Beyond Planarity: Testing Full Outer-2-Planarity in Linear Time

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Abstract. A graph is 1-planar, if it admits a 1-planar embedding, where each edge has at most one crossing. Unfortunately, testing *1-planarity* of a graph is known as NP-complete.
This paper considers the problem of testing *2-planarity* of a graph, in particular,testing *full outer-2-planarity* of a graph. A graph is *fully-outer-2-planar*, if it admits a *fully-outer-2-planar embedding* such that every vertex is on the outer boundary, no edge has more than two crossings, and no crossing appears along the outer boundary. We present several structural properties of triconnected outer-2-planar graphs and fully-outer-2-planar graphs, and prove that triconnected fully-outer-2-planar graphs have constant number of fully-outer-2-planar embeddings. Based on these properties, we present a linear-time algorithm for testing fully outer-2-planarity of a graph G, where G is triconnected, biconnected and oneconnected. The algorithm also produce a fully outer-2-planar embedding of a graph, if it exists. We also show that every fully-outer-2-planar embedding admits a straight-line drawing.

1 Introduction

A recent research topic in topological graph theory generalises the notion of planarity to *almost planar graphs*, i.e., non-planar graphs with some specific crossings, or with some forbidden crossing patterns. Examples include k*-planar graphs* (i.e., graphs can be embedded with at most k crossings per edge), k*-quasi- planar graphs* (i.e., graphs can be embedded without k mutually crossing edges), *RAC graphs* (i.e., graphs can be embedded with right angle crossings), and *fan-crossing-free graphs* (i.e., graphs can be embedded without fan-crossings) [2, 5, 7, 19].

Some mathematical results are known for these graphs, for example, linear *density* of such graphs. Pach and Toth [19] proved that a 1-planar graph with n vertices has at most $4n - 8$ edges. Agarwal et al. [2] (Ackerman [1]) showed that 3-(4-) quasi-planar graphs have linear number of edges. Fox et al. [9] proved that k-quasi-planar graphs have at most $O(n \log^{1+o(1)} n)$ edges. Didimo et al. [7] showed that RAC graphs have at most $4n - 10$ edges. Cheong et al. [5] showed that fan-crossing free graphs have at most $4n - 8$ edges.

Recently, algorithmics and complexity for such graphs have been investigated. Grigoriev and Bodlaender, and Kohrzik and Mohar proved that testing 1-planarity of a graph is NP-complete [13, 17]. Argyriou et al. proved that testing whether a given graph is a RAC graph is NP-hard [3]. On the positive side, Eades et al. [8] showed that the problem of testing *maximal 1-planarity* (i.e., addition of an edge destroys 1-planarity) of a graph can be solved in linear time, if a *rotation system* (i.e., the circular ordering of edges for each vertex) is given. Hong et al. [15], and Auer et al. [4] independently proved that testing *outer-1-planarity* (i.e., 1-planar graphs with every vertex is on the outer face, introduced by Eggleton [10]) of a graph, can be solved in linear time.

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This paper considers the problem of testing *2-planarity* of a graph, in particular, testing *fully outer-2-planarity* of a graph. An embedding γ of a graph G in the plane is *2-planar*, if no edge has more than two crossings. A 2-planar embedding of G is called *outer-2-planar* (O2PE), if every vertex is on the outer boundary. An outer-2-planar embedding of G is called *fully outer-2-planar* (FO2PE), if no edge crossings appear along the outer boundary. A graph G is *2-planar* (resp., *outer-2-planar*, *fully outer-2-planar*) if it admits a 2-planar (resp., *outer-2-planar, fully outer-2-planar*) embedding (see Fig. 1).

