy the following conditions:*

- *For each $(k, l) \in \{1, \ldots, m_i\} \times \{1, \ldots, n+1\}$,*

$$
(A^{ij}, b^{ij}) = \rho_i e_k^{(m_i)} (e_l^{(n+1)})^\top \quad \text{with } j := m_i l + k,
$$

  where ρ_i is a given nonnegative constant, and $e_r^{(p)}$ is a p-dimensional unit vector with 1 at the r-th component and 0 elsewhere.
- *For each $(k, l) \in \{1, \ldots, t_i\} \times \{1, \ldots, t_i\}$,*

$$
(\gamma^{ik})^\top \gamma^{il} = \sigma_i^2 \delta_{kl},
$$

  where σ_i is a given nonnegative constant, and δ_{kl} denotes Kronecker's delta, i.e., $\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kl} = 1$ for $k = l$.

Note that Assumption B claims that $\mathcal{U}_i$ is an $m_i(n+1)$-dimensional sphere with radius ρ_i in the $m_i(n+1)$-dimensional space and $\mathcal{V}_i$ is a t_i-dimensional sphere with radius σ_i in the m_i-dimensional space, i.e.,

$$
\begin{aligned}
\mathcal{U}_i &= \left\{ (\hat{A}^i, \hat{b}^i) \;\middle|\; (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + (\delta A^i, \delta b^i),\ \|(\delta A^i, \delta b^i)\|_F \leq \rho_i \right\} \subset \mathbb{R}^{m_i(n+1)}, \\
\mathcal{V}_i &= \left\{ \hat{\gamma}^i \;\middle|\; \hat{\gamma}^i = \gamma^{i0} + \delta\gamma^i,\ \|\delta\gamma^i\| \leq \sigma_i, \delta\gamma^i \in \operatorname{span}\{\gamma^{ik}\}_{k=1}^{t_i} \right\} \subset \mathbb{R}^{m_i}.
\end{aligned}
$$

Under this assumption, we have the following theorem.

Theorem 2.4. [19, Theorem 3.4] *Suppose Assumption* B *holds. Then, x^* solves RC* (2.6) *if and only if there exists $(\alpha^*, \beta^*, \lambda_0^*)$ such that $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ is an optimal solution of SDP* (2.7)*.*

3 SDCP reformulation of robust Nash equilibrium problems

In this subsection, we focus on the game in which each player takes a mixed strategy and minimizes a convex quadratic cost function with respect to his own strategy. We show that each player's optimization problem can be reformulated as an SDP, and the robust Nash equilibrium problem reduces to an SDCP.

Originally, the SDCP is to find, for a given mapping $F : \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m \to \mathcal{S}^n \times \mathbb{R}^m$, a triple $(X, Y, z) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ such that

$$\mathcal{S}^n_+ \ni X \perp Y \in \mathcal{S}^n_+, \quad F(X, Y, z) = 0,$$

where $X \perp Y$ means $\langle X, Y \rangle := \mathrm{tr}(X^\top Y) = 0$. An SDCP can be solved by using Newton type algorithms based on the merit function reformulation [9, 20].

Throughout this section, the cost functions and the strategy sets of all players satisfy the following conditions:

(i) Player i's cost function $f_i^{\hat{u}^i}$ is defined by*2

$$f_i^{\tilde{u}^i}(x^i, \hat{x}^{-i}) = \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij} \hat{x}^j, \tag{3.1}$$

where $\hat{A}_{ij} \in \mathbb{R}^{m_i \times m_j}$ $(j \in \mathcal{I}_{-i})$ are matrices involving uncertainties.

(ii) Player i takes a mixed strategy, i.e.,

$$S_i = \left\{ x^i \in \mathbb{R}^{m_i} \;\middle|\; x^i \geq 0,\ \mathbf{1}_{m_i}^\top x^i = 1 \right\}, \tag{3.2}$$

where $\mathbf{1}_{m_i}$ denotes $(1, 1, \ldots, 1)^\top \in \mathbb{R}^{m_i}$.

