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Robust Portfolio Selection with a Combined WCVaR and Factor Model

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Abstract In this paper, a portfolio selection model with a combined Worst-case Conditional Value-at-Risk (WCVaR) and Multi-Factor Model is proposed. It is shown that the probability distributions in the definition of WCVaR can be determined by specifying the mean vectors under the assumption of multivariate normal distribution with a fixed variance-covariance matrix. The WCVaR minimization problem is then reformulated as a linear programming problem. In our numerical experiments, to compare the proposed model with the traditional mean variance model, we solve the two models using the real market data in Japan and present the efficient frontiers to illustrate the difference. The comparison reveals that the WCVaR minimization model is more robust than the traditional one in a market recession period.

Keywords Portfolio Selection, Worst-Case Conditional Value-at-Risk, Multi-Factor Model, Linear Programming

1 Introduction

In portfolio selection problems, it is commonly accepted that investors must deal with a trade-off between expected returns and variance of returns. In Markowitz's paper [12], the formula of calculating the expected return on a portfolio was proposed, together with the variance and covariance of the portfolio. By incorporating the theory of Markowitz [12], Sharp [16] investigated the market equilibrium under conditions of risk and gave an asset pricing theory known as CAPM. Ross [15] generalized the Security Market Line in CAPM to a multi-factor case, which served as a basis for the Multi-Factor Model. After that, Fama and French [8] showed a Multi-Factor Model containing three factors; the market index, the firm size and the book to market equity.

On the other hand, it is noted that in the process of portfolio selection, the original data brought to the model are not always accurate, i.e., it may be subject to some errors. Thus the result may be influenced by perturbations in the parameters. To deal with such a situation, Ben-Tal and Nemirovski [3] introduced a framework that can handle linear programming problems with contaminated data. Assuming that the perturbations lie within an uncertainty

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set, they proposed a robust counterpart of the original optimization model to produce a solution immune to the contaminated data.

The combination of the Multi-Factor Model with robust optimization was given by Goldfarb and Iyengar [9], where they showed robust formulations of portfolio selection problems by assuming statistical and modeling errors in the estimates of market parameters. Besides considering the robust formulation of Markowitz's mean-variance model, they studied the robust portfolio selection with Value-at-Risk (VaR) constraints. However, because of the existence of probability distribution with a long-tail on the loss side, VaR may fail to be a proper measure of potential extreme loss. According to Artzner et al. [1], it does not satisfy the rule of subadditivity. Instead of VaR, Rockafellar and Uryasev [13, 14] considered a new measure called the conditional Value-at-Risk (CVaR). They focused on minimizing CVaR in the portfolio optimization problem by constructing a new function F_β, which is actually minimized instead of dealing with CVaR directly. The model has the advantage of calculating CVaR efficiently while calculating VaR simultaneously. However, the function F_β is constructed on the bases of a certain single probability distribution, which represents the future joint probability distribution of some random variables and thus must be approximated by simulations. Therefore, it may not be the same probability distribution as the one expected by the investors in advance. At this point, it is necessary to impose an uncertainty on the probability distribution that allows us to include some worst-case scenarios in a more general form of the uncertainty set.

This idea led El Ghaoui, Oks and Oustry [4] to assume partial information on the return's distribution in the portfolio optimization problem and extend their approach to uncertainty in a factor model. Specifically, they considered the set of allowable distributions, and defined the worst-case VaR with respect to the set of probability distributions. Then they showed that the optimization of the worst-case VaR can be reformulated as a semidefinite programming problem (SDP), which can be solved efficiently by the interior point method.

As another attempt to deal with the set of probability distributions, Zhu and Fukushima [17] proposed the concept of worst-case CVaR (WCVaR), which is proved to be a coherent measure, and investigated the minimization of WCVaR under mixture distribution uncertainty, box uncertainty, and ellipsoidal uncertainty. The problems are formulated as linear programs and second-order cone programs (SOCPs). Then by applying the model to portfolio management problems, they showed that the model is robust in practice, through some numerical experiments using real market data. Because of the inadequate knowledge of the set of probability distributions in the definition of WCVaR and the possibility of predicting distribution parameters such as mean through a Multi-Factor Model, we may expect that if there is a bridge that links the two fields together, then the model developed by Zhu and Fukushima [17] can further be applied in the financial area.

This paper is devoted to the following three points. First, as an extension of the work of Zhu and Fukushima [17], we aim at specifying the shape of probability distributions with some assumptions on the portfolio selection problem. Second, by estimating the mean vector of the underlying probability distribution through the Multi-Factor Model, we relate the WCVaR minimization problem to the asset pricing theory. Third, to show the robustness of our model for real market data, we compare it with Markowitz's mean variance model.

The remainder of this paper consists of five sections. In Section 2, we introduce the definition of WCVaR and propose a general formulation for the WCVaR minimization problem. In Section 3, we explain the structure of Single Index and Multi-Factor Models based on some previous studies and investigate the estimation of the mean vector of a probability distribution by the Multi-Factor Model. In Section 4, we present a formulation for the WCVaR minimization problem based on the Factor-Model in portfolio management. In Section 5, in order to obtain stable investments in different assets, we generate samples from the underlying probability distributions by quasi-Monte Carlo approach instead of the traditional one and show some numerical results on the WCVaR model. In Section 6, the conclusion and some future studies are mentioned.

2 WCVaR Formulation

Consider a portfolio selection problem, where a decision is represented by vector $\mathbf{x}$ and it is restricted to a feasible set $\mathcal{X}$. By using some measure M which is a function of $\mathbf{x}$, the *risk* of one portfolio associated with a decision vector $\mathbf{x}$ can be quantified. Thus the minimization of the risk for a portfolio is generally formulated as

$$\begin{array}{ll} \min & M(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \mathcal{X}. \end{array} \tag{1}$$

In practice, it is possible to construct and use different measures M. Before describing in detail some measures and their minimization, we first give several important definitions.

Let $f(\mathbf{x}, \mathbf{y})$ denote the loss associated with the decision vector $\mathbf{x} \in \mathcal{X} \subseteq R^n$ and the random vector $\mathbf{y} \in R^m$. Specifically, each component of $\mathbf{x}$ refers to the fraction of the fund allocated to an asset and $\mathbf{y}$ is the vector of future rates of returns of all assets, i.e., $m = n$. In the rest of this section, we assume that $\mathbf{y}$ follows a continuous distribution and denote its density function as $p(\cdot)$. For the proper definition of CVaR and WCVaR, We also assume $\mathrm{E}[|f(\mathbf{x}, \mathbf{y})|] < +\infty$ for each $\mathbf{x} \in \mathcal{X}$.

Given a decision vector $\mathbf{x} \in \mathcal{X}$, the probability that $f(\mathbf{x}, \mathbf{y})$ does not exceed a threshold α is given by

$$\Psi(\mathbf{x}, \alpha) \stackrel{def}{=} \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y}) d\mathbf{y}.$$

For a fixed $\mathbf{x}$, $\Psi(\mathbf{x}, \alpha)$ is a cumulative distribution function of $f(\mathbf{x}, \mathbf{y})$ for the loss associated with $\mathbf{x}$. It is obvious that $\Psi(\mathbf{x}, \alpha)$ is non-decreasing with respect to α.