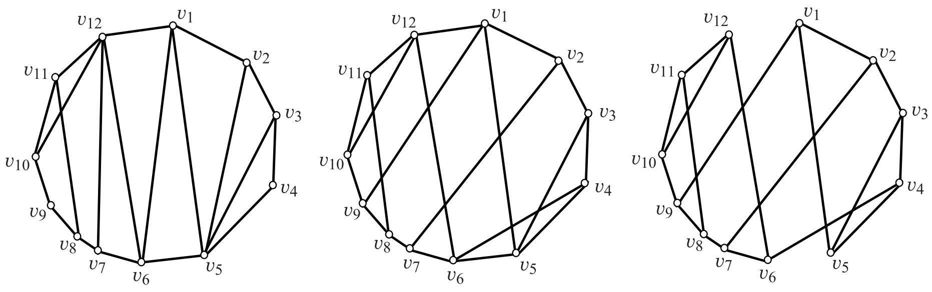

Fig. 1. Illustration for outer-2-plane embeddings of graphs: (a) An FO2PE γ_1 of a biconnected graph G_1; (b) An FO2PE γ_2 of a triconnected graph G_2; (c) An O2PE γ_3 of a triconnected graph G_3.

The problem of testing outer-2-planarity seems much harder than testing outer-1-planarity. In fact, it was shown that outer-1-planar graphs are indeed planar graphs [4], however K_5 is an outer-2-planar graph, which is not planar. Note that there is only one triconnected outer-1-planar graph, K_4, and it has unique outer-1-planar embedding [4, 15]. However, we can show that there is a triconnected outer-2-planar graph which has exponentially many outer-2-planar embeddings.

Moreover, the outer boundary of an FO2PE of a biconnected graph G is a Hamiltonian cycle of G. Note that testing whether a given graph has a Hamiltonian cycle is known to be NP-complete, even for cubic graphs [12].

We first study several structural properties of outer-2-planar graphs and fully outer-2-planar graphs. Based on these properties, we present a linear-time algorithm for testing fully outer-2-planarity of a graph G. The following theorem summarizes our main results.

Theorem 1. *There is a linear-time algorithm that tests whether a given graph is fully outer-2-planar, and produces a fully outer-2-planar embedding of the graph if it exists.*

We use connectivity approach to prove Theorem 1. The *oneconnected* case is easy; see Theorem 4 in Section 3. The *biconnected* case is more involved; see Theorem 5 in Section 4. The main thrust of this paper is to solve the *triconnected* case, described in Section 5. The following theorem is the key to design linear-time algorithm for FO2PE.

Theorem 2. *The number of all FO2PEs of a triconnected graph G is constant, and the set of all FO2PEs of G can be generated in linear time.*

The well-known Fary's theorem [11] proved that every plane graph admits a straight-line drawing. However, Thomassen [20] presented two forbidden subgraphs for straight-line drawings of 1-plane graphs. Hong et al. [16] gave a linear-time testing and drawing algorithm to construct a straight-line 1-planar drawing, if it exists. Recently, Nagamochi solved the more general problem of straight-line drawability for wider class of embedded graphs [18]. On the otherhand, Eggleton [10] showed that every outer-1-plane graph admits a straight-line drawing. We also show that every outer-2-plane graph admits a straight-line drawing.

Theorem 3. *Every outer-2-plane embedding admits a straight-line drawing.*

2 Preliminaries

Let $G = (V, E)$ be a graph, where n denotes $|V|$ unless stated otherwise. Let $X, Y \subseteq V$ be subsets of vertices and $F \subseteq E$ be a subset of edges. For a vertex v, let $E(v)$ denote the set of edges vu incident to v, $\deg(v)$ denote the degree $|E(v)|$ of v, $N(v)$ denote the set of neighbors u of v, and $N[v] = N(v) \cup \{v\}$. We may indicate the underlying graph G in these notations in such a way that $E(v)$ is written as $E(v)$. Let $G - F$ denote the graph obtained from G by removing the edges in F, and $G - X$ denote the graph obtained from G by removing the vertices in X together with the edges in $\cup_{v \in X} E(v)$. Let G/X denote the graph obtained from a graph G by contracting the vertices in a subset X of vertices into a single vertex, where any resulting loops and multiple edges are removed. A vertex of degree d is called a *degree-d vertex*. A simple cycle of length k is called a *k-cycle*, where a 3-cycle is called a *triangle*.