(iii) $m_i \geq 3$ for all $i \in \mathcal{I}$.

We call $\hat{A}_{ij}$ a cost matrix. Note that these matrices correspond to the parameters $\hat{u}^i$ involved in the cost functions of game (2.2), i.e.,

$$\hat{u}^i = \mathrm{vec}\left[\hat{A}_{i1} \quad \ldots, \hat{A}_{iN}\right] \in \mathbb{R}^{m_i m},$$

where vec denotes the vectorization operator that creates an nm-dimensional vector $[(p_1^c)^\top \cdots (p_m^c)^\top]^\top$ from a matrix $P \in \Re^{n \times m}$ with column vectors $p_1^c, \ldots, p_m^c \in \mathbb{R}^n$.

For the robust Nash equilibrium problem with the above cost functions and strategy sets, Nishimura et al. [18] show that it can be reformulated as an SOCCP under the assumption that the players can exactly estimate either the opponents' strategies or their own cost matrices. In this subsection, we consider the more general case where both of them are uncertain. We first establish the existence of a robust Nash equilibrium, and then, show that the robust Nash equilibrium problem can be reformulated as an SDCP. To this end, we make the following assumption.

Assumption C. *For each $i \in \mathcal{I}$, the uncertainty sets $X_i(\cdot)$ and U_i are given as follows:*

(a) $X_i(x^{-i}) = \prod_{j \in \mathcal{I}_{-i}} X_{ij}(x^j)$, *where* $X_{ij}(x^j) := \{x^j + \delta x^{ij} \mid \|\delta x^{ij}\| \leq \sigma_{ij},\ \mathbf{1}_{m_j}^\top \delta x^j = 0\ (j \in \mathcal{I}_{-i})\}$ *for some nonnegative scalars* σ_{ij}.

(b) $U_i = \prod_{j \in \mathcal{I}_{-i}} D_{ij}$, *where* $D_{ij} := \{A_{ij} + \delta A_{ij} \in \mathbb{R}^{m_i \times m_j} \mid \|\delta A_{ij}\|_F \leq \rho_{ij}\}$ *for some nonnegative scalars* ρ_{ij}. *Moreover,* $A_{ii} + \rho_{ii} I$ *is symmetric and positive semidefinite.*

Assumption C claims that $X_{ij}(x^j)$ is the closed sphere with center x^j and radius σ_{ij} in the subspace $\{x \in \mathbb{R}^{m_j} \mid \mathbf{1}_{m_j}^\top x = 0\}$, and D_{ij} is also the closed sphere with center A_{ij} and radius ρ_{ij} in $\mathbb{R}^{m_i \times m_i}$. Note

*2 Although we can include the additional linear term $c^\top x$ in (3.1), we suppress it for simplicity.

that Assumption C is weaker than the assumptions in Nishimura et al. [18]. Indeed, Assumption C with either $\rho_{ij} = 0$ or $\sigma_{ij} = 0$ for all $(i, j) \in \mathcal{I} \times \mathcal{I}$ corresponds to the assumptions in [18].

Under Assumption C, we rewrite each player' optimization problem (2.4). Note that the worst cost function $\tilde{f}_i$ can be written as

$$
\begin{aligned}
&\tilde{f}_i(x^i, x^{-i}) \\
&= \max \left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij} \hat{x}^j \,\middle|\, \begin{array}{l} \hat{A}_{ii} \in D_{ii}, \\ \hat{A}_{ij} \in D_{ij},\ \hat{x}^j \in X_{ij}(x^j)\ (j \in \mathcal{I}_{-i}) \end{array} \right\} \\
&= \max \left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \,\middle|\, \hat{A}_{ii} \in D_{ii} \right\} + \sum_{j \in \mathcal{I}_{-i}} \max \left\{ (x^i)^\top \hat{A}_{ij} \hat{x}^j \,\middle|\, \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\} \\
&= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j \in \mathcal{I}_{-i}} \max \left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \,\middle|\, \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\},
\end{aligned} \tag{3.3}
$$

where the last equality holds since

$$
\begin{aligned}
\max \left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \,\middle|\, \hat{A}_{ii} \in D_{ii} \right\} &= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max \left\{ \frac{1}{2}(x^i)^\top \delta A_{ii} x^i \,\middle|\, \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max \left\{ \frac{1}{2}(x^i \otimes x^i)\,\mathrm{vec}(\delta A_{ii}) \,\middle|\, \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \frac{1}{2}\rho_{ii} \|x^i\|^2 \\
&= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i.
\end{aligned}
$$