Given a confidence level β (commonly set as $0.90, 0.95, 0.99$) and a fixed $\mathbf{x} \in \mathcal{X}$, the Value-at-Risk (VaR) is defined as

$$\mathrm{VaR}_\beta(\mathbf{x}) \stackrel{def}{=} \min\{\alpha \in R : \Psi(x, \alpha) \geq \beta\}.$$

VaR has been considered a proper measure for the risk management. However, according to Bedar [2], the value of VaR is sensitive to small perturbations of data. On the other hand, VaR fails to capture the quantity of loss when the distribution of some random variable has a flat tail on the loss side. Rockafellar and Uryasev [13, 14] proposed a new risk measure called the Conditional Value-at-Risk (CVaR), which is an alternative measure of risk for VaR and defined as

$$\mathrm{CVaR}_\beta(\mathbf{x}) \stackrel{def}{=} \frac{1}{1-\beta} \int_{f(\mathbf{x},\mathbf{y}) \geq \mathrm{VaR}_\beta(\mathbf{x})} f(\mathbf{x},\mathbf{y}) p(\mathbf{y}) d\mathbf{y}. \tag{2}$$

Since $\mathrm{CVaR}_\beta(\mathbf{x})$ represents the average of loss that exceeds $\mathrm{VaR}_\beta(\mathbf{x})$, the extreme value of loss on the flat tail of a probability distribution can be gauged by CVaR. In fact, CVaR has been considered one of the measures to be used instead of the objective function in problem (1). From definition (2), for a fixed β, we have the following formulation for the minimization of CVaR:

$$\begin{array}{ll} \min & \mathrm{CVaR}_\beta(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \mathcal{X}. \end{array} \tag{3}$$

This is the optimization problem that Rockafellar and Uryasev have discussed in their papers [13, 14]. In the definition (2) of CVaR, the random vector $\mathbf{y} \in R^m$ is assumed to follow a probability distribution represented by a density function $p(\cdot)$. Thus the randomness in $\mathbf{y} \in R^m$ is characterized by the density function $p(\cdot)$. However, in practice, we usually do not have enough information on the probability distribution $p(\cdot)$. Thus it may be reasonable to suppose that there exists an uncertainty about $p(\cdot)$, which can be expressed as

$$p(\cdot) \in \mathcal{P}, \tag{4}$$

where $\mathcal{P}$ is a set consisting of different probability distributions. It implies an uncertainty about the future probability distribution $p(\cdot)$ and also reflects our reliance on a set of several possible scenarios of distribution. According to the work of Zhu and Fukushima [17], worst-case CVaR (WCVaR) may be used as a new risk measure based on the condition (4) in portfolio selection. The concept is defined as

$$\mathrm{WCVaR}_\beta(\mathbf{x}) \stackrel{def}{=} \sup_{p(\cdot) \in \mathcal{P}} \mathrm{CVaR}_\beta(\mathbf{x}). \tag{5}$$

Generally, a proper risk measure ρ mapping random loss X or Y to a real number should satisfy the following rules:

(a) Subadditivity: For all random losses X and Y, $\rho(X+Y) \leq \rho(X) + \rho(Y)$;

(b) Positive homogeneity: For a positive constant λ, $\rho(\lambda X) = \lambda \rho(X)$;

(c) Monotonicity: If $X \leq Y$ for each outcome, then $\rho(X) \leq \rho(Y)$;

(d) Translation invariance: For a constant m, $\rho(X+m) = \rho(X) + m$.

In portfolio selection, these rules imply that (a) the risk of a portfolio constituted by two assets is less than or equal to the sum of the total risk if the two assets are measured individually, (b) if the asset grows λ times larger, the corresponding risk should also be λ times greater, (c) if one of the assets has less loss or equal loss to the other one, its risk should be less than or equal to the risk of the other asset, (d) the risk of the portfolio remains constant if the risk-free asset or cash is added to it. These intuitive rules exclude many risk measures such as the standard deviation and VaR. The measure satisfying all the rules is called a *coherent* risk measure. Actually, WCVaR is shown to be a coherent risk measure [17]. This fact allows WCVaR to be a proper risk measure that can be used as the objective function in problem (1), just as CVaR does in formulation (3):

$$\begin{array}{ll} \min & \mathrm{WCVaR}_\beta(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \mathcal{X}. \end{array} \tag{6}$$

In this problem, minimization of $\mathrm{WCVaR}_\beta(\mathbf{x})$ becomes the most critical step, which may not be easy to accomplish at first glance. However, according to Rockafellar and Uryasev [13, 14], the calculation of CVaR can be achieved by minimizing the following function:

$$F_\beta(\mathbf{x}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mathbf{y} \in R^m} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ p(\mathbf{y}) d\mathbf{y},$$

where $[t]^+ = \max\{0, t\}$. It means that

$$\mathrm{CVaR}_\beta(\mathbf{x}) = \min_{\alpha \in R} F_\beta(\mathbf{x}, \alpha).$$

Thus the problem of minimizing CVaR (3) is rewritten as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha) \in R^n \times R} & F_\beta(\mathbf{x}, \alpha) \\ \text{s.t.} & \mathbf{x} \in \mathcal{X}. \end{array} \tag{7}$$

An important fact is that problem (7) is a convex optimization problem [13, 14]. This suggests that the WCVaR minimization problem (6) can also be reduced to a tractable one. As the definition of WCVaR (5) shows, in order to deal with WCVaR, we need a probability distribution set $\mathcal{P}$. In this paper, we assume that the set $\mathcal{P}$ consists of the mixtures of some predetermined distributions

$$\mathcal{P}_\mathcal{M} \stackrel{def}{=} \left\{ \sum_{i=1}^{l} \lambda_i p^i(\cdot) : \sum_{i=1}^{l} \lambda_i = 1, \lambda \geq 0, i = 1, \cdots, l \right\},$$

where $p^i(\cdot)$ denotes the i-th likelihood distribution and l denotes the number of the likelihood distributions. Let

$$F_\beta^i(\mathbf{x}, \alpha) \stackrel{def}{=} \alpha + \frac{1}{1-\beta} \int_{\mathbf{y} \in R^m} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ p^i(\cdot) d\mathbf{y},$$

which corresponds to the density function $p^i(\cdot)$, $i = 1, \cdots, l$. Moreover, let $\mathcal{L} \stackrel{def}{=} \{1, \cdots, l\}$ and

$$F_\beta^\mathcal{L}(\mathbf{x}, \alpha) \stackrel{def}{=} \max_{i \in \mathcal{L}} F_\beta^i(\mathbf{x}, \alpha), \quad i = 1, \cdots, l.$$

From Zhu and Fukushima [17], we have the following theorem and corollary.

