Technical Report #2012-001

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January 12, 2012

Robustness of Consensus Systems

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Abstract– This paper considers robust consensus problem of a multi-agent systems under the communication constraints described by a communication graph. The uncertainty of each agent is modeled as a norm-bounded multiplicative uncertainty. A necessary and sufficient condition for achieving the robust consensus is characterized in terms of the eigenvalues of the weighted Laplacian of the communication graph. In the case where the nominal transfer function shared by all the agents is positive real, the robust consensus condition turns out to depend only on the largest eigenvalue. It is also shown that, if the nominal transfer function has a pole on the imaginary axis, the stability margin of the closed-loop multi-agent system is independent of the graph topology nor the number of the agents.

Key Words: multi-agent system, consensus, graph topology, robustness

1 Introduction

Multi-agent coordinations have recently been attracting a great attention in the area of systems and control, since such phenomena can be encountered in many applications in physics, biology, robotics, computer science, etc (see e.g. the references 1) and 2)).

There have recently been several works on the robustness analysis of multi-agent systems[4)-8)]. In particular, Takaba[6)] derived a sufficient condition for robust output synchronization against gain-bounded uncertainties for the case where the nominal agents are incrementally passive nonlinear systems. For general LTI multi-agent systems, Hara and his coworkers[7)-9)] derived conditions for stability and robust stability by using a generalized frequency variable representation.

In this paper, we will consider the necessary and sufficient condition for achieving robust consensus of an LTI multi-agent system with norm-bounded uncertainties. We will show that the robust consensus condition can be simplified in the case where each agent is nominally passive. We will also study the stability margin for several specific graph topologies and their asymptotic properties as the number of agents goes to infinity.

2 Problem Formulation

2.1 System description

In this paper, we consider a multi-agent system consisting of N agents (Fig. 1). Each agent is described by the input-output equations

$$x_i = \hat{p}_i(s)u_i, \quad y_i = x_i + v_i, \quad i = 1, 2, \ldots, N, \quad (1)$$

where x_i, y_i, u_i, and v_i are the controlled output, the measured output, the control input, and the measurement noise, respectively. The dynamics of each agent is given by the SISO transfer function $\hat{p}_i(s)$. This transfer function contains multiplicative uncertainty $\Delta_i(s)$ which represents modeling errors and/or heterogeneity in the individual agents:

$$\hat{p}_i(s) = (1 + \Delta_i(s))p(s),$$

where $p(s)$ denotes the nominal dynamics shared by all agents. Then, the system equations in (1) is reduced to (see Fig. 2)

$$z_i = p(s)u_i, \quad (2a)$$
$$x_i = z_i + w_i, \quad (2b)$$
$$y_i = x_i + v_i, \quad (2c)$$
$$w_i = \Delta_i(s)z_i, \quad (2d)$$
$$i = 1, 2, \ldots, N$$

Throughout this paper, we will assume that $\Delta_i(s)$ is a stable transfer function whose $\mathscr{H}_\infty$-norm is bounded by a constant δ, namely, it belongs to the set B_δ defined by

$$\mathsf{B}_\delta := \{\Delta \in \mathscr{H}_\infty \mid \|\Delta\|_\infty \leq \delta\}$$

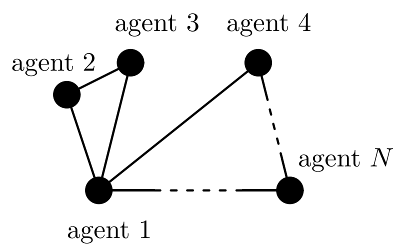

Fig. 1: Multi-agent system

Define

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \; z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix},$$

