A differentiable merit function for the shifted perturbed KKT conditions of the nonlinear semidefinite programming

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Abstract.
In this paper we consider a primal-dual interior point method for solving nonlinear semidefinite programming problems, which is based on the shifted perturbed KKT conditions. The main task of the interior point method is to get a point approximately satisfying the shifted perturbed KKT conditions. We first propose a differentiable merit function whose stationary points always satisfy the conditions. The function is an extension of that proposed by Forsgren and Gill for the nonlinear programming problem. Then, we develop a Newton type method that finds a stationary point of the merit function. We show the global convergence of the proposed Newton type method under some mild conditions. Finally, we report some numerical results which show that the proposed method is competitive to the existing primal-dual interior point method based on the perturbed KKT conditions.

Key Words.
The nonlinear semidefinite programming, a primal-dual interior point method, the shifted perturbed KKT conditions, a merit function, the Newton type method

1 Introduction

In this paper we consider the following nonlinear semidefinite programming (SDP) problem:

$$\begin{array}{ll} \underset{x \in \mathbf{R}^n}{\text{minimize}} & f(x), \\ \text{subject to} & g(x) = 0, \quad X(x) \succeq 0, \end{array} \tag{1.1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}^m$ and $X : \mathbf{R}^n \to \mathbf{S}^d$ are twice continuously differentiable functions, and $\mathbf{S}^d$ denotes the set of $d \times d$ real symmetric matrices. Let $\mathbf{S}^d_{++}$ ($\mathbf{S}^d_{+}$) denote the set of $d \times d$ real symmetric positive (semi)definite matrices. For a matrix $M \in \mathbf{S}^d$, $M \succeq 0$ and

$M \succ 0$ mean that $M \in \mathbf{S}^d_+$ and $M \in \mathbf{S}^d_{++}$, respectively. If the functions f, g and X are linear, then the nonlinear SDP (1.1) is reduced to the linear SDP.

The nonlinear SDP is a wide class of the mathematical programming problems, and has many applications [5, 7, 11, 20, 24]. The linear programming, the second order cone programming, the linear SDP and the nonlinear programming can be cast as the nonlinear SDP. The linear SDP has been studied extensively by many researchers [1, 3, 6, 21, 22, 23]. However there exist important applications that are formulated as the nonlinear SDP, but cannot be reduced to the linear SDP. For example, the Gaussian channel capacity problems [24], the minimization (or maximization) of the minimal (or maximal) eigenvalue problems [16], the nearest correlation matrix problems [17] and the static output feedback problems [18] are such applications. Thus it is worth to develop solution methods for the nonlinear SDP.

Until now several solution methods for the nonlinear SDP have been proposed [5, 9, 19, 26]. Basically, these methods are extensions of the existing methods for the nonlinear programming, such as the sequential quadratic programming method, the successive linearization method, the augmented Lagrangian method and the interior point method.

Freund, Jarre and Vogelbush [5] proposed the sequential semidefinite programming method for the nonlinear SDP. However, they consider only the case where the objective function is a quadratic function and the constraint functions are affine. Kanzow, Nagel, Kato and Fukushima [9] extended the successive linearization method with a certain exact penalty function and the trust region-type technique. They show that the extended method is globally convergent under rather strong assumptions on the generated sequence, which are not verified in advance. Stingl [19] presented the augmented Lagrangian method for the nonlinear SDP. The method needs to calculate the eigenvalue decomposition of a matrix, and hence, it may not be suitable for solving some large-scale problems. Yamashita, Yabe and Harada [26] applied the primal-dual interior point method for the nonlinear SDP, and they exploit a nondifferentiable L_1 merit function to determine a step length. They showed the global convergence of their algorithm under some unclear assumptions on the generated sequences. These assumptions are discussed in Section 4.3.

The purpose of this work is to propose an interior point method for (1.1) that converges globally under milder conditions than the above existing methods. In particular, we give concrete conditions related to the problem data, e.g., f, g and X only. We show that these conditions hold for the linear SDP.

Recently, Kato, Yabe and Yamashita [10] proposed a primal-dual interior point method based on the shifted perturbed KKT conditions, which are an extension of the method proposed by Forsgren and Gill [4] for the nonlinear programming. The method generates points satisfying the shifted perturbed KKT conditions at each iteration. In order to find such a point, Kato, Yabe and Yamashita [10] exploit a merit function which is an extension of [25]. However, since the merit function is rather complicated, it might be difficult to implement it appropriately. In this paper, we propose a new merit function F whose stationary points satisfy the shifted perturbed KKT conditions. It is an extension the merit function of [4] for the nonlinear programming. It consists of simple functions of matrices, such as log-determinant and trace, and hence it is easy to implement. We show the following important properties of the merit function F.

(i) The merit function F is differentiable;

(ii) Any stationary point of the merit function F is a shifted perturbed KKT point;

(iii) The level set of the merit function F is bounded under some reasonable assumptions.

Note that Kato, Yabe and Yamashita [10] showed that their merit function also enjoys (i) and (ii), but they did not show the property (iii). Due to these properties, we can find a point satisfying the shifted perturbed KKT conditions by minimizing the merit function F. For the minimization of F, we further propose a Newton type method based on the nonlinear equations in the shifted perturbed KKT conditions. We show that the Newton direction is sufficiently descent for the merit function F. As a result, we prove the global convergence of the proposed Newton type method. These details are discussed in Section 4.

The present paper is organized as follows. In Section 2, we introduce some operators in $\mathbf{S}^d$ and important concepts, which are used in the subsequent sections. In Section 3, we present a primal-dual interior point algorithm based on the shifted perturbed KKT conditions. In Section 4, we first propose a merit function F for the shifted perturbed KKT point and present its properties. Secondly, we propose a Newton type algorithm for minimization of the merit function. Moreover, we prove the global convergence of the Newton type algorithm. In Section 5, we report some numerical results for the proposed method. Finally, we make some concluding remarks in Section 6.

Throughout this paper, we use the following notations. Let p and q be positive integers. For matrices $A, B \in \mathbf{R}^{p\times q}$, $\langle A, B\rangle$ denotes the inner product of A and B defined by

$$\langle A, B\rangle \equiv \mathrm{tr}(A^\top B),$$

where $\mathrm{tr}(M)$ denotes the trace of a square matrix M, and the superscript $\top$ denotes the transpose of a vector or a matrix. Note that if $q = 1$, then $\langle\cdot,\cdot\rangle$ denotes the inner product of vectors in $\mathbf{R}^p$. For a given vector $w \in \mathbf{R}^p$ and a matrix $W \in \mathbf{R}^{p\times q}$, w_i denotes the i-th element of the vector w, and W_{ij} denotes the (i, j)-th element of the matrix W. Moreover, $\|w\|$ denotes the Euclidean norm of the vector w defined by

$$\|w\| \equiv \sqrt{\langle w, w\rangle},$$

and $\|W\|_F$ denotes the Frobenius norm of the matrix W defined by

$$\|W\|_F \equiv \sqrt{\langle W, W\rangle}.$$

Let $\mathcal{V} \equiv \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{S}^d$. For a given $v \in \mathcal{V}$, we use the following notations for simplicity.

$$v = \begin{bmatrix} x \\ y \\ Z \end{bmatrix} \quad \text{or} \quad v = (x, y, Z),$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and $Z \in \mathbf{S}^d$, respectively. We further define the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ on $\mathcal{V}$ as $\langle v_1, v_2\rangle \equiv \langle x_1, x_2\rangle + \langle y_1, y_2\rangle + \langle Z_1, Z_2\rangle$ and $\|v\| \equiv \sqrt{\langle v, v\rangle}$, where $v_1 = (x_1, y_1, Z_1) \in \mathcal{V}$ and $v_2 = (x_2, y_2, Z_2) \in \mathcal{V}$. For a given matrix $U \in \mathbf{S}^d$, $\lambda_1(U), \dots, \lambda_d(U)$ denote eigenvalues of the matrix U. In particular, $\lambda_{\min}(U)$ and $\lambda_{\max}(U)$ denote the minimum and the maximum eigenvalues of the matrix U, respectively. For a given matrix $V \in \mathbf{S}^d_+$, $V^{\frac{1}{2}} \in \mathbf{S}^d_+$ denotes the matrix such that $V = V^{\frac{1}{2}}V^{\frac{1}{2}}$. Note that $V^{\frac{1}{2}} \equiv Q\Lambda Q^\top$, where

$$\Lambda = \begin{bmatrix} \sqrt{\lambda_1(V)} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \sqrt{\lambda_d(V)} \end{bmatrix},$$

and Q is a certain orthogonal matrix such that $V = Q\Lambda^2 Q^\top$. Let $\Phi : P_1 \times P_2 \to P_3$, where P_1 and P_2 are open sets. We denotes a Fréchet derivative of Φ as $\nabla\Phi$. We further denote a Fréchet derivative of Φ with respect to a variable $Z \in P_1$ as $\nabla_Z \Phi$. Moreover, if Φ is a vector-valued function, then J_Φ denotes a Jacobian of Φ.

2 Preliminaries

In this section, we first introduce some operators. Then we present some useful properties of the log-determinant function on $\mathbf{S}^d$. Moreover, we introduce the (approximate) KKT conditions related to the primal-dual interior point methods for the nonlinear SDP.

2.1 Some operators and their properties

Let $U, V \in \mathbf{S}^d, P, Q \in \mathbf{R}^{d\times d}$ and $x, w \in \mathbf{R}^n$. We use the following notations.

(i) A product $\circ$ of the matrices U and V is defined by

$$U \circ V \equiv \frac{UV + VU}{2}.$$

(ii) A partial derivative of $X(x)$ with respect to x_i is denoted by $A_i(x) \in \mathbf{S}^d$, that is,

$$A_i(x) \equiv \frac{\partial}{\partial x_i} X(x) \quad \text{for } i = 1, \dots, n.$$

(iii) An operator $\mathcal{A}(x)$ from $\mathbf{R}^n$ to $\mathbf{S}^d$ is defined by

$$\mathcal{A}(x)w \equiv \sum_{i=1}^{n} w_i A_i(x).$$

(iv) The adjoint operator of $\mathcal{A}(x)$ is denoted by $\mathcal{A}^*(x)$, that is,

$$\mathcal{A}^*(x)U = \begin{bmatrix} \langle A_1(x), U\rangle \\ \vdots \\ \langle A_n(x), U\rangle \end{bmatrix}.$$

(v) An operator $P \odot Q$ from $\mathbf{S}^d$ to $\mathbf{S}^d$ is defined by

$$(P \odot Q)U \equiv \frac{1}{2}(PUQ^\top + QUP^\top). \tag{2.1}$$

If $X(x) = \sum_{i=1}^n x_i A_i$ with some constant matrices $A_i \in \mathbf{S}^d, i = 1, \dots, n$, then $A_i(x) = A_i, i = 1, \dots, n$. Note that $U \circ V = 0$ is equivalent to $UV = 0$ if U and V are symmetric positive semidefinite. Note also that a gradient of $\langle X(x), U\rangle$ with respect to x is given by

$$\nabla_x \langle X(x), U\rangle = \mathcal{A}^*(x)U. \tag{2.2}$$

We list some useful properties of the operator $P \odot Q$. See [22] and [26] for their proofs.

Proposition 1 *Let P and Q be nonsingular matrices in $\mathbf{R}^{d\times d}$. Then the following statements hold.*

(a) *The operator $P \odot Q$ is invertible.*

(b) $\langle U, (P\odot Q)V\rangle = \left\langle (P^\top \odot Q^\top)U, V\right\rangle$ for all $U, V \in \mathbf{S}^d$,
$\langle U, (P\odot Q)^{-1}V\rangle = \left\langle (P^\top \odot Q^\top)^{-1}U, V\right\rangle$ for all $U, V \in \mathbf{S}^d$.

(c) $(P\odot P)^{-1} = (P^{-1}\odot P^{-1})$. □

Some interior point methods for SDP exploit a scaling of $X(x)$ and Z, where $Z \in \mathbf{S}^d$ corresponds to a Lagrangian multiplier matrix for $X(x) \succeq 0$ in (1.1). (The details of Z are given in Subsection 2.3. We will exploit the scaling in the proposed method.) Let T be a nonsingular matrix in $\mathbf{R}^{d\times d}$. We consider the scaled matrices $\widetilde{X}(x)$ and $\widetilde{Z}$ defined by

$$\widetilde{X}(x) \equiv (T\odot T)X(x) \quad \text{and} \quad \widetilde{Z} \equiv (T^{-\top}\odot T^{-\top})Z.$$

We show some useful properties of $\widetilde{X}(x)$ and $\widetilde{Z}$.

Proposition 2 *The following statements hold.*

(a) *Let $\psi(x,Z) \equiv \widetilde{X}(x)\circ\widetilde{Z}$. Then we have $\nabla_x\psi(x,Z) = \frac{1}{2}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x)$ and $\nabla_Z\psi(x,Z) = \frac{1}{2}(\widetilde{X}(x)\odot I)(T^{-\top}\odot T^{-\top})$.*

(b) *Suppose that $X(x)$ and Z are symmetric positive definite. Suppose also that $\widetilde{X}(x)$ and $\widetilde{Z}$ commute. Then we have*

$$\left\langle (\widetilde{Z}\odot I)(\widetilde{X}(x)\odot I)U, U\right\rangle \geq 0 \quad \text{for all } U \in \mathbf{S}^d.$$

Furthermore, the strict inequality holds in the above if and only if $U \neq 0$.

