The integrable discrete hungry systems and their related matrix eigenvalues

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Abstract

Some of the authors design an algorithm, named the dhLV algorithm, for computing complex eigenvalues of a certain band matrix. The recursion formula of the dhLV algorithm is derived from the discrete hungry Lotka-Volterra system which is an integrable system. One of the authors proposes an algorithm, named the multiple dqd algorithm, for eigenvalues of totally nonnegative (TN) matrix. In this paper, by introducing a similarity transformation and a theorem for matrix eigenvalues, we show that eigenvalues of the TN matrix are computable by the dhLV algorithm. Based on the integrable discrete hungry Toda equation, we design a new algorithm for TN matrix eigenvalues. We also describe a close relationship among the above three algorithms. The numerical stabilities of two algorithm, based on the integrable discrete hungry systems, are investigated through an error analysis of them. Some numerical examples are given.

Keywords: discrete hungry Toda equation, discrete hungry Lotka-Volterra

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1. Introduction

Several integrable systems have the profound relationships with various algorithms. In [26], Symes finds that 1-step of the QR algorithm, for computing matrix eigenvalues, corresponds to a time evolution of the continuous-time Toda equation. Hirota's discretization technique [11] leads to the discrete-time version of the Toda equation,

$$\begin{cases} q_k^{(n+1)} = q_k^{(n)} - e_{k-1}^{(n+1)} + e_k^{(n)}, \quad k = 1, 2, \ldots, m, \\ e_k^{(n+1)} = e_k^{(n)} \dfrac{q_{k+1}^{(n)}}{q_k^{(n+1)}}, \quad k = 1, 2, \ldots, m-1, \\ e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \quad n = 0, 1, \ldots, \end{cases} \tag{1}$$

where $q_k^{(n)}, e_k^{(n)}$ denote the values of q_k, e_k at the discrete time n, respectively. The discrete Toda equation (1) is just equal to the recursion formula of Rutishauser's quotient difference (qd) algorithm [25] for eigenvalue of a symmetric tridiagonal matrix. The qd algorithm, namely, the discrete Toda equation (1) is applicable for singular value of a bidiagonal matrix [5, 6]. The applications of the discrete Toda equation (1) are also observed in many fields such as the BCH-Goppa decoding [21], Laplace transformation [20], and so on. In [19], it is shown that the discrete Toda equation (1) is related to the ϵ-algorithm for accelerating the convergence rate of sequence.

Another integrable discrete system yields some numerical algorithms related to matrix eigenvalues or singular values. Iwasaki and Nakamura in [14, 15] design an algorithm for singular values from the integrable discrete-time discrete Lotka-Volterra (dLV) system [12],

$$\begin{cases} u_k^{(n+1)}(1 + \delta^{(n+1)} u_{k-1}^{(n+1)}) = u_k^{(n)}(1 + \delta^{(n)} u_{k+1}^{(n)}), \quad k = 1, 2, \ldots, 2m-1, \\ u_0^{(n)} \equiv 0, \quad u_{2m}^{(n)} \equiv 0, \quad n = 0, 1, \ldots, \end{cases} \tag{2}$$

which is known as a time discretization of the continuous-time Lotka-Volterra (LV) system, where $\delta^{(n)}$ denotes the nth discrete step size and $u_k^{(n)}$ denotes the number of kth species at the discrete time $\sum_{j=0}^{n-1} \delta^{(j)}$. The LV system originally describes the struggle for survival of $2m-1$ species such that the kth species preys upon the $(k+1)$th species and becomes the food for the $(k-1)$th species [30]. The dLV system (2) is also given from the discrete Toda equation (1) through the Miura transformation,

$$\begin{cases} q_k^{(n)} = \dfrac{1}{\delta^{(n)}}\left(1+\delta^{(n)}u_{2k-2}^{(n)}\right)\left(1+\delta^{(n)}u_{2k-1}^{(n)}\right), \quad k=1,2,\ldots,m, \\ e_k^{(n)} = \delta^{(n)}u_{2k-1}^{(n)}u_{2k}^{(n)}, \quad k=1,2,\ldots,m-1. \end{cases} \tag{3}$$

A remarkable property of the dLV system (2) is that, for the suitable initial $u_k^{(0)}$, the dLV variable $u_{2k-1}^{(n)}$ converges to singular value of bidiagonal matrix, as $n \to \infty$. This asymptotic convergence immediately brings an algorithm, named the dLV algorithm, for singular value. In [3], the dLV algorithm is reviewed by Chu. In order to accelerate the convergence rate, Iwasaki and Nakamura in [16] introduce shift of origin into the dLV algorithm. The shifted version of the dLV algorithm is called the modified dLV with shift (mdLVs) algorithm. The computed singular values by the mdLVs algorithm are shown to have high relative accuracy.

The LV system is naturally extended to the continuous-time hungry LV (hLV) system by considering the case where the kth species prays not only the $(k+1)$th species but also the $(k+2)$th, the $(k+3)$th, …, the $(k+M)$th ones [2, 13]. The hLV system with $M=1$ coincides with the LV system. A time-discretization [22] of the hLV (dhLV) system is given as

$$\begin{cases} u_k^{(n+1)}\displaystyle\prod_{j=1}^{M}(1+\delta^{(n+1)}u_{k-j}^{(n+1)}) = u_k^{(n)}\prod_{j=1}^{M}(1+\delta^{(n)}u_{k+j}^{(n)}), \quad k=1,2,\ldots,M_m, \\ u_{1-M}^{(n)} \equiv 0,\ldots,u_0^{(n)} \equiv 0, \quad u_{M_m+1}^{(n)} \equiv 0,\ldots,u_{M_m+M}^{(n)} \equiv 0, \quad n=0,1,\ldots, \end{cases} \tag{4}$$

where $M_k := (M+1)k - M$ and the notations k, $\delta^{(n)}$ and $u_k^{(n)}$ in (4) are the same as those in the dLV system (2). From the dhLV system (4), Fukuda, Ishiwata, Iwasaki and Nakamura in [7] derive an algorithm, named the dhLV algorithm, for computing complex eigenvalue of a band matrix. Though it is verified by some numerical examples that eigenvalues are computed with high relative accuracy, any error analysis for the dhLV algorithm has not been theoretically clarified.

The box and ball system (BBS) is found by Takahashi and Matsukidaira in [23] from the viewpoint of integrable systems. The BBS represents a movement of finite number of balls in an array of boxes. Rule of the BBS is that the leftmost ball is moved to the nearest right empty box. The dynamics of the BBS is related to the discrete Toda equation (1). Tokihiro, Nagai and Satsuma in [24] propose a different BBS, named the numbered BBS, by numbering the balls from 1 to M. Of course, every ball is distinguished by its index. The numbered BBS imposes that the leftmost ball with index 1 is moved to the nearest right empty box, and do this procedure for the balls with index from 2 to M. The numbered BBS is also associated with the integrable discrete hungry Toda (dhToda) equation,

$$\begin{cases} Q_k^{(n+M)} = Q_k^{(n)} + E_k^{(n)} - E_{k-1}^{(n+1)}, \quad k = 1, 2, \ldots, m, \\ E_k^{(n+1)} = \dfrac{Q_{k+1}^{(n)} E_k^{(n)}}{Q_k^{(n+M)}}, \quad k = 1, 2, \ldots, m-1, \\ E_0^{(n)} := 0, \quad E_m^{(n)} := 0, \end{cases} \tag{5}$$

which is regarded as an extension of the discrete Toda equation (1). The authors of [24] expect that the dhToda equation (5) has some interesting relationship with matrix eigenvalue. However, to the best of our knowledge, the dhToda equation (5) has not been shown to be related to what kind of matrices, much less achieves a new algorithm for matrix eigenvalue.

In [29], Yamamoto and Fukaya propose an algorithm, named multiple dqd algorithm, for eigenvalues of totally nonnegative (TN) band matrices [1, 18] where all the minors are nonnegative. TN matrices appear in many fields of mathematical subjects and applications including combinatorics, probability, stochastic processes, and inverse problems [4, 8, 9, 17]. It is also shown in [29] that the multiple dqd variable corresponds to the dhLV one.

The main purpose of this paper is threefold. The first is to clarify that the eigenvalues of TN matrix are computable by the dhLV algorithm. The second is to design a new algorithm for matrix eigenvalue through a matrix representation and an asymptotic analysis for the dhToda equation (5). The relationship among the three algorithms: the dhLV algorithm, the algorithm based on the dhToda equation (5), and the multiple dqd algorithm is also shown. The third is to perform an error analysis of the dhLV algorithm and the algorithm based on the dhToda equation (5). Some numerical examples are given.

This paper is organized as follows. In Section 2, we briefly explain how to derive the dhLV algorithm proposed in [7] from the dhLV system (4). We expand the class of matrices where the dhLV algorithm is applicable by considering a similarity transformation. And then, with the help of a theorem in [28] for matrix eigenvalue we especially show that eigenvalues of TN matrix are computable by the dhLV algorithm. We also clarify the relationship of the dhLV algorithm with the multiple dqd algorithm. In Section 3, we investigate a matrix representation, named the Lax form, for the dhToda equation (5) and an asymptotic behavior of the dhToda variable as time variable $n \to \infty$. Based on the dhToda equation (5), we design a new algorithm for eigenvalues of TN matrix. We besides describe the relationship of the dhLV algorithm, the algorithm designed in Section 3, and the multiple dqd algorithm. In Section 4, we present the error analyses for the dhLV algorithm and the algorithm in Section 3, in order to show the numerical stabilities of them. In Section 5, we confirm the theoretical results in Sections 2–4 through some numerical examples. Finally, we give concluding remarks in Section 6.

2. The dhLV algorithm for band matrices

A matrix A is said to be totally nonnegative (TN), if every minor of A is nonnegative [1, 18]. The main purpose of this section is to show that the eigenvalues of some TN matrices are computable by the dhLV algorithm proposed in [7].

The dhLV algorithm and its basic properties are briefly reviewed in Section 2.1. In Section 2.2, we give a different aspect on the dhLV algorithm. Not only the band matrices $\mathcal{L}^{(n)}+dI$ appearing in Section 2.1 but also the TN matrices in Section 2.2 are shown to be the targets of the dhLV algorithm. A theorem on matrix eigenvalue [28] plays a key role for this proof. In Section 2.3, we explain the multiple dqd algorithm designed in [29] for computing the eigenvalues of TN matrices. We also clarify a relationship between the dhLV algorithm and the multiple dqd algorithm.

2.1. The dhLV system and matrix eigenvalues

We first survey some important properties of the dhLV system, which are the basis of the dhLV algorithm. One of the essential properties of integrable systems is a matrix representation called Lax form. The Lax form is an idea which arises from the study of integrable systems. It is fruitful to reconsider the Lax form from the viewpoint of matrix analysis. The QR and the qd algorithms are actually related to the integrable systems, called the Toda equation and the discrete Toda equation (1), respectively, through the discussion of the Lax form. The dLV algorithm is designed with the help of a Lax form for the dLV system.