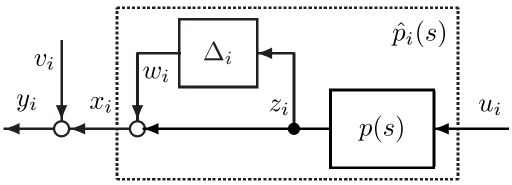

Fig. 2: Agent with multiplicative uncertainty

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \; v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \; w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix},$$

$$\Delta(s) = \mathrm{diag}(\Delta_1(s), \ldots, \Delta_N(s)).$$

Then, the equation (2) is equivalently rewritten as

$$z = p(s)u, \tag{3a}$$
$$x = z + w, \tag{3b}$$
$$y = x + v, \tag{3c}$$
$$w = \Delta(s)z. \tag{3d}$$

2.2 Communication Graph

Communication among the agents is performed through the network defined by the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, N\}$ is the set of nodes, and $\mathcal{E} \subseteq \{(i,j) |\, i, j \in \mathcal{V}\}$ are the set of edges (see Fig. 1). The node $i \in \mathcal{V}$ represents the i-th agent. The edge $(i, j) \in \mathcal{E}$ represents the two-way communication link between the agents i and j. At each time instant, the agent i transmits y_i to its neighbors j, $(i, j) \in \mathcal{E}$.

Assumption 1 *The communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is time-invariant, i.e. the network topology of $\mathcal{G}$ is fixed all the time.*

We now introduce some useful matrices from the algebraic graph theory[10]. Denote with B the incidence matrix of $\mathcal{G}$. Then, the weighted Laplacian L of $\mathcal{G}$ is defined by $L = BKB^\top$, where K is a positive definite diagonal matrix whose diagonal elements denote the weight on the edges of $\mathcal{G}$. Note that $\mathrm{rank}L = \mathrm{rank}B \leq N - 1$, and the equality is attained when $\mathcal{G}$ is a connected graph. By definition, L has at least one zero eigenvalue with the eigenvector $\mathbf{1}_N$, *i.e.* $L\mathbf{1}_N = 0$, where $\mathbf{1}_n := (1, 1, \cdots, 1)^\top \in \mathbb{R}^n$.

Let U be the orthogonal matrix such that

$$U^\top L U = \begin{bmatrix} \hat{L} & 0 \\ 0 & 0 \end{bmatrix} := \left[\begin{array}{ccc|c} \lambda_2 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \lambda_N & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right],$$

where $\lambda_1 (= 0)$, $\lambda_2, \ldots, \lambda_N$ are the eigenvalues of L ordered as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Note that $\mathcal{G}$ is connected iff $\lambda_2 > 0$.

Since $L\mathbf{1}_N = 0$ holds, U can be expressed as

$$U = \begin{bmatrix} Q^\top & \frac{1}{\sqrt{N}}\mathbf{1}_N \end{bmatrix} \tag{4}$$

where $Q \in \mathbb{R}^{(N-1)\times N}$ satisfies

$$Q^\top Q + \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top = I_N, \tag{5a}$$
$$Q\mathbf{1}_N = 0, \tag{5b}$$
$$QQ^\top = I_{N-1}. \tag{5c}$$

Lemma 1 For u_i, $v_i \in \mathscr{L}_2$, $i = 1, \ldots, N$, we define

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad \tilde{u} = Qu, \quad \tilde{v} = Qv.$$

Then,

$$\langle \tilde{u}, \, \tilde{v} \rangle = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle u_i - u_j, \, v_i - v_j \rangle,$$

where $\langle u, \, v \rangle := \int_0^\infty u(t)^\top v(t) dt$.

2.3 Consensus Protocol

The consensus problem is to design a protocol (distributed control law) which drives the outputs of all agents towards the same value, i.e., $x_i - x_j$, $i, j \in \mathcal{V}$ should converge to zero, or should be sufficiently small in the presence of measurement noises.