Hence, each player's optimization problem (2.4) can be rewritten as follows:

$$
\begin{aligned}
\underset{x^i}{\text{minimize}} \quad & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j \in \mathcal{I}_{-i}} \max \left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \,\middle|\, \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\} \\
\text{subject to} \quad & \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \ge 0.
\end{aligned} \tag{3.4}
$$

Now we show the existence of a robust Nash equilibrium under Assumption C.

Theorem 3.1. *Suppose that the cost functions and the strategy sets are given by* (3.1) *and* (3.2)*, respectively. Suppose further that Assumption* C *holds. Then, there exists at least one robust Nash equilibrium.*

Proof. It suffices to show that the worst cost function $\tilde{f}_i$ and the strategy set S_i satisfy the three conditions given in Theorem 2.2. S_i is obviously nonempty, convex and compact from (3.2). $\tilde{f}_i$ is continuous from (3.3), [3, Theorem 1.4.16] and the continuity of the set-valued mapping X_{ij}. Moreover, from (3.3), $A_{ii} + \rho_{ii} I \succeq 0$, and [8, Proposition 1.2.4(c)], $\tilde{f}_i(\cdot, x^{-i})$ is convex for arbitrarily fixed $x^{-i} \in S_{-i}$. □

Next we show that problem (3.4) can be rewritten as an SDP. We note that problem (3.4) has a structure analogous to problem (2.6), and $X_{ij}(x^j)$ and D_{ij} satisfy Assumption B. Indeed, $X_{ij}(x^j)$ can be constructed by means of some vectors γ^{ijk} $(k = 1, \dots, m_j - 1)$ which form orthogonal bases of the subspace $\{x \mid \mathbf{1}_{m_j}^\top x = 0\}$ with $\|\gamma^{ijk}\| = \sigma_{ij}$ for all k, i.e., $X_{ij}(x^j) = \{\hat{x}^j \mid \hat{x}^j = x^j + \sum_{k=1}^{m_j-1} v_k^{ij} \gamma^{ijk},\ (v_k^{ij})^\top v_k^{ij} \le 1\}$.

Thus, by Theorem 2.4, problem (3.4) can be rewritten as the following SDP:

$$\begin{array}{ll} \underset{x^i,\alpha^{-i},\beta^{-i},\lambda^{-i}}{\text{minimize}} & \frac{1}{2}(x^i)^\top(A_{ii}+\rho_{ii}I)x^i - \sum_{j\in\mathcal{I}_{-i}}\lambda_{ij} \\ \text{subject to} & \begin{bmatrix} P_0^{ij}(x^i) & q^{ij}(x^i,x^j) \\ q^{ij}(x^i,x^j)^\top & r^{ij}(x^i,x^j)-\lambda_{ij}\end{bmatrix} \succeq \alpha_{ij}\begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1\end{bmatrix} + \beta_{ij}\begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1\end{bmatrix}, \ (j\in\mathcal{I}_{-i}) \\ & \alpha^{-i} := (\alpha_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}^{N-1}_+, \quad \beta^{-i} := (\beta_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}^{N-1}_+, \\ & \lambda^{-i} := (\lambda_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}^{N-1}, \\ & \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \geq 0, \end{array} \tag{3.5}$$