A *topological graph* or *embedding* γ of a graph G is a representation of a graph (possibly with multiple edges) in the plane, where each vertex is a point and each edge is a Jordan arc between the points representing its endpoints. Two edges *cross* if they have a point in common, other than their endpoints. The point in common is a *crossing*. To avoid pathological cases, standard non-degeneracy conditions apply: (i) two edges intersect in at most one point; (ii) an edge does not contain a vertex other than its endpoints; (iii) no edge crosses itself; (iv) edges must not meet tangentially; (v) no three edges share a crossing point; and (vi) no two edges that share an endpoint cross.

For an O2PE γ of a graph $G = (V, E)$, we denote by $\partial\gamma$ the outer boundary of γ, which may pass though a crossing point made by two edges. An edge $e \in E$ is called an *outer* (resp., *inner*) edge of γ if the whole drawing of e is part of $\partial\gamma$ (resp., $\partial\gamma$ passes though only the end-vertices of e). An edge may not be outer or inner when a crossing on it appears along $\partial\gamma$. Let $V_{\partial\gamma}$, $E_{\partial\gamma}$ and $C_{\partial\gamma}$ denote the sets of vertices, outer edges and crossings in $\partial\gamma$.

For two vertices $u, v \in V$, the boundary path traversed from u to v in the clockwise order is denoted by $\partial\gamma[u, v]$. Let $V_{\partial\gamma}[u, v]$, $E_{\partial\gamma}[u, v]$ and $C_{\partial\gamma}[u, v]$ denote the sets of vertices, outer edges and crossings in $\partial\gamma[u, v]$. Also let $V_{\partial\gamma}(u, v] = V_{\partial\gamma}[u, v] - \{u\}$, $V_{\partial\gamma}[u, v) = V_{\partial\gamma}[u, v] - \{v\}$, $V_{\partial\gamma}(u, v) = V_{\partial\gamma}[u, v] - \{u, v\}$. We call the boundary path $\partial\gamma[u, v]$ *crossing-free* if $C_{\partial\gamma}[u, v] = \emptyset$, i.e., it consists of outer edges.

3 Connected Graphs

We first observe that we can focus on biconnected graphs to design algorithms for testings (full) outer-2-planarity.

Theorem 4. *A graph is outer-2-planar (resp., fully outer-2-planar) if and only if its biconnected components are outer-2-planar (resp., fully outer-2-planar).*

Proof. Let γ be an O2PE (resp., FO2PE) of a graph G. Then the embedding γ_H induced by a biconnected component H of the graph is an O2PE (resp., FO2PE) since no new crossing is introduced and the vertices in the component stay on the boundary of γ_H. Conversely assume that each biconnected component H of the graph G admits an O2PE (resp., FO2PE) γ_H. Starting with $\gamma^* := \gamma_H$ for some biconnected component H, we combine γ^* with $\gamma_{H'}$ for a biconnected component H' which shares a cut-vertex with one of the scanned biconnected components. Since such a cut-vertex remains on the outer boundary of γ^* and no new cycle through the cut-vertex is created, the newly combined embedding is also an O2PE (resp., FO2PE). By repeating this, we can obtain an O2PE (resp., FO2PE) of G. □

Thus, in what follows, we treat only biconnected graphs G as input. For a permutation $[v_1, v_2, \ldots, v_n]$ of the vertices of a biconnected graph G, let $\gamma = (G, [v_1, v_2, \ldots, v_n])$ denote an embedding of G such that vertices $v_1, v_2, \ldots, v_n$ appear along $\partial\gamma$ in the clockwise manner. We can easily observe that the number of crossings on each edge in an O2PE γ is determined only by the ordering of all vertices along $\partial\gamma$.