A Lax form for the dhLV system (4) is presented in [27] as follows.

$$\mathcal{R}^{(n)}\mathcal{L}^{(n+1)} = \mathcal{L}^{(n)}\mathcal{R}^{(n)}, \tag{6}$$

$$\mathcal{L}^{(n)} = \left(\begin{array}{ccccccc} \multicolumn{3}{c}{\overbrace{0 \quad \cdots \quad 0}^{M}} & U_1^{(n)} & & & \\ 1 & 0 & \cdots & 0 & U_2^{(n)} & & \\ & 1 & \ddots & & \ddots & \ddots & \\ & & \ddots & \ddots & & \ddots & U_{M_m}^{(n)} \\ & & & \ddots & \ddots & & 0 \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 1 & 0 \end{array}\right), \tag{7}$$

$$\mathcal{R}^{(n)} = \left(\begin{array}{ccccccc} V_1^{(n)} & & & & & & \\ 0 & V_2^{(n)} & & & & & \\ \vdots & 0 & \ddots & & & & \\ 0 & \vdots & \ddots & \ddots & & & \\ \delta^{(n)} & 0 & & \ddots & \ddots & & \\ & \ddots & \ddots & & \ddots & \ddots & \\ & & \delta^{(n)} & \multicolumn{3}{c}{\underbrace{0 \quad \cdots \quad 0}_{M}} & V_{M_m+M}^{(n)} \end{array}\right), \tag{8}$$

$$U_k^{(n)} := u_k^{(n)} \prod_{j=1}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}), \tag{9}$$

$$V_k^{(n)} := \prod_{j=0}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}). \tag{10}$$

The equality in each entry of (6) is equivalent to the dhLV system (4).

Assume that

$$0 < u_k^{(0)} < K_0, \quad k = 1, 2, \ldots, M_m, \tag{11}$$

where K_0 is some positive constant. Then it is obvious from (10) that $V_k^{(n)} \geq 1$ in $\mathcal{R}^{(n)}$ for $k = 1, 2, \ldots, M_m + M$. Hence there exists the inverse matrix of $\mathcal{R}^{(n)}$, and then (6) can be transformed as

$$\mathcal{L}^{(n+1)} = (\mathcal{R}^{(n)})^{-1} \mathcal{L}^{(n)} \mathcal{R}^{(n)}. \tag{12}$$

This is a similarity transformation from $\mathcal{L}^{(n)}$ to $\mathcal{L}^{(n+1)}$. Namely, the eigenvalues of $\mathcal{L}^{(n)}$ are invariant under the time evolution from n to $n+1$. Therefore, the matrices $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$, $\mathcal{L}^{(2)}, \ldots$ are similar to each other. For a unit matrix I and an arbitrary constant d, the matrices $\mathcal{L}^{(0)} + dI$ and $\mathcal{L}^{(1)} + dI$, $\mathcal{L}^{(2)} + dI, \ldots$ are also similar.

There exist other invariants under the time evolution of the dhLV systems. The sum and the product concerning the variable $U_k^{(n)}$,

$$\sum_{k=1}^{M_m} U_k^{(n)} = \sum_{k=1}^{M_m} U_k^{(n+1)}, \tag{13}$$

$$\prod_{k=1}^{m} U_{M_k}^{(n)} = \prod_{k=1}^{m} U_{M_k}^{(n+1)}, \tag{14}$$

are invariants. From the assumption (11), it holds that $0 < \sum_{k=1}^{M_m} U_k^{(0)} < K_1$ and $0 < \prod_{k=1}^{m} U_{M_k}^{(0)} < K_2$, where K_1 and K_2 are positive constants. Taking account of (13) and (14), we derive $0 < u_k^{(n)} < K$ for a positive constant K. An asymptotic behavior of $u_k^{(n)}$ with (11) is also given as

$$\lim_{n\to\infty} u_{M_k}^{(n)} = c_k, \quad k = 1, 2, \ldots, m, \tag{15}$$

$$\lim_{n\to\infty} u_{M_k+p}^{(n)} = 0, \quad k = 1, 2, \ldots, m-1, \quad p = 1, 2, \ldots, M, \tag{16}$$

where $c_1, c_2, \ldots, c_m$ are positive constants such that

$$c_1 \geq c_2 \geq \cdots \geq c_m. \tag{17}$$

See [7] for the proof of (13), (14) and (15), (16).

Next we explain how to apply the dhLV system (4) to a matrix eigenvalue computation. Obviously, from (9), (10) and (15), (16), the limits of $U_k^{(n)}$ and

$V_k^{(n)}$ also exist as $n \to \infty$. The limit of the matrix $\mathcal{L}^{(n)} + dI$ becomes

$$\mathcal{L}(d) := \lim_{n\to\infty} (\mathcal{L}^{(n)} + dI)$$
$$= \begin{pmatrix} \mathcal{L}_1(d) & & & \\ \mathcal{E}_M & \mathcal{L}_2(d) & & \\ & \ddots & \ddots & \\ & & \mathcal{E}_M & \mathcal{L}_m(d) \end{pmatrix}, \tag{18}$$

where $\mathcal{L}_k(d)$ and $\mathcal{E}_M$ are $(M+1) \times (M+1)$ block matrices

$$\mathcal{L}_k(d) := \begin{pmatrix} d & & & c_k \\ 1 & d & & \\ & \ddots & \ddots & \\ & & 1 & d \end{pmatrix}, \quad \mathcal{E}_M := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}. \tag{19}$$

It is of significance to note that, by cofactor expansion,

$$\det(\lambda I - \mathcal{L}(d)) = \prod_{k=1}^{m} \det(\lambda I - \mathcal{L}_k(d)),$$
$$\det(\lambda I - \mathcal{L}_k(d)) = (\lambda - d)^{M+1} - c_k.$$

Then, the characteristic polynomial of $\mathcal{L}(d)$ is given as

$$\det(\lambda I - \mathcal{L}(d)) = \prod_{k=1}^{m} \left[(\lambda - d)^{M+1} - c_k \right].$$

Consequently, we obtain the eigenvalues $\lambda_{k,\ell}$ of $\mathcal{L}^{(0)} + dI$ as follows.

$$\lambda_{k,\ell} = c_k^{\frac{1}{M+1}} \left(\exp\left(\frac{2\pi i}{M+1} \right) \right)^{\ell} + d, \quad k = 1, 2, \ldots, m \quad \ell = 0, 1, \ldots, M. \tag{20}$$

Namely, the eigenvalues of $\mathcal{L}^{(0)} + dI$ are given by using the $(M+1)$th root of c_k derived from the time evolution of the dhLV system (4). Since, for a sufficiently large N, $u_{M_k}^{(N)}$ is an approximation to c_k, the $(M+1)$th root of $u_{M_k}^{(n)}$ leads to the approximation of the eigenvalues of $\mathcal{L}^{(0)} + dI$.

The above discussion is a brief review of [7]. We finally expand the applicable range of the dhLV algorithm. Let us introduce a diagonal matrix

$$D := \mathrm{diag}\,(1, \alpha_1, \alpha_1\alpha_2, \ldots, (\alpha_1\alpha_2\cdots\alpha_{M_m+M-1}))\,, \tag{21}$$

with arbitrary positive constants $\alpha_1, \alpha_2, \ldots, \alpha_{M_m+M-1}$. Then the similarity transformation by D yields

$$\hat{\mathcal{L}}^{(n)} + dI := D(\mathcal{L}^{(n)} + dI)D^{-1} \tag{22}$$

$$= \begin{pmatrix} d & & & \hat{U}_1^{(n)} & & \\ \alpha_1 & d & & & \ddots & \\ & \alpha_2 & \ddots & & & \hat{U}_{M_m}^{(n)} \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \alpha_{M_m+M-1} & d \end{pmatrix}, \tag{23}$$

where $\hat{U}_k^{(n)} = U_k^{(n)}/(\alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+M-1})$. Obviously, the eigenvalues of $\hat{\mathcal{L}}^{(n)} + dI$ coincide with those of $\mathcal{L}^{(n)} + dI$. Hence the eigenvalues of $\hat{\mathcal{L}}^{(0)} + dI$ are given as (20), if the initial $U_1^{(0)}$, $U_2^{(0)}, \ldots,$ $U_{M_m}^{(0)}$ are set, in accordance with $\hat{U}_1^{(0)}$, $\hat{U}_2^{(0)}, \ldots,$ $\hat{U}_{M_m}^{(0)}$ and α_1, $\alpha_2, \ldots, \alpha_{M_m+M-1}$, as

$$U_k^{(0)} = \hat{U}_k^{(0)}(\alpha_k\alpha_{k+1}\cdots\alpha_{k+M-1}). \tag{24}$$

To sum up, the dhLV algorithm for computing the eigenvalues of $\hat{\mathcal{L}}^{(0)} + dI$ is shown in Table 1. The lines from the 7th to the 11th of the dhLV algorithm is performed until $\max_{k\neq M_1,M_2,\ldots,M_m} u_k \le eps$ or $n > n_{\max}$ is satisfied for a sufficiently small $eps > 0$.

2.2. The dhLV algorithm for TN matrix

We show that eigenvalues of TN matrix are computable by the dhLV algorithm. It has been already reported in [7] that, as is described in Section 2.1, the band matrix $\mathcal{L}^{(n)} + dI$ is a target of the dhLV algorithm. Of course $\mathcal{L}^{(n)} + dI$ is not TN. For simplicity, we hereinafter discuss the case where $d = 0$.