Theorem 1. *For each* $\mathbf{x}$*,* WCVaR *with respect to* $\mathcal{P}_\mathcal{M}$ *is given by*

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{\alpha \in R} F_\beta^\mathcal{L}(\mathbf{x}, \alpha).$$

Corollary 1. *Minimizing* $\mathrm{WCVaR}_\beta(\mathbf{x})$ *over* $\mathcal{X}$ *can be achieved by minimizing* $F_\beta^{\mathcal{L}}(\mathbf{x},\alpha)$ *over* $\mathcal{X}\times R$*, i.e.,*

$$\min_{\mathbf{x}\in\mathcal{X}} \mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{(\mathbf{x},\alpha)\in\mathcal{X}\times R} F_\beta^{\mathcal{L}}(\mathbf{x},\alpha).$$

More specifically, if $(\mathbf{x}^*,\alpha^*)$ *attains the right-hand side minimum, then* $\mathbf{x}^*$ *attains the left-hand side minimum and* α^* *attains the minimum of* $F_\beta^{\mathcal{L}}(\mathbf{x}^*,\alpha)$*, and vice versa.*

The above results indicate that the minimization of $F_\beta^{\mathcal{L}}(\mathbf{x},\alpha)$ over $\mathcal{X}\times R$ is equivalent to the minimization of $\mathrm{WCVaR}_\beta(\mathbf{x})$ on $\mathcal{X}$. In view of the definition of $F_\beta^{\mathcal{L}}(\mathbf{x},\alpha)$, the WCVaR minimization problem (6) can thus be reformulated as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha)\in\mathcal{X}\times R}\max_{i\in\mathcal{L}} & F_\beta^i(\mathbf{x},\alpha) \\ \text{s.t.} & \mathbf{x}\in\mathcal{X}. \end{array} \tag{8}$$

By introducing an auxiliary variable θ, problem (8) can be rewritten as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha,\theta)\in\mathcal{X}\times R\times R} & \theta \\ \text{s.t.} & F_\beta^i(\mathbf{x},\alpha)\le\theta, \quad i=1,\cdots,l, \\ & \mathbf{x}\in\mathcal{X}. \end{array} \tag{9}$$

Note that $\mathcal{P}_{\mathcal{M}}$ is the set consisting of all the mixtures of the density functions $p^i(\cdot)$, and each $F_\beta^i(\mathbf{x},\alpha)$ corresponds to the i-th likelihood distribution. Thus for a clear understanding of the correspondence of $F_\beta^i(\mathbf{x},\alpha)$ to $p^i(\cdot)$, we define

$$\mathcal{P}_{Original} \overset{def}{=} \{p^i(\cdot): i=1,\cdots,l\},$$

i.e., $\mathcal{P}_{Original}$ is the collection of the original predetermined density functions $p^i(\cdot)$ used to construct $\mathcal{P}_{\mathcal{M}}$. Thus the WCVaR minimization problem (9) can be discussed under $\mathcal{P}_{Original}$.

To reformulate problem (9), it is necessary to determine the characteristics of each $p^i(\cdot)$ in $\mathcal{P}_{Original}$. As mentioned in the beginning of this section, $p^i(\cdot)$ represents a density function of vector $\mathbf{y}$ of asset returns. Thus if we have some information that can help us to specify the shape of the probability distribution of asset returns, then by reducing the WCVaR formulation (9) to a standard frame, we may expect to apply it in practice. To do so, we assume that $\mathbf{y}$ follows a multivariate normal distribution (Gaussian distribution) with a mean vector μ and a variance-covariance matrix Σ, i.e., $\mathbf{y}\sim N(\mu,\Sigma)$.

There are two reasons for imposing the multivariate normal distribution assumption. First, it is convenient for numerical calculation because Gaussian distribution can be approximated by generating data through Monte-Carlo simulations. Second, the property 'multivariate normal' is common for the distribution of asset returns and employed by a number of previous studies.

In this paper, we assume that Σ is constant and focus on the situation where the mean vector μ is subject to uncertainty. We note that in portfolio selection the mean vector μ in $N(\mu,\Sigma)$ refers to the expected returns of stocks in the financial market and that there are several important theories offering us with an access to the prediction of the future asset returns. Therefore, to proceed our work, it is necessary to examine some of these theories in advance.

3 Estimation of Expected Returns by Factor-Model

As shown in the previous section, the set of original probability distributions $\mathcal{P}_{Original}$ consists of different components $p^i(\cdot), i = 1, \cdots, l$. This implies that each $p^i(\cdot)$ reflects the characteristics of the future probability distribution of the asset returns. Here we only collect a set of distribution scenarios from past information for the prediction of future probability distribution. However, we notice that the uncertainty about the probability distribution is determined by the unknown mean vector μ, with Σ fixed, according to our multivariate normal distribution assumption given at the end of Section 2. This uncertainty regarding asset returns arises from the future unknown environment surrounding stocks in the financial market and the macro economy system. Thus with some approaches that can help us to grasp such uncertainty, it is then possible to make a description of the set $\mathcal{P}_{Original}$. Therefore, the intention of dealing with uncertainty on asset returns relies naturally on the theory of Asset Pricing Model, which is one of the most important branches in financial theory during the past half century.

3.1 Single-Index Model

The fact that most stocks tend to increase in price when the market goes up leads to the idea that the correlation of asset returns may be due to a common response to the market changes. Thus to describe the movement of asset returns, one action is to relate the individual security returns to the return on a market index, which can be expressed as follows:

$$y_{jt} = \alpha_j + \beta_j y_{mt} + \varepsilon_{jt}, \tag{10}$$

where

y_{jt} : the return on one particular stock j at time t,
y_{mt} : the return on the market index at time t,
α_j : a constant component independent of the market performance,
β_j : a constant that measures the expected change in y_{jt} given a change in y_{mt},
ε_{jt} : the random residual error term for asset j at time t.

The model shows that the performance of a stock is divided into two components. One is due to the performance of the whole market, regared as the macro effect, and the other is due to itself, uncorrelated with the market performance, regarded as the micro effect which is merely related to the stock itself. By construction, since ε_{jt} is a random residual error, its expectation and variance are given by

$$\mathrm{E}(\varepsilon_{jt}) = 0, \quad \mathrm{VAR}(\varepsilon_{jt}) = \sigma^2_{\varepsilon_j} \tag{11}$$

for some $\sigma^2_{\varepsilon_j}$. By assumption, we have

$$\mathrm{COV}(\varepsilon_{jt}, y_{mt}) = 0 \tag{12}$$

and

$$\mathrm{COV}(\varepsilon_{jt}, \varepsilon_{kt}) = 0, \quad j \neq k, \tag{13}$$

where COV stands for covariance between two random variables. Assumption (12) indicates that the return on any asset is independent of the return on the market index. Moreover, assumption (13) implies that ε_{jt} is independent of ε_{kt} for $j \neq k$, i.e., the assets vary together because of the co-movement with the market index.

We note that y_{mt} and ε_{jt} are both random variables in (10). Thus if we take the expectation of equation (10), we obtain

$$\mathrm{E}(y_{jt}) = \alpha_j + \beta_j \mathrm{E}(y_{mt}) + \mathrm{E}(\varepsilon_{jt}). \tag{14}$$

Since the last term in (14) is null by (11), it follows

$$\mathrm{E}(y_{jt}) = \alpha_j + \beta_j \mathrm{E}(y_{mt}). \tag{15}$$

As the investors are more concerned with *risk premium*, defined as the excess of the risk-free rate r_f, i.e., $y_{jt} - r_f$ for security and $y_{mt} - r_f$ for market, we put the risk premium instead of the return in (15) and obtain

$$\mathrm{E}(y_{jt} - r_f) = \alpha_j + \beta_j \mathrm{E}(y_{mt} - r_f). \tag{16}$$

This is one of the most famous models in the financial field, named the Capital Asset Pricing Model (CAPM). It was developed by Sharp [16] and Lintner [11] and built on the work [12] of Markowitz on the portfolio selection theory. CAPM provides us with a way to deal with the equilibrium relationship between the risk and the expected return on assets. Here y_{mt} represents a well-diversified market portfolio. The model determines appropriately the required return of a risky asset if that asset is added to the market portfolio. It also offers another concept β_j, which quantifies the sensitivity of the asset to the whole market. The value of β_j acts as an indicator of the risk of one asset; the higher the value of β_j is, the more sensitive the asset is to the movement of the market.