Under the communication constraints due to $\mathcal{G}$, a typical strategy for achieving coordinative tasks such as consensus, synchronization, formation, etc, is to apply the relative output feedback law

$$u_i = -\sum_{(i,j)\in\mathcal{E}} k_{ij}(y_i - y_j), \quad i \in \mathcal{V}, \tag{6}$$

or equivalently,

$$u = -Ly = -L(x + v), \tag{7}$$

where $k_{ij} > 0$, $(i, j) \in \mathcal{E}$ are constant feedback gains corresponding to weights in the graph $\mathcal{G}$, and $K = \mathrm{diag}\{k_{ij} : (i, j) \in \mathcal{E}\}$. It then follows from (3) and (7) that the overall multi-agent system is described as

$$x = -p(s)L(x + v) + w, \tag{8a}$$
$$z = -p(s)L(x + v), \tag{8b}$$
$$w = \Delta(s)z. \tag{8c}$$

We define

$$\tilde{x} = Qx, \quad x' = \frac{1}{\sqrt{N}}\mathbf{1}_N^\top x, \quad \bar{x} = \frac{1}{N}\mathbf{1}_N^\top x.$$

We also apply analogous definitions to other variables.

Since

$$\|\tilde{x}\|_2 = \left(\frac{1}{2N} \sum_{i,j \in \mathcal{V}} \|x_i - x_j\|_2^2 \right)^{1/2} \tag{9}$$

immediately follows from Lemma 1, we adopt the (worst-case) $\mathscr{L}_2$-gain of the closed-loop map from $\tilde{v}$ to $\tilde{x}$ as the performance measure of the consensus task.

Definition 1 The multi-agent system (8) is said to *achieve the robust consensus* if the closed-loop transfer function from $\tilde{v}$ to $\tilde{x}$ is stable for all Δ_i's in B_δ.

3 Robustness Analysis

3.1 Robust consensus condition

It follows from (8) that

$$x = -[I_N + p(s)L]^{-1} p(s) L v + [I_N + p(s)L]^{-1} w,$$
$$z = -[I_N + p(s)L]^{-1} p(s) L v - [I_N + p(s)L]^{-1} p(s) L w.$$

By applying the coordinate transformations

$$\begin{bmatrix} \tilde{x} \\ x' \end{bmatrix} = U^\top x, \quad \begin{bmatrix} \tilde{v} \\ v' \end{bmatrix} = U^\top v$$

to the above equations, we obtain

$$\begin{bmatrix} z \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} -Q^\top G(s) Q & -Q^\top G(s) \\ H(s) Q & -G(s) \end{bmatrix} \begin{bmatrix} w \\ \tilde{v} \end{bmatrix} \tag{10a}$$

$$w = \Delta(s) z. \tag{10b}$$

where

$$G(s) := [I_{N-1} + p(s)\hat{L}]^{-1} p(s) \hat{L}, \tag{11}$$

$$H(s) := [I_{N-1} + p(s)\hat{L}]^{-1}. \tag{12}$$

Therefore, the consensus analysis for multi-agent system of (8) is equivalently reduced to the robust stability analysis of the LFT system in (10). That is, the consensus is achieved for the nominal case ($\Delta(s) = 0$) if and only if $G(s)$ is stable, and a condition for the robust consensus is equivalent to the robust stability condition of (10) against the diagonally structured uncertainty.

A necessary and sufficient condition for the nominal consensus immediately follows from Theorem 1 in the reference[7].

Proposition 1 The multi-agent system in (8) achieves the nominal consensus if and only if

$$d(s) + n(s)\lambda_i \tag{13}$$

is a Hurwitz polynomial, where $(d,\, n)$ is a pair of coprime polynomials satisfying $p(s) = n(s)/d(s)$.

Remark 1 By definition of $\hat{L}$, $G(s)$ is a diagonal transfer matrix of the form

$$G(s) = \operatorname{diag}(g_2(s),\, g_3(s), \dots,\, g_N(s)), \tag{14a}$$

$$g_i(s) = \frac{p(s)\lambda_i}{1 + p(s)\lambda_i}, \quad i = 2, \dots, N. \tag{14b}$$