Let us introduce a technique for matrix permutation which is a special case of [28]. Let P be the permutation matrix such that $P\mathcal{L}^{(n)}$ becomes the

Table 1: dhLV algorithm for $\hat{\mathcal{L}}^{(0)} + dI$

for $k := 1, 2, \ldots, M_m$ **do**
  $U_k^{(0)} = \hat{U}_k^{(0)} \prod_{j=0}^{M-1} \alpha_{k+j}$
end for
for $k := 1, 2, \ldots, M_m$ **do**
  $u_k^{(0)} = U_k^{(0)} / \prod_{j=1}^{M} (1 + \delta^{(0)} u_{k-j}^{(0)})$
end for
for $n := 1, 2, \ldots, n_{\max}$ **do**
  for $k := 1, 2, \ldots, M_m$ **do**
    $u_k^{(n+1)} := u_k^{(n)} \left[\prod_{j=1}^{M} (1 + \delta^{(n)} u_{k+j}^{(n)}) / \prod_{j=1}^{M} (1 + \delta^{(n+1)} u_{k-j}^{(n+1)}) \right]$
  end for
end for
for $k := 1, 2, \ldots, m$ **do**
  for $\ell := 1, 2, \ldots, M+1$ **do**
    $\lambda_{k,\ell} := \sqrt[M+1]{u_{M_k}^{(n)}} \{\cos[2\ell\pi/(M+1)] + i \sin[2\ell\pi/(M+1)]\} + d$
  end for
end for

matrix which is given by interchanging the $[(k-1)(M+1)+j]$th row of $\mathcal{L}^{(n)}$ with the $[(j-1)m+k]$th one for $j = 1, 2, \ldots, M+1$ and $k = 1, 2, \ldots, m$. Namely, P is the matrix whose $((j-1)m+k,\ (k-1)(M+1)+j)$ entry is 1 and the others are 0. Since $\mathcal{L}^{(n)} P^{-1}$ is the matrix given by interchanging the $[(k-1)(M+1)+j]$th column of $\mathcal{L}^{(n)}$ with the $[(j-1)m+k]$th one, it follows that

$$\begin{aligned} \mathcal{B}^{(n)} &:= P \mathcal{L}^{(n)} P^{-1} \\ &= \begin{pmatrix} & & & & \mathcal{L}_1^{(n)} \\ \mathcal{R}_M^{(n)} & & & & \\ & \ddots & & & \\ & & \mathcal{R}_2^{(n)} & & \\ & & & \mathcal{R}_1^{(n)} & \end{pmatrix}, \end{aligned} \tag{25}$$

where

$$\mathcal{L}_1^{(n)} = \begin{pmatrix} U_{M_1}^{(n)} & & & & \\ 1 & U_{M_2}^{(n)} & & & \\ & \ddots & \ddots & & \\ & & & 1 & U_{M_m}^{(n)} \end{pmatrix}, \quad \mathcal{R}_j^{(n)} = \begin{pmatrix} 1 & U_{M_2-j}^{(n)} & & \\ & 1 & \ddots & \\ & & \ddots & U_{M_m-j}^{(n)} \\ & & & 1 \end{pmatrix}. \tag{26}$$

As concerned with matrix eigenvalues, we take up the following theorem.

Theorem 2.1 (Watkins [28]). *The nonzero complex number λ is an eigenvalue of X if and only if its kth roots $\lambda^{1/k}, \lambda^{1/k}\omega, \lambda^{1/k}\omega^2, \dots, \lambda^{1/k}\omega^{k-1}$ are all eigenvalues of $\hat{X}$, where $\omega = \exp(2\pi i/k)$ and*

$$X = X_k X_{k-1} \cdots X_1 \in \mathbf{C}^{m\times m}, \quad X_j \in \mathbf{C}^{m\times m}, \quad j = 1, 2, \dots, k, \tag{27}$$

$$\hat{X} = \begin{pmatrix} & & & & X_k \\ X_1 & & & & \\ & X_2 & & & \\ & & \ddots & & \\ & & & X_{k-1} & \end{pmatrix}. \tag{28}$$

Let $k = M + 1$ in Theorem 2.1. Then $\mathcal{B}^{(n)}$ in (25) has the same form as $\hat{X}$ in (28). The blocks $\mathcal{R}_M^{(n)}, \mathcal{R}_{M-1}^{(n)}, \cdots, \mathcal{R}_1^{(n)}$ and $\mathcal{L}_1^{(n)}$ correspond to $X_1, X_2, \cdots, X_M$ and X_{M+1}, respectively. So, $\mathcal{A}^{(n)} := \mathcal{L}_1^{(n)}\mathcal{R}_1^{(n)}\mathcal{R}_2^{(n)} \cdots \mathcal{R}_M^{(n)}$ has the same form as X in (27). Let us assume that $u_1^{(n)}, u_2^{(n)}, \dots, u_{M_m}^{(n)}$, appearing in $\mathcal{R}_1^{(n)}, \mathcal{R}_2^{(n)}, \dots, \mathcal{R}_M^{(n)}$ and $\mathcal{L}_1^{(n)}$, are positive. Obviously, $\mathcal{L}_1^{(n)}, \mathcal{R}_1^{(n)}, \mathcal{R}_2^{(n)}, \dots, \mathcal{R}_M^{(n)}$ are the TN matrices, and then $\mathcal{A}^{(n)}$ is also. As is shown in Section 2.1, the eigenvalues of $\mathcal{B}^{(n)}$ are the $(M+1)$th roots of $c_1, c_2, \dots, c_m$. Hence it turns out that the eigenvalues of $\mathcal{A}^{(n)}$ become $c_1, c_2, \dots, c_m$. In the case where $\mathcal{A}^{(n)}$ is decomposed as $\mathcal{A}^{(n)} = \mathcal{L}_1^{(n)}\mathcal{R}_1^{(n)}\mathcal{R}_2^{(n)} \cdots \mathcal{R}_M^{(n)}$, the eigenvalues of $\mathcal{A}^{(n)}$ are accordingly computed by the dhLV algorithm.

In [1], it is shown that any strictly sign regular matrix has real and distinct eigenvalues. TN matrix is one of strictly sign regular matrices. In other words, TN matrix does not have multiple eigenvalues. Since $\mathcal{A}^{(n)}$ is the TN matrix, it is concluded that $c_1, c_2 \dots, c_m$ are distinct. By taking account of (17), we have the following theorem.

Theorem 2.2. *Let $u_k^{(0)} > 0$ for $k = 1, 2, \ldots, M_m$. As $n \to \infty$, the dhLV variable $u_{M_k}^{(n)}$ converges to c_k, where*

$$c_1 > c_2 > \cdots > c_m. \tag{29}$$

More precisely, c_k becomes the eigenvalue of the TN matrix $\mathcal{A}^{(0)}$.

It is to be noted that (29) holds in the eigenvalue computation of $\hat{\mathcal{L}}^{(0)} + dI$. The discussion in this section leads to the sorting property (29), which is more precise than (17).

2.3. Relationship with the multiple dqd algorithm

Recently, one of the authors proposes the multiple dqd algorithm for computing eigenvalues of TN band matrix [29].

Let $L_1, L_2, \ldots, L_{m_L}$ and $R_1, R_2, \ldots, R_{m_R}$ be the $m \times m$ lower and upper bidiagonal matrices, respectively, defined by

$$L_j = \begin{pmatrix} q_{j,1} & & & \\ 1 & q_{j,2} & & \\ & \ddots & \ddots & \\ & & 1 & q_{j,m} \end{pmatrix}, \quad R_j = \begin{pmatrix} 1 & e_{j,1} & & \\ & 1 & \ddots & \\ & & \ddots & e_{j,m-1} \\ & & & 1 \end{pmatrix}, \tag{30}$$

where $q_{j,1}, q_{j,2}, \ldots, q_{j,m} > 0$ and $e_{j,1}, e_{j,2}, \ldots, e_{j,m-1} > 0$. Then the target matrix of the multiple dqd algorithm is represented as

$$A_{\mathrm{TN}} = L_1 L_2 \cdots L_{m_L} R_1 R_2 \cdots R_{m_R}. \tag{31}$$

Since it is obvious that $L_1, L_2, \ldots, L_{m_L}$ and $R_1, R_2, \ldots, R_{m_R}$ are the TN matrices, A_{TN} is also. Besides the TN matrix A_{TN} has no multiple eigenvalues. The basic idea of the multiple dqd algorithm is to employ the dqd algorithm $(m_L \times m_R)$ times for one LR transformation of A_{TN}. See [29] for the details concerning the convergence theorem.

Let us consider the case where $m_L = 1, m_R = M$ in the multiple dqd algorithm. Note that L_j and R_j in (30) have the same form as $\mathcal{L}_j^{(n)}$ and $\mathcal{R}_j^{(n)}$ (26), respectively. Then the form of A_{TN} in (31) coincides with that of $\mathcal{A}^{(n)} = \mathcal{L}_1^{(n)} \mathcal{R}_1^{(n)} \mathcal{R}_2^{(n)} \cdots \mathcal{R}_M^{(n)}$.

The target TN matrix of the dhLV algorithm is accordingly equal to that of the multiple dqd algorithm with $m_L = 1$ and $m_R = M$.

3. dhToda equation and matrix eigenvalue

We here investigate some properties of the dhToda equation (5), and then design a new algorithm for computing matrix eigenvalues in terms of the dhToda equation (5). The dhToda equation (5) with $M = 1$ becomes the discrete Toda equation (1). The discrete Toda equation (1) has a close relationship to the qd algorithm for tridiagonal matrix eigenvalues. It is known that the discrete Toda equation (1) is just equal to the recursion formula of the qd algorithm. No wonder that the dhToda equation (5) is also related to matrix eigenvalue problem. The main purpose of this section is to design a matrix eigenvalue algorithm in terms of the dhToda equation (5).

The dqd algorithm is an improvement version of the qd algorithm, and algebraically equivalent to the qd algorithm. The dqd algorithm differs from the qd algorithm in that its recursion formula, called the differential form, has no subtraction. In other words, the dqd algorithm employs the differential form of the discrete Toda equation (1). In Section 3.1, we first derive a differential form of the dhToda equation, and we next show the positivity and the asymptotic behavior of the dhToda variables. In Section 3.2, based on the differential form of the dhToda equation, we finally design a new algorithm for computing eigenvalues.

3.1. Properties of the dhToda equation

Let us begin our analysis by deriving a differential form without subtraction from the dhToda equation (5). Let us introduce a new variable $D_k^{(n)}$ defined by

$$\begin{aligned} D_1^{(n)} &:= Q_1^{(n)}, \\ D_k^{(n)} &:= Q_k^{(n)} - E_{k-1}^{(n+1)}, \quad k = 2, 3, \ldots, m. \end{aligned}$$

Then, by combining it with (5), we obtain the relationship between $D_k^{(n)}$ and $D_{k+1}^{(n)}$

$$D_{k+1}^{(n)} = \frac{Q_{k+1}^{(n)}}{Q_k^{(n+M)}} D_k^{(n)}.$$

Note that the ratio $Q_{k+1}^{(n)}/Q_k^{(n+M)}$ also appears in the 2nd equation of (5). Moreover, let

$$F_{k+1}^{(n)} := \frac{Q_{k+1}^{(n)}}{Q_k^{(n+M)}},$$

then the differential form without subtraction of (5) is given by

$$\begin{cases} Q_k^{(n+M)} = E_k^{(n)} + D_k^{(n)}, \quad k = 1, 2, \ldots, m, \\ E_k^{(n+1)} = F_{k+1}^{(n)} E_k^{(n)}, \quad k = 1, 2, \ldots, m-1, \\ D_{k+1}^{(n)} = F_{k+1}^{(n)} D_k^{(n)}, \quad D_1^{(n)} = Q_1^{(n)}, \quad F_{k+1}^{(n)} = \dfrac{Q_{k+1}^{(n)}}{Q_k^{(n+M)}}. \end{cases} \tag{32}$$

Though the recursion formula employed in (32) is different from that in (5), the sequences of $Q_k^{(n)}$ and $E_k^{(n)}$ generated by (32) coincide with those by (5). The differential form (32) is useful for clarifying the positivity of the dhToda variables $Q_k^{(n)}$ and $E_k^{(n)}$. If $Q_k^{(0)}, Q_k^{(1)}, \ldots, Q_k^{(M-1)}$ for $k = 1, 2, \ldots, m$ and $E_k^{(0)}$ for $k = 1, 2, \ldots, m-1$ are positive, then $Q_k^{(M)}$ and $E_k^{(1)}$ are also positive. For $n = 1, 2, \ldots,$ by induction, we obtain the following proposition on the positivity of the dhToda variables.