If α_j, β_j and $\sigma^2_{\varepsilon_j}$ are assumed to be constant through time, then for each stock j, equation (10) is expected to hold at every time t. Thus by time series regression analysis for one past period, referred to as sample period (SP) in this paper, α_j, β_j, and $\sigma^2_{\varepsilon_j}$ can be estimated. Furthermore, if $y_{mt} - r_f$ is estimated based on historical data, it is possible to calculate the expected return of asset j for one future period, referred to as Test Period (TP) in this paper, through equation (16). A procedure for estimating the expected return can be summarized as follows:

Procedure 1.

Step 1. Calculate α_j and β_j by time-series regression analysis for stock j in SP.

Step 2. Estimate the market risk premium and r_f based on the data in SP.

Step 3. Calculate the expected return of asset j by equation (16).

3.2 Multi-Factor Model

As suggested by the Single Index Model in the previous subsection, the stocks move together because of a common movement with the market. Therefore it can be extended to a multi-factor model consisting of different types of systematic risk. The Arbitrage Pricing Theory (APT), proposed by Ross [15], emerged as an alternative asset pricing model to CAPM. It combines a factor model with a no-arbitrage condition (Law of One Price) and posits a relationship between expected return and risk. Since the assumptions in APT are weaker than those of CAPM, it is considered more general and can be used to support the Multi-Factor Model from a theoretical aspect. The model is shown as follows:

$$y_{jt} = \alpha_j + \sum_{h=1}^{Q} \beta_{jh} f_{ht} + \varepsilon_{jt},$$

where

f_{ht} :	the different factors at time t,
α_j :	the constant component independent of the factors' performance,
β_{jh} :	the factor-loadings,
ε_{jt} :	the residual error term at time t,
Q :	the number of factors.

The explanation is similar to the Single Index Model except that there are several factors besides the market index. Each factor represents the exposure to one systematic risk, such as inflation risk, business-cycle risk and so on. As in the Single Factor Model, it is convenient to assume that the factors are uncorrelated to each other and the residual error term is also independent of the factors. To guarantee that the stocks vary together because of the co-movement with a set of factors, assumption (13) in the Single Factor Model should remain in the Multi-Factor Model.

Since the Multi-Factor Model incorporates a set of factors, the determination of the risk factors becomes important. Furthermore the selected systematic factors should have considerable ability to explain the asset returns. One example of the multi-factor approach is presented by Fama and French [8], known as the *Fama-French Three-Factor Model*. This model is based on firm characteristics such as the market capitalization and the ratio of the book value of equity to the market value of equity. These factor variables seem to give a prediction of average asset returns by empirical studies. Considering the risk premium, the model can be shown as

$$y_{jt} - r_f = \alpha_j + \beta_{jm}(y_{mt} - r_f) + \beta_{jSMB} SMB_t + \beta_{jHML} HML_t + \varepsilon_{jt}, \tag{17}$$

where

$y_{jt} - r_f$: the risk premium for asset j at time t,
β_{jM}, β_{jSMB}, β_{jHML} : the factor-loadings of the corresponding factors,
α_j : the constant component independent of factors' performance,
$y_{mt} - r_f$: the risk premium for the market index at time t,
SMB_t : the abbreviation for *Small Minus Big*, which stands for the return of a portfolio of small stocks in excess of the return of a portfolio of large stocks at time t,
HML_t : the abbreviation for *High Minus Low*, which represents the return of a portfolio of stocks with a high book-to-market ratio in excess of the return of a portfolio of stocks with a low book-to-market ratio at time t,
ε_{jt} : the random residual error term at time t.

The model indicates that the risk premium for one asset can be explained by the risk premium for the market index, the firm size and the book-to-market equity. If we take into account the expectation of equation (17), we obtain

$$\mathrm{E}[y_{jt} - r_f] = \alpha_j + \beta_{jm}\mathrm{E}[y_{mt} - r_f] + \beta_{jSMB}\mathrm{E}[SMB_t] + \beta_{jHML}\mathrm{E}[HML_t] + \mathrm{E}[\varepsilon_{jt}].$$

Since $\mathrm{E}[\varepsilon_{jt}]$ is null, we have

$$\mathrm{E}[y_{jt} - r_f] = \alpha_j + \beta_{jm}\mathrm{E}[y_{mt} - r_f] + \beta_{jSMB}\mathrm{E}[SMB_t] + \beta_{jHML}\mathrm{E}[HML_t]. \tag{18}$$

This equation is similar to equation (16) except that it considers two additional factors. Note that α_j, β_{jM}, β_{jSMB}, β_{jHML} and $\sigma^2_{\varepsilon_j}$ are assumed to be constant through time. Then Procedure 1 in the Single Index Model can be extended to Procedure 2 in the Fama-French Three Factor Model as follows:

Procedure 2.

Step 1. Calculate the constant component α_j, the factor-loadings β_{jM}, β_{jSMB} and β_{jHML} by time series regression analysis for stock j in SP.

Step 2. Estimate the market risk premium, r_f, *SMB* and *HML* based on the data in SP.

Step 3. Calculate the expected return on asset j by equation (18).

4 Combination of WCVaR Formula with Factor Model

In Section 2, it has been shown that the formulation of minimizing WCVaR (9) can be discussed with $\mathcal{P}_{Original} = \{p^i(\cdot) : i = 1, \cdots, l\}$. Moreover, in Section 3, it has been suggested that the prediction of the expected asset return can be performed via the Multi-Factor Model in one SP. Note that in portfolio selection, under the assumption of *multivariate normal* distribution with a fixed Σ, the original probability distribution $p^i(\cdot)$ in $\mathcal{P}_{Original}$ can be determined by specifying the mean vector μ^i whose components are the expected returns of the stocks. Thus, to combine WCVaR with the Multi-Factor Model, SP should be chosen properly to estimate the parameter μ^i.

4.1 Falling Period

In the previous section, Procedure 2 for calculating the expected asset return via the Multi-Factor Model has been discussed. In Step 1 of Procedure 2, we need to specify SP. To make our model more robust to the future recession, it may be useful to determine the probability distribution $p^i(\cdot)$ based on the data in a market depression period.

Thus it is natural to find some proxy that can reflect the condition of the whole market and pick up the falling tendency based on that proxy. This motivates us to consider the market index (such as TOPIX) as the proxy and examine its trace in a time range. We define $I_{Market} \in R$ as an index of the financial market. Obviously, it is a function of time t, and can be denoted as $I_{Market}(t)$. We also define the time period during which the value of $I_{Market}(t)$ has a falling tendency as the Falling Period (FP). Just for simplicity, suppose that $I_{Market}(t)$ is a differentiable function with respect to time t. Then FP can be expressed as

$$\mathrm{FP} \stackrel{def}{=} \left\{ t : \frac{dI_{Market}(t)}{dt} < 0 \right\}.$$

By determining $p^i(\cdot)$ based on FP, Procedure 2 can be tailored to FP:

Procedure 2'.