We next consider a condition for robust consensus. The situation here is slightly different from that of the reference[8], since there exists a non-square matrix Q between $G(s)$ and $\Delta(s)$. Since $\Delta(s)$ has a diagonal structure, the robust stability condition for the multi-agent system in (10) is reduced to a so-called scaled small gain condition on $F(s) := Q^\top G(s) Q$:

$$\inf_{D \in \mathsf{D}} \|D(s) F(s) D(s)^{-1}\|_\infty < \delta^{-1},$$
$$\mathsf{D} := \{D(s) \mid \ N \times N \text{ diagonal},\ D(s),\, D(s)^{-1} \in \mathscr{H}_\infty\}.$$

However, since $F(s) = F(s)^\top$ holds due to the diagonal structure of $G(s)$, the above scaled small gain condition is equivalent to the unscaled small gain condition $\|F\|_\infty < \delta^{-1}$[11]. Moreover, since $QQ^\top = I_{N-1}$, we have $\|F\|_\infty = \|G\|_\infty < \delta^{-1}$. Finally, by making use of the diagonal structure (14) of $G(s)$, we obtain the following robust consensus condition.

Proposition 2 The multi-agent system of (8) achieves the robust consensus if and only if it achieves the nominal consensus, and

$$\left\| \frac{p(s)\lambda_i}{1 + p(s)\lambda_i} \right\|_\infty < \delta^{-1}, \quad i = 2, \dots, N. \tag{15}$$

From this condition, we can compute the stability margin $\bar{\delta}$ of the multi-agent system (10) as

$$\begin{aligned} \bar{\delta} &= \frac{1}{\displaystyle\max_{i \in [2, \dots, N]} \left\| \frac{p(s)\lambda_i}{1 + p(s)\lambda_i} \right\|_\infty} \\ &= \min_{i \in [2, \dots, N]} \inf_{\omega \in \mathbb{R}} \left| 1 + \frac{1}{p(j\omega)\lambda_i} \right| \end{aligned} \tag{16}$$

3.2 Consensus for nominally passive agents

In this subsection, we assume that, each agent is nominally passive, which is the case for many practical situations such as vehicle formation, robotic coordination, e.t.c. This is equivalent to saying that the transfer function $p(s)$ is positive real, namely, it is analytic in $\mathbb{C}_+$ and $\operatorname{Re} p(j\omega) \geq 0$ for all ω.

By the well-known passivity theorem, $d(s) + n(s)\lambda$ is Hurwitz for all $\lambda > 0$. Thus, the nominal consensus condition (Proposition 1) is reduced to Proposition 3.

Proposition 3 Assume that $p(s)$ is positive real. Then, the multi-agent system (8) achieves the nominal consensus iff either one of the following is satisfied.

(i) $d(s)$ is Hurwitz.

(ii) $d(s)$ has a root on the imaginary axis, and $\lambda_2 > 0$.

The following lemma plays an important role for the robust consensus.

Lemma 2 Assume that $p(s)$ is positive real. Then,

$$\left| \frac{p(j\omega)\lambda}{1 + p(j\omega)\lambda} \right| \geq \left| \frac{p(j\omega)\lambda'}{1 + p(j\omega)\lambda'} \right| \quad \forall \omega \in \mathbb{R}.$$

holds for $\lambda > \lambda' \geq 0$.

Proof: The inequality in the lemma is equivalent to

$$\Phi(\omega) := |1 + p(j\omega)\lambda'|^2\lambda^2 - |1 + p(j\omega)\lambda|^2\lambda'^2 \geq 0 \quad (17)$$

By direct calculation, we obtain

$$\Phi(\omega) = 2\lambda\lambda'(\lambda - \lambda')\mathrm{Re}\, p(j\omega) + (\lambda^2 - \lambda'^2).$$

Since $\mathrm{Re}\, p(j\omega) \geq 0$ holds by the positive realness, and since $\lambda > \lambda'$, we conclude that (17) is satisfied. This completes the proof. ■

Therefore, we obtain the following proposition regarding the robust consensus for the positive real case.