(c) *Suppose that $\widetilde{X}(x)$ and $\widetilde{Z}$ commute. Then we have*

$$(\widetilde{X}(x)\odot I)(\widetilde{Z}\odot I) = (\widetilde{Z}\odot I)(\widetilde{X}(x)\odot \widetilde{Z}).$$

Proof. (a) Let $x \in \mathbf{R}^n$ be fixed. Since the function X is differentiable, we have

$$X(x+h) = X(x) + \sum_{i=1}^n h_i A_i(x) + o(\|h\|) \quad \text{for } h \in \mathbf{R}^n,$$

and hence

$$\begin{aligned}\widetilde{X}(x+h) &= \widetilde{X}(x) + (T\odot T)\sum_{i=1}^n h_i A_i(x) + o(\|h\|).\\ &= \widetilde{X}(x) + (T\odot T)\mathcal{A}(x)h + o(\|h\|).\end{aligned}$$

It then follows that

$$\begin{aligned}
\psi(x+h,Z) &= \widetilde{X}(x+h)\widetilde{Z} + \widetilde{Z}\widetilde{X}(x+h) \\
&= \{\widetilde{X}(x) + (T\odot T)\mathcal{A}(x)h + o(\|h\|)\}\widetilde{Z} + \widetilde{Z}\{\widetilde{X}(x) + (T\odot T)\mathcal{A}(x)h + o(\|h\|)\} \\
&= \widetilde{X}(x)\widetilde{Z} + \widetilde{Z}\widetilde{X}(x) + (T\odot T)\mathcal{A}(x)h\widetilde{Z} + \widetilde{Z}(T\odot T)\mathcal{A}(x)h + o(\|h\|) \\
&= \psi(x,Z) + \frac{1}{2}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x)h + o(\|h\|),
\end{aligned}$$

which implies that $\nabla_x\psi(x,Z) = \frac{1}{2}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x)$. Next we give $\nabla_Z\psi(x,Z)$. Let $H\in\mathbf{S}^d$. Then we have

$$\begin{aligned}
\psi(x,Z+H) &= \widetilde{X}(x)(T^{-\top}\odot T^{-\top})(Z+H) + (T^{-\top}\odot T^{-\top})(Z+H)\widetilde{X}(x) \\
&= \widetilde{X}(x)\widetilde{Z} + \widetilde{Z}\widetilde{X}(x) + \widetilde{X}(x)(T^{-\top}\odot T^{-\top})H + (T^{-\top}\odot T^{-\top})H\widetilde{X}(x) \\
&= \psi(x,Z) + \frac{1}{2}(\widetilde{X}(x)\odot I)(T^{-\top}\odot T^{-\top})H,
\end{aligned}$$

which implies that $\nabla_Z\psi(x,Z) = \frac{1}{2}(\widetilde{X}(x)\odot I)(T^{-\top}\odot T^{-\top})$. Thus, (a) is proved.
(b) Since the matrices $X(x)$ and Z are symmetric positive definite, $\widetilde{X}(x)$ and $\widetilde{Z}$ are also symmetric positive definite. It then follows from the commutativity of $\widetilde{X}(x)$ and $\widetilde{Z}$ that $\widetilde{X}(x)\widetilde{Z}$ is symmetric positive definite. Thus, there exists $(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}$ such that $\widetilde{X}(x)\widetilde{Z} = (\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}$. Let $U\in\mathbf{S}^d$. Then we have

$$\begin{aligned}
\Big\langle(\widetilde{Z}\odot I)(\widetilde{X}(x)\odot I)U,U\Big\rangle &= \frac{1}{4}\mathrm{tr}((\widetilde{Z}\widetilde{X}(x)U + \widetilde{X}(x)U\widetilde{Z} + \widetilde{Z}U\widetilde{X}(x) + U\widetilde{X}(x)\widetilde{Z})U) \\
&= \frac{1}{4}\mathrm{tr}(\widetilde{X}(x)U\widetilde{Z}U) + \frac{1}{4}\mathrm{tr}(\widetilde{Z}U\widetilde{X}(x)U) + \frac{1}{4}\mathrm{tr}(U\widetilde{X}(x)\widetilde{Z}U) + \frac{1}{4}\mathrm{tr}(U\widetilde{Z}\widetilde{X}(x)U) \\
&= \frac{1}{4}\mathrm{tr}(\widetilde{X}(x)^{\frac{1}{2}}U\widetilde{Z}^{\frac{1}{2}}\widetilde{Z}^{\frac{1}{2}}U\widetilde{X}(x)^{\frac{1}{2}}) + \frac{1}{4}\mathrm{tr}(\widetilde{Z}^{\frac{1}{2}}U\widetilde{X}(x)^{\frac{1}{2}}\widetilde{X}(x)^{\frac{1}{2}}U\widetilde{Z}^{\frac{1}{2}}) \\
&\qquad + \frac{1}{4}\mathrm{tr}(U\widetilde{X}(x)\widetilde{Z}U) + \frac{1}{4}\mathrm{tr}(U\widetilde{X}(x)\widetilde{Z}U) \\
&= \frac{1}{2}\mathrm{tr}(\widetilde{X}(x)^{\frac{1}{2}}U\widetilde{Z}^{\frac{1}{2}}\widetilde{Z}^{\frac{1}{2}}U\widetilde{X}(x)^{\frac{1}{2}}) + \frac{1}{2}\mathrm{tr}(U(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}U) \\
&= \frac{1}{2}\|\widetilde{X}(x)^{\frac{1}{2}}U\widetilde{Z}^{\frac{1}{2}}\|_F^2 + \frac{1}{2}\|(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}U\|_F^2 \\
&\geq 0,
\end{aligned}$$

where the third equality follows from the commutativity of $\widetilde{X}(x)$ and $\widetilde{Z}$. Note that, since $\widetilde{X}(x)^{\frac{1}{2}}$, $\widetilde{Z}^{\frac{1}{2}}$ and $(\widetilde{X}(x)\widetilde{Z})^{\frac{1}{2}}$ are positive definite, the strict inequality holds in the above if and only if $U\neq 0$.
(c) For any $U\in\mathbf{S}^d$, we have

$$\begin{aligned}
(\widetilde{X}(x)\odot I)(\widetilde{Z}\odot I)U &= \frac{1}{4}(\widetilde{X}(x)\widetilde{Z}U + \widetilde{Z}U\widetilde{X}(x) + \widetilde{X}(x)U\widetilde{Z} + U\widetilde{Z}\widetilde{X}(x)) \\
&= \frac{1}{4}(\widetilde{Z}\widetilde{X}(x)U + \widetilde{X}(x)U\widetilde{Z} + \widetilde{Z}U\widetilde{X}(x) + U\widetilde{X}(x)\widetilde{Z}) \\
&= (\widetilde{Z}\odot I)(\widetilde{X}(x)\odot I)U,
\end{aligned}$$

where the second equality follows from the commutativity of $\widetilde{X}(x)$ and $\widetilde{Z}$. Hence, we obtain $(\widetilde{X}(x)\odot I)(\widetilde{Z}\odot I) = (\widetilde{Z}\odot I)(\widetilde{X}(x)\odot I)$. □

2.2 Properties of the log-determinant function

Let $\phi : \mathbf{S}^d_{++} \to \mathbf{R}$ be defined by $\phi(M) \equiv -\log\det M$. Let Ω be defined by $\Omega \equiv \{x \in \mathbf{R}^n | X(x) \succ 0\}$, and let $\varphi : \Omega \to \mathbf{R}$ be defined by

$$\varphi(x) \equiv \phi(X(x)). \tag{2.3}$$

We first give the differentiability and convexity of φ.

Proposition 3

(a) *The function φ is differentiable on Ω, and its derivative is given by $\nabla\varphi(x) = -\mathcal{A}^*(x)X(x)^{-1}$.*

(b) *Suppose that*

$$X(\lambda u + (1-\lambda)v) - \lambda X(u) - (1-\lambda)X(v) \succeq 0 \quad \text{for } \lambda \in [0,1] \text{ and } u, v \in \Omega. \tag{2.4}$$

Then φ is convex on Ω. Moreover, if X is injective on Ω, then φ is strictly convex.

(c) *Suppose that (2.4) holds. Suppose also that $A_1(x), \dots, A_n(x)$ are linearly independent for all $x \in \Omega$. Then φ is strictly convex.*

Proof. (a) It follows from [21, Section 5] that

$$\nabla\phi(M) = -M^{-1}. \tag{2.5}$$

Then, we have from the chain rule that

$$\nabla\varphi(x) = -\mathcal{A}^*(x)X(x)^{-1}. \tag{2.6}$$

(b) First note that $\det A \le \det B$ if $0 \preceq A$ and $0 \preceq B - A$ from [8, Corollary 7.7.4]. It then follows from (2.4) that for any $\lambda \in [0,1]$ and $u, v \in \Omega$ such that $u \neq v$,

$$\det[\lambda X(u) + (1-\lambda)X(v)] \le \det[X(\lambda u + (1-\lambda)v)].$$

Since $-\log$ is a decreasing function on $(0,\infty)$ and ϕ is strictly convex from [8, Theorem 7.6.7], we have

$$\begin{aligned}
\varphi(\lambda u + (1-\lambda)v) &= -\log\det[X(\lambda u + (1-\lambda)v)] \\
&\le -\log\det[\lambda X(u) + (1-\lambda)X(v)]. \\
&= \phi(\lambda X(u) + (1-\lambda)X(v)) \\
&\le \lambda\phi(X(u)) + (1-\lambda)\phi(X(v)) \\
&= \lambda\varphi(u) + (1-\lambda)\varphi(v),
\end{aligned}$$

which shows that φ is convex on Ω.

Suppose that $u \neq v$. Then, since X is injective on Ω, $X(u) \neq X(v)$. Moreover, since ϕ is strictly convex,

$$\begin{aligned}
\varphi(\lambda u + (1-\lambda)v) &\le \phi(\lambda X(u) + (1-\lambda)X(v)) \\
&< \lambda\phi(X(u)) + (1-\lambda)\phi(X(v)) \\
&\le \lambda\varphi(u) + (1-\lambda)\varphi(v)
\end{aligned}$$

for $\lambda \in (0,1)$. Thus, φ is strictly convex.
(c) Since X is twice differentiable, $X(v + \lambda(u - v)) - X(v) = \lambda \mathcal{A}(v)(u - v) + o(\lambda)$ for $u, v \in \Omega$ and $\lambda \in (0,1)$. Then (2.4) can be written as $\lambda \mathcal{A}(v)(u - v) - \lambda(X(u) - X(v)) + o(\lambda) \succeq 0$. Dividing both sides by λ, we have $\mathcal{A}(v)(u-v) - X(u) + X(v) + \frac{o(\lambda)}{\lambda} \succeq 0$. Letting $\lambda \to 0$ yields

$$\mathcal{A}(v)(u - v) - X(u) + X(v) \succeq 0.$$

Let $M \equiv \mathcal{A}(v)(u-v) - X(u) + X(v)$. Since M and $X(v)^{-1}$ are symmetric positive semidefinite, there exist $M^{\frac{1}{2}}$ and $X(v)^{-\frac{1}{2}}$. Then we have

$$\left\langle X(v)^{-1}, M \right\rangle = \mathrm{tr}(X(v)^{-1}M) = \mathrm{tr}(X(v)^{-\frac{1}{2}} M^{\frac{1}{2}} M^{\frac{1}{2}} X(v)^{-\frac{1}{2}}) = \|M^{\frac{1}{2}} X(v)^{-\frac{1}{2}}\|_F^2.$$

It then follows from the convexity of ϕ, (2.3) and (2.5) that

$$\begin{aligned}
\varphi(u) - \varphi(v) &= \phi(X(u)) - \phi(X(v)) \\
&\geq \left\langle -X(v)^{-1}, X(u) - X(v) \right\rangle \\
&= \left\langle X(v)^{-1}, M \right\rangle + \left\langle X(v)^{-1}, -\mathcal{A}(v)(u - v) \right\rangle \\
&= \|M^{\frac{1}{2}} X(v)^{-\frac{1}{2}}\|_F^2 + \left\langle -\mathcal{A}^*(v) X(v)^{-1}, u - v \right\rangle \\
&\geq \langle \nabla \varphi(v), u - v \rangle, \qquad (2.7)
\end{aligned}$$

where the last inequality follows from (2.6).

Since φ is convex by (b), it suffices for (c) to show that $u = v$ if and only if $\varphi(u) - \varphi(v) = \langle \nabla\varphi(v), u - v \rangle$. If $u = v$, then it is clear that $\varphi(u) - \varphi(v) = \langle \nabla\varphi(v), u - v \rangle$. Conversely, suppose that $\varphi(u) - \varphi(v) = \langle \nabla\varphi(v), u - v \rangle$, then the equality holds in (2.7). It follows from (2.7) that $\|M^{\frac{1}{2}} X(v)^{-\frac{1}{2}}\|_F = 0$, and $\phi(X(u)) - \phi(X(v)) = \langle -X(v)^{-1}, X(u) - X(v) \rangle$. Then, we have $\mathcal{A}(v)(u-v) = 0$ from the definition of M. Since $A_1(x), \ldots, A_n(x)$ are linearly independent for all $x \in \Omega$, we have $u = v$. □

Note that Proposition 3 (b) does not assume the differentiability of X.

We next show that matrices in a level set of ϕ is uniformly positive definite, which is a key property for the level boundedness of the merit function proposed in Section 4.