Step 1. Calculate the constant component α_j, the factor-loadings β_{jM}, β_{jSMB} and β_{jHML} by time series regression analysis for stock j in FP.

Step 2. Estimate the market risk premium, r_f, *SMB* and *HML* based on the data in FP.

Step 3. Calculate the expected return on asset j by equation (18).

Note that there may exist many FPs during the past time span. Hence it is convenient to index them as FP_i, $i = 1, \cdots, l$. During each FP_i, by applying Procedure 2', the expected returns can be obtained for all assets $j = 1, \cdots, n$. In other words, we can form a vector μ^i consisting of n components of the expected returns. Then we can obtain a density function $p^i(\cdot)$ for each FP_i. Moreover, by collecting $p^i(\cdot), i = 1, \cdots, l$, together, the set $\mathcal{P}_{Original}$ can be constructed.

This process can be summarized as follows:

Procedure 3.

Step 0. Set $i := 1$, $j := 1$; Go to **Step 1**.

Step 1. Calculate the constant component α_j, the factor-loadings β_{jM}, β_{jSMB} and β_{jHML} by time series regression analysis for stock j in FP_i.

Step 2. Estimate market risk premium, r_f, *SMB* and *HML* based on the data in FP_i.

Step 3. Calculate the expected return on asset j by equation (18), and set $j := j + 1$. If $j > n$, then go to **Step 4**. Otherwise, return to **Step 1**.

Step 4. Form a mean vector μ^i, obtain $p^i(\cdot)$, and set $i := i + 1$. If $i > l$, then go to **Step 5**. Otherwise, set $j := 1$ and go to **Step 1**.

Step 5. Collect $p^i(\cdot)$ together to form $\mathcal{P}_{Original}$.

Here, n refers to the number of assets in the portfolio. For a clear understanding of the WCVaR minimization formulation under the Multi-Factor Model, we use $\mathcal{P}_{Factor-Model}$ to denote the set $\mathcal{P}_{Original}$ obtained by using the Multi-Factor Model approach. Then the WCVaR minimization formulation (9) in Section 2 can be reformulated as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha,\theta)\in\mathcal{X}\times R\times R} & \theta \\ \text{s.t.} & F_\beta^i(\mathbf{x},\alpha) \le \theta, \\ & \mathbf{x} \in \mathcal{X}, \end{array}$$

where $p^i(\cdot) \in \mathcal{P}_{Factor-Model}, i = 1, \cdots, l$.

According to Rockafellar and Uryasev [13, 14], for a single distribution $p(\cdot)$, the approximation of $F_\beta(\mathbf{x},\alpha)$ can be given as

$$\tilde{F}_\beta(\mathbf{x},\alpha) = \alpha + \frac{1}{S(1-\beta)} \sum_{k=1}^{S} [f(\mathbf{x},\mathbf{y}_{[k]}) - \alpha]^+,$$

where S denotes the number of samples and $\mathbf{y}_{[k]}$ refers to the k-th sample. If the number of samples is large enough, $\tilde{F}_\beta(\mathbf{x},\alpha)$ approximates $F_\beta(\mathbf{x},\alpha)$ by the Law of Large Numbers in statistics. Thus for each distribution $p^i(\cdot) \in \mathcal{P}_{Factor-Model}, i = 1, \cdots, l$, we can approximate $F_\beta^i(\mathbf{x},\alpha)$ by

$$\tilde{F}_\beta^i(\mathbf{x},\alpha) = \alpha + \frac{1}{S^i(1-\beta)} \sum_{k=1}^{S^i} [f(\mathbf{x},\mathbf{y}_{[k]}^i) - \alpha]^+,$$

where S^i denotes the number of samples from distribution $p^i(\cdot)$ and $\mathbf{y}_{[k]}^i$ denotes the k-th sample. The WCVaR minimization model can then be reformulated as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha,\theta)\in\mathcal{X}\times R\times R} & \theta \\ \text{s.t.} & \alpha + \frac{1}{S^i(1-\beta)} \sum_{k=1}^{S^i} [f(\mathbf{x},\mathbf{y}_{[k]}^i) - \alpha]^+ \le \theta, \quad k = 1, \cdots, S^i, \quad i = 1, \cdots, l, \\ & \mathbf{x} \in \mathcal{X}. \end{array} \tag{19}$$

By introducing the auxiliary variables $\mathbf{u}^i \in R^{S_i}$, $i = 1, \cdots, l$, we can rewrite (19) as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha,\theta,\mathbf{u})\in R^n\times R\times R\times R^N} & \theta \\ \text{s.t.} & \alpha + \frac{1}{S^i(1-\beta)} (\mathbf{e}^i)^T \mathbf{u}^i \le \theta, \quad i = 1, \cdots, l, \\ & \mathrm{u}_{[k]}^i \ge f(\mathbf{x},\mathbf{y}_{[k]}^i) - \alpha, \quad k = 1, \cdots, S^i,\ i = 1, \cdots, l, \\ & \mathbf{u}^i \ge 0, \quad i = 1, \cdots, l, \\ & \mathbf{x} \in \mathcal{X}, \end{array} \tag{20}$$

where $\mathbf{e}^i = (1, \cdots, 1)^T \in R^{S_i}$, $\mathbf{u}^i = (\mathrm{u}_{[1]}^i, \cdots, \mathrm{u}_{[S^i]}^i)^T$, $i = 1, \cdots, l$, $\mathbf{u} = ((\mathbf{u}^1)^T, \cdots, (\mathbf{u}^l)^T)^T$, and $N = \sum_{i=1}^l S^i$. Note that if the loss function $f(\mathbf{x},\mathbf{y})$ is linear with respect to $\mathbf{x}$ and $\mathcal{X}$ is a convex polyhedron, then problem (20) becomes a linear programming problem.

Remark. When we consider a WCVaR minimization problem under the condition $p^i(\cdot) \in \mathcal{P}_{Factor-Model}$, we in fact deal with the problem under the condition $p(\cdot) \in \mathcal{P}_M$ which is the mixtures of the probability distributions $p^i(\cdot)$ obtained from the Multi-Factor Model.

4.2 Portfolio Selection Based on WCVaR

Now we are in a position to describe the set $\mathcal{X}$ which is formed by some constraints in the portfolio selection problem. The j-th component of vector $\mathbf{x}$ represents the fraction of the fund allocated to the j-th stock. Thus the sum of all components of vector $\mathbf{x}$ is equal to 1, namely

$$\mathbf{e}^T\mathbf{x} = 1, \tag{21}$$

where $\mathbf{e} \in R^n$ is the vector of ones and n is the number of assets.