To solve the problem of finding an FO2PE γ of a graph G, we consider the problem with an additional constraint such that a set B of specified edges is required to appear along the boundary; i.e., $B \subseteq E_{\partial\gamma}$, and denote such an instance by (G, B). An FO2PE of γ of G such that $B \subseteq E_{\partial\gamma}$ is called an *FO2PE extension* of (G, B), and an instance (G, B) is called *extendible* if it admits an FO2PE extension.

4 Biconnected Graphs

Our algorithm for biconnected case uses the decomposition of a biconnected graph G into triconnected components, alternatively known as the *SPQR tree*, defined by di Battista and Tamassia [6], which can be computed in linear time [14]. Each triconnected component consists of *real edges* (i.e., edges in the original graph) and *virtual edges*. (i.e., edges introduced during the decomposition process, which represents the other triconnected components, sharing the same virtual edges defined by cut-pairs).

Each node ν in the SPQR tree is associated with a graph called the *skeleton* of ν, denoted by $\sigma(\nu)$, which corresponds to a triconnected component. There are four types of nodes ν in the SPQR tree: (i) S-node, where $\sigma(\nu)$ is a simple cycle with at least three vertices; (ii) P-node, where $\sigma(\nu)$ consists of two vertices connected by at least three edges; (iii) Q-node, where $\sigma(\nu)$ consists of two vertices connected by two (real and virtual) edges; and (iv) R-node, where $\sigma(\nu)$ is a simple triconnected graph with at least four vertices. The set of virtual edges in the skeleton of a node ν by $E_{\mathrm{vir}}(\nu)$.

In this paper, we use the SPR tree, a simplified version of the SPQR tree *without* Q-nodes, and treat the SPR tree as a *rooted tree* by choosing an arbitrary node as its root. Let ρ be the parent node of an internal node ν. The graph $\sigma(\rho)$ has exactly one virtual edge e in common with $\sigma(\nu)$; e is called the *parent virtual edge* in $\sigma(\nu)$, and a *child virtual edge* in $\sigma(\rho)$. We denote the graph formed from $\sigma(\nu)$ by deleting its parent virtual edge as $\sigma^-(\nu)$, and denote the graph formed from the union of $\sigma^-(\nu)$ over all descendants ν of ρ by G^-_ρ. We also denote the graph G^-_ρ together with the parent virtual edge in $\sigma(\rho)$ by G_ρ. Note that $E_{\mathrm{vir}}(\nu)$ is the set of virtual edges in $\sigma(\rho)$ including the parent virtual edge when ν is a non-root node.

For a given biconnected graph G, we establish a recurrence relationship of FO2PE problem instances (G, B) based on the SPR decomposition of G. In fact we prove that G admits an FO2PE if and only if for each node ν in the SPR decomposition of G, the instance $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ is extendible. We easily see that for S-node ν $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ are cycles and always extendible. More specifically, we prove the following Theorem.

Theorem 5. *A biconnected graph $G = (V, E)$ admits an FO2PE if and only if the following holds: for each P-node ν, $|E_{\mathrm{vir}}(\nu)| \le 2$; and for each R-node ν, $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ is extendible. Moreover, there is a linear-time algorithm for constructing a FO2PE of G, if it exists.*

Before we prove Theorem 5, we first show the following lemma.

Lemma 1. *Let γ be an arbitrary FO2PE of $G = (V, E)$, and H be a component in $G - \{u, v\}$ for a cut-pair $\{u, v\}$. Then γ is given by a cyclic order $[v_1, v_2, \ldots, v_n]$ such that $v_1 = u$, $\{v_2, v_3, \ldots, v_i\} = V(H)$ and $v_{i+1} = v$ appear in this order.*

Proof for Necessity of Theorem 5: Let γ be an arbitrary FO2PE of $G = (V, E)$. To derive a contradiction, first assume that $|E_{\mathrm{vir}}(\nu)| \ge 3$ for some P-node ν. Thus for the two vertices u, v in the skeleton $\sigma(\nu)$, $G - \{u, v\}$ has at least three components, say H_1, H_2 and H_3. By Lemma 1, for each $i = 1, 2, 3$, the vertices u, $V(H_i)$ and v must appear consecutively along $\partial\gamma$. However, this is impossible unless the vertex u appear more than once along $\partial\gamma$.