Proposition 4 *For a given $\gamma \in \mathbf{R}$, let $\mathcal{L}_\phi(\gamma) = \{U \in \mathbf{S}^d_{++} | \phi(U) \leq \gamma\}$. Let Γ be a bounded subset of $\mathbf{S}^d$. Then, there exists $\underline{\lambda} > 0$ such that $\lambda_{\min}(U) \geq \underline{\lambda}$ for all $U \in \mathcal{L}_\phi(\gamma) \cap \Gamma$.*

Proof. Suppose the contrary, that is, there exists a sequence $\{U_j\} \subset \mathcal{L}_\phi(\gamma) \cap \Gamma$ such that $\lambda_{\min}(U_j) \to 0$ as $j \to \infty$. Then

$$-\log \lambda_{\min}(U_j) \to \infty. \qquad (2.8)$$

Since $U_j \in \mathcal{L}_\phi(\gamma)$, we have $\gamma \geq \phi(U_j) = -\log\det U_j = -\sum_{i=1}^d \log \lambda_i(U_j)$. It then follows from (2.8) that there exists an index k and an infinite subset $\mathcal{J}$ such that $\lim_{j\to\infty, j\in\mathcal{J}} -\log \lambda_k(U_j) = -\infty$, that is, $\lim_{j\to\infty, j\in\mathcal{J}} \lambda_k(U_j) = \infty$. However, this is contrary to the boundedness of $\{U_j\}$. Therefore, there exists $\underline{\lambda} > 0$ such that $\lambda_{\min}(U) \geq \underline{\lambda}$ for all $U \in \mathcal{L}_\phi(\gamma) \cap \Gamma$. □

2.3 The shifted perturbed KKT conditions for the nonlinear SDP

We first introduce optimality conditions for the nonlinear SDP (1.1). Let $v = (x, y, Z)$. The Lagrangian function L of (1.1) is given by

$$L(v) \equiv f(x) - g(x)^\top y - \langle X(x), Z \rangle ,$$

where $y \in \mathbf{R}^m$ and $Z \in \mathbf{S}^d$ are the Lagrange multiplier vector and matrix for $g(x) = 0$ and $X(x) \succeq 0$, respectively. From (2.2), a gradient of the Lagrangian function L with respect to x is given by

$$\nabla_x L(v) = \nabla f(x) - J_g(x)^\top y - \mathcal{A}^*(x)Z.$$

The Karush-Kuhn-Tucker (KKT) conditions of (1.1) are written as

$$\begin{bmatrix} \nabla_x L(v) \\ g(x) \\ X(x)Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.9}$$

and

$$X(x) \succeq 0, \qquad Z \succeq 0. \tag{2.10}$$

Most of the solution methods for the nonlinear SDP is developed to find a point $v = (x, y, Z)$ that satisfies the KKT conditions. However, it is difficult to get such a point directly due to the complementarity condition $X(x)Z = 0$ with $X(x) \succeq 0$ and $Z \succeq 0$. To overcome this difficulty, the primal-dual interior point method proposed by Yamashita, Yabe and Harada [26] exploit the following perturbed KKT conditions with a parameter $\mu > 0$.

$$\begin{bmatrix} \nabla_x L(v) \\ g(x) \\ X(x)Z - \mu I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.11}$$

and

$$X(x) \succ 0, \qquad Z \succ 0. \tag{2.12}$$

They [26] proposed the Newton type algorithm to get a point satisfying the perturbed KKT conditions.

In this paper, we focus on the following shifted perturbed KKT conditions. For $\mu > 0$,

$$\begin{bmatrix} \nabla_x L(v) \\ g(x) + \mu y \\ X(x)Z - \mu I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.13}$$

and

$$X(x) \succ 0, \qquad Z \succ 0. \tag{2.14}$$

The above shifted perturbed KKT conditions are derived by Forsgren and Gill [4] for the nonlinear programming. In what follows, we call a point v satisfying the shifted perturbed KKT conditions *a shifted perturbed KKT point*. Furthermore, we define a set $\mathcal{W} \subset \mathcal{V}$ by

$$\mathcal{W} \equiv \{(x, y, Z) \in \mathcal{V} \mid X(x) \succ 0, Z \succ 0\}.$$

We call a point $v \in \mathcal{W}$ an *interior point*.

3 A primal-dual interior point method based on the shifted perturbed KKT conditions

In this section, we introduce a prototype of an interior point algorithm based on the shifted perturbed KKT conditions (2.13) and (2.14). Note that the prototype has been already proposed in [10].

The primal-dual interior point method generates a sequence $\{v_k\} \subset \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{S}^d$ such that the point v_k approximately satisfies the shifted perturbed KKT conditions (2.13) and (2.14) with $\mu = \mu_k > 0$, where $\{\mu_k\}$ is a positive sequence such that $\mu_k \to 0$ $(k \to \infty)$.

To construct a concrete algorithm, it is important to define the approximate shifted perturbed KKT point, and to provide a method for finding the approximate shifted perturbed KKT point.

We first give a concrete definition of the approximate shifted perturbed KKT point. To this end, let

$$r(v;\mu) \equiv \begin{bmatrix} \nabla_x L(v) \\ g(x) + \mu y \\ X(x)Z - \mu I \end{bmatrix}.$$

Moreover, let

$$\rho(v;\mu) \equiv \sqrt{\left\| \begin{bmatrix} \nabla_x L(v) \\ g(x) + \mu y \end{bmatrix} \right\|^2 + \|X(x)Z - \mu I\|_F^2}.$$

For a given $\varepsilon > 0$, we define the *approximate shifted perturbed KKT point* as a point v satisfying $\rho(v;\mu) \le \varepsilon$ and $v \in \mathcal{W}$. Note that $\rho(v;\mu) = 0$ and $v \in \mathcal{W}$ if and only if v is the shifted perturbed KKT point. Note also that $\rho(v;0) = 0, X(x) \succeq 0$ and $Z \succeq 0$ if and only if v is an original KKT point of the nonlinear SDP (1.1).

Now, we give the framework of the primal-dual interior point method.

Algorithm 1

Step 0. Let $\{\mu_k\}$ be a positive sequence such that $\mu_k \to 0$ as $k \to \infty$. Choose constants $\sigma, \epsilon > 0$. Set $k = 0$.

Step 1. Find an approximate shifted perturbed KKT point v_{k+1} with $\varepsilon = \sigma\mu_k$, that is, $v_{k+1} \in \mathcal{W}$ such that $\rho(v_{k+1};\mu_k) \le \sigma\mu_k$.

Step 2. If $\rho(v_{k+1};0) \le \epsilon$, then stop.

Step 3. Set $k = k+1$ and go to Step 1. □

The following theorem gives conditions for the global convergence of Algorithm 1. It can be proven in a way similar to [26, Theorem 1]. Thus, we omit the proof.

Theorem 1 *Suppose that the approximate shifted perturbed KKT point v_{k+1} is found in Step 1 at every iteration. Moreover suppose that the sequence $\{x_k\}$ is bounded, and that the MFCQ condition holds at any accumulation point of $\{x_k\}$, i.e., for any accumulation point x^* of $\{x_k\}$, the matrix $J_g(x^*)$ is of full rank and there exists a nonzero vector $w \in \mathbf{R}^n$ such that*

$$J_g(x^*)w = 0 \quad \text{and} \quad X(x^*) + \sum_{i=1}^{n} A_i(x^*) \succ 0.$$

Then, the sequences $\{y_k\}$ and $\{Z_k\}$ are bounded, and any accumulation point of $\{v_k\}$ satisfies the KKT conditions (2.9) and (2.10). □

The theorem guarantees the global convergence if the approximate shifted perturbed KKT point v_{k+1} is found at each iteration. Thus it is important to present concrete algorithm that finds the point. In the next section, we will propose a merit function for the shifted perturbed KKT point and a Newton type algorithm for solving the unconstrained minimization problem of the merit function.

4 Finding a shifted perturbed KKT point

In order to find the approximate shifted perturbed KKT point in Step 1 of Algorithm 1, we may solve the following unconstrained minimization problem:

$$\begin{array}{ll} \text{minimize} & \rho(v;\mu)^2, \\ \text{subject to} & v \in \mathcal{V}, \end{array}$$

where recall that $\mathcal{V} = \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{S}^d$. Unfortunately, a stationary point of the problem is not necessarily a shifted perturbed KKT point unless $\nabla r(v;\mu)$ is invertible. In this section, we first construct a differentiable merit function F whose stationary point is a shifted perturbed KKT point. Moreover, we show that a Newton direction for the nonlinear equations $r(v;\mu) = 0$ is a descent direction of the merit function F. Next, we propose a Newton type algorithm for solving the unconstrained minimization of the merit function F. Finally, we show that the proposed algorithm finds a shifted perturbed KKT point under some mild assumptions.

4.1 Merit function and its properties

We propose the following merit function $F : \mathcal{W} \to \mathbf{R}$ for the shifted perturbed KKT point.

$$F(x, y, Z) \equiv F_{BP}(x) + \nu F_{PD}(x, y, Z),$$

where ν is a positive constant, and the functions $F_{BP} : \Omega \to \mathbf{R}$ and $F_{PD} : \mathcal{W} \to \mathbf{R}$ are

$$F_{BP}(x) \equiv f(x) + \frac{1}{2\mu}\|g(x)\|^2 - \mu \log\det X(x),$$

and

$$F_{PD}(x, y, Z) \equiv \frac{1}{2\mu}\|g(x) + \mu y\|^2 + \langle X(x), Z\rangle - \mu \log\det X(x) \det Z,$$

respectively. The functions F_{BP} and F_{PD} are called the primal barrier penalty function and the primal-dual barrier penalty function, respectively. Note that F is convex with respect to x when f is convex and g, X are linear. The merit function F is an extension of that proposed by Forsgren and Gill [4] for the nonlinear programming.

Remark 1 *For the perturbed KKT conditions, Kato, Yabe and Yamashita [10] also proposed the following merit function* $\widetilde{F} : \mathcal{W} \to \mathbf{R}$.

$$\widetilde{F}(x, y, Z) \equiv F_{BP}(x) + \nu \widetilde{F}_{PD}(x, y, Z),$$

where $\widetilde{F}_{PD}(w)$ *is defined by*

$$\widetilde{F}_{PD}(x, y, Z) \equiv \frac{1}{2}\|g(x) + \mu y\|^2 + \log \frac{\frac{1}{d}\langle X(x), Z\rangle + \|Z^{\frac{1}{2}}X(x)Z^{\frac{1}{2}} - \mu I\|_F^2}{(\det(X(x)Z))^{\frac{1}{d}}}.$$

They showed that $\widetilde{F}$ *has nice properties as the merit function* F*. However,* $\widetilde{F}$ *is more complicated than* F*, and hence it might not be easy to implement the Newton type method based on* $\widetilde{F}$ *in [10]. Furthermore, even if* f *is convex and* g*,* X *are linear,* $\widetilde{F}$ *is not necessarily convex with respect to* x*.*

In the rest of this subsection, we present some useful properties of the merit function F such as the differentiability, the equivalence between a stationary point of F and a shifted perturbed KKT point, and the level boundedness.

First of all, we present a concrete formula of the derivatives of the merit function F.

Theorem 2 *The merit function* F *is differentiable at* $w = (x, y, Z) \in \mathcal{W}$*. Moreover, its derivative is given by*

$$\nabla F(w) = \begin{bmatrix} \nabla F_{BP}(x) + \nu \nabla_x F_{PD}(w) \\ \nu \nabla_y F_{PD}(w) \\ \nu \nabla_Z F_{PD}(w) \end{bmatrix},$$

where

$$\begin{aligned}
\nabla F_{BP}(x) &= \nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X(x)^{-1}, \\
\nabla_x F_{PD}(w) &= \frac{1}{\mu} J_g(x)^\top (g(x) + \mu y) + \mathcal{A}^*(x)(Z - \mu X(x)^{-1}), \\
\nabla_y F_{PD}(w) &= g(x) + \mu y, \\
\nabla_Z F_{PD}(w) &= X(x) - \mu Z^{-1}.
\end{aligned}$$

Proof. By the definition of the merit function F, we have

$$\nabla_x F(w) = \nabla F_{BP}(x) + \nu \nabla_x F_{PD}(w), \quad \nabla_y F(w) = \nu \nabla_y F_{PD}(w), \quad \nabla_Z F(w) = \nu \nabla_Z F_{PD}(w).$$

Thus, the derivative of F is given by

$$\nabla F(w) = \begin{bmatrix} \nabla_x F(w) \\ \nabla_y F(w) \\ \nabla_Z F(w) \end{bmatrix} = \begin{bmatrix} \nabla F_{BP}(x) + \nu \nabla_x F_{PD}(w) \\ \nu \nabla_y F_{PD}(w) \\ \nu \nabla_Z F_{PD}(w) \end{bmatrix}.$$

By Proposition 3 (a), we obtain

$$\nabla F_{BP}(x) = \nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X(x)^{-1},$$
$$\nabla_x F_{PD}(w) = \frac{1}{\mu} J_g(x)^\top (g(x) + \mu y) + \mathcal{A}^*(x)(Z - \mu X(x)^{-1}).$$

From [21, Section 5], we also get $\nabla_Z F_{PD}(w) = X(x) - \mu Z^{-1}$. Moreover, $\nabla_y F_{PD}(w) = g(x) + \mu y$. □

Next, we show the equivalence between a stationary point of the merit function F and a shifted perturbed KKT point.

Theorem 3 *A point $w^* \in \mathcal{W}$ is a stationary point of the merit function F if and only if w^* is a shifted perturbed KKT point.*

Proof. First, let $w^* = (x^*, y^*, Z^*) \in \mathcal{W}$ be a stationary point of the merit function F. It then follows from Theorem 2 that

$$\nabla f(x^*) + \tfrac{1}{\mu} J_g(x^*)^\top (2g(x^*) + \mu y^*) + \mathcal{A}^*(x^*)(Z^* - 2\mu X(x^*)^{-1}) = 0, \quad (4.1)$$
$$g(x^*) + \mu y^* = 0, \quad X(x^*) - \mu (Z^*)^{-1} = 0. \quad (4.2)$$

Thus we have

$$\begin{aligned}
\nabla_x L(w^*) &= \nabla f(x^*) - J_g(x^*)^\top y^* - \mathcal{A}^*(x^*) Z^* \\
&= \nabla f(x^*) + \frac{1}{\mu} J_g(x^*)^\top g(x^*) - \frac{1}{\mu} \mathcal{A}^*(x^*) X(x^*)^{-1} \\
&= -\frac{1}{\mu} J_g(x^*)^\top (g(x^*) + \mu y^*) - \mathcal{A}^*(x^*)(Z^* - \mu X(x^*)^{-1}) \\
&= 0,
\end{aligned}$$

where the second and third equalities follow from (4.2) and (4.1), respectively. Therefore, w^* is a shifted perturbed KKT point.