where

$$\begin{aligned} P_0^{ij}(x^i) &:= -\frac{1}{2}\begin{bmatrix} 0 & \rho_{ij}(\Gamma_{ij}^\top((x^i)^\top\otimes I_{m_j}))^\top \\ \rho_{ij}\Gamma_{ij}^\top((x^i)^\top\otimes I_{m_j}) & 0 \end{bmatrix}, \\ q^{ij}(x^i,x^j) &:= -\frac{1}{2}\begin{bmatrix} \rho_{ij}((x^i)^\top\otimes I_{m_j})^\top x^j \\ \Gamma_{ij}^\top A_{ij}^\top x^i \end{bmatrix}, \quad r^{ij}(x^i,x^j) := -(x^j)^\top A_{ij}^\top x^i, \\ P_1^{ij} &:= \begin{bmatrix} -I_{m_i m_j} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^{ij} := \begin{bmatrix} 0 & 0 \\ 0 & -I_{m_j-1} \end{bmatrix}, \\ \Gamma_{ij} &:= \begin{bmatrix} \gamma^{ij1} & \cdots & \gamma^{ij(m_j-1)} \end{bmatrix}. \end{aligned} \tag{3.6}$$

Finally, we show that the robust Nash equilibrium problem reduces to an SDCP. Since the semidefinite constraints in (3.5) are linear with respect to $x^i, \alpha^{-i}, \beta^{-i}$ and λ^{-i}, we can rewrite the constraints as

$$\sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} \succeq \alpha_{ij}M_\alpha^{ij} + \beta_{ij}M_\beta^{ij}, \quad (j\in\mathcal{I}_{-i}),$$

where $M_k^{ij}(x^j) \in \mathcal{S}^{m_j(m_i+1)}$ $(k=1,\dots,m_i)$, $M_\lambda^{ij}, M_\alpha^{ij}$, and $M_\beta^{ij} \in \mathcal{S}^{m_j(m_i+1)}$ are defined by

$$\begin{aligned} M_k^{ij}(x^j) &:= \begin{bmatrix} P_0^{ij}(e_k^{(m_i)}) & q^{ij}(e_k^{(m_i)},x^j) \\ q^{ij}(e_k^{(m_i)},x^j)^\top & r^{ij}(e_k^{(m_i)},x^j) \end{bmatrix} \quad (k=1,\dots,m_i), \\ M_\lambda^{ij} &:= -e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)}\left(e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)}\right)^\top, \quad M_\alpha^{ij} := \begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1 \end{bmatrix}, \quad M_\beta^{ij} := \begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Then, the Karush-Kuhn-Tucker (KKT) conditions for (3.5) are given by

$$\begin{gathered} ((A_{ii}+\rho_{ii}I)x^i)_k - \sum_{j\in\mathcal{I}_{-i}}\left\langle M_k^{ij}(x^j), Z^{ij}\right\rangle - (\mu_x^i)_k + \nu^i = 0, \quad (k=1,\dots,m_i), \\ \left\langle M_\alpha^{ij}, Z^{ij}\right\rangle - (\mu_\alpha^i)_j = 0, \quad (j\in\mathcal{I}_{-i}), \\ \left\langle M_\beta^{ij}, Z^{ij}\right\rangle - (\mu_\beta^i)_j = 0, \quad (j\in\mathcal{I}_{-i}), \\ \left\langle M_\lambda^{ij}, Z^{ij}\right\rangle + 1 = 0, \quad (j\in\mathcal{I}_{-i}), \\ \left\langle \sum_{k=1}^{m_1} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} - \alpha_{ij}M_\alpha^{ij} - \beta_{ij}M_\beta^{ij},\, Z^{ij} \right\rangle = 0, \\ (\mu_\alpha^i)^\top\alpha^{-i} = 0, \quad (\mu_\beta^i)^\top\beta^{-i} = 0, \quad (\mu_x^i)^\top x^i = 0, \\ \sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} \succeq \alpha_{ij}M_\alpha^{ij} + \beta_{ij}M_\beta^{ij}, \quad (j\in\mathcal{I}_{-i}), \\ \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \geq 0, \quad \alpha^{-i} \geq 0, \quad \beta^{-i} \geq 0, \\ Z^{ij} \succeq 0, \quad \mu_x^i \geq 0, \quad \mu_\alpha^i \geq 0, \quad \mu_\beta^i \geq 0, \end{gathered}$$