Proposition 3.1. *Let $Q_k^{(0)} > 0, Q_k^{(1)} > 0, \ldots, Q_k^{(M-1)} > 0$ for $k = 1, 2, \ldots, m$ and $E_k^{(0)} > 0$ for $k = 1, 2, \ldots, m-1$. Then the variables $Q_k^{(n)}$, $E_k^{(n)}$ and $D_k^{(n)}$ in the differential form* (32) *satisfy the positivity,*

$$Q_k^{(n)} > 0, \quad k = 1, 2, \ldots, m, \quad n = M, M+1, \ldots, \tag{33}$$

$$E_k^{(n)} > 0, \quad k = 1, 2, \ldots, m-1, \quad n = 1, 2, \ldots, \tag{34}$$

$$D_k^{(n)} > 0, \quad k = 1, 2, \ldots, m, \quad n = 0, 1, \ldots. \tag{35}$$

With the help of Proposition 3.1, we have a theorem on an asymptotic convergence of the dhToda variables $Q_k^{(n)}$ and $E_k^{(n)}$ as $n \to \infty$.

Theorem 3.2. *Let $Q_k^{(0)} > 0, Q_k^{(1)} > 0, \ldots, Q_k^{(M-1)} > 0$ for $k = 1, 2, \ldots, m$ and $E_k^{(0)} > 0$ for $k = 1, 2, \ldots, m-1$. As $n \to \infty$, the limits of $Q_k^{(n)}$ and $E_k^{(n)}$ are given by*

$$\lim_{n \to \infty} \prod_{j=0}^{M-1} Q_k^{(n-j)} = C_k, \quad k = 1, 2, \ldots, m, \tag{36}$$

$$\lim_{n \to \infty} E_k^{(n)} = 0, \quad k = 1, 2, \ldots, m-1, \tag{37}$$

where C_k is a positive constant and $C_1 \geq C_2 \geq \cdots \geq C_m$.

Proof. We first give a proof of (37). Let us take summation of the both sides of the 1st equation of (5) of the superscript from 0 to n,

$$\sum_{j=0}^{n} Q_k^{(j+M)} = \sum_{j=0}^{n} Q_k^{(j)} + \sum_{j=0}^{n} E_k^{(j)} - \sum_{j=0}^{n} E_{k-1}^{(j+1)}. \tag{38}$$

In order to consider the limit $n \to \infty$, we may assume that $n > M$ without loss of generality. Noting that $Q_k^{(M)}, Q_k^{(M+1)}, \cdots, Q_k^{(n)}$ appears in the both sides of (38), we derive

$$\sum_{j=n-M+1}^{n} Q_k^{(j+M)} = \sum_{j=0}^{M-1} Q_k^{(j)} + \sum_{j=0}^{n} E_k^{(j)} - \sum_{j=0}^{n} E_{k-1}^{(j+1)}. \tag{39}$$

From Proposition 3.1, it is obvious that $\sum_{j=n-M+1}^{n} Q_k^{(j+M)} > 0$. This implies that the right hand side of (39) is positive. Hence it follows that

$$\sum_{j=0}^{n} E_{k-1}^{(j+1)} < \sum_{j=0}^{M-1} Q_k^{(j)} + \sum_{j=0}^{n} E_k^{(j)}. \tag{40}$$

The case where $k = m$ and $n \to \infty$ in (40) with $\sum_{j=0}^{\infty} E_m^{(j)} = 0$ leads to, for positive constant $\bar{K}_0$,

$$\sum_{j=1}^{\infty} E_{m-1}^{(j)} < \sum_{j=0}^{M-1} Q_m^{(j)} < \bar{K}_0. \tag{41}$$

Successively, by considering the cases where $k = m-1, m-2, \ldots, 1$, we have, for positive constant $\bar{K}_{m-k}$,

$$\sum_{j=0}^{\infty} E_k^{(j)} < \bar{K}_{m-k}, \quad k = m-1, m-2, \ldots, 1. \tag{42}$$

From the positivity $E_k^{(n)} > 0$, it is concluded that $E_k^{(n)} \to 0$ as $n \to \infty$.

We next prove (36) with the help of (37). Let $n = \ell \times M + j$ in the 1st equation of (5), then

$$Q_k^{((\ell+1)\times M+j)} = Q_k^{(\ell\times M+j)} + E_k^{(\ell\times M+j)} - E_{k-1}^{(\ell\times M+j+1)}. \tag{43}$$

Moreover, let us take summation of the both sides of (43) of ℓ from ℓ_1 to $\ell_2 - 1$ where $\ell_2 \geq \ell_1$. Then it follows that

$$Q_k^{(\ell_2 \times M+j)} = Q_k^{(\ell_1 \times M+j)} + \sum_{\ell=\ell_1}^{\ell_2-1} E_k^{(\ell \times M+j)} - \sum_{\ell=\ell_1}^{\ell_2-1} E_{k-1}^{(\ell \times M+j+1)}. \tag{44}$$

By combining it with the positivity $E_k^{(n)} > 0$, we derive

$$\begin{aligned} \left| Q_k^{(\ell_2 \times M+j)} - Q_k^{(\ell_1 \times M+j)} \right| &\leq \left| \sum_{\ell=\ell_1}^{\ell_2-1} E_k^{(\ell \times M+j)} \right| + \left| \sum_{\ell=\ell_1}^{\ell_2-1} E_{k-1}^{(\ell \times M+j+1)} \right| \\ &\leq \left| \sum_{\ell=\ell_1}^{\infty} E_k^{(\ell \times M+j)} \right| + \left| \sum_{\ell=\ell_1}^{\infty} E_{k-1}^{(\ell \times M+j+1)} \right|. \end{aligned} \tag{45}$$

Noting that the right hand side of (45) converges to zero as $\ell_1 \to \infty$, we have

$$\lim_{\ell_1, \ell_2 \to \infty} \left| Q_k^{(\ell_2 \times M+j)} - Q_k^{(\ell_1 \times M+j)} \right| = 0, \tag{46}$$

which implies that $\{Q_k^{(0 \times M+j)}, Q_k^{(1 \times M+j)}, Q_k^{(2 \times M+j)}, \dots\}$ is a Cauchy sequence. Since $\{Q_k^{(0 \times M+j)}, Q_k^{(1 \times M+j)}, Q_k^{(2 \times M+j)}, \dots\}$ is a real positive sequence, it turns out that $Q_k^{(\ell \times M+j)}$, for each j, converges to some positive constant $C_{k,j}$ as $\ell \to \infty$. It is concluded that $\prod_{j=0}^{M-1} Q_k^{(n-j)}$ converges to some positive constant $C_k = \prod_{j=0}^{M-1} C_{k,j}$ as $n \to \infty$.

We finally show the inequality of C_k for $k = 1, 2, \dots, m$. From the 2nd equation of the dhToda equation (5),

$$\begin{aligned} E_k^{(n)} &= E_k^{(0)} \prod_{N=0}^{n-1} \frac{Q_{k+1}^{(N)}}{Q_k^{(N+M)}} \\ &= E_k^{(0)} \prod_{\ell=0}^{n'} \frac{\mathcal{Q}_{k+1}^{(\ell)}}{\mathcal{Q}_k^{(\ell+1)}}, \end{aligned} \tag{47}$$

where $\mathcal{Q}_k^{(\ell)} = \prod_{j=0}^{M-1} Q_k^{(\ell \times M-j)}$ and $n > n' \in \mathbf{N}$. Note that $E_k^{(0)}$ is bounded and $\lim_{n\to\infty} E_k^{(n)} = 0$ for $k = 1, 2, \dots, m-1$, then from (47) we have

$$\lim_{n\to\infty} \prod_{\ell=0}^{n} \frac{\mathcal{Q}_{k+1}^{(\ell)}}{\mathcal{Q}_k^{(\ell+1)}} = 0, \quad k = 1, 2, \dots, m-1. \tag{48}$$

If $\lim_{\ell\to\infty} \mathcal{Q}_{k+1}^{(\ell)}/\mathcal{Q}_k^{(\ell+1)} > 1$, then it contradicts (48). From (36), we obtain

$$C_k \geq C_{k+1}, \quad k = 1, 2, \ldots, m-1. \tag{49}$$

□

To sum up, the suitable initial setting of the dhToda variables yields that as n grows larger, $\prod_{j=0}^{M-1} Q_k^{(n-j)}$ and $E_k^{(n)}$, with keeping the positivity $Q_k^{(n)} > 0$ and $E_k^{(n)} > 0$, converge to some positive constant and zero, respectively.

3.2. The dhToda algorithm for TN matrix

In order to find the conserved quantities for the dhToda equation, we consider the Lax form,

$$L^{(n+1)} R^{(n+M)} = R^{(n)} L^{(n)}, \tag{50}$$

$$L^{(n)} := \begin{pmatrix} 1 & & & & \\ E_1^{(n)} & 1 & & & \\ & E_2^{(n)} & \ddots & & \\ & & \ddots & 1 & \\ & & & E_{m-1}^{(n)} & 1 \end{pmatrix}, \tag{51}$$

$$R^{(n)} := \begin{pmatrix} Q_1^{(n)} & 1 & & & \\ & Q_2^{(n)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & Q_m^{(n)} \end{pmatrix}. \tag{52}$$

Let us introduce the matrix, given by the matrix products of $L^{(n)}$, $R^{(n)}$, $R^{(n+1)}$, $\cdots$, and $R^{(n+M-1)}$,

$$A^{(n)} := L^{(n)} R^{(n+M-1)} R^{(n+M-2)} \cdots R^{(n+1)} R^{(n)}. \tag{53}$$

Note here that the entries of $A^{(n)}$ consist of the dhToda variables. Then from (50) we derive $A^{(n+1)} = R^{(n)} A^{(n)} (R^{(n)})^{-1}$, which implies that the eigenvalues of $A^{(n)}$ are invariant under the time evolution from n to $n+1$. So, it should be emphasized here that $A^{(n)}$, for any n, has the same eigenvalues as $A^{(0)}$.

In [24], the conserved quantities of so-called numbered box and ball systems are presented based on the Lax form (50) for the dhToda equation (5).