Conversely, let $w^* = (x^*, y^*, Z^*)$ be a shifted perturbed KKT point. Then, we obtain that

$$\nabla_x L(w^*) = \nabla f(x^*) - J_g(x^*)^\top y^* - \mathcal{A}^*(x^*) Z^* = 0,$$
$$g(x^*) + \mu y^* = 0, \quad X(x^*) Z^* - \mu I = 0.$$

It then follows from Theorem 2 that

$$
\begin{aligned}
\nabla_x F(w^*) &= \nabla f(x^*) + \frac{1}{\mu} J_g(x^*)(2g(x^*) + \mu y^*) + \mathcal{A}^*(x^*)(Z^* - 2\mu X(x^*)^{-1}) \\
&= \nabla f(x^*) + \frac{1}{\mu} J_g(x^*)^\top g(x^*) - \mu \mathcal{A}^*(x^*) X(x^*)^{-1} \\
&\qquad + \frac{1}{\mu} J_g(x^*)^\top (g(x^*) + \mu y^*) + \mathcal{A}^*(x^*)(Z^* - \mu X(x^*)^{-1}) \\
&= \nabla f(x^*) - J_g(x^*)^\top y^* - \mathcal{A}^*(x^*) Z^* \\
&\qquad + \frac{1}{\mu} J_g(x^*)^\top (g(x^*) + \mu y^*) + \mathcal{A}^*(x^*)(Z^* - \mu X(x^*)^{-1}) \\
&= 0,
\end{aligned}
$$

$$
\nabla_y F(w^*) = g(x^*) + \mu y^* = 0,
$$

$$
\nabla_Z F(w^*) = X(x^*) - \mu (Z^*)^{-1} = (X(x^*)(Z^*) - \mu I)(Z^*)^{-1} = 0.
$$

Therefore, we have $\nabla F(w^*) = 0$, that is, w^* is a stationary point of F. □

This theorem is an extension of [4, Lemma 3.1] for the nonlinear programming. From this theorem, we can find an approximate shifted perturbed KKT point by solving the following unconstrained minimization problem.

$$
\begin{array}{ll} \text{minimize} & F(w), \\ \text{subject to} & w \in \mathcal{W}. \end{array} \tag{4.3}
$$

One of the sufficient conditions under which descent methods find a stationary point is that a level set of the objective function is bounded. Thus, it is worth providing sufficient conditions for the level boundedness of the merit function F. For a given $\alpha \in \mathbf{R}$, we define the level set $\mathcal{L}(\alpha)$ of F by

$$
\mathcal{L}(\alpha) = \{ w \in \mathcal{W} \mid F(w) \le \alpha \}.
$$

We first give two lemmas.

Lemma 1 *Let $w = (x, y, Z) \in \mathcal{W}$ and $\mu > 0$. Then the following properties hold.*

(a) $\langle X(x), Z \rangle - \mu \log\det X(x) Z \ge d\mu(1 - \log \mu)$,

(b) $F_{PD}(w) \ge d\mu(1 - \log \mu)$. *The equality holds if and only if $g(x) + \mu y = 0$ and $X(x)Z - \mu I = 0$.*

(c) $\displaystyle\lim_{\langle X(x), Z\rangle \downarrow 0} F_{PD}(w) = \infty$ and $\displaystyle\lim_{\langle X(x), Z\rangle \uparrow \infty} F_{PD}(w) = \infty$.

Proof. The properties (a), (b) and (c) directly follow from [26, Lemma 1]. □

Lemma 2 *Suppose that an infinite sequence $\{w_j = (x_j, y_j, Z_j)\}$ is included in $\mathcal{L}(\alpha)$. Suppose also that the sequence $\{x_j\}$ is bounded. Then, the sequences $\{y_j\}$ and $\{Z_j\}$ are also bounded. In addition, the sequences $\{X(x_j)\}$ and $\{Z_j\}$ are uniformly positive definite.*

Proof. Since $\{x_j\}$ is bounded, the sequence $\{-\log\det X(x_j)\}$ is bounded below. Thus, there exists a real number M_1 such that

$$M_1 \le f(x_j) + \frac{1}{2\mu}\|g(x_j)\|^2 - \log\det X(x_j) = F_{BP}(x_j) \quad \text{for all } j. \tag{4.4}$$

It then follows from $w_j \in \mathcal{L}(\alpha)$ and the definition of F that

$$F_{PD}(w_j) = \frac{1}{\nu}(F(w_j) - F_{BP}(x_j)) \le \frac{1}{\nu}(\alpha - M_1) \quad \text{for all } j, \tag{4.5}$$

which can be rewritten as

$$\begin{aligned} \frac{1}{2\mu}\|g(x_j) + \mu y_j\|^2 &\le \frac{1}{\nu}(\alpha - M_1) - \langle X(x_j), Z_j\rangle + \mu\log\det X(x_j)Z_j \\ &\le \frac{1}{\nu}(\alpha - M_1) - d\mu(1 - \log\mu), \end{aligned}$$

where the last inequality follows from Lemma 1 (a). Hence, the sequence $\{y_j\}$ is bounded.

Next we show that $\{X(x_j)\}$ is uniformly positive definite. From Lemma 1 (b) and (4.4), we have

$$M_1 \le F_{BP}(x_j) = F(w_j) - \nu F_{PD}(w_j) \le \alpha - \nu F_{PD}(w_j) \le \alpha - d\mu(1 - \log\mu) \quad \text{for all } j,$$

and hence the sequence $\{F_{BP}(x_j)\}$ is bounded. It then follows from the boundedness of $\{x_j\}$ and $F_{BP}(x_j) = f(x_j) + \frac{1}{2\mu}\|g(x_j)\|^2 - \mu\log\det X(x_j)$ that $\{-\log\det X(x_j)\}$ is also bounded. From Proposition 4, the boundedness of $\{-\log\det X(x_j)\}$ and $\{X(x_j)\}$ implies that $\{X(x_j)\}$ is uniformly positive definite, that is, there exists $\underline{\lambda}$ such that $\lambda_{\min}(X(x_j)) \ge \underline{\lambda} > 0$ for all j.

Next we show that $\{Z_j\}$ is bounded. From Lemma 1 (b) and (4.5), we have

$$d\mu(1 - \log\mu) \le F_{PD}(w_j) \le \frac{1}{\nu}(\alpha - M_1) \quad \text{for all } j,$$

and hence the sequence $\{F_{PD}(w_j)\}$ is bounded. It then follows from Lemma 1 (c) that $\{\langle X(x_j), Z_j\rangle\}$ is bounded. Thus, there exists a real number M_2 such that for all j,

$$M_2 \ge \mathrm{tr}(X(x_j)Z_j) \ge \lambda_{\min}(X(x_j))\mathrm{tr}(Z_j) \ge \underline{\lambda}\mathrm{tr}(Z_j) = \underline{\lambda}\sum_{k=1}^{d}\lambda_k(Z_j) \tag{4.6}$$

where the second inequality follows from [2, Proposition 8.4.13]. Since $\{Z_j\}$ is positive definite, $\lambda_k(Z_j) > 0$ for $k = 1, \ldots, d$. It then follows from (4.6) that $\{\lambda_k(Z_j)\}$ is bounded for $k = 1, \ldots, d$, and hence $\{Z_j\}$ is bounded.

Finally, we show that $\{Z_j\}$ is uniformly positive definite. Recall that

$$F_{PD}(w_j) = \frac{1}{2\mu}\|g(x_j) + \mu y_j\|^2 + \langle X(x_j), Z_j\rangle - \mu\log\det X(x_j) - \mu\log\det Z_j,$$

and that the sequences $\{x_j\}, \{y_j\}, \{\langle X(x_j), Z_j\rangle\}, \{-\log\det X(x_j)\}$ and $\{F_{PD}(w_j)\}$ are bounded. Therefore, $\{-\log\det Z_j\}$ is also bounded. It then follows from Proposition 4 and the boundedness of $\{Z_j\}$ that $\{Z_j\}$ is uniformly positive definite. $\square$

We now give sufficient conditions under which any level set of the merit function F is bounded.

Theorem 4 *Suppose that the following four assumptions hold.*
(i) The function f is convex;
(ii) The functions $g_1, \ldots, g_m$ are affine;
(iii) The function X satisfies $X(\lambda u + (1-\lambda)v) - \lambda X(u) - (1-\lambda)X(v) \succeq 0$ for $\lambda \in [0,1]$ and $u, v \in \Omega$;
(iv) The matrices $A_1(x), \ldots, A_n(x)$ are linearly independent for all $x \in \Omega$;
(v) There exists a shifted perturbed KKT point.
Then, the level set $\mathcal{L}(\alpha)$ of F is bounded for all $\alpha \in \mathbf{R}$.

Proof. Let $\{(x_k, y_k, Z_k)\}$ be infinite sequence in $\mathcal{L}(\alpha)$. We first show that the sequence $\{x_k\}$ is bounded. In order to prove this by contradiction, we suppose that there exists a subset $\mathcal{I} \subset \{0, 1, \ldots\}$ such that $\lim_{k\to\infty, k\in\mathcal{I}} \|x_k\| = \infty$. Without loss of generality, we suppose that $\|x_k\| > 1$ for all $k \in \mathcal{I}$. From the assumption (v), there exists a shifted perturbed KKT point $w^* = (x^*, y^*, Z^*)$. Let $u : \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$u(x_k) \equiv \frac{1}{\|x_k\|} x_k + \left(1 - \frac{1}{\|x_k\|}\right) x^*. \tag{4.7}$$

Then, since $\|x_k\| > 1$, we have

$$\|u(x_k)\| = \left\| \frac{1}{\|x_k\|} x_k + \left(1 - \frac{1}{\|x_k\|}\right) x^* \right\| \le 1 + \left(1 - \frac{1}{\|x_k\|}\right) \|x^*\| < 1 + \|x^*\| \quad \text{for all } k \in \mathcal{I},$$

which implies that the sequence $\{u(x_k)\}_{\mathcal{I}}$ is bounded. Therefore, there exists at least one accumulation point of $\{u(x_k)\}_{\mathcal{I}}$. Let u^* be an accumulation point of $\{u(x_k)\}_{\mathcal{I}}$. Then, there exists a subset $\mathcal{J} \subset \mathcal{I}$ such that $\lim_{k\to\infty, k\in\mathcal{J}} u(x_k) = u^*$. From the definition (4.7) of u, we obtain

$$\|u(x_k) - x^*\| = \frac{\|x_k - x^*\|}{\|x_k\|} \ge \left| \frac{\|x_k\| - \|x^*\|}{\|x_k\|} \right| = \left| 1 - \frac{\|x^*\|}{\|x_k\|} \right| \quad \text{for all } k \in \mathcal{J},$$

and hence

$$\|u^* - x^*\| = \lim_{k\to\infty, k\in\mathcal{J}} \|u(x_k) - x^*\| \ge \lim_{k\to\infty, k\in\mathcal{J}} \left| 1 - \frac{\|x^*\|}{\|x_k\|} \right| = 1. \tag{4.8}$$

Since w^* is the shifted perturbed KKT point, we have from Theorem 2 that

$$0 = \nabla_x L(w^*) = \nabla f(x^*) + \frac{1}{\mu} J_g(x^*)^\top g(x^*) - \frac{1}{\mu} \mathcal{A}^*(x^*) X(x^*)^{-1} = \nabla F_{BP}(x^*). \tag{4.9}$$

Note that F_{BP} is strictly convex from Proposition 3 (c) and the assumptions (i)–(iv). It then follows from (4.9) that x^* is the unique global minimizer of F_{BP}. Thus, (4.8) implies that

$$0 < F_{BP}(u^*) - F_{BP}(x^*). \tag{4.10}$$

Let $\gamma_k \equiv F_{BP}(u(x_k)) - F_{BP}(x^*)$ and $\gamma^* \equiv F_{BP}(u^*) - F_{BP}(x^*)$. Note that γ^* is positive from (4.10) and $\lim_{k\to\infty, k\in\mathcal{J}} \gamma_k = \gamma^*$. Therefore, for any $\varepsilon \in (0, \gamma^*)$, there exists a positive integer k_0 such that $|\gamma_k - \gamma^*| < \varepsilon$ for all $k \in \mathcal{J}$ such that $k \geq k_0$, which yields that

$$0 < \gamma^* - \varepsilon < \gamma_k = F_{BP}(u(x_k)) - F_{BP}(x^*) < \gamma^* + \varepsilon \quad \text{for all } k \in \mathcal{J} \text{ such that } k \geq k_0. \tag{4.11}$$

By the definition of F and Lemma 1 (b), we have

$$F_{BP}(x_k) + \nu d\mu(1 - \log \mu) \leq F_{BP}(x_k) + \nu F_{PD}(w_k) = F(w_k) \leq \alpha \quad \text{for all } k \in \mathcal{J},$$

which implies that

$$F_{BP}(x_k) = f(x_k) + \frac{1}{2\mu}\|g(x_k)\|^2 - \mu \log \det X(x_k) \leq \beta \quad \text{for all } k \in \mathcal{J}, \tag{4.12}$$

where $\beta \equiv \alpha - \nu d\mu(1 - \log \mu)$. From the convexity of F_{BP}, we obtain

$$F_{BP}(u(x_k)) \leq \frac{1}{\|x_k\|} F_{BP}(x_k) + \left(1 - \frac{1}{\|x_k\|}\right) F_{BP}(x^*) \quad \text{for all } k \in \mathcal{J},$$

which means that

$$F_{BP}(x^*) + \left(F_{BP}(u(x_k)) - F_{BP}(x^*)\right) \|x_k\| \leq F_{BP}(x_k) \quad \text{for all } k \in \mathcal{J}. \tag{4.13}$$

It then follows from (4.11), (4.12) and (4.13) that

$$F_{BP}(x^*) + (\gamma^* - \varepsilon)\|x_k\| < F_{BP}(x^*) + \gamma_k \|x_k\| \leq \beta \quad \text{for all } k \in \mathcal{J} \text{ such that } k \geq k_0.$$

Rearranging the above inequality, we have

$$\|x_k\| \leq \frac{\beta - F_{BP}(x^*)}{\gamma^* - \varepsilon} < \infty \quad \text{for all } k \in \mathcal{J} \text{ such that } k \geq k_0.$$

The inequality contradicts $\|x_k\| \to \infty$ $(k \to \infty)$. Hence, for any sequence $\{x_k, y_k, Z_k\} \subset \mathcal{L}(\alpha)$, the sequence $\{x_k\}$ is bounded. Since $\{x_k\}$ is bounded and $\{F(w_j)\}$ is bounded above, it follows from Lemma 2 that the sequences $\{y_k\}$ and $\{Z_k\}$ are also bounded. □

Due to Theorems 2–4, we can solve the unconstrained minimization problem (4.3) by any descent method, such as the quasi-Newton method and the steepest descent method, and hence get an approximate shifted perturbed KKT point v_{k+1} in Step 1 of Algorithm 1.