Proposition 4 Assume that $p(s)$ is positive real. Then, the multi-agent system (8) achieves the robust consensus if and only if it achieves the nominal consensus, and

$$\left\| \frac{p(s)\lambda_N}{1 + p(s)\lambda_N} \right\|_\infty < \delta^{-1} \quad (18)$$

Thus, the stability margin in (16) becomes

$$\bar{\delta} = = \inf_{\omega \in \mathbb{R}} \left| 1 + \frac{1}{p(j\omega)\lambda_N} \right| \quad (19)$$

We summarize the observations from Propositions 3 and 4 as follows.

(i) The nominal consensus is achieved as long as the graph $\mathcal{G}$ is connected ($\lambda_2 > 0$).

(ii) The robust consensus condition and the stability margin depend only on the largest eigenvalue of the weighted Laplacian L.

(iii) Since $p(s)$ is positive real and $\lambda_N > 0$, the lower bound of the stability margin is given by $\bar{\delta} \geq 1$.

(iv) In the same way as Lemma 2, it is seen that the stability margin $\bar{\delta}$ increases for a smaller λ_N. Hence, when $p(s)$ is fixed, the optimal stability margin can be obtained by minimizing λ_N. The minimization of the eigenvalue of the weighted Laplacian was considered by Boyd [3]. In particular, if the weights (feedback gains) k_{ij}, $(i, j) \in \mathcal{V}$ are confined in a convex set, this minimization problem reduces to a convex programming.

(v) If $p(s)$ has a pole on the imaginary axis, say $s = j\omega_o$, then the infimum in (19) is achieved by taking $\omega \to \omega_o$. In this case, the stability margin $\bar{\delta} = 1$ depend on neither the graph topologies nor the number of agents.

3.3 Stability margins for specific graphs

In this subsection, we will consider the relation between the robustness and the graph topologies when N is very large. For simplicity, we assume that $p(s)$ is positive real, and that $k_{ij} = 1$ for all $(i, j) \in \mathcal{E}$.

The list of Laplacian eigenvalues below is taken from the references 2) and 12).

- **Line graph:**
  $\lambda_i = 2 - 2\cos\frac{\pi(i-1)}{N}, \quad i = 1, 2, \ldots, N$
- **Cycle graph:**
  $\left\{0,\ 2 - 2\cos\frac{2\pi}{N},\ 2 - 2\cos\frac{4\pi}{N}, : \ldots,\ 2 - 2\cos\frac{2(N-1)\pi}{N}\right\}$
  In particular,
  $$\lambda_2 = 2 - 2\cos\frac{2\pi}{N},\ \lambda_N = \begin{cases} 4 & (N\text{: even}) \\ 2\left(1 + \cos\frac{\pi}{N}\right) & (N\text{: odd}) \end{cases}$$
- **Star graph:**
  $$\lambda_1 = 0,\ \lambda_2 = \cdots = \lambda_{N-1} = 1,\ \lambda_N = N$$
- **Complete graph:**
  $$\lambda_1 = 0, \quad \lambda_2 = \cdots = \lambda_N = N$$

It is seen from the above list that λ_N is monotone increasing with respect to N, and λ_N converges to 4 for the line and cycle graphs, and λ_N goes to $+\infty$ for the star and complete graphs, as N goes to infinity. Thus, the stability margin $\bar{\delta}$ is monotone decreasing with respect to N, and

$$\lim_{N\to\infty} \bar{\delta} = \begin{cases} \inf_{\omega \in \mathbb{R}} \left| 1 + \frac{1}{4p(j\omega)} \right| \geq 1 & \text{(line, cycle)}, \\ 1 & \text{(star, complete)}. \end{cases}$$

4 Conclusions

A necessary and sufficient condition for achieving the robust consensus against norm-bounded uncertainties is characterized in terms of the eigenvalues of the associated Laplacian matrix of the communication graph. In the case where the nominal transfer function $p(s)$ is positive real, the consensus condition turned out to depend only on the largest eigenvalue. In addition, if $p(s)$ has a pole on the imaginary axis, the stability margin of the overall multi-agent system is independent of the graph topology nor the number of the agents. We have also compared the stability margins for several specific graph topologies and their asymptotic properties as N goes to infinity.

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