where $Z^{ij} \in \mathcal{S}^{m_j(m_i+1)}$, $\mu_x^i \in \mathbb{R}^{m_i}$, $\mu_\alpha^i, \mu_\beta^i \in \mathbb{R}^{N-1}$ and $\nu^i \in \mathbb{R}$ are Lagrange multipliers. Eliminating μ_x^i, μ_α^i

and μ^i_β, we obtain the following conditions for each $i \in \mathcal{I}$:

$$\begin{aligned}
\mathcal{S}_+^{m_i(m_j+1)} \ni Z^{ij} \perp \sum_{k=1}^{m_i} x^i_k M^{ij}_k(x^j) + \lambda_{ij} M^{ij}_\lambda - \alpha_{ij} M^{ij}_\alpha - \beta_{ij} M^{ij}_\beta \in \mathcal{S}_+^{m_i(m_j+1)} \quad (j \in \mathcal{I}_{-i}), \\
\mathbb{R}^{m_i}_+ \ni x^i \perp \Big(\big((A_{ii} + \rho_{ii} I)x^i\big)_k - \sum_{j \in \mathcal{I}_{-i}} \langle M^{ij}_k(x^j), Z^{ij} \rangle + \nu^i \Big)_{k=1,\dots,m_i} \in \mathbb{R}^{m_i}, \\
\mathbb{R}^{N-1}_+ \ni \alpha^{-i} \perp \big(\langle M^{ij}_\alpha, Z^{ij} \rangle \big)_{j \in \mathcal{I}_{-i}} \in \mathbb{R}^{N-1}_+, \quad \mathbb{R}^{N-1}_+ \ni \beta^{-i} \perp \big(\langle M^{ij}_\beta, Z^{ij} \rangle \big)_{j \in \mathcal{I}_{-i}} \in \mathbb{R}^{N-1}_+, \\
\langle M^{ij}_\lambda, Z^{ij} \rangle = -1 \ \ (j \in \mathcal{I}_{-i}), \quad \mathbf{1}^\top_{m_i} x^i = 1.
\end{aligned} \tag{3.7}$$

Since the above KKT conditions hold for all players simultaneously, the robust Nash equilibrium problem can be reformulated as the problem of finding $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j \in \mathcal{I}_{-i}}, \nu^i)_{i \in \mathcal{I}}$ that satisfies (3.7) for all $i \in \mathcal{I}$. Thus, we obtain the following theorem.

Theorem 3.2. *Suppose that the cost functions and the strategy sets are given by* (3.1) *and* (3.2)*, respectively. Suppose further that Assumption* C *holds. Then, x is a robust Nash equilibrium if and only if $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j \in \mathcal{I}_{-i}}, \nu^i)_{i \in \mathcal{I}}$ solves SDCP* (3.7).

4 Numerical example

In this section, we solve a robust Nash equilibrium problem with uncertainties in both the cost matrices and the opponents' strategies, by using the SDCP reformulation approach proposed in Section 3. In particular, by changing the size of uncertainty sets, we observe the behavior of equilibria. For solving the reformulated SDCPs, we apply the Fischer-Burmeister type merit function approach proposed by Yamashita and Fukushima [20]. In minimizing the merit function, we use `fminunc` in MATLAB Optimization toolbox. All programs are coded in MATLAB 7.4.0 and run on a machine with Intel® Core 2 DUO 3.00GHz CPU and 3.20GB memories.

We consider the two-person robust Nash equilibrium problem where the cost functions and the strategy sets are given by (3.1) and (3.2), respectively. We also suppose that Assumption C holds with

$$A_{11} = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 8 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & -12 & 0 \\ 2 & 1 & 1 \\ 4 & 3 & 1 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 6 & -1 \\ 2 & 1 & 9 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 1 & -2 \\ -11 & 6 & -4 \end{bmatrix},$$

$\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = \rho$, and $\sigma_{12} = \sigma_{21} = \sigma$. If $(\rho, \sigma) = (0, 0)$, then we have the ordinary Nash equilibrium problem with $\hat{A}_{ij}$ and $\hat{x}^j$ $(i, j = 1, 2)$ in (3.1) replaced by A_{ij} and x^j, respectively. The Nash equilibrium of this game is $\big(x^1_{\rm Nash}, x^2_{\rm Nash}\big) = \big((0.6203,\ 0.1020,\ 0.2777),\ (0.0000,\ 0.0748,\ 0.9252)\big)$.