Since the eigenvalues of $A^{(n)}$ is invariant, conserved quantities of the dhToda equation (5) are given by

$$\mathrm{Tr}\{(A^{(n)})^j\}, \quad j = 1, 2, \dots, m. \tag{54}$$

The authors of [24] suggest that the dhToda equation (5) has an interesting relationship with matrix eigenvalue. However, to the best of our knowledge, the matrix eigenvalue algorithm has not been derived from the dhToda equation (5).

Now we design a new algorithm for computing eigenvalues of $m \times m$ band matrix $A^{(0)}$, given by the matrix products of lower bidiagonal $L^{(0)}$ and upper bidiagonal $R^{(M-1)}, R^{(M-2)}, \dots, R^{(0)}$ such that $A^{(0)} = L^{(0)} R^{(M-1)} R^{(M-2)} \cdots R^{(0)}$. If $m > M$, then the form of $A^{(n)}$ is as follows.

$$A^{(n)} = \left(\begin{array}{cccccc} \multicolumn{3}{c}{\overbrace{* \quad \cdots \quad *}^{M}} & 1 & & \\ * & * & \cdots & * & \ddots & \\ & * & \ddots & & \ddots & 1 \\ & & \ddots & \ddots & & * \\ & & & \ddots & \ddots & \vdots \\ & & & & * & * \end{array} \right), \tag{55}$$

where $*$ denotes a nonzero entry. The $(i, i+M)$ entry of $A^{(n)}$ is fixed to 1 and the other nonzeros consist of the dhToda variables $Q_k^{(n)}$ and $E_k^{(n)}$. If $m \leq M$, $A^{(n)}$ becomes the upper Hessenberg form without the entries fixed to 1. In both cases, the eigenvalues of $A^{(0)}$ are computable with the dhToda equation (5), as is shown in the following theorem.

Theorem 3.3. *Let $Q_k^{(0)} > 0, Q_k^{(1)} > 0, \dots, Q_k^{(M-1)} > 0$, and $E_k^{(0)} > 0$. Then for $k = 1, 2, \dots, m$, $C_k = \lim_{n\to\infty} \prod_{j=0}^{M-1} Q_k^{(n-j)}$ coincides with an eigenvalue of $A^{(0)}$.*

Proof. As is shown in the proof of Theorem 3.2, the subsequence $\{Q_k^{(0\times M+j)}, Q_k^{(1\times M+j)}, Q_k^{(2\times M+j)}, \dots\}$ is a Cauchy sequence for all $k = 1, 2, \dots, m$ and $j = 1, 2, \dots, M$. Obviously, the Cauchy sequence $\{Q_k^{(0\times M+j)}, Q_k^{(1\times M+j)}, Q_k^{(2\times M+j)}, \dots\}$ is bounded. For arbitrary k and n, $Q_k^{(n)}$ is also bounded.

By combining it with the convergence of $E_k^{(n)}$ shown in Theorem 3.2, we find that the $(i+1,i)$ entry of $A^{(n)}$, written as $E_i^{(n)}\prod_{j=0}^{M-1}Q_i^{(n+j)}$, converges to zero as $n\to\infty$. Let $(A_k^{(n)})_{i,i}$ be the diagonal (i,i) entry of $A_k^{(n)}=L^{(n)}R^{(n+M-1)}R^{(n+M-2)}\cdots R^{(n+M-k)}$. Then $(A_k^{(n)})_{i,i}$ is given by

$$(A_k^{(n)})_{i,i}=E_{i-1}^{(n)}\prod_{j=M+1-k}^{M-1}Q_{i-1}^{(n+j)}+Q_i^{(n+M-k)}(A_{k-1}^{(n)})_{i,i},\quad k=2,3,\dots,M,$$
$$(A_0^{(n)})_{i,i}=1,\quad (A_1^{(n)})_{i,i}=E_{i-1}^{(n)}+Q_i^{(n+M-1)}.$$

Noting that $A_M^{(n)}=A^{(n)}$, we derive from the limit (37) and the boundedness of $Q_k^{(n)}$ that the limit of the diagonal entry becomes $C_k=\lim_{n\to\infty}\prod_{j=0}^{M-1}Q_k^{(n+j)}$. Consequently, as $n\to\infty$, the matrix $A^{(n)}$ converges to the following upper triangular matrix, namely,

$$\lim_{n\to\infty}A^{(n)}=\begin{pmatrix} C_1 & * & \cdots & * & 1 & & \\ & C_2 & * & \cdots & * & \ddots & \\ & & \ddots & \ddots & & \ddots & 1 \\ & & & \ddots & \ddots & & * \\ & & & & \ddots & \ddots & \vdots \\ & & & & & C_{m-1} & * \\ & & & & & & C_m \end{pmatrix},\tag{56}$$

and the diagonal entries C_k for $k=1,2,\dots,m$ are the eigenvalues of $A^{(0)}$. $\square$

Suppose that the positive sequences $\{E_1^{(0)},E_2^{(0)},\dots,E_{m-1}^{(0)}\}$ and $\{Q_1^{(0)},Q_2^{(0)},\dots,Q_m^{(0)}\}$, $\{Q_1^{(1)},Q_2^{(1)},\dots,Q_m^{(1)}\}$,…,$\{Q_1^{(M-1)},Q_2^{(M-1)},\dots,Q_m^{(M-1)}\}$ are given. Then, from (51)–(53), we have the band matrix $A^{(0)}$. Theorem 3.3 claims that, for sufficiently large n, $\prod_{j=0}^{M-1}Q_k^{(n-j)}$ becomes an approximate eigenvalue of $A^{(0)}$ through the dhToda equation (5). The above procedure for matrix eigenvalues is called the dhToda algorithm, and is shown in Table 2. The values M and m are given from the form of $A^{(0)}$ and the parameter $n_{\max}$ is set as the maximal iteration number. As the inequality $\max_k E_k^{(n)}<eps$ is employed as the stopping criterion for sufficiently small $eps>0$.

Table 2: dhToda algorithm

for $n := 0, 1, 2, \ldots, n_{\max}$ **do**
    $D_1^{(n)} = Q_1^{(n)}$
    for $k := 1, 2, \ldots, m-1$ **do**
        $Q_k^{(n+M)} = E_k^{(n)} + D_k^{(n)}$
        $F_{k+1}^{(n)} = Q_{k+1}^{(n)} / Q_k^{(n+M)}$
        $E_k^{(n+1)} = F_{k+1}^{(n)} E_k^{(n)}$
        $D_{k+1}^{(n)} = F_{k+1}^{(n)} D_k^{(n)}$
    end for
    $Q_m^{(n+M)} = D_m^{(n)}$
end for
for $k := 0, 1, 2, \ldots, m$ **do**
    $C_k = \prod_{j=0}^{M-1} Q_k^{(n-j)}$
end for

3.3. Relationships between the dhLV and the multiple dqd algorithms

We clarify a relationship of the dhToda algorithm to the dhLV algorithm. Let us introduce the block matrix $B^{(n)} \in \mathbf{R}^{m(M+1)\times m(M+1)}$, composed by the matrices $L^{(n)}$ in (51) and $R^{(n+M-1)}, \ldots, R^{(n+1)}, R^{(n)}$ in (52), such that

$$B^{(n)} = \begin{pmatrix} & & & & L^{(n)} \\ R^{(n+M-1)} & & & & \\ & R^{(n+M-2)} & & & \\ & & \ddots & & \\ & & & R^{(n)} & \end{pmatrix}. \qquad (57)$$

Let us recall here the permutation technique shown in Section 2.2. We consider the inverse of the permutation. Since the permutation matrix P is such that $PB^{(n)}$ becomes the matrix given by interchanging the $((j-1)m+k)$th row of $B^{(n)}$ with the $((k-1)(M+1)+j)$th one, $P^\top$ is such that $P^\top B^{(n)}$

becomes the matrix given by interchanging the $((k-1)(M+1)+j)$th row of $B^{(n)}$ with the $((j-1)m+k)$th one. It follows that

$$P^{-1}B^{(n)}P = \begin{pmatrix} S_1^{(n)} & J & & \\ H_1^{(n)} & S_2^{(n)} & \ddots & \\ & \ddots & \ddots & J \\ & & H_{m-1}^{(n)} & S_m^{(n)} \end{pmatrix}, \tag{58}$$

$$S_k^{(n)} := \begin{pmatrix} 0 & & & 1 \\ Q_k^{(n+M-1)} & 0 & & \\ & \ddots & \ddots & \\ & & Q_k^{(n)} & 0 \end{pmatrix}, \tag{59}$$

$$H_k^{(n)} := \begin{pmatrix} 0 & \cdots & 0 & E_k^{(n)} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{60}$$

The band matrix $P^{-1}B^{(n)}P$ in (58) has the same form as $\hat{\mathcal{L}}^{(n)}$ in (23) whose eigenvalues are computable by the dhLV algorithm. Just as the discussion in Section 2.2, the band matrices $A^{(n)}$ in (55) and $B^{(n)}$ in (57) correspond to the matrices X and $\hat{X}$ in Theorem 2.1, respectively. So, it is realized that all the eigenvalues of $B^{(n)}$ are given as the $(M+1)$th root of those of $A^{(n)}$. On the other hand, $B^{(n)}$ and $P^{-1}B^{(n)}P$ in (58) are similar and they have the same eigenvalues. So it is concluded that all the eigenvalues of $P^{-1}B^{(n)}P$ are given as the $(M+1)$th root of those of $A^{(n)}$. Namely, the target matrix of the dhLV algorithm becomes that of the dhToda algorithm by a suitable initial setting.

The dhToda algorithm is also related to the multiple dqd algorithm. By comparing $L^{(n)}$ in (51), $R^{(n)}$ in (52) with L_j, R_j in (30), we see that the transpose of $L^{(n)}$ and $R^{(n)}$ have the same form of R_j and L_j, respectively. So, let

$$(L^{(0)})^\top = R_1, \quad (R^{(0)})^\top = L_1, \quad (R^{(1)})^\top = L_2, \dots, (R^{(M-1)})^\top = L_M.$$

Let us recall that $A^{(0)} = L^{(0)}R^{(M-1)}R^{(M-2)}\cdots R^{(0)}$ is the target matrix of the dhToda algorithm. Then the transpose of $A^{(0)}$ becomes

$$\begin{aligned}(A^{(0)})^\top &= (R^{(0)})^\top (R^{(1)})^\top \cdots (R^{(M-1)})^\top (L^{(0)})^\top \\ &= L_1 L_2 \cdots L_M R_1.\end{aligned}$$

This implies that the target matrices of the dhToda algorithm and the multiple dqd algorithm with $m_L = M, m_R = 1$ are similar to each other.