Assume that $|E_{\mathrm{vir}}(\nu)| \le 2$ for each P-node ν. Next we show that that $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ is extendible for any R-node ν. For each virtual edge $e = st \in E_{\mathrm{vir}}(\nu)$, there are exactly two components H^*_e and H_e in $G - \{s, t\}$ by the assumption of P-nodes, where $V(H^*_e) \cup \{s, t\} \subseteq V(\sigma(\nu))$. Clearly H_e and $H_{e'}$ are disjoint for any two virtual edges $e, e' \in E_{\mathrm{vir}}(\nu)$. Hence by Lemma 1, the vertices in H_e for each virtual edge $e = st \in E_{\mathrm{vir}}(\nu)$ appear consecutively between s and t along $\partial\gamma$. Hence we can obtain an FO2PE extension ξ_ν of $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ from γ by shortening the subsequence for the vertices in $V(H_e) \cup \{s, t\}$ for each virtual edge $e = st \in E_{\mathrm{vir}}(\nu)$ into s, t. This proves the necessity of Theorem 5. □

Proof for Sufficiency of Theorem 5: We construct an FO2PE γ of G by an induction along the parent-child relationship of the rooted SPR tree T of G, as shown in the algorithm below. For a given graph G, we have computed the SPR tree T of G and computed an FO2PE extension ξ_ν of $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$ for each node ν in T, and assume that the necessary condition in Theorem 5 holds. Note that for a P- and S-node ν, its skeleton $\sigma(\nu)$ is a pair of real and virtual edges with the same end-vertices, two virtual edges (possibly with one real edge) with the same end-vertices, and a simple cycle of length at least 3, respectively, each of which admits an FO2PE extension ξ_ν of $(\sigma(\nu), E_{\mathrm{vir}}(\nu))$.

Let ν be a P-, R- or S-node chosen in the for-loop of lines 7-12, where we have obtained an FO2PE extension $\xi_\nu = [v_1, v_2, \dots, v_{n'}]$ of $(\sigma(\nu), E_{\rm vir}(\nu))$ in line 2 and an FO2PE extension γ_μ of $(G_\mu, \{st\})$ for each child μ of ν and the corresponding child virtual edge $st \in E_{\rm vir}(\nu)$ in the previous iterations of the for-loop. Since the parent edge st of μ is contained in $E_{\rm vir}(\mu)$, γ_μ is given by a cyclic order $[u_1 = s, u_2, u_3, \dots, u_{p-1}, u_p = t]$ of the vertices in G_μ. Also in ξ_ν, the virtual edge st appears as an outer edge; i.e., vertices s and t appear consecutively as $[v_i = s, v_{i+1} = t]$ in ξ_ν. Therefore by replacing each child virtual edge st in ξ_ν with the corresponding FO2PE extension γ_μ, i.e., replacing the subsequence $[v_i = s, v_{i+1} = t]$ in ξ_ν with $[u_1 = s, u_2, u_3, \dots, u_{p-1}, u_p = t]$, we can obtain an FO2PE extension γ_ν of $(G_\nu, \{ab\})$ with the parent virtual edge ab of ν or of $(G, \emptyset)$ when $\nu = \nu^*$. This proves the sufficiency of Theorem 5. □

See below for the detailed description of **Algorithm BICONNECTED FO2PE** and time complexity analysis. Essentially, the algorithm can be implemented to run in linear time, if the R-node (i.e., triconnected) case can be solved in linear time.