Remark 2 *The level boundedness of the merit function for the nonlinear programming is not given in the original paper [4]. Applying Theorem 4, it is easy to show that the merit function M in [4] is level bounded if the objective function f is convex, the constraint functions c_i $(i \in \mathcal{E})$ are affine, and* $\mathrm{rank}(J_c) = n$.

Remark 3 *Kato, Yabe and Yamashita [10] showed that their merit function $\widetilde{F}$ is differentiable and its stationary point is shifted perturbed KKT point. However, they did not present the level boundedness of their merit function.*

4.2 Newton algorithm for minimization of the merit function

In this subsection, we propose a Newton type method for the unconstrained minimization problem (4.3) of the merit function F.

We exploit the scaling of $X(x)$ and Z discussed in Subsection 2.1. Let $T \in \mathbf{R}^{d\times d}$ be a nonsingular matrix such that

$$TX(x)T^\top T^{-\top} Z T^{-1} = T^{-\top} Z T^{-1} T X(x) T^\top. \tag{4.14}$$

Let $\widetilde{X}(x)$ and $\widetilde{Z}$ be defined by

$$\begin{aligned} \widetilde{X}(x) &= TX(x)T^\top = (T \odot T)X(x), \\ \widetilde{Z} &= T^{-\top} Z T^{-1} = (T^{-\top} \odot T^{-\top})Z, \end{aligned}$$

respectively. Note that $\widetilde{X}(x)$ and $\widetilde{Z}$ commute, that is, $\widetilde{X}(x)\widetilde{Z} = \widetilde{Z}\widetilde{X}(x)$ from (4.14). As seen later, the scaling enables us to analyze and calculate a Newton direction easily. In the subsequent discussions, for simplicity, we denote $X(x)$ and $\widetilde{X}(x)$ by X and $\widetilde{X}$, respectively.

Next, we give a Newton direction, and show that it is descent direction for the merit function F. The Newton direction is derived from the nonlinear equations $r(w;\mu) = 0$ in the shifted perturbed KKT conditions (2.13). However, the matrix ΔZ of a pure Newton direction $(\Delta x, \Delta y, \Delta Z)$ for $r(w;\mu) = 0$ is not necessarily symmetric due to $XZ - \mu I = 0$. Thus, we consider the following symmetrized shifted perturbed KKT conditions with the scaling.

$$r_S(w;\mu) \equiv \begin{bmatrix} \nabla_x L(w) \\ g(x) + \mu y \\ \widetilde{X} \circ \widetilde{Z} - \mu I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.15}$$

and

$$\widetilde{X} \succ 0, \qquad \widetilde{Z} \succ 0.$$

Note that $\widetilde{X} \circ \widetilde{Z} - \mu I = 0$ is equivalent to $XZ - \mu I = 0$ if X and Z are symmetric positive semidefinite [26]. Moreover, $\widetilde{X}(x) \succ 0$ and $\widetilde{Z} \succ 0$ if and only if $X(x) \succ 0$ and $Z \succ 0$. Therefore, the symmetrized shifted perturbed KKT conditions (4.15) are essentially same as the original shifted perturbed KKT conditions (2.13).

We apply the Newton method to the equation (4.15). Before we give a concrete Newton equations, we provide a first order approximation of $\widetilde{X} \circ \widetilde{Z} - \mu I$ at $(x + \Delta x, Z + \Delta Z)$. From Proposition 2 (a), it is written as

$$\widetilde{X} \circ \widetilde{Z} - \mu I + \frac{1}{2}(\widetilde{Z} \odot I)(T \odot T)\mathcal{A}(x)\Delta x + \frac{1}{2}(\widetilde{X} \odot I)(T^{-\top} \odot T^{-\top})\Delta Z. \tag{4.16}$$

Let ΔX be defined as

$$\Delta X \equiv \sum_{i=1}^{n} \Delta x_i A_i(x) = \mathcal{A}(x)\Delta x,$$

and let $\Delta\widetilde{X}$ and $\Delta\widetilde{Z}$ be the scaled matrices of ΔX and ΔZ with T, that is,

$$\Delta\widetilde{X} \equiv T\Delta X T^\top = (T \odot T)\Delta X \quad \text{and} \quad \Delta\widetilde{Z} \equiv T^{-\top}\Delta Z T^{-1} = (T^{-\top} \odot T^{-\top})\Delta Z,$$

respectively. Then, the first order approximation (4.16) can be written as

$$\begin{aligned}&\widetilde{X}\circ\widetilde{Z}-\mu I+\frac{1}{2}(\widetilde{Z}\odot I)\Delta\widetilde{X}+\frac{1}{2}(\widetilde{X}\odot I)\Delta\widetilde{Z}\\&=\frac{1}{2}(\widetilde{X}\widetilde{Z}+\widetilde{Z}\widetilde{X})-\mu I+\frac{1}{2}(\widetilde{Z}\Delta\widetilde{X}+\Delta\widetilde{X}\widetilde{Z})+\frac{1}{2}(\widetilde{X}\Delta\widetilde{Z}+\Delta\widetilde{Z}\widetilde{X})\end{aligned}$$

Consequently the Newton equations for the nonlinear equation (4.15) are written as

$$G\Delta x-J_g(x)^\top\Delta y-\mathcal{A}^*(x)\Delta Z=-\nabla_x L(w), \tag{4.17}$$

$$J_g(x)\Delta x+\mu\Delta y=-g(x)-\mu y, \tag{4.18}$$

$$\widetilde{Z}\Delta\widetilde{X}+\Delta\widetilde{X}\widetilde{Z}+\widetilde{X}\Delta\widetilde{Z}+\Delta\widetilde{Z}\widetilde{X}=2\mu I-\widetilde{X}\widetilde{Z}-\widetilde{Z}\widetilde{X}, \tag{4.19}$$

where G denotes a Hessian matrix of the Lagrangian function L with respect to x or its approximation. In what follows, we call the solution $\Delta w\equiv(\Delta x,\Delta y,\Delta Z)$ of the Newton equations (4.17)–(4.19) the *Newton direction*.

Next, we give the explicit form of the Newton direction Δw. From (4.18), we have

$$\Delta y=-\frac{1}{\mu}(g(x)+\mu y+J_g(x)\Delta x). \tag{4.20}$$

Moreover, since $(\widetilde{X}\odot I)\Delta\widetilde{Z}=\frac{1}{2}(\widetilde{X}\Delta\widetilde{Z}+\Delta\widetilde{Z}\widetilde{X}),(\widetilde{X}\odot I)(\mu\widetilde{X}^{-1}-\widetilde{Z})=\frac{1}{2}(2\mu I-\widetilde{Z}\widetilde{X}-\widetilde{X}\widetilde{Z})$ and $(\widetilde{Z}\odot I)\Delta\widetilde{X}=\frac{1}{2}(\widetilde{Z}\Delta\widetilde{X}+\Delta\widetilde{X}\widetilde{Z})$, the equation (4.19) can be rewritten as

$$(\widetilde{X}\odot I)\Delta\widetilde{Z}+(\widetilde{Z}\odot I)\Delta\widetilde{X}=(\widetilde{X}\odot I)(\mu\widetilde{X}^{-1}-\widetilde{Z}) \tag{4.21}$$

Since the matrix X is positive definite and the scaling matrix T is nonsingular, the matrix $\widetilde{X}=TXT^\top$ is also positive definite. Therefore, the operator $(\widetilde{X}\odot I)$ is invertible from Proposition 1 (a). Moreover, $\widetilde{X}^{-1}=(TXT^\top)^{-1}=T^{-\top}X^{-1}T^{-1}=(T^{-\top}\odot T^{-\top})X^{-1}$. It then follows from (4.21) that

$$\begin{aligned}\Delta\widetilde{Z}&=(\mu\widetilde{X}^{-1}-\widetilde{Z})-(\widetilde{X}\odot I)^{-1}(\widetilde{Z}\odot I)\Delta\widetilde{X}\\&=(T^{-\top}\odot T^{-\top})(\mu X^{-1}-Z)-(\widetilde{X}\odot I)^{-1}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x)\Delta x,\end{aligned} \tag{4.22}$$

where the last equality follows from the definition of the scale matrices $\widetilde{Z}$ and $\Delta\widetilde{X}$. Since $\Delta\widetilde{Z}=(T^{-\top}\odot T^{-\top})\Delta Z$ and $(T^{-\top}\odot T^{-\top})^{-1}=(T^\top\odot T^\top)$ from Proposition 1 (c), multiplying both side of (4.22) by $(T^{-\top}\odot T^{-\top})^{-1}$ yields

$$\Delta Z=\mu X^{-1}-Z-(T^\top\odot T^\top)(\widetilde{X}\odot I)^{-1}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x)\Delta x. \tag{4.23}$$

Finally, we give the concrete form of Δx. Substituting (4.20) and (4.23) into (4.17), we obtain

$$\begin{aligned}\left(G+H+\frac{1}{\mu}J_g(x)^\top J_g(x)\right)\Delta x&=-\nabla_x L(w)-\frac{1}{\mu}J_g(x)^\top(g(x)+\mu y)+\mathcal{A}^*(x)(\mu X^{-1}-Z)\\&=-\left(\nabla f(x)+\frac{1}{\mu}J_g(x)^\top g(x)-\mu\mathcal{A}^*(x)X^{-1}\right),\end{aligned} \tag{4.24}$$

where

$$H=\mathcal{A}^*(x)(T^\top\odot T^\top)(\widetilde{X}\odot I)^{-1}(\widetilde{Z}\odot I)(T\odot T)\mathcal{A}(x).$$

Note that H is the linear operator from $\mathbf{R}^n$ to $\mathbf{R}^n$, and

$$Hu = \mathcal{A}^*(x)(T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)\mathcal{A}(x)u \quad \text{for all } u \in \mathbf{R}^n.$$

From the definitions of $\mathcal{A}(x)$ and $\mathcal{A}^*(x)$, the linear operator H is regarded as the matrix whose (i,j)-th element is written as

$$H_{ij} = \left\langle A_i(x), (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_j(x) \right\rangle. \tag{4.25}$$

Since $J_g(x)^\top J_g(x)$ is positive semidefinite, we can solve the linear equation (4.24) with respect to Δx if $G + H$ is positive definite. Fortunately, H is positive semidefinite as shown below.

Lemma 3 *Suppose that X and Z are symmetric positive definite. Then, H is symmetric positive semidefinite. Furthermore, if $A_1(x), \ldots, A_n(x)$ are linearly independent for all $x \in \mathbf{R}^n$, then H is symmetric positive definite.*

Proof. Since $\widetilde{X}$ is positive definite, the operator $\widetilde{X} \odot I$ is invertible from Proposition 1 (a). Let $u \in \mathbf{R}^n$ and $V = (\widetilde{X} \odot I)^{-1}(T \odot T)\mathcal{A}(x)u$. Then, we have

$$\begin{aligned}
\langle Hu, u \rangle &= \left\langle \mathcal{A}^*(x)(T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)\mathcal{A}(x)u, u \right\rangle \\
&= \left\langle (\widetilde{Z} \odot I)(T \odot T)\mathcal{A}(x)u, (\widetilde{X} \odot I)^{-1}(T \odot T)\mathcal{A}(x)u \right\rangle \\
&= \left\langle (\widetilde{Z} \odot I)(\widetilde{X} \odot I)(\widetilde{X} \odot I)^{-1}(T \odot T)\mathcal{A}(x)u, (\widetilde{X} \odot I)^{-1}(T \odot T)\mathcal{A}(x)u \right\rangle \\
&= \left\langle (\widetilde{Z} \odot I)(\widetilde{X} \odot I)V, V \right\rangle \\
&\geq 0,
\end{aligned} \tag{4.26}$$

where the second equality follows from Proposition 1 (b) and the last inequality follows from Proposition 2 (b). Therefore, H is positive semidefinite.