4. Error analysis

Error analysis for the dqds algorithm has been reported in [6, 18]. The zero-shift dqd algorithm is a numerically improved version of original qd algorithm, however, the both algorithms generate the same sequence of similarity transformations theoretically. Let us recall here that the dhToda equation (5) is a generalization of the discrete Toda equation (1) which is just equal to the recursion formula of the qd algorithm. As is shown in the previous sections, the 1-step of the dhLV algorithm is algebraically equivalent to that of the dhToda algorithm based on the dhToda equation (5). Here the 1-step of algorithm means the procedure of a similarity transformation. To sum up, the dhLV and the dhToda algorithms have some relationships with the dqd algorithm. So, along the line similar to [6, 18], we present the error analysis of the dhLV and the dhToda algorithms in finite precision arithmetic.

In order to estimate the relative perturbation of matrix eigenvalue after the 1-step of the dhLV and the dhToda algorithms, our analysis in this section is twofold. The first is to estimate the rounding errors appearing in the 1-step of the dhLV and the dhToda algorithms, respectively. Let ψ be the mapping from a set of floating point numbers $\mathbf{F}$ to itself $\mathbf{F}$ which generates the 1-step of algorithm in floating point arithmetic. Suppose that the target matrix of similarity transformation has the entry x, then, after the 1-step of algorithm, we get a floating point number $\hat{x}$ such that $\hat{x} = \psi(x)$. Here we may regard ψ as a composition of three mappings ψ_1, ψ_2 and ψ_3, where $\psi_1 : x \mapsto \vec{x}$ gives a relative perturbation of x, $\psi_2 : \vec{x} \mapsto \breve{x}$ represents the 1-step of algorithm for transforming $\vec{x}$ to $\breve{x}$ exactly, and $\psi_3 : \breve{x} \mapsto \hat{x}$ gives a relative perturbation of $\breve{x}$. From the viewpoint of mixed stability in [6], it is concluded that the algorithm is better if relative perturbations given by ψ_1 and ψ_3 are small.

The second is to analyse the relative perturbations of the computed eigenvalues in accordance with the results for mixed stability. As concerns of the relative perturbation of matrix eigenvalue, Koev [18] proves the following theorem.

$$\begin{tikzcd} x \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\psi_1"'] & \hat{x} \\ \vec{x} \arrow[r, mapsto, "\psi_2"'] & \breve{x} \arrow[u, mapsto, "\psi_3"'] \end{tikzcd}$$

Figure 1: x diagram

Theorem 4.1 (Koev [18]). *Let $B_1, B_2, \ldots, B_s$ be nonnegative nonsingular bidiagonal matrices, and let $A = B_1 B_2 \cdots B_s$. Let x be an entry in some B_r, for $r = 1, 2, \ldots, s$, and $\hat{A}$ be obtained from A by replacing x in B_r by $\hat{x} = x(1+\delta)$, where $|\delta| \ll 1$. Let the eigenvalues of A and $\hat{A}$ be $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(\hat{A}) \geq \lambda_2(\hat{A}) \geq \cdots \geq \lambda_n(\hat{A})$, respectively. Then for all $k = 1, 2, \ldots, n$*

$$|\hat{\lambda}_k - \lambda_k| \leq \frac{2|\delta|}{1 - 2|\delta|} \lambda_k. \tag{61}$$

In this section, we hereinafter use the notation $\lambda_k(\cdot)$ as the eigenvalue. Let $\mathbf{u}$ be the unit roundoff of the floating point system. In order to analyse the rounding error, for the floating point arithmetic of two floating point numbers x and y, we use the standard model

$$\begin{aligned} fl(x \circ y) &= (x \circ y)(1 + \eta_1) && (62) \\ &= (x \circ y)/(1 + \eta_2), && (63) \end{aligned}$$

where $\circ \in \{+, -, \times, /\}$ and $|\eta_1|, |\eta_2| < \mathbf{u}$. The following lemma by Higham [10] is useful in relative error analysis of eigenvalues.

Lemma 4.2 (Higham [10]). *For any positive integer k let θ_k denote a quantity bounded according to $|\theta_k| \leq \gamma_k = k\mathbf{u}/(1 - k\mathbf{u})$. The following relation holds.*

$$\gamma_k + \gamma_j + \gamma_k \gamma_j \leq \gamma_{k+j}. \tag{64}$$

In [18], Koev rewrites (64) as the inequality

$$|(1 + \delta_1)(1 + \delta_2) - 1| \leq \frac{(m_1 + m_2)\delta}{1 - (m_1 + m_2)\delta}, \tag{65}$$

for clarifying the relative accuracy of his algorithm, where $\delta_k = m_k \mathbf{u}/(1 - m_k \mathbf{u})$. We also use (65) for our analysis of relative perturbation of eigenvalues.

4.1. Error analysis for the dhLV algorithm

The target matrices of the dhLV algorithm are the band $\mathcal{L}^{(0)}$ in (7), $\mathcal{L}^{(0)} + dI$, $\hat{\mathcal{L}}^{(0)} + dI$ in (23) and the TN $\mathcal{A}^{(0)}$ in (31). Let us recall that the 1-step for a similarity transformation in the dhLV algorithm are not essentially different in any cases of computing the eigenvalues of $\mathcal{L}^{(0)}$, $\mathcal{L}^{(0)}+dI$, $\hat{\mathcal{L}}^{(0)}+dI$ and $\mathcal{A}^{(0)}$. So, in the former of this section, we only discuss the dhLV algorithm for the eigenvalues of $\mathcal{L}^{(0)}$ in (7). Let $\delta^{(0)}, \delta^{(1)}, \dots$ be fixed as the positive constant δ for simplicity. Moreover, let us introduce a new variable

$$w_k^{(n)} := u_k^{(n)} \prod_{j=1}^{M} (\zeta + u_{k-j}^{(n)}), \tag{66}$$

where $\zeta = 1/\delta$. Note that the entry $U_k^{(n)}$, given as (9) in $\mathcal{L}^{(n)}$ coincides with $\delta^M w_k^{(n)}$. Hence the time evolution from $u_k^{(n)}$ to $u_k^{(n+1)}$ of the dhLV system (4) is also rewritten as that from $w_k^{(n)}$ to $w_k^{(n+1)}$ as follows.

$$\begin{cases} u_k^{(n)} = \dfrac{w_k^{(n)}}{\prod_{j=1}^{M} (\zeta + u_{k-j}^{(n)})}, \quad k = 1, 2, \dots, M_m, \\ w_k^{(n+1)} = u_k^{(n)} \prod_{j=1}^{M} (\zeta + u_{k+j}^{(n)}), \quad k = 1, 2, \dots, M_m. \end{cases} \tag{67}$$

Let us simplify $u_k^{(n)}, w_k^{(n)}$ as u_k, w_k, respectively. In (67), by taking account of (62) and (63), $u_k^{(n)}$ and $w_k^{(n+1)}$ are not computed without rounding errors. Namely, in the floating point arithmetic, we get the floating point numbers $\vec{u}_k$ and $\hat{w}_k$ instead of $u_k^{(n)}$ and $w_k^{(n+1)}$, respectively, after the 1-step of the dhLV algorithm. Actually, $\vec{u}_k$ and $\hat{w}_k$ with rounding errors are given as

$$\begin{cases} \vec{u}_k = \dfrac{w_k}{\prod_{j=1}^{M} (\zeta + \vec{u}_{k-j})} \dfrac{1 + \varepsilon_/(k)}{(1 + \varepsilon_+(k, M)) \prod_{i=1}^{M-1} (1 + \varepsilon_+(k, i))(1 + \varepsilon_\times(k, i))}, \\ \hat{w}_k = \vec{u}_k \prod_{i=1}^{M} (\zeta + \vec{u}_{k+j})) \prod_{i=1}^{M} (1 + \varepsilon_+^\star(k, i))(1 + \varepsilon_\times^\star(k, i)), \end{cases} \tag{68}$$

where $\varepsilon_/(k), \varepsilon_\circ(k, i), \varepsilon_\circ^\star(k, i)$ for $\circ \in \{+, \times, /\}$ denote the relative perturbations arisen from the addition, the multiplication, the division, and they satisfy $|\varepsilon_\circ(k,i)| < \mathbf{u}$. Let Ψ be the mapping from $W := \{w_1, w_2, \ldots, w_{M_m}\}$ to $\hat{W} := \{\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{M_m}\}$. Then, it may be regarded that the 1-step of the dhLV algorithm is performed by Ψ.

Let us introduce two mappings $\Psi_1 : W \to \vec{W}$ and $\Psi_3 : \breve{W} \to \hat{W}$, where $\vec{W} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{M_m}\}$, $\breve{W} = \{\breve{w}_1, \breve{w}_2, \ldots, \breve{w}_{M_m}\}$ and

$$\vec{w}_k = w_k \frac{1 + \varepsilon_/(k)}{(1 + \varepsilon_+(k,M)) \prod_{i=1}^{M-1} (1 + \varepsilon_+(k,i))(1 + \varepsilon_\times(k,i))}, \tag{69}$$

$$\hat{w}_k = \breve{w}_k (1 + \varepsilon_+^\star(k,M)) \prod_{i=1}^{M-1} (1 + \varepsilon_+^\star(k,j))(1 + \varepsilon_\times^\star(k,i)). \tag{70}$$

Moreover, let $\Psi_2 : \vec{W} \to \breve{W}$ be the mapping where

$$\begin{cases} \vec{u}_k = \dfrac{\vec{w}_k}{\prod_{j=1}^{M} (\zeta + \vec{u}_{k-j})}, \quad k = 1, 2, \ldots, M_m, \\ \breve{w}_k = \vec{u}_k \prod_{j=1}^{M} (\zeta + \vec{u}_{k+j}), \quad k = 1, 2, \ldots, M_m. \end{cases} \tag{71}$$

Then the mapping Ψ is regarded as the composition of Ψ_1, Ψ_2 and Ψ_3, namely, $\Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1$. It is remarkable that (71) becomes (67) by replacing $\vec{u}_k, \vec{w}_k$ and $\breve{w}_k$ with $u_k^{(n)}, w_k^{(n)}$, and $w_k^{(n+1)}$, respectively. This implies that $\vec{w}_k = w_k, \breve{w}_k = \hat{w}_k$ by Ψ_1, Ψ_3, respectively, and $\Psi = \Psi_2$ in the case where all ε are zero. We show the relationship of Ψ with Ψ_1, Ψ_2, Ψ_3, and then have the following theorem.