In our model, we suppose that the short selling position is not allowed, i.e., each component of vector $\mathbf{x}$ should be greater than or equal to 0. Thus we have

$$\mathbf{x} \geq 0. \tag{22}$$

Since the investors in the market will require a minimum level, given by η, of performance achieved by portfolio investment, we impose the condition

$$\mathbf{R}^T\mathbf{x} \geq \eta, \tag{23}$$

where $\mathbf{R}$ refers to the vector of the expected returns for all stocks and η stands for the minimum performance of the portfolio investment. In Section 3, based on one SP, $\mathbf{R}$ can be estimated by Procedure 2 and is equal to the estimated mean vector. Note that there are other mean vectors that are estimated based on FPs through Procedure 2'. In order to compare our model with the mean-variance model in Section 5, we fix a SP, estimate $\mathbf{R}$ by Procedure 2, and use the constraint (23) in the two models. To determine $p^i(\cdot)$ in the WCVaR minimization problem, we estimate the mean vectors from the fixed SP and FPs.

Since the return on one portfolio can be expressed as $\mathbf{x}^T\mathbf{y}$, the loss function is given by

$$f(\mathbf{x},\mathbf{y}) = -\mathbf{x}^T\mathbf{y}.$$

Consequently, the portfolio selection model with a combined WCVaR and the Multi-Factor Model can be formulated as

$$\begin{array}{ll} \min_{(\mathbf{x},\alpha,\theta,\mathbf{u})\in R^n\times R\times R\times R^N} & \theta \\ \text{s.t.} & \alpha + \frac{1}{S^i(1-\beta)}(\mathbf{e}^i)^T\mathbf{u}^i \leq \theta, \quad i = 1,\cdots,l, \\ & \mathrm{u}^i_{[k]} \geq -\mathbf{x}^T\mathbf{y}^i_{[k]} - \alpha, \quad k = 1,\cdots,S^i, \quad i = 1,\cdots,l, \\ & \mathbf{u}^i \geq 0, \quad i = 1,\cdots,l, \\ & \mathbf{e}^T\mathbf{x} = 1, \\ & \mathbf{x} \geq 0, \\ & \mathbf{R}^T\mathbf{x} \geq \eta, \end{array} \tag{24}$$

where $\mathbf{u}^i = (\mathrm{u}^i_{[1]},\cdots,\mathrm{u}^i_{[S^i]})^T$, $i = 1,\cdots,l$, $\mathbf{u} = ((\mathbf{u}^1)^T,\cdots,(\mathbf{u}^l)^T)^T$, $N = \sum_{i=1}^l S^i$ and $\mathbf{y}^i_{[k]}$ are generated with the distributions $p^i(\cdot) \in \mathcal{P}_{Factor-Model}$ for $i = 1,\cdots,l$.

5 Numerical Experiments

In this section, we compare the WCVaR minimization model (24) with the traditional mean variance model through numerical experiments using real market data in Japan. In particular, we examine how robust the portfolio will be if the investors make a decision based on the model (24), compared with the traditional method. In the process, the Fama-French Three-Factor Model is used to estimate the expected return on each stock and quasi-Monte Carlo simulation is applied to generate samples from each probability distribution. The efficient frontiers are drawn to examine the difference between the two models. All the data are obtained from Nikkei Quick News Inc. and Daiwa Institute of Research Holdings Ltd. We use MatLab 7.8.0 for solving linear programming problems on a PC with Intel(R) Core(TM)2 Duo CPU 1.20GHz and 2.0 GB RAM.

5.1 Quasi-Monte Carlo Simulation

Since we assume a normal distribution of $\mathbf{y}$, i.e., $\mathbf{y} \sim N(\mu, \Sigma)$, it is convenient to generate samples from the joint distribution of $\mathbf{y}$ by Monte Carlo simulation (MC simulation) approach. However, as Rockafellar and Uryasev [13] pointed out, in the CVaR optimization, the conventional Monte Carlo approach may lead to an unstable value of CVaR. Thus they worked with the Sobol quasi-random sequence besides pseudo-random sequences (MC simulation). We also found that the investment ratio (each component of decision vector $\mathbf{x}$) for each asset varies heavily among different simulations, which makes it difficult to compare our model with other models. Therefore, in this paper, to obtain a stable decision vector $\mathbf{x}$, we generate samples from the multivariate normal distribution by the quasi-random approach.

The quasi-random method has strong advantages in the risk management field. In this approach, deterministic sequences are generated instead of random sequences. These sequences are known as low discrepancy sequences which cover the space *evenly*, so that the clustering brought by the pseudo-random approach can be avoided. Another advantage is that quasi-Monte Carlo simulations perform more efficiently than the traditional MC simulation. As pointed out by Kreinin [10], in Monte Carlo simulations, errors E for estimates in any two simulations satisfy

$$E \sim O\left(\frac{1}{\sqrt{\mathrm{NUM}}}\right),$$

where NUM refers to the number of samples in one simulation. It shows that in order to reduce the error E by half, NUM should be increased by four times the original value. In contrast to this fact, star discrepancy D_{NUM} in the quasi-Monte Carlo approach has the following property:

$$D_{\mathrm{NUM}} \sim O\left(\frac{(\log \mathrm{NUM})^d}{\mathrm{NUM}}\right),$$

where d refers to the dimension of each sample generated in the simulation. It is clear that with a small d, D_{NUM} decreases asymptotically with $\frac{1}{\mathrm{NUM}}$.

Table 1: Codes and Industries for Stocks in Portfolio

Code	Company Name	Industry
4502	Takeda Pharmaceutical Co., Ltd.	Pharmaceutical Drug
4503	Astellas Pharma Inc.	Pharmaceutical Drug
4063	Shin-Etsu Chemical Co., Ltd.	Chemical
4901	FUJIFILM Holdings Co.	Chemical
3382	7-Eleven Japan Inc.	Retail Trade
9983	Fast Retail Co., Ltd	Retail Trade
7974	Nintendo Co., Ltd.	Service
4689	Yahoo Japan Co.	Service
7203	Toyota Motor Co.	Automotive
7267	Honda Motor Co., Ltd.	Automotive
8058	Mitsubishi Co.	Trading Business
8031	Mitsui & Co., Ltd.	Trading Business
9437	NTT Docomo Co.	Telecommunications
5401	Nippon Steel Co.	Steel
5411	JFE Steel Co.	Steel
9022	Central Japan Railway Co.	Railway Business
7751	Canon Inc.	Electric Appliance
6752	Panasonic Co.	Electric Appliance
9501	Tokyo Electric Power Co., Inc.	Electric Power
9503	Kansai Electric Power Co., Inc.	Electric Power
8802	Mitsubishi Estate Co., Ltd.	Real Estate
8801	Mitsui Fudosan Co., Ltd.	Real Estate
8604	Nomura Holdings Inc.	Securities

5.2 Data Description and Real Market Data Analysis

To select the stocks in our experiments, we pick up the first 50 listed stocks in the Tokyo Stock Exchange by ranking the total market values. Then by selecting the first and second market value stocks in each industry, 23 stocks are obtained as each asset in our portfolio. The code, the name, and the corresponding industry of the security are shown in Table 1. The time period for the data is from 1/11/2005 to 1/11/2010, which provides us with 1228 samples in the experiments. The daily closing prices are used for calculating the daily returns.