Next we show that H is symmetric. From (4.25), we have

$$\begin{aligned}
H_{ij} &= \left\langle A_i(x), (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_j(x) \right\rangle \\
&= \operatorname{tr}(A_i(x)(T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_j(x)) \\
&= \operatorname{tr}((T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_j(x)A_i(x)) \\
&= \left\langle (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_j(x), A_i(x) \right\rangle \\
&= \left\langle (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(\widetilde{X} \odot I)(\widetilde{X} \odot I)^{-1}(T \odot T)A_j(x), A_i(x) \right\rangle \\
&= \left\langle (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{X} \odot I)(\widetilde{Z} \odot I)(\widetilde{X} \odot I)^{-1}(T \odot T)A_j(x), A_i(x) \right\rangle \\
&= \left\langle (T^\top \odot T^\top)(\widetilde{Z} \odot I)(\widetilde{X} \odot I)^{-1}(T \odot T)A_j(x), A_i(x) \right\rangle \\
&= \left\langle A_j(x), (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)A_i(x) \right\rangle \\
&= H_{ji},
\end{aligned}$$

where the sixth equality follows from Proposition 2 (c) and the eighth equality follows from Proposition 1 (b).

Furthermore, suppose that $A_1(x), \ldots, A_n(x)$ are linearly independent for all $x \in \mathbf{R}^n$ and $u \neq 0$. Then, we have $V = (\widetilde{X} \odot I)^{-1}(T \odot T)\mathcal{A}(x)u \neq 0$. It follows from Proposition 2 (b) and (4.26) that $\langle Hu, u\rangle > 0$, i.e., H is positive definite. □

Remark 4 *In the case of the linear SDP, $A_1(x), \ldots, A_n(x)$ are usually supposed to be linearly independent for $x \in \mathbf{R}^n$. Then, H is positive definite from Lemma 3.*

To sum up the above discussion, we give the concrete formulae of the Newton direction Δw in the following theorem.

Theorem 5 *Let $\mu > 0$ and $w = (x, y, Z) \in \mathcal{W}$. Suppose that the matrix $G + H$ is positive definite. Then, the Newton equations (4.17)–(4.19) have the unique solution $\Delta w = (\Delta x, \Delta y, \Delta Z)$ such that*

$$\begin{aligned}
\Delta x &= -\left(G + H + \frac{1}{\mu}J_g(x)^\top J_g(x)\right)^{-1}\left(\nabla f(x) + \frac{1}{\mu}J_g(x)^\top g(x) - \mu\mathcal{A}^*(x)X^{-1}\right), \quad (4.27)\\
\Delta y &= -\frac{1}{\mu}(g(x) + \mu y + J_g(x)\Delta x),\\
\Delta Z &= \mu X^{-1} - Z - (T^\top \odot T^\top)(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)(T \odot T)\mathcal{A}(x)\Delta x.
\end{aligned}$$

Proof. It is clear that $\frac{1}{\mu}J_g(x)^\top J_g(x)$ is positive semidefinite. Thus, the positive definiteness of $G + H$ and (4.24) yield that

$$\Delta x = -\left(G + H + \frac{1}{\mu}J_g(x)^\top J_g(x)\right)^{-1}\left(\nabla f(x) + \frac{1}{\mu}J_g(x)^\top g(x) - \mu\mathcal{A}^*(x)X^{-1}\right).$$

Furthermore, Δy and ΔZ directly follow from (4.20) and (4.23), respectively. □

One of the main burdens on the computations of the Newton direction Δw is the calculation of the operator $(\widetilde{X} \odot I)^{-1}$ in (4.23) and (4.25). Note that $(\widetilde{X} \odot I)^{-1}$ in (4.23) and (4.25) appears as $(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)$. Hence, when $\widetilde{X} = I$, it is clear that $(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I) = \widetilde{Z} \odot I$. On the other hand, when $\widetilde{X} = \widetilde{Z}$, $(\widetilde{X} \odot I)^{-1}(\widetilde{Z} \odot I)$ is the identity mapping. Thus, if we choose the scaling matrix T such that $\widetilde{X} = I$ or $\widetilde{X} = \widetilde{Z}$, we do not have to explicitly handle the operator $(\widetilde{X} \odot I)^{-1}$. This is one of the reasons why we exploit the scaling. Note that the choices of T such that $\widetilde{X} = I$ or $\widetilde{X} = \widetilde{Z}$ is well-known as HRVW/KSH/M direction or NT direction.

(i) HRVW/KSH/M choice

Let $T = X^{-\frac{1}{2}}$. Then we have $\widetilde{X} = I$ and $\widetilde{Z} = X^{\frac{1}{2}}ZX^{\frac{1}{2}}$. This choice corresponds to the dual HRVW/KSH/M choice for the linear SDP [6, 12, 13].

(ii) NT choice

Let $T = W^{-\frac{1}{2}}$, where $W = X^{\frac{1}{2}}(X^{\frac{1}{2}}ZX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}$. Then we have $\widetilde{X} = W^{-\frac{1}{2}}XW^{-\frac{1}{2}} = W^{\frac{1}{2}}ZW^{\frac{1}{2}} = \widetilde{Z}$. This choice corresponds to the NT choice for the linear SDP [14, 15].

Next, we show that the Newton direction is a descent direction for the merit function F. For this purpose, we first show the following two lemmas.

Lemma 4 *Let $\mu > 0$ and $w = (x, y, Z) \in \mathcal{W}$. Suppose that $G + H$ is positive definite. Let Δx be given by (4.27). Then we have*

$$\nabla F_{BP}(x)^\top \Delta x = -\Delta x^\top \left(G + H + \frac{1}{\mu} J_g(x)^\top J_g(x) \right) \Delta x \le 0.$$

Furthermore, $\nabla F_{BP}(x)^\top \Delta x = 0$ if and only if $\Delta x = 0$.

Proof. We easily see that $G + H + \frac{1}{\mu} J_g(x)^\top J_g(x)$ is positive definite from the positive definiteness of $G + H$. Since $\nabla F_{BP}(x) = \nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X^{-1}$ from Theorem 2, it then follows from (4.24) that

$$\begin{aligned}
\nabla F_{BP}(x)^\top \Delta x &= \Delta x^\top \left(\nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X^{-1} \right) \\
&= -\Delta x^\top \left(G + H + \frac{1}{\mu} J_g(x)^\top J_g(x) \right) \Delta x \\
&\le 0.
\end{aligned}$$

Furthermore, since $G + H + \frac{1}{\mu} J_g(x)^\top J_g(x)$ is positive definite, $\nabla F_{BP}(x)^\top \Delta x = 0$ if and only if $\Delta x = 0$. □

Lemma 5 *Let $\mu > 0$ and $w = (x, y, Z) \in \mathcal{W}$. Let $\Delta w = (\Delta x, \Delta y, \Delta Z)$ be given in Theorem 5. Then we have*

$$\langle \nabla F_{PD}(w), \Delta w \rangle = -\frac{1}{\mu} \| g(x) + \mu y \|^2 - \| (\widetilde{X} \widetilde{Z})^{-\frac{1}{2}} (\mu I - \widetilde{X} \widetilde{Z}) \|_F^2 \le 0.$$

Furthermore, $\langle \nabla F_{PD}(w), \Delta w \rangle = 0$ if and only if $g(x) + \mu y = 0$ and $XZ - \mu I = 0$.

Proof. From Theorem 2, we obtain

$$\begin{aligned}
\langle \nabla F_{PD}(w), \Delta w \rangle &= \langle \nabla_x F_{PD}(w), \Delta x \rangle + \langle \nabla_y F_{PD}(w), \Delta y \rangle + \langle \nabla_Z F_{PD}(w), \Delta Z \rangle \\
&= \frac{1}{\mu} \Delta x^\top J_g(x)^\top (g(x) + \mu y) + \Delta x^\top \mathcal{A}^*(x)(Z - \mu X^{-1}) \\
&\quad + (g(x) + \mu y)^\top \Delta y + \left\langle X - \mu Z^{-1}, \Delta Z \right\rangle .
\end{aligned} \tag{4.28}$$

On the other hand, we have from the definitions of $\mathcal{A}^*(x)$ and ΔX that

$$\begin{aligned}
\Delta x^\top \mathcal{A}^*(x)(Z - \mu X^{-1}) &= \sum_{i=1}^n \Delta x_i \left\langle A_i(x), Z - \mu X^{-1} \right\rangle \\
&= \left\langle \sum_{i=1}^n \Delta x_i A_i(x), Z - \mu X^{-1} \right\rangle \\
&= \left\langle \mathcal{A}(x) \Delta x, Z - \mu X^{-1} \right\rangle \\
&= \left\langle \Delta X, Z - \mu X^{-1} \right\rangle .
\end{aligned} \tag{4.29}$$

From Proposition 1 (c) and (b), we have

$$\begin{aligned}\left\langle \Delta X, Z-\mu X^{-1}\right\rangle &= \left\langle (T\odot T)^{-1}(T\odot T)\Delta X, Z-\mu X^{-1}\right\rangle \\ &= \left\langle (T^{-1}\odot T^{-1})(T\odot T)\Delta X, Z-\mu X^{-1}\right\rangle \\ &= \left\langle (T\odot T)\Delta X, (T^{-\top}\odot T^{-\top})(Z-\mu X^{-1})\right\rangle .\end{aligned}$$

Moreover since $\widetilde{X}^{-1} = ((T\odot T)X)^{-1} = (TXT^\top)^{-1} = T^{-\top}X^{-1}T^{-1} = (T^{-\top}\odot T^{-\top})X^{-1}$, we further have

$$\begin{aligned}\left\langle \Delta X, Z-\mu X^{-1}\right\rangle &= \left\langle \Delta\widetilde{X}, \widetilde{Z}-\mu\widetilde{X}^{-1}\right\rangle \\ &= \left\langle \Delta\widetilde{X}, (I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\widetilde{Z}\right\rangle \\ &= \mathrm{tr}(\Delta\widetilde{X}(I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\widetilde{Z}) \\ &= \mathrm{tr}((I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\widetilde{Z}\Delta\widetilde{X}). \end{aligned} \tag{4.30}$$

Since $\widetilde{X}$ and $\widetilde{Z}$ commute, $\widetilde{X}^{-1}$ and $\widetilde{Z}^{-1}$ also commute. Then we get

$$\begin{aligned}\mathrm{tr}((I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\widetilde{Z}\Delta\widetilde{X}) &= \mathrm{tr}(\widetilde{Z}(I-\mu\widetilde{Z}^{-1}\widetilde{X}^{-1})\Delta\widetilde{X}) \\ &= \mathrm{tr}(\widetilde{Z}(I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\Delta\widetilde{X}) \\ &= \mathrm{tr}((I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\Delta\widetilde{X}\widetilde{Z}). \end{aligned} \tag{4.31}$$

From (4.30) and (4.31), we obtain

$$\begin{aligned}\left\langle \Delta X, Z-\mu X^{-1}\right\rangle &= \frac{1}{2}\mathrm{tr}((I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\widetilde{Z}\Delta\widetilde{X}) + \frac{1}{2}\mathrm{tr}((I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1})\Delta\widetilde{X}\widetilde{Z}) \\ &= \frac{1}{2}\left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \widetilde{Z}\Delta\widetilde{X}\right\rangle + \frac{1}{2}\left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \Delta\widetilde{X}\widetilde{Z}\right\rangle . \end{aligned} \tag{4.32}$$

Note that $\langle X-\mu Z^{-1}, \Delta Z\rangle = \langle \Delta Z, X-\mu Z^{-1}\rangle$. In a way similar to prove (4.32), we also have

$$\left\langle X-\mu Z^{-1}, \Delta Z\right\rangle = \frac{1}{2}\left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \widetilde{X}\Delta\widetilde{Z}\right\rangle + \frac{1}{2}\left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \Delta\widetilde{Z}\widetilde{X}\right\rangle . \tag{4.33}$$

From (4.28), (4.29), (4.32) and (4.33), we obtain

$$\begin{aligned}\langle \nabla F_{PD}(w), \Delta w\rangle &= \frac{1}{\mu}(g(x)+\mu y)^\top(J_g(x)\Delta x + \mu\Delta y) \\ &\quad + \frac{1}{2}\left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \widetilde{Z}\Delta\widetilde{X} + \Delta\widetilde{X}\widetilde{Z} + \widetilde{X}\Delta\widetilde{Z} + \Delta\widetilde{Z}\widetilde{X}\right\rangle . \end{aligned} \tag{4.34}$$

Note that since $\widetilde{X}$ and $\widetilde{Z}$ are symmetric positive definite and commute, $\widetilde{X}\widetilde{Z}$ is symmetric positive definite, and hence there exists $(\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}$. Then, by substituting (4.18) and (4.19) into (4.34), we have

$$\begin{aligned}\langle \nabla F_{PD}(w), \Delta w\rangle &= -\frac{1}{\mu}\|g(x)+\mu y\|^2 + \left\langle I-\mu\widetilde{X}^{-1}\widetilde{Z}^{-1}, \mu I - \widetilde{X}\widetilde{Z}\right\rangle \\ &= -\frac{1}{\mu}\|g(x)+\mu y\|^2 - \left\langle (\widetilde{Z}\widetilde{X})^{-1}(\mu I - \widetilde{Z}\widetilde{X}), \mu I - \widetilde{X}\widetilde{Z}\right\rangle \\ &= -\frac{1}{\mu}\|g(x)+\mu y\|^2 - \left\langle (\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}(\mu I - \widetilde{X}\widetilde{Z}), (\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}(\mu I - \widetilde{X}\widetilde{Z})\right\rangle \\ &= -\frac{1}{\mu}\|g(x)+\mu y\|^2 - \|(\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}(\mu I - \widetilde{X}\widetilde{Z})\|_F^2 \\ &\le 0, \end{aligned}$$

where the third equality follows from the commutativity of $\widetilde{X}$ and $\widetilde{Z}$. Moreover, it is clear that $\langle \nabla F_{PD}(w), \Delta w \rangle = 0$ if and only if $g(x) + \mu y = 0$ and $XZ - \mu I = 0$. □

Now, we show that the Newton direction Δw is the descent direction for the merit function F.