Theorem 4.3. *There exist two sets $\vec{W}$ and $\breve{W}$, given by adding the small relative perturbations to W and $\hat{W}$, respectively, such that $\vec{W}$ is theoretically transformed into $\breve{W}$ by the 1-step of the dhLV algorithm. Each entry of $\vec{W}$ and $\breve{W}$ differs from the corresponding ones of W and $\hat{W}$ by at most $2M\mathbf{u}$, respectively.*

In the following, we mainly analyse the relative perturbation of eigenvalues in the 1-step of the dhLV algorithm for transforming the TN matrix $\mathcal{A}^{(n)}$ into $\mathcal{A}^{(n+1)}$. Without any numerical errors, the eigenvalues of $\mathcal{A}^{(n)}$ coincide

$$
\begin{tikzcd}
W \arrow[r, "\Psi"] \arrow[d, "\Psi_1"'] & \hat{W} \\
\vec{W} \arrow[r, "\Psi_2"'] & \breve{W} \arrow[u, "\Psi_3"']
\end{tikzcd}
$$

Figure 2: W diagram

with those of $\mathcal{A}^{(n+1)}$. Let us recall that the block matrix $\mathcal{B}^{(n)}$ is given from $\mathcal{L}^{(n)}$ by only permutation and the entry $U_k^{(n)}$ in $\mathcal{L}^{(n)}$ is equal to $\delta^M w_k^{(n)}$. This implies that the entries of $\mathcal{B}^{(n)}$ are written by $w_k^{(n)}$. Since $\mathcal{A}^{(n)}$ becomes the products of the blocks $\mathcal{L}_1^{(n)}$ and $\mathcal{R}_1^{(n)}, \mathcal{R}_2^{(n)}, \ldots, \mathcal{R}_M^{(n)}$ in $\mathcal{B}^{(n)}$, it is also obvious that entries of $\mathcal{A}^{(n)}$ are represented by $w_k^{(n)}$. If a small relative perturbation is added to $W^{(n)} = \{w_1^{(n)}, w_2^{(n)}, \ldots, w_{M_m}^{(n)}\}$, the entries of $\mathcal{A}^{(n)}$ undergo the same amount of relative perturbation. Let us denote the matrices corresponding to $W, \vec{W}, \breve{W}$ and $\hat{W}$ by $\mathcal{A}, \vec{\mathcal{A}}, \breve{\mathcal{A}}$ and $\hat{\mathcal{A}}$, respectively. Then, by combining Theorem 4.3 with Theorem 4.1 and (65), the differences between $\lambda_k(\vec{\mathcal{A}})$ and $\lambda_k(\mathcal{A})$ and between $\lambda_k(\hat{\mathcal{A}})$ and $\lambda_k(\breve{\mathcal{A}})$ are evaluated as

$$
|\lambda_k(\vec{\mathcal{A}}) - \lambda_k(\mathcal{A})| \leq \frac{4MM_m\mathbf{u}}{1 - 4MM_m\mathbf{u}}\lambda_k(\mathcal{A}), \tag{72}
$$

$$
|\lambda_k(\hat{\mathcal{A}}) - \lambda_k(\breve{\mathcal{A}})| \leq \frac{4MM_m\mathbf{u}}{1 - 4MM_m\mathbf{u}}\lambda_k(\hat{\mathcal{A}}). \tag{73}
$$

Noting that the similarity transformation is exactly performed by Ψ_2, we get $\lambda_k(\vec{\mathcal{A}}) = \lambda_k(\breve{\mathcal{A}})$. By combining it with (72), (73) and Lemma 4.2, we have the following theorem.

Theorem 4.4. *In floating point arithmetic, after the 1-step of the dhLV algorithm, the relative perturbation of eigenvalues is estimated as*

$$
|\lambda_k(\hat{\mathcal{A}}) - \lambda_k(\mathcal{A})| \leq \frac{8MM_m\mathbf{u}}{1 - 8MM_m\mathbf{u}}\lambda_k(\mathcal{A}). \tag{74}
$$

In [18], Koev shows that his algorithm for $m \times m$ TN matrix gives rise to the relative perturbation of eigenvalues as

$$
|\hat{\lambda}_k - \lambda_k| \leq \frac{\left(\frac{32}{3}m^3 + O(m^2)\right)\mathbf{u}}{1 - \left(\frac{32}{3}m^3 + O(m^2)\right)\mathbf{u}}\lambda_k, \tag{75}
$$

and hence claim that his algorithm has high relative accuracy. Comparing Theorem 4.4 with (75), we therefore conclude that the upper bound of relative perturbation in (74) is smaller than in (75), and then the dhLV algorithm is high relative accurate in the process of computing eigenvalues of TN matrix.

Now we consider transformation from $\mathcal{L}^{(n)}$ into $\mathcal{L}^{(n+1)}$ by the dhLV algorithm. Note that the $\lambda_k(\mathcal{L}^{(n)})$ becomes the $(M+1)$th root of the eigenvalue of $\mathcal{B}^{(n)}$. Then we derive the following corollary.

Corollary 4.5. *In the floating point arithmetic, after the 1-step of the dhLV algorithm, the relative perturbation of eigenvalues is estimated as*

$$|\lambda_k(\mathcal{L}^{(n+1)}) - \lambda_k(\mathcal{L}^{(n)})| \leq \frac{1}{M+1} \cdot \frac{8MM_m\mathbf{u}}{1-8MM_m\mathbf{u}} \lambda_k(\mathcal{L}^{(n)}). \tag{76}$$

The upper bound of relative perturbation in (76) is smaller than that in (74). We see that the dhLV algorithm for $\mathcal{L}^{(n)}$ is also high relative accurate even though $\mathcal{L}^{(n)}$ has the complex eigenvalues.

4.2. Error analysis for the dhToda algorithm

We here remark that the dhToda equation (5) with $M = 1$ is just the discrete Toda equation (1) which is the recursion formula of the qd algorithm.

We also observe in [6] an error analysis for the dqd algorithm, which is the differential form of the qd algorithm. It is emphasized that the qd recursion formula (1) is transformed into the dhToda equation (5) by replacing $e_k^{(n)}, e_k^{(n+1)}, q_k^{(n)}$ and $q_k^{(n+1)}$ in (1) with $E_k^{(n)}, E_k^{(n+1)}, Q_k^{(n)}$ and $Q_k^{(n+M)}$, respectively. Fortunately, in Section 3.1, we already get a differential form of (5) similar to the recursion formula of the dqd algorithm. So, along the same line of [6], we give an error analysis for the dhToda algorithm. For simplicity, let us introduce eight kinds of sets,

$$\begin{aligned}
&Q := \{Q_1^{(n)}, Q_2^{(n)}, \ldots, Q_m^{(n)}\}, && \vec{Q} := \{\vec{Q}_1^{(n)}, \vec{Q}_2^{(n)}, \ldots, \vec{Q}_m^{(n)}\}, \\
&\breve{Q} := \{\breve{Q}_1^{(n+M)}, \breve{Q}_2^{(n+M)}, \ldots, \breve{Q}_m^{(n+M)}\}, && \hat{Q} := \{\hat{Q}_1^{(n+M)}, \hat{Q}_2^{(n+M)}, \ldots, \hat{Q}_m^{(n+M)}\}, \\
&E := \{E_1^{(n)}, E_2^{(n)}, \ldots, E_{m-1}^{(n)}\}, && \vec{E} := \{\vec{E}_1^{(n)}, \vec{E}_2^{(n)}, \ldots, \vec{E}_{m-1}^{(n)}\}, \\
&\breve{E} := \{\breve{E}_1^{(n+1)}, \breve{E}_2^{(n+1)}, \ldots, \breve{E}_{m-1}^{(n+1)}\}, && \hat{E} = \{\hat{E}_1^{(n+1)}, \hat{E}_2^{(n+1)}, \ldots, \hat{E}_{m-1}^{(n+1)}\}.
\end{aligned}$$

Here we use $\hat{Q}$ and $\hat{E}$ as the sets given by transforming from Q and E through the dhToda algorithm in floating point arithmetic. The sets $\hat{Q}$ and $\hat{E}$ are distinguished from the sets theoretically given by the dhToda algorithm. Then we immediately have the following theorem on error analysis for the dhToda algorithm.

Theorem 4.6. *There exist $\vec{Q}, \vec{E}$ and $\breve{Q}, \breve{E}$, given by adding the small relative perturbation of Q, E and $\hat{Q}, \hat{E}$, respectively, such that $\vec{Q}$ and $\vec{E}$ are theoretically transformed into $\breve{Q}$ and $\breve{E}$ by the 1-step of the dhLV algorithm, respectively. Each entry of $\vec{Q}, \vec{E}$ and $\breve{Q}, \breve{E}$ differs from the corresponding ones of Q, E and $\hat{Q}, \hat{E}$ by at most $3\mathbf{u}, \mathbf{u}$ and $2\mathbf{u}, 2\mathbf{u}$, respectively.*

$$\begin{tikzcd}
\{Q, E\} \arrow[r] \arrow[d, "\text{Change each } Q_k \text{ by } 3\mathbf{u},\ E_k \text{ by } 1\mathbf{u}"'] & \{\hat{Q}, \hat{E}\} \\
\{\vec{Q}, \vec{E}\} \arrow[r] & \{\breve{Q}, \breve{E}\} \arrow[u, "\text{Change each } \breve{Q}_k,\ \breve{E}_k \text{ by } 2\mathbf{u}"']
\end{tikzcd}$$

Figure 3: $\{Q, E\}$ diagram

Let $A = A^{(n)}$. Moreover, let $\vec{A}, \breve{A}$ and $\hat{A}$ denote the matrices given by replacing $\{Q, E\}$ with $\{\vec{Q}, \vec{E}\}, \{\breve{Q}, \breve{E}\}$ and $\{\hat{Q}, \hat{E}\}$ in the entries of A. With the help of Theorem 4.1 and Lemma 4.2, we clarify a relative perturbation of matrix eigenvalue after the 1-step of the dhToda algorithm in floating point arithmetic. The differences of $\lambda_k(\vec{A})$ and $\lambda_k(\hat{A})$ from $\lambda_k(A)$ and $\lambda_k(\breve{A})$ are evaluated as

$$|\lambda_k(\vec{A}) - \lambda_k(A)| \leq \frac{8m\mathbf{u}}{1 - 8m\mathbf{u}} \lambda_k(A), \tag{77}$$

$$|\lambda_k(\hat{A}) - \lambda_k(\breve{A})| \leq \frac{8m\mathbf{u}}{1 - 8m\mathbf{u}} \lambda_k(\hat{A}). \tag{78}$$

By taking account that $\lambda_k(\vec{A}) = \lambda_k(\breve{A})$, from (77), (78) and also (4.2), we obtain a perturbation theorem for eigenvalues in the dhToda algorithm.

Theorem 4.7. *In floating point arithmetic, after the 1-step of the dhToda algorithm, the relative perturbation of eigenvalue is evaluated as*

$$|\lambda_k(\hat{A}) - \lambda_k(A)| \leq \frac{16m\mathbf{u}}{1 - 16m\mathbf{u}} \lambda_k(A). \tag{79}$$

Similarly as the dhLV algorithm, it is concluded that the dhToda algorithm is high relative accurate with respect to eigenvalues of TN matrix.

5. Numerical experiments

In this section, we numerically confirm our results shown in the previous sections. Numerical experiments have been carried out on our computer with OS: Windows XP, CPU: Genuine Intel (R) CPU L2400 @ 1.66GHz, RAM: 2GB, compiler: Microsoft(R) C/C++ Optimizing Compiler Version 15.00.30729.01.