The purpose of the experiments is to compare the model (24) with the traditional mean variance model, which is shown as follows:

$$\begin{aligned} \min \quad & \mathrm{VAR}(R_{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{25}$$

where $R_{\mathbf{x}}$ represents the return on portfolio and VAR stands for the variance of a random variable. By replacing the set $\mathcal{X}$ in the constraints with the portfolio constraints (21)–(23), problem (25) can be reformulated as

$$\begin{array}{ll} \min & \mathbf{x}^T \Sigma \mathbf{x} \\ \text{s.t.} & \mathbf{e}^T \mathbf{x} = 1, \\ & \mathbf{x} \geq 0, \\ & \mathbf{R}^T \mathbf{x} \geq \eta, \end{array} \tag{26}$$

where Σ is the variance-covariance matrix of the stocks. For convenience, we let RbM stand for the robust model (24) and MVM stand for the mean-variance model (26) in the subsequent discussions. To proceed, we pick up two periods; one is the sample period (SP) as defined in Section 3, and the other is the test period (TP). By applying RbM (24) and MVM (26) in SP and by solving the two models, two decision vectors are obtained, denoted as $\mathbf{x}_{RbM}$ for RbM and $\mathbf{x}_{MVM}$ for MVM. By observing the actual performance of the two portfolios invested in the two decision vectors $\mathbf{x}_{RbM}$ and $\mathbf{x}_{MVM}$ in TP, the movement of the return on the two portfolios is analyzed. The data in SP are used to estimate Σ in the objective function of MVM (26), and the expected return $\mathbf{R}$ in the constraints of MVM (26) is estimated based on the data in SP through Procedure 2. Note that we estimate the return on the three factors by averaging the values in the SP during Step 2 of Procedure 2.

On the other hand, to compare with the MVM (26), we fix the same SP for calculating $\mathbf{R}$ in the constraint (23) of RbM. Moreover TOPIX is used as the index of the market to determine the FP for RbM. By observing the falling tendency of TOPIX value in the time range, two FPs are specified, namely

$$\begin{array}{l} \mathrm{FP}_1 : 18/6/2007 \sim 17/8/2007, \\ \mathrm{FP}_2 : 3/12/2007 \sim 22/1/2008. \end{array}$$

Since we have fixed one SP for comparison, the same SP is also included to generate one probability distribution in RbM (24). As shown by the definition of $\mathcal{P}_{Factor-Model}$ in Section 3, the set $\mathcal{P}_{Factor-Model}$ in this case consists of three probability distributions $p^i(\cdot)$, $i = 1, 2, 3$, calculated from FP_1, FP_2 and SP, respectively, that is,

$$\mathcal{P}_{Factor-Model} = \{p^i(\cdot) : i = 1, 2, 3\}.$$

Note that both models incorporate the same data in SP while additionally, the RbM (24) considers FP_1 and FP_2 as the *past* information input. Thus we expect that RbM (24) can perform more robustly than the MVM in the experiment. We set $l = 3$ and $n = 23$ in Procedure 3 to construct $\mathcal{P}_{Factor-Model}$. We also set $\beta = 0.90$ and $S_1 = S_2 = S_3 = 800$ in the RbM (24) in calculating the decision vector $\mathbf{x}_{RbM}$.

For an accurate comparison of the two models, we select two pairs of SP and TP, denoted $(\mathrm{SP}_\mathrm{I}, \mathrm{TP}_\mathrm{I})$, $(\mathrm{SP}_\mathrm{II}, \mathrm{TP}_\mathrm{II})$, which represent the following time periods:

$$\begin{array}{ll} \mathrm{SP}_\mathrm{I} : 21/10/2009 \sim 18/3/2010, & \mathrm{TP}_\mathrm{I} : 8/6/2010 \sim 29/10/2010, \\ \mathrm{SP}_\mathrm{II} : 23/1/2008 \sim 17/6/2008, & \mathrm{TP}_\mathrm{II} : 1/9/2008 \sim 30/1/2009. \end{array}$$

The reason why we take TP_II into account is that there was a financial crisis (Rehman Shock) in TP_II. Thus by including such a recession period, the efficiency of the RbM (24) can be tested thoroughly.

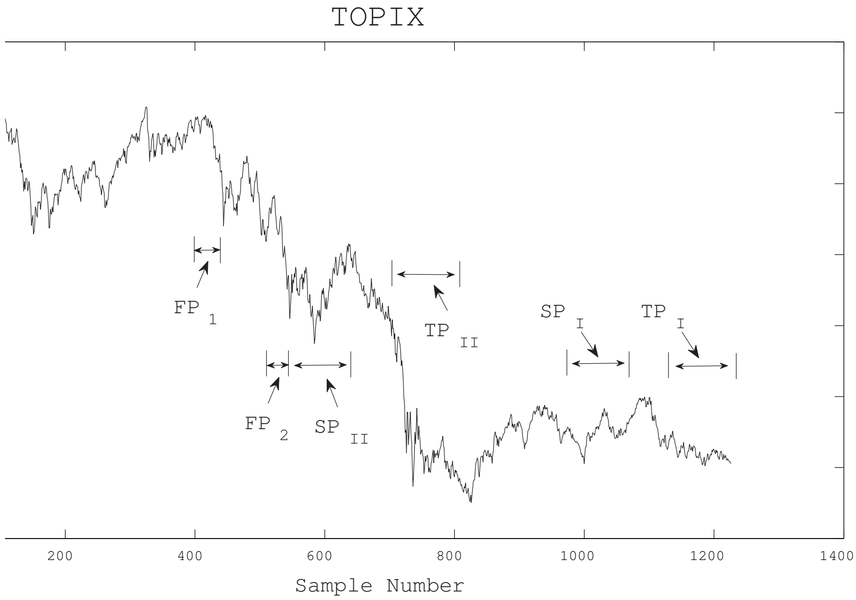

Figure 1: Trace of TOPIX Value

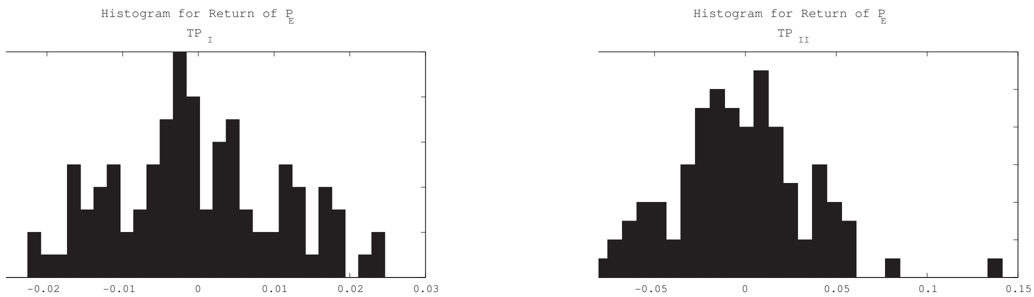

Figure 2: Comparison of the Return on $\mathbf{P}_E$ in $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$

The selected FP_1, FP_2, $\mathrm{SP_I}$, $\mathrm{SP_{II}}$, $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$ are shown in Figure. 1. Note that FP_1 and FP_2 are taken before the two pairs of the periods $(\mathrm{SP_I}, \mathrm{TP_I})$ and $(\mathrm{SP_{II}}, \mathrm{TP_{II}})$. This is because construction of probability distributions in RbM (24) should be based on the historical data before SP and TP.