Theorem 6 *Let $\mu > 0$ and $w = (x, y, Z) \in \mathcal{W}$. Assume that $G + H$ is positive definite. Then, $\Delta w = (\Delta x, \Delta y, \Delta Z)$ be given in Theorem 5 is a descent direction for the merit function F, i.e.,*

$$\begin{aligned} \langle \nabla F(w), \Delta w \rangle &= -\Delta x^\top \left(G + H + \frac{1}{\mu} J_g(x)^\top J_g(x) \right) \Delta x \\ &\quad - \frac{\nu}{\mu} \|g(x) + \mu y\|^2 - \nu \|(\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}(\mu I - \widetilde{X}\widetilde{Z})\|_F^2 \\ &\leq 0. \end{aligned}$$

Furthermore, $\langle \nabla F(w), \Delta w \rangle = 0$ if and only if w is a shifted perturbed KKT point.

Proof. From Lemmas 4 and 5, we have

$$\begin{aligned} \langle \nabla F(w), \Delta w \rangle &= \nabla F_{BP}(x)^\top \Delta x + \nu \langle \nabla F_{PD}(w), \Delta w \rangle \\ &= -\Delta x^\top \left(G + H + \frac{1}{\mu} J_g(x)^\top J_g(x) \right) \Delta x \\ &\quad - \frac{\nu}{\mu} \|g(x) + \mu y\|^2 - \nu \|(\widetilde{X}\widetilde{Z})^{-\frac{1}{2}}(\mu I - \widetilde{X}\widetilde{Z})\|_F^2 \\ &\leq 0. \end{aligned} \tag{4.35}$$

Now, we show the second part of this theorem. Suppose that w is a shifted perturbed KKT point, i.e., $\nabla f(x) - J_g(x)^\top y - \mathcal{A}^*(x)Z = 0, g(x) + \mu y = 0$ and $XZ - \mu I = 0$. Then we have

$$\nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X^{-1} = \nabla f(x) - J_g(x)^\top y - \mathcal{A}^*(x)Z = 0.$$

It then follows from (4.27) that $\Delta x = 0$. Moreover, we have $\langle \nabla F(w), \Delta w \rangle = 0$ from (4.35).

Conversely, suppose that $\langle \nabla F(w), \Delta w \rangle = 0$. Since it follows from Lemmas 4 and 5 that $\nabla F_{BP}(x)^\top \Delta x \leq 0$ and $\langle \nabla F_{PD}(w), \Delta w \rangle \leq 0$, we have $\nabla F_{BP}(x)^\top \Delta x = 0$ and $\langle \nabla F_{PD}(w), \Delta w \rangle = 0$. It further follows from Lemmas 4 and 5 that $\Delta x = 0, g(x) + \mu y = 0$ and $XZ - \mu I = 0$. Then we have from (4.27) that

$$\nabla_x L(w) = \nabla f(x) - J_g(x)^\top y - \mathcal{A}^*(x)Z = \nabla f(x) + \frac{1}{\mu} J_g(x)^\top g(x) - \mu \mathcal{A}^*(x) X^{-1} = 0.$$

Thus, w is a shifted perturbed KKT point. □

Theorem 6 guarantees that $F(w + \alpha \Delta w) < F(w)$ for sufficiently small $\alpha > 0$ if w is not a shifted perturbed KKT point.

Now, we discuss how to choose an appropriate step size α such that $F(w + \alpha\Delta w) < F(w)$. Since the merit function F and the Newton equations (4.17)–(4.19) are well-defined only on $\mathcal{W}$. Therefore, the new point $w + \alpha\Delta w$ is required to be an interior point. Thus, we must choose the step size $\alpha \in (0, 1]$ such that $X(x + \alpha\Delta x) \succ 0$ and $Z + \alpha\Delta Z \succ 0$. To this end, we first calculate

$$\bar{\alpha}_x = \begin{cases} -\frac{\tau}{\lambda_{\min}(X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}})} & \text{if } \lambda_{\min}(X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}) < 0 \\ 1 & \text{otherwise} \end{cases}$$

and

$$\bar{\alpha}_z = \begin{cases} -\frac{\tau}{\lambda_{\min}(Z^{-\frac{1}{2}}\Delta Z Z^{-\frac{1}{2}})} & \text{if } \lambda_{\min}(Z^{-\frac{1}{2}}\Delta Z Z^{-\frac{1}{2}}) < 0 \\ 1 & \text{otherwise,} \end{cases}$$

where $\tau \in (0, 1)$ is a given constant. Set

$$\bar{\alpha} = \min\{1, \bar{\alpha}_x, \bar{\alpha}_z\}. \tag{4.36}$$

Then $Z + \alpha\Delta Z \succ 0$ for any $\alpha \in (0, \bar{\alpha}]$. Moreover, $X(x + \alpha\Delta x) \succ 0$ for any $\alpha \in (0, \bar{\alpha}]$ if X is linear. Note that if X is nonlinear, $X(x + \alpha\Delta x)$ is not necessarily positive definite for any $\alpha \in (0, \bar{\alpha}]$.

Next we choose a step size $\alpha \in (0, \bar{\alpha}]$ such that $F(w + \alpha\Delta w) < F(w)$ and $X(x + \alpha\Delta x) \succ 0$. For this purpose, we adopt the following Armijo's line search rule: Find the smallest nonnegative integer l such that

$$F(w + \bar{\alpha}\beta^l\Delta w) \le F(w) + \varepsilon_0\bar{\alpha}\beta^l \langle \nabla F(w), \Delta w \rangle,$$
$$X(x + \bar{\alpha}\beta^l\Delta x) \succ 0$$

and set $\alpha = \bar{\alpha}\beta^l$, where $\beta, \varepsilon_0 \in (0, 1)$. Note that the second condition is not necessary when X is linear.

Now, we describe a concrete Newton type method for Step 1 of Algorithm 1. Recall that the script k denotes the k-th iteration of Algorithm 1.

Algorithm 2 (for Step 2 of Algorithm 1)

Step 0. Choose $\beta, \varepsilon_0, \tau \in (0, 1)$ and set $j = 0$ and $w_0 = v_k$.

Step 1. If $\rho(w_j, \mu_k) \le \sigma\mu_k$, then set $v_{k+1} = w_j$ and return.

Step 2. Obtain the Newton direction $\Delta w_j = (\Delta x_j, \Delta y_j, \Delta Z_j)$ by solving the Newton equations (4.17)–(4.19).

Step 3. Set $\alpha_j = \bar{\alpha}_j\beta^{l_j}$, where $\bar{\alpha}_j$ is given by (4.36) and l_j is the smallest nonnegative integer such that

$$F(w_j + \bar{\alpha}_j\beta^{l_j}\Delta w_j) \le F(w_j) + \varepsilon_0\bar{\alpha}_j\beta^{l_j} \langle \nabla F(w_j), \Delta w_j \rangle,$$
$$X(x_j + \bar{\alpha}_j\beta^{l_j}\Delta x_j) \succ 0.$$

Step 4. Set $w_{j+1} = w_j + \alpha_j\Delta w_j$ and $j = j + 1$, and go to Step 1.

4.3 Global convergence of Algorithm 2

In this subsection, we prove the global convergence of Algorithm 2. For this purpose, we make the following assumptions.

Assumptions

(A1) The functions $f, g_1, \ldots, g_m$ and X are twice continuously differentiable.

(A2) The sequence $\{x_j\}$ generated by Algorithm 2 remains in some compact set Ω of $\mathbf{R}^n$.

(A3) The matrix $G_j + H_j + \frac{1}{\mu} J_g(x_j)^\top J_g(x_j)$ is uniformly positive definite and the sequence $\{G_j\}$ is bounded.

(A4) The sequences $\{T_j\}$ and $\{T_j^{-1}\}$ are bounded.

Note that Assumption (A2) holds under the assumptions of Theorem 4. Assumption (A3) guarantees that the Newton equations (4.17)–(4.19) have an unique solution.

Remark 5 *Assumptions (A1)–(A3) hold for the linear SDP such that $A_1(x_j), \ldots, A_n(x_j)$ are linearly independent. In fact, it is clear that Assumption (A1) holds. Theorem 4 guarantees that Assumption (A2) holds. Moreover H_j is positive definite from Remark 4 and $G_j = 0$. Thus, Assumption (A3) holds.*

Remark 6 *Yamashita, Yabe and Harada [26] showed that their Newton type algorithm globally converges to a perturbed KKT point satisfying (2.11) and (2.12) under the boundedness of the sequence $\{y_j\}$ in addition to Assumptions (A1)–(A4). However they do not give sufficient conditions for the boundedness of $\{y_j\}$.*

Remark 7 *Kato, Yabe and Yamashita [10] also showed that the Newton type algorithm with the merit function $\widetilde{F}$ can find a shifted perturbed KKT point under the same assumptions. However, there do not give concrete sufficient conditions for Assumption (A2).*

First of all, we show that the sequence $\{w_j\}$ generated by Algorithm 2 is bounded.

Lemma 6 *Suppose that Assumptions (A2) holds. Then, the sequence $\{w_j = (x_j, y_j, Z_j)\}$ generated by Algorithm 2 is bounded. Furthermore, the matrices $\{X_j\}$ and $\{Z_j\}$ are uniformly positive definite.*

Proof. Since the sequence $\{F(w_j)\}$ is monotonically decreasing, we have $F(w_j) \leq F(w_0)$ for all j. It then follows from Assumption (A2) and Lemma 2 that we have the desired results. □

Note that the above lemma guarantees that Assumption (A4) holds if the scaling matrix T is given by HRVW/KSH/M choice or NT choice.

Lemma 7 *Suppose that Assumptions (A2)–(A4) hold. Then, the sequence $\{\Delta w_j\}$ generated by Algorithm 2 is bounded.*

Proof. It follows from Assumptions (A2)–(A4), Lemma 6 and Theorem 5 that the sequence $\{\Delta w_j\}$ generated by Algorithm 2 is bounded. □

We now show the global convergence of Algorithm 2. Here, we suppose that Algorithm 2 generates an infinite sequence and w_j is not shifted perturbed KKT point for all j.

Theorem 7 *Suppose that Assumptions (A1)–(A4) hold. Then, the sequence $\{w_j = (x_j, y_j, Z_j)\}$ generated by Algorithm 2 has an accumulation point $w^* = (x^*, y^*, Z^*)$. Moreover, the accumulation point w^* is a shifted perturbed KKT point.*

Proof. Since the sequence $\{w_j\}$ is bounded from Lemma 6, it has at least one accumulation point w^*.

Next, we prove that w^* is a shifted perturbed KKT point. To this end, we first show that the sequence $\{\bar{\alpha}_j\}$ given in Step 3 of Algorithm 2 is away from zero, that is, there exists a real number $\bar{\alpha}$ such that $0 < \bar{\alpha} \leq \bar{\alpha}_j$ for all j. Note that from Lemmas 6 and 7, the sequences $\{X_j\}$, $\{Z_j\}$, $\{\Delta X_j\}$ and $\{\Delta Z_j\}$ are bounded. Moreover the matrices $\{X_j\}$ and $\{Z_j\}$ are uniformly positive definite. Hence, the sequence $\{\lambda_{\min}(X_j^{-\frac{1}{2}} \Delta X_j X_j^{-\frac{1}{2}})\}$ and $\{\lambda_{\min}(Z_j^{-\frac{1}{2}} \Delta Z_j Z_j^{-\frac{1}{2}})\}$ are also bounded. It then follows from the definition of $\bar{\alpha}_j$ that there exists a real number $\bar{\alpha}$ such that $0 < \bar{\alpha} \leq \bar{\alpha}_j$ for all j.

Next, we show $\langle \nabla F(w_j), \Delta w_j \rangle \to 0$ as $j \to \infty$. From the Armijo's line search strategy in Step 3, we have

$$F(w_{j+1}) - F(w_j) \leq \varepsilon_0 \bar{\alpha}_j \beta^{l_j} \langle \nabla F(w_j), \Delta w_j \rangle,$$
$$X(x_j + \bar{\alpha}_j \beta^{l_j} \Delta x_j) \succ 0.$$

Summing up the above inequality from $j = 1$ to $j = \widetilde{j}$, we have

$$F(w_{\widetilde{j}+1}) - F(w_1) \leq \varepsilon_0 \sum_{j=1}^{\widetilde{j}} \bar{\alpha}_j \beta_j \langle \nabla F(w_j), \Delta w_j \rangle.$$

It then follows from $\langle \nabla F(w_j), \Delta w_j \rangle \leq 0$ from Theorem 6 and $\bar{\alpha} \leq \bar{\alpha}_j$ that

$$F(w_{\widetilde{j}+1}) - F(w_1) \leq \varepsilon_0 \bar{\alpha} \sum_{j=1}^{\widetilde{j}} \beta^{l_j} \langle \nabla F(w_j), \Delta w_j \rangle.$$

Since the sequence $\{w_j\}$ is bounded, the sequence $\{F(w_j)\}$ is also bounded, and hence

$$-\infty < \sum_{j=1}^{\infty} \beta^{l_j} \langle \nabla F(w_j), \Delta w_j \rangle \leq 0.$$

Therefore, we have

$$\lim_{j \to \infty} \beta^{l_j} \langle \nabla F(w_j), \Delta w_j \rangle = 0.$$

Now we consider two cases: $\liminf_{j \to \infty} \beta^{l_j} > 0$ and $\liminf_{j \to \infty} \beta^{l_j} = 0$.