First, we fix $\rho = 1$ and vary σ from 0 to 1 to observe the behavior of robust Nash equilibria. Table 1 and Figure 1 show the values and the trajectories of obtained robust Nash equilibria, respectively. In Figure 1, two large gray circles represent the Nash equilibria with $(\rho, \sigma) = (0, 0)$, and white circles represent the robust Nash equilibrium with $\rho = 1$ and $\sigma = 0,\ 0.1,\ 0.2, \dots, 0.9,\ 1.0$. We can observe that, when σ is small, player 1 tends to choose strategy 1,[*3] since he knows that player 2 chooses strategy 1 with low possibility (i.e., x^2_1 is small) and the first row vector of A_{12} contains small values $(A_{12})_{12} = -12$ and $(A_{12})_{13} = 0$. However, as σ increases, player 1 tends to choose strategy 2 instead of strategy 1 because of the following facts:

[*3] In other words, x^1_1 is larger than x^1_2 and x^1_3 at the robust Nash equilibria.

Table. 1 Robust Nash equilibria with various choices of σ

ρ	σ	player 1	player 2
1	0	(0.5930, 0.1625, 0.2445)	(0.0717, 0.1560, 0.7723)
1	0.1	(0.4664, 0.2961, 0.2375)	(0.3424, 0.2184, 0.4392)
1	0.2	(0.3202, 0.4353, 0.2444)	(0.5592, 0.2489, 0.1919)
1	0.5	(0.0710, 0.6500, 0.2790)	(0.6834, 0.2260, 0.0905)
1	1.0	(0.0000, 0.7349, 0.2651)	(0.7470, 0.1506, 0.1024)

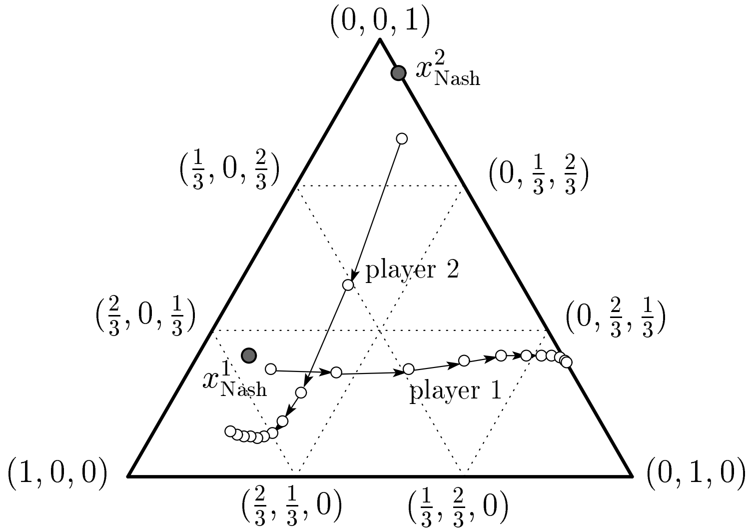

Fig. 1 Trajectory of robust Nash equilibria with various choices of σ

(i) $(A_{11})_{22}$ is smaller than $(A_{11})_{11}$ and $(A_{11})_{33}$;

(ii) the largest component of the second row vector of A_{12} is smaller than the largest components of the first and the third row vectors, i.e., $(A_{12})_{21} < (A_{12})_{11}$, $(A_{12})_{31}$.