As an example matrix, we adopt the TN matrix $A_0 = L^{(0)}R^{(2)}R^{(1)}R^{(0)}$ with

$$L^{(0)} = \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \end{pmatrix}, \quad R^{(0)} = R^{(1)} = R^{(2)} = \begin{pmatrix} 5 & 1 & & \\ & 5 & 1 & \\ & & 5 & 1 \\ & & & 5 \end{pmatrix}.$$

Note here that $M = 3$ and $m = 4$ in our dhLV and the dhToda algorithms. We first discuss the behavior of the dhToda variables $Q_k^{(n)}$ and $E_k^{(n)}$. See also [7] for the behavior of the dhLV variables $u_k^{(n)}$. It turns out from Figure 4 that, as is shown in Theorem 3.2, $E_k^{(n)}$ converges to zero. Figure 5 shows that the behavior of $Q_k^{(n)}$ becomes periodic gradually as n grows larger. It is obvious from Figure 6 that the product $p_k := \prod_{j=0}^{M-1} Q_k^{(n-j)}$ converges to some positive constant. This numerical convergence also agrees with Theorem 3.2.

Next we demonstrate that, for two kinds of matrices, the eigenvalues are computable by the dhToda and the dhLV algorithms. Let us set $eps = 1.0E-16$ in the both algorithms. Let us here introduce the block matrix

$$B_0 = \begin{pmatrix} & & & L^{(0)} \\ R^{(2)} & & & \\ & R^{(1)} & & \\ & & R^{(0)} & \end{pmatrix}. \tag{80}$$

For the suitable permutation matrix P_0 shown in Section 3.3, the band matrix

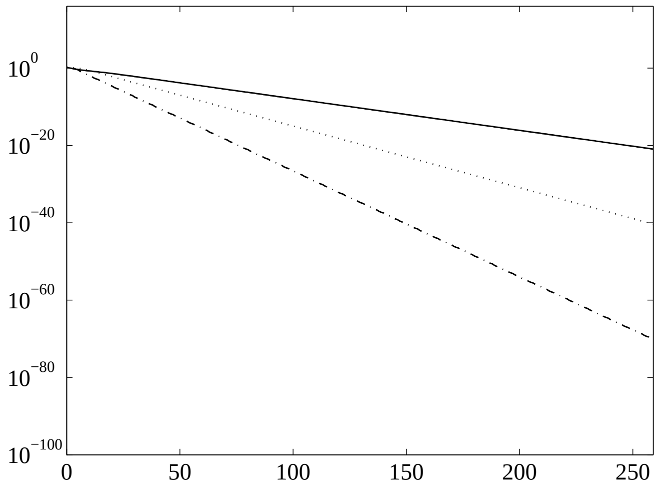

Figure 4: A graph of the iteration number n (x-axis) and the values of $E_1^{(n)}$, $E_2^{(n)}$, $E_3^{(n)}$ (y-axis) in the dhToda algorithm. Solid line : $E_1^{(n)}$, dotted line : $E_2^{(n)}$ and dashed line : $E_3^{(n)}$.

$L_0 := P_0^\top B_0 P_0$ has two diagonals as

$$L_0 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & 1 & & & & & & & & & & & & \\
5 & & & & 1 & & & & & & & & & & & \\
 & 5 & & & & 1 & & & & & & & & & & \\
 & & 5 & & & & 1 & & & & & & & & & \\
\hline
 & & & 2 & & & & 1 & & & & & & & & \\
 & & & & 5 & & & & 1 & & & & & & & \\
 & & & & & 5 & & & & 1 & & & & & & \\
 & & & & & & 5 & & & & 1 & & & & & \\
\hline
 & & & & & & & 2 & & & & 1 & & & & \\
 & & & & & & & & 5 & & & & 1 & & & \\
 & & & & & & & & & 5 & & & & 1 & & \\
 & & & & & & & & & & 5 & & & & 1 & \\
\hline
 & & & & & & & & & & & 2 & & & & 1 \\
 & & & & & & & & & & & & 5 & & & \\
 & & & & & & & & & & & & & 5 & & \\
 & & & & & & & & & & & & & & 5 &
\end{array}\right). \tag{81}$$

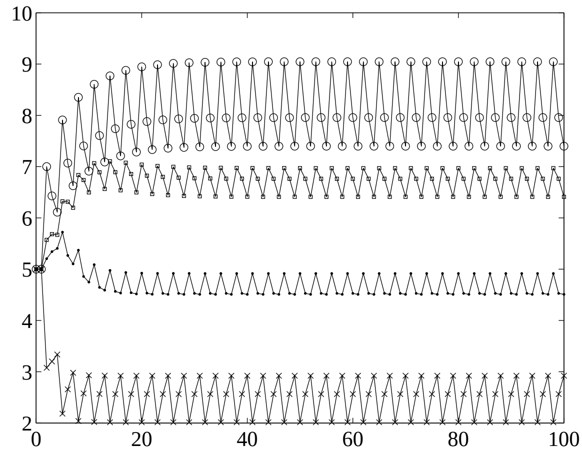

Figure 5: A graph of the iteration number n (x-axis) and the values of $Q_1^{(n)}$, $Q_2^{(n)}$, $Q_3^{(n)}$, $Q_4^{(n)}$ (y-axis) in the dhToda algorithm. Solid line with $\circ$: $Q_1^{(n)}$, with $\bigtriangledown$: $Q_2^{(n)}$, with $\star$: $Q_3^{(n)}$, with $\times$: $Q_4^{(n)}$

Table 3: Computed eigenvalues of A_0

	Mathematica	dhLV algorithm	dhToda algorithm
λ_1	532.35140651953578	532.35140651953509	532.35140651953520
λ_2	302.15799192937254	302.15799192937277	302.15799192937300
λ_3	100.36858294952133	100.36858294952130	100.36858294952131
λ_4	15.122018601570330	15.122018601570332	15.122018601570332

In order to get the eigenvalues of A_0 and L_0 with high relative accuracy, we employ the Mathematica function `eigenvalues[ ]` with 100 digits arithmetic. We also use our dhLV and dhToda algorithms in double precision arithmetic. Tables 3 and 4, respectively, show the eigenvalues of A_0 and L_0 computed by `eigenvalues[ ]` and our algorithms. The 1st columns of the both tables display the results by rounding `eigenvalues[`A_0`]` and `eigenvalues[`L_0`]` into double precision numbers. The computed eigenval-

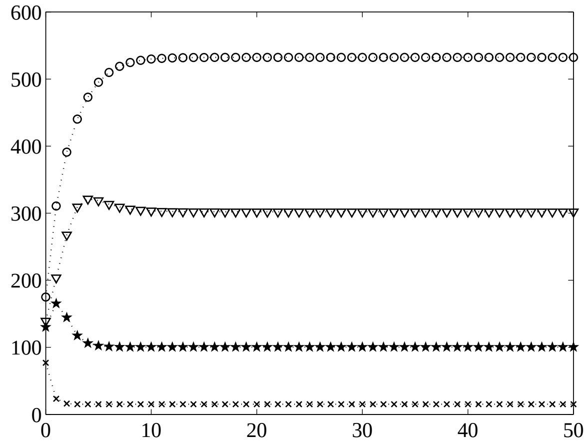

Figure 6: A graph of the ratio of the iteration number n to the parameter $M = 3$ (x-axis) and the values of $p_1^{(n)}$, $p_2^{(n)}$, $p_3^{(n)}$, $p_4^{(n)}$ (y-axis) in the dhToda algorithm. Dotted line with $\circ$: p_1, with $\bigtriangledown$: p_2, with $\star$: p_3, with $\times$: p_4

ues by the dhLV and the dhToda algorithms, respectively, are shown in the 2nd and the 3rd columns of the both tables. By comparing the 2nd and the 3rd columns with the 1st columns in both tables, we conclude that the eigenvalues of both matrices by the dhLV and the dhToda algorithm with high relative accuracy. This numerical results are consistent with our error analysis shown in Section 4.

6. Concluding remarks

In this paper, we first survey the dhLV algorithm in [7] derived from the integrable dhLV system, and then expand the target matrix of the dhLV algorithm by considering TN matrix. We next investigate some properties of the integrable dhToda equation. It is found that the dhToda variables become periodic and their products converge to matrix eigenvalues as the time variable $n \to \infty$. By taking account of this asymptotic convergence, we design a new algorithm, named the dhToda algorithm, for computing eigenvalues of TN matrix. We describe the relationship of the dhLV algorithm to the dhToda algorithm, namely, the class of matrices, whose eigenvalues are computable by both the dhLV algorithm and the dhToda algorithm, are

Table 4: Computed eigenvalues of L_0

	Mathematica	dhLV algorithm	dhToda algorithm
$\lambda_{1,1}$	4.803409376080853	4.803409376080851	4.803409376080851
$\lambda_{1,2}$	4.803409376080853i	4.803409376080851i	4.803409376080851i
$\lambda_{1,3}$	-4.803409376080853	-4.803409376080851	-4.803409376080851
$\lambda_{1,4}$	-4.803409376080853i	-4.803409376080851i	-4.803409376080851i
$\lambda_{2,1}$	4.169255606169454	4.169255606169454	4.169255606169455
$\lambda_{2,2}$	4.169255606169454i	4.169255606169454i	4.169255606169455i
$\lambda_{2,3}$	-4.169255606169454	-4.169255606169454	-4.169255606169455
$\lambda_{2,4}$	-4.169255606169454i	-4.169255606169454i	-4.169255606169455i
$\lambda_{3,1}$	3.165187545316407	3.165187545316406	3.165187545316406
$\lambda_{3,2}$	3.165187545316407i	3.165187545316406i	3.165187545316406i
$\lambda_{3,3}$	-3.165187545316407	-3.165187545316406	-3.165187545316406
$\lambda_{3,4}$	-3.165187545316407i	-3.165187545316406i	-3.165187545316406i
$\lambda_{4,1}$	1.971979709463506	1.971979709463506	1.971979709463506
$\lambda_{4,2}$	1.971979709463506i	1.971979709463506i	1.971979709463506i
$\lambda_{4,3}$	-1.971979709463506	-1.971979709463506	-1.971979709463506
$\lambda_{4,4}$	-1.971979709463506i	-1.971979709463506i	-1.971979709463506i

essentially the same. It is remarkable here that the transformation, such as the Miura transformation (3), from the dhLV variable to the dhToda one or its inverse has not been reported yet, but the dhLV algorithm is nevertheless related to the dhToda algorithm from the viewpoint of matrix eigenvalue. It is also shown that our two algorithms are related to the multiple dqd algorithm, which is proposed for eigenvalues of TN matrix in [29]. In order to clarify numerical stability of our two algorithms, we give the error analysis for them. Through some numerical experiments, we confirm that the dhToda variables have the asymptotic convergence discussed theoretically and computed eigenvalues by our two algorithms are with high relative accuracy.

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