Case 1 : $\liminf_{j\to\infty} \beta^{l_j} > 0$. Then, we have

$$\lim_{j\to\infty} \langle \nabla F(w_j), \Delta w_j \rangle = 0.$$

Case 2 : $\liminf_{j\to\infty} \beta^{l_j} = 0$. In this case, there exists a subset $\mathcal{J} \subset \{0, 1, \cdots\}$ such that $\lim_{j\to\infty, j\in\mathcal{J}} l_j = \infty$. Since $\{X(x_j)\}$ is uniformly positive definite and $\{\Delta x_j\}$ is bounded, there exists $\bar{l}$ such that $X(x_j + \bar{\alpha}_j \beta^l \Delta x_j) \succ 0$ for all $l > \bar{l}$. Therefore, without loss of generality, we suppose that $X(x_j + \bar{\alpha}_j \beta^{l_j - 1} \Delta x_j) \succ 0$ for all $j \in \mathcal{J}$. Furthermore, since $l_j - 1$ does not satisfy the Armijo rule in Step 3, we have

$$\varepsilon_0 t_j \langle \nabla F(w_j), \Delta w_j \rangle < F(w_j + t_j \Delta w_j) - F(w_j),$$

where $t_j \equiv \bar{\alpha}_j \beta^{l_j - 1}$. Let $h(t) \equiv F(w_j + t\Delta w_j)$. It then follows from the mean value theorem for h that there exists $\theta_j \in (0, 1)$ such that

$$\begin{aligned} \varepsilon_0 t_j \langle \nabla F(w_j), \Delta w_j \rangle &< F(w_j + t_j \Delta w_j) - F(w_j) \\ &= h(t_j) - h(0) \\ &= t_j h'(\theta_j t_j) \\ &= t_j \langle \nabla F(w_j + \theta_j t_j \Delta w_j), \Delta w_j \rangle, \end{aligned}$$

which yields that

$$\begin{aligned} 0 < (\varepsilon_0 - 1) \langle \nabla F(w_j), \Delta w_j \rangle &< \langle \nabla F(w_j + \theta_j t_j \Delta w_j) - \nabla F(w_j), \Delta w_j \rangle \\ &\leq \|\nabla F(w_j + \theta_j t_j \Delta w_j) - \nabla F(w_j)\| \|\Delta w_j\|, \end{aligned} \tag{4.37}$$

where the last inequality follows from the Cauchy-Schwarz inequality. Since $\{w_j\}$ and $\{\Delta w_j\}$ are bounded and $\lim_{j\to\infty, j\in\mathcal{J}} t_j = 0$, we have from Assumption (A1)

$$\lim_{j\to\infty, j\in\mathcal{J}} \|\nabla F(w_j + \theta_j t_j \Delta w_j) - \nabla F(w_j)\| = 0.$$

It then follows from (4.37) that

$$\lim_{j\to\infty, j\in\mathcal{J}} \langle \nabla F(w_j), \Delta w_j \rangle = 0.$$

From both cases, we can conclude that

$$\lim_{j\to\infty} \langle \nabla F(w_j), \Delta w_j \rangle = 0. \tag{4.38}$$

From the boundedness of $\{w_j\}$ and Assumptions (A3) and (A4), there exists a subset $\mathcal{K} \subset \{0, 1, \ldots\}$ such that

$$\lim_{j\to\infty, j\in\mathcal{K}} w_j = w^*, \quad \lim_{j\to\infty, j\in\mathcal{K}} G_j = G^*, \quad \lim_{j\to\infty, j\in\mathcal{K}} T_j = T^*.$$

Moreover from (2.1), the sequences $\{T_j \odot T_j\}_{\mathcal{K}}$ and $\{T_j^\top \odot T_j^\top\}_{\mathcal{K}}$ converge to $T^* \odot T^*$ and $(T^*)^\top \odot (T^*)^\top$, respectively. Then we have from (4.25) that

$$\lim_{j\to\infty, j\in\mathcal{K}} H_j = H^*.$$

Note that the matrix $G^* + H^* + \frac{1}{\mu} J_g(x^*)^\top J_g(x^*)$ is positive definite from Assumption (A3). It then follows from (4.27) that the subsequence $\{\Delta x_j\}_\mathcal{K}$ converges to Δx^*, where

$$\Delta x^* = -\left(G^* + H^* + \frac{1}{\mu} J_g(x^*)^\top J_g(x^*)\right)^{-1} \left(\nabla f(x^*) + \frac{1}{\mu} J_g(x^*)^\top g(x^*) - \mu \mathcal{A}^*(x^*) X(x^*)^{-1}\right).$$

Similarly, $\{\Delta y_j\}_\mathcal{K}$ and $\{\Delta Z_j\}_\mathcal{K}$ converge to Δy^* and ΔZ^*, where

$$\begin{aligned} \Delta y^* &= -\frac{1}{\mu}(g(x^*) + \mu y^* + J_g(x^*)\Delta x^*), \\ \Delta Z^* &= \mu X(x^*)^{-1} - Z^* - ((T^*)^\top \odot (T^*)^\top)(\widetilde{X}(x^*) \odot I)^{-1}(\widetilde{Z}^* \odot I)(T^* \odot T^*)\mathcal{A}(x^*)\Delta x^*, \end{aligned}$$

and $\widetilde{Z}^* = ((T^*)^{-\top} \odot (T^*)^{-\top})Z^*$. It then follows from (4.38) that

$$\langle \nabla F(w^*), \Delta w^* \rangle = 0.$$

Then, from Theorem 6, we have

$$\nabla_x L(w^*) = 0, \quad g(x^*) + \mu y^* = 0 \quad \text{and} \quad X(x^*)Z^* - \mu I = 0,$$

which means that w^* is a shifted perturbed KKT point. □

5 Numerical experiments

In this section, we report some numerical experiments for the proposed algorithm (Algorithm 1 with Algorithm 2). We compare the proposed algorithm with the interior point method [26] based on the perturbed KKT conditions. We present the number of iterations and the CPU time of both algorithms. The programs are written in MATLAB R2010a and run on a machine with an Intel Core i7 920 2.67GHz CPU and 3.00GB RAM. The parameter μ_k used in the both algorithms is updated by $\mu_{k+1} = \mu_k/10$ with $\mu_0 = 0.1$. Moreover, we exploit the approximate Hessian G_k updated by the Levenberg-Marquardt type algorithm [26, Remark 3]. We adopt the scaling matrix $T = X^{-\frac{1}{2}}$, and use the following parameters.

$$\begin{gathered} M_c = 3.5, \quad \nu = 1.0, \quad \tau = 0.95, \\ \beta = 0.95, \quad \varepsilon_0 = 0.50. \end{gathered}$$

We solved the following four test problems used in [26] from the initial points indicated in [26].

Gaussian channel capacity problem:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\sum_{i=1}^{n} \log(1 + t_i), \\ \text{subject to} \quad & \frac{1}{n}\sum_{i=1}^{n} X_{ii} \le P, \ X_{ii} \ge 0, \ t_i \ge 0, \\ & \begin{bmatrix} 1 - a_i t_i & \sqrt{r_i} \\ \sqrt{r_i} & a_i X_{ii} + r_i \end{bmatrix} \succeq 0, \ (i = 1, \ldots, n), \end{aligned}$$

where the decision variables are X_{ii} and t_i for $i = 1, \ldots, n$. In the experiment, the constants r_i and a_i for $i = 1, \ldots, n$ are randomly chosen from the interval $[0, 1]$, and P is set to 1. Note that the objective function of the problem is nonconvex and the constraint functions are linear.

Minimization of the minimal eigenvalue problem:

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(\Pi M(q)), \\ \text{subject to} & \mathrm{tr}(\Pi) = 1, \\ & \Pi \succeq 0, \\ & q \in Q, \end{array}$$

where $Q \subset \mathbf{R}^p$, and M is a function from $\mathbf{R}^p$ to $\mathbf{S}^n$, and decision variables are $q \in \mathbf{R}^p$ and $\Pi \in \mathbf{S}^n$. In the experiment, p is set to 2, and the function M is given by $M(q) \equiv q_1 q_2 M_1 + q_1 M_2 + q_2 M_3$, where $M_1, M_2, M_3 \in \mathbf{S}^n$ are given constant matrices whose elements are randomly chosen from the interval $[-1, 1]$. The constraint region Q is set to $[-1, 1] \times [-1, 1]$. Note that the objective function is nonconvex and the constraint functions are linear.

Nearest correlation matrix problem:

$$\begin{array}{ll} \underset{X \in \mathbf{S}^n}{\text{minimize}} & \frac{1}{2} \|X - A\|_F^2, \\ \text{subject to} & X \succeq \epsilon I, \\ & X_{ii} = 1, \ (i = 1, \ldots, n), \end{array}$$

where $A \in \mathbf{S}^n$ is a given constant matrix, and $\epsilon \in \mathbf{R}$ is a given constant. Note that $X \succeq \epsilon I$ is equivalent to $X - \epsilon I \succeq 0$. In the experiment, elements of the matrix A are randomly chosen from the interval $[-1, 1]$ with $A_{ii} = 1$ for $i = 1, \ldots, n$. Moreover, we set $\epsilon = 10^{-3}$. Note that the objective function is quadratic and the constraint functions are linear. Therefore, the problem is convex.

Static output feedback (SOF) problem:

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}(X), \\ \text{subject to} & P \succeq 0, \\ & F(Q)P + PF(Q)^\top + DD^\top \preceq 0, \\ & \begin{bmatrix} X & G(Q)P \\ PG(Q)^\top & P \end{bmatrix} \succeq 0, \end{array}$$

where $X \in \mathbf{S}^{n_z \times n_z}, P \in \mathbf{S}^{n_x \times n_x}$ and $Q \in \mathbf{R}^{n_u \times n_y}$ are decision variables, and the functions F and G are defined by

$$F(Q) = A + MQC \quad \text{and} \quad G(Q) = B + NQC.$$

Moreover, the matrices $A \in \mathbf{R}^{n_x \times n_x}, B \in \mathbf{R}^{n_z \times n_x}, C \in \mathbf{R}^{n_y \times n_x}, D \in \mathbf{R}^{n_x \times n_w}, M \in \mathbf{R}^{n_x \times n_u}$ and $N \in \mathbf{R}^{n_z \times n_u}$ are given constant matrices, and the elements of these matrices are randomly

Table 1: Gaussian channel capacity problem

	Algorithm 1		SDPIP	
n	iteration	time(s)	iteration	time(s)
5	19	0.37	19	0.42
10	17	1.82	17	1.78
15	22	9.52	21	8.41
20	22	28.82	21	28.03
25	39	129.14	36	130.51
30	29	196.47	24	181.20
35	31	443.46	27	388.13
40	32	848.94	27	785.54

Table 2: Minimization of the minimal eigenvalue problem

	Algorithm 1		SDPIP	
n	iteration	time(s)	iteration	time(s)
5	6	0.23	9	0.28
10	7	1.16	10	1.60
15	7	7.19	10	10.09
20	8	39.03	10	46.88
25	8	108.23	11	162.18
30	8	241.76	14	443.60
35	8	560.41	16	1161.47
40	10	1289.72	16	2092.33

Table 3: Nearest correlation matrix problem

	Algorithm 1		SDPIP	
n	iteration	time(s)	iteration	time(s)
5	8	0.13	9	0.15
10	8	1.52	10	1.79
15	10	10.33	11	11.02
20	11	37.47	12	40.68
25	10	151.93	11	180.84
30	9	307.40	10	328.88
35	11	875.31	11	872.60
40	11	1503.82	11	1461.04

Table 4: SOF-$\mathcal{H}_2$ problem

		Algorithm 1		SDPIP	
Problem	n	iteration	time(s)	iteration	time(s)
AC1	27	191	6.14	191	6.10
AC2	39	142	9.32	142	9.28
AC3	38	162	10.30	162	10.19
AC6	64	182	51.16	182	52.25
AC17	22	11	0.27	11	0.28
HE1	15	12	0.19	12	0.19
HE2	24	22	0.60	22	0.60
HE3	115	245	223.74	245	223.52
REA1	26	98	2.88	98	2.90
DIS1	88	257	127.47	257	127.95
DIS2	16	10	0.21	10	0.17
DIS3	58	99	14.85	99	15.41
DIS4	66	16	3.05	16	3.34
AC4	13	54	0.80	54	0.78
BDT1	96	145	96.54	145	102.84
MFP	26	167	4.90	167	4.91
EB1	59	9	2.24	9	2.37
NN15	20	13	0.27	13	0.28
PSM	49	87	11.29	87	11.27

chosen from the interval $[0, 1]$. Since the objective function is linear and the constraint functions are nonconvex, the problem is nonconvex.

For the termination criteria, we set $\varepsilon = 1.0e - 4$ for Gaussian channel capacity problem, Minimization of the minimal eigenvalue problem and Nearest correlation matrix problem, and $\varepsilon = 1.0e - 3$ for SOF-$\mathcal{H}_2$ problem.

We show the numerical results in Tables 1–4. In these tables, SDPIP denotes the interior point algorithm in [26]. From Tables 1–4, we see that Algorithm 1 is competitive to SDPIP.

6 Concluding remarks

In this paper, we have proposed the new merit function F for the shifted perturbed KKT conditions. We have shown the properties of the merit function. In particular, we gave the level boundedness of the merit function F, which is not given in other related papers for the nonlinear SDP. Moreover, we have proposed the Newton type method (Algorithm 2) to find an approximate shifted perturbed KKT point. We further have proved the global convergence under weaker assumptions than those in [26]. In the numerical experiments, we have shown that Algorithm 1 is competitive to Algorithm SDPIP.

As future research, it is worth to show that Algorithm 1 converges superlinearly under appropriate conditions.

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