Particularly, the fact (ii) is characteristic of the robust Nash equilibrium. When σ is large, player 1 cannot estimate the effect of player 2's strategy. He therefore chooses his strategy by taking all unfavorable cases into consideration. Indeed, if player 2 chooses a large value for x_1^2, then it is unfavorable for player 1 since the first column vector of A_{12} is $(5, 2, 4)^\top$, which is bigger than the other two column vectors.

Next, we fix $\sigma = 0.01$ and vary ρ from 0 to 30. Table 2 and Figure 2 show the values and the trajectories of obtained robust Nash equilibria, respectively. In Figure 2, two large gray circles represent the Nash equilibrium with $(\rho, \sigma) = (0, 0)$, and white circles represent the robust Nash equilibria with $\sigma = 0.01$ and ρ = 0, 1, 2, 3, 5, 10, 20, 30. In contrast with the previous result, we can observe that the both players' robust Nash equilibria approach $(1/3, 1/3, 1/3)$ as ρ increases. This is a convincing result since, when ρ is very large, each player can hardly evaluate his cost function, and hence takes the three pure strategies with almost equal probability so as to mitigate the damage due to the worst possible case.

5 Concluding remarks

In this paper, using the SDP reformulation technique for robust LP, we have showed that the robust Nash equilibrium problem reduces to an SDCP under the assumption that *both* the opponents' strategies and each player's cost parameters belong to Euclidean uncertainty sets. We have also provided some numerical examples to show the validity of the SDCP reformulation and the behavior of robust Nash equilibria.

Table. 2 Robust Nash equilibria with various choices of ρ

ρ	σ	player 1	player 2
0	0.01	(0.6194, 0.1040, 0.2766)	(0.0000, 0.0838, 0.9162)
1	0.01	(0.5811, 0.1750, 0.2439)	(0.0985, 0.1623, 0.7393)
2	0.01	(0.5377, 0.2256, 0.2365)	(0.2039, 0.1936, 0.6024)
5	0.01	(0.4773, 0.2747, 0.2479)	(0.2913, 0.2492, 0.4594)
10	0.01	(0.4313, 0.2990, 0.2696)	(0.3200, 0.2854, 0.3945)
30	0.01	(0.3754, 0.3206, 0.3039)	(0.3322, 0.3172, 0.3505)

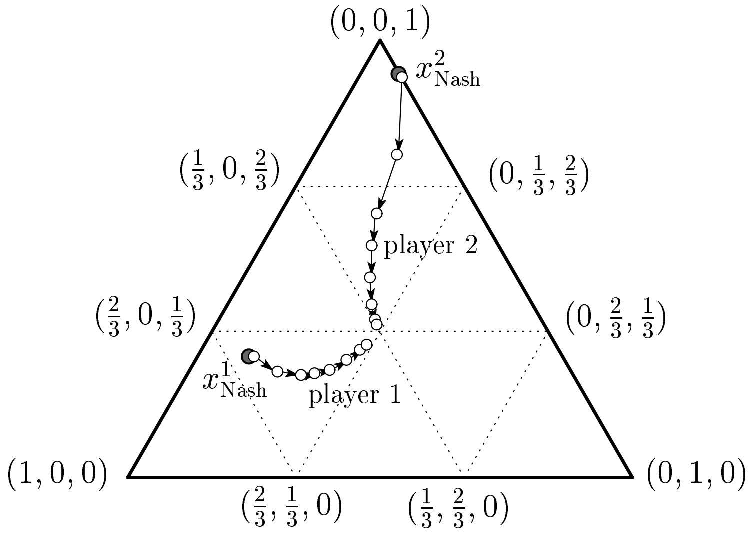

Fig. 2 Trajectory of robust Nash equilibria with various choices of ρ

We still have some future issues to be addressed. In this paper, we have reformulated the robust Nash equilibrium problem as a *nonlinear* SDCP. Since many efficient algorithms have been proposed for *linear* SDCPs, it would be more meritorious to reformulate the robust Nash equilibrium problem as a linear SDCP. Also, it is a challenging subject to extend the SDCP reformulation results to the ellipsoidal uncertainty case, since condition (2.9) is no longer guaranteed when the uncertainty sets are ellipsoidal.

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