Before discussing the comparison of the two models, we first examine the characteristics of the data in $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$. By constructing a portfolio by investing equally in the 23 assets, denoted $\mathbf{P}_E$, the histogram of the return on $\mathbf{P}_E$ in $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$ can be observed. From Figure 2, the distribution of the return on $\mathbf{P}_E$ in $\mathrm{TP_{II}}$ has a longer tail on the loss side than the one in $\mathrm{TP_I}$. Specifically, Table 2 indicates that in $\mathrm{TP_I}$ the mean of $\mathbf{P}_E$ is higher than that of $\mathbf{P}_E$ in $\mathrm{TP_{II}}$, and the variance of $\mathbf{P}_E$ in $\mathrm{TP_{II}}$ turns out to be greater than that of $\mathbf{P}_E$ in $\mathrm{TP_I}$. These facts can be considered the evidence of a depressed market condition in $\mathrm{TP_{II}}$.

We obtain two portfolios from the two models (24) and (26) based on the data in SP and compare the performance of the two portfolios in TP. For a fixed η in (24) and (26), we obtain

Table 2: Mean and Standard Deviation of $\mathbf{P}_E$ in $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$

	$\mathrm{TP_I}$	$\mathrm{TP_{II}}$
Mean	-0.00046	-0.0036
Std.	0.0113	0.0373

a pair of vectors $(\mathbf{x}_{RbM}, \mathbf{x}_{MVM})$ based on the data in SP. Then we allocate the total wealth to the assets by the decision vectors $\mathbf{x}_{RbM}$ and $\mathbf{x}_{MVM}$ respectively in TP. The two portfolios are denoted as $\mathbf{P}_{RbM}$ and $\mathbf{P}_{MVM}$. Thus based on the asset returns in TP, the standard deviation (σ) of the return on $\mathbf{P}_{RbM}$ and $\mathbf{P}_{MVM}$ can be calculated. Then by changing the value of η in the formulation of the two models, several pairs of $(\eta,\ \sigma)$ are obtained and two curves, known as the efficient frontiers, are drawn for $\mathbf{P}_{RbM}$ and $\mathbf{P}_{MVM}$.

Figure 3 shows the efficient frontiers for the two TPs. In both $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$, the standard deviation of returns on $\mathbf{P}_{RbM}$ and $\mathbf{P}_{MVM}$ increases as η grows. This reflects the basic *high risk high return* principle in investment when the investors are risk-averse. To see the difference between the two models, let us have a closer look at Figure 3. In $\mathrm{TP_I}$, for a fixed η, the standard deviation of $\mathbf{P}_{RbM}$ lies on the right of that of $\mathbf{P}_{MVM}$, which indicates that the risk of $\mathbf{P}_{MVM}$ is lower than that of $\mathbf{P}_{RbM}$. However, in $\mathrm{TP_{II}}$, $\mathbf{P}_{RbM}$ is more *robust* than $\mathbf{P}_{MVM}$ since the efficient frontier for $\mathbf{P}_{RbM}$ lies on the left of the one for $\mathbf{P}_{RbM}$. Moreover, as the value of η grows, the standard deviations in the two models merge into a same value. This corresponds to the fact that the unsystematic risk can be diversified through portfolio selection while the systematic risk remains and is equal to the market risk. From Figure 3, it can be concluded that the RbM (24) may fail to outperform the MVM (26) in the normal period, but shows some advantage over the MVM (26) in a recession case where the market is falling down heavily, as when the financial crisis took place. The explanation for this lies in two points: (1) The return on the assets in a depressed market period usually shows a flat tail on the loss side, which reflects the existence of some extreme large loss values. (2) Since our model generates samples from probability distributions of the asset returns using the historical data in the falling periods, we have included the past information on the structures of asset returns. Also because of dealing with the minimization of the risk measure WCVaR, which is the minimization of the worst case of the conditional expected value of the loss exceeding a threshold on the flat tail, the risk of the portfolio in the RbM (24) can be controlled in a market recession period.

6 Conclusion

In this paper, we have proposed a robust portfolio selection model with a combination of WCVaR and Multi-Factor Model. We have shown that the proposed model yields a more robust portfolio than the traditional mean variance model in a market recession period. However, there are three points that need to be highlighted.

(1) In constructing the uncertainty set $\mathcal{P}_{Factor-Model}$, the multivariate normal distribution with a *fixed* Σ is assumed. However, in practice, the structure of the correlation be-

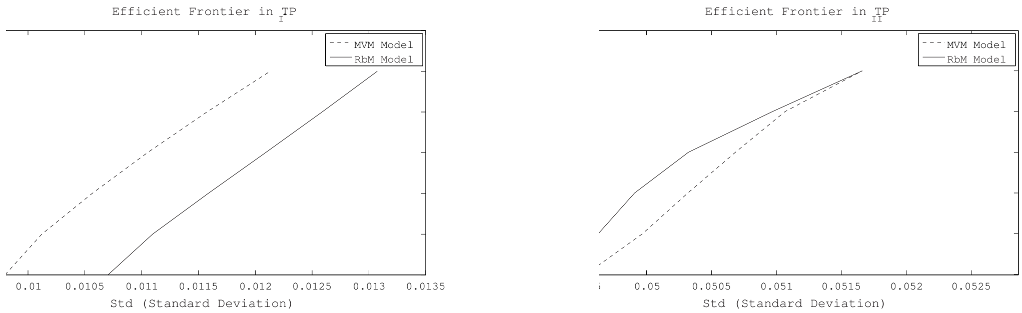

Figure 3: Efficient Frontiers in $\mathrm{TP_I}$ and $\mathrm{TP_{II}}$

tween different stocks is known to be unstable. Thus, there is also an uncertainty about the variance-covariance matrix that can be taken into account in the robust model. Moreover, the distribution of the asset return will often appear to be not normal, e.g., an asymmetric distribution with a skew.

(2) In the procedure of estimating the expected asset return, several factor loadings are assumed to be constant through time, which may not remain true as suggested by some studies [8]. Also, to gain a more precise estimation of the return of the factors, an advanced statistical approach can be applied in the procedure.

(3) The last point is that the available information in the market may have already been reflected on the price of stocks, which is known as efficient market hypothesis (EMH). Fama [6, 7] distinguishes three types of EMH; the weak form, the semi-strong form and the strong form, where the classification is based on the degree of the amount of information reflected on the price. The key point in EMH is that in an efficient market, it is very difficult to make prediction of future price by using past information, because the price has already absorbed it and the only thing that could give an influence to the stock price is future events which have not happened yet. Those facts may induce that our model which has incorporated the past information will not function in the future period. However, as anomalies regarding fundamental analysis exist in the market from empirical studies, EMH is still a matter of debate. Thus by using past information, we can at least obtain some profound insights into the characteristics of the investment. Furthermore, since the proposed model is shown to be more robust than the traditional model for real market data, using WCVaR as a risk measure casts some effects in the portfolio selection.

As new financial instruments spread out, many new approaches will be developed to meet the needs of the financial world. There is still much space to combine the field of operations research with the area of finance; see a recent survey paper [5]. Of course, as the amount of information is growing faster and faster, computer skills are required to solve large scale problems in financial investment, and new methods should be developed in future studies.

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