Algorithm BICONNECTED FO2PE
Input: A biconnected simple graph G.
Output: An FO2PE γ of G if any or $\emptyset$ otherwise.
1: Construct the SPR tree T of G;
2: Compute an FO2PE extension ξ_ν of $(\sigma(\nu), E_{\rm vir}(\nu))$ for each node ν in T;
3: **if** $|E_{\rm vir}(\nu)| \geq 3$ for some P-node ν or $(\sigma(\nu), E_{\rm vir}(\nu))$ is not extendible
  for some R-node ν **then**
4:   Return $\emptyset$
5: **else**
6:   Regard a node as the root ν^* of T;
7:   **for** each non-root node ν of T chosen from the bottom to the top along T **do**
8:     Compute an FO2PE extension γ_ν of $(G_\nu, \{ab\})$ with the parent virtual edge $ab$of ν
     (or $(G, \emptyset)$ when $\nu = \nu^*$) from $\xi_\nu = [v_1, v_2, \dots, v_{n'}]$ as follows:
9:     **for** each child μ of ν and the corresponding child virtual edge $st \in E_{\rm vir}(\nu)$ **do**
10:       Replace the subsequence $[v_i = s, v_{i+1} = t]$ in ξ_ν with an FO2PE extension
       $\gamma_\mu = [u_1 = s, u_2, u_3, \dots, u_{p-1}, u_p = t]$ of $(G_\mu, \{st\})$
11:     **end for**
12:   **end for**;
13:   Return $\gamma := \gamma_{\nu^*}$
14: **end if**

We show that when Theorem 2 holds the above algorithm can be implemented to run in linear time. The time complexity of the Algorithm for line 1 is linear [14]. After this, we see that any operation on a node ν in T takes in $O(|\sigma(\nu)|)$ time. In lines 2-3, we can test whether there is no P-node ν with $|E_{\rm vir}(\nu)| \geq 3$ in $O(|\sigma(\nu)|) = O(1)$ time, and finding an FO2PE extension ξ_ν of $(\sigma(\nu), E_{\rm vir}(\nu))$ takes $O(|\sigma(\nu)|)$ time for a P- or S-node ν (since the structure of $\sigma(\nu)$ is nearly a cycle) and $O(|\sigma(\nu)|)$ time for an R-node ν by Theorem 2. The for-loop of lines 7-12 takes in $O(n)$ time in total, because inserting a subsequence $\gamma_\mu = [u_1 = s, u_2, u_3, \dots, u_{p-1}, u_p = t]$ into $\xi_\nu = [v_1, v_2, \dots, v_{n'}]$ at the position $[v_i = s, v_{i+1} = t]$ can be carried out in $O(1)$ time using doubly-liked lists for storing all sequences such as ξ_ν and γ_μ. Hence to prove Theorem 1, it suffices to show Theorem 2.

5 Triconnected Graphs

In this section, we prove Theorem 2, i.e., every triconnected graph G has a constant number of FO2PEs, and they can be generated in linear time. Note that Theorems 5 and 2 imply that FO2PE testing for biconnected graphs can be done in linear time.

To prove Theorem 2, we derive a recurrence relationship over FO2PE problem instances (G, B) for special local structures B, called "rims." First, we prove several structural results on triconnected O2PE and FO2PE.

5.1 Structural results on triconnected O2PE and FO2PE

We first present structural results on triconnected O2PE.

Lemma 2. *Every O2PE of a triconnected graph G is quasi-planar unless G is $K_{3,3}$.*

Proof. Let γ be an O2PE with three pairwise crossing edges $e_i = u_i v_i$, $i = 1, 2, 3$, where u_1, u_2, u_3, v_1, v_2 and v_3 appear in this order along $\partial\gamma$. Note that each of these three edges already has two crossings on it. Hence if $V_{\partial\gamma}(u_1, u_2) \neq \emptyset$, then there must be an edge $e = ab$ that joins a vertex $a \in V_{\partial\gamma}(u_1, u_2)$ and a vertex $b \in V_{\partial\gamma}(u_2, u_3)$, since otherwise $\{u_1, u_2\}$ would be a cut-pair in a triconnected graph. However, γ cannot admit such an edge $e = ab$, since it would cross one of the three pairwise crossing edges. Hence $V_{\partial\gamma}(u_1, u_2) = \emptyset$. Analogously we have $V_{\partial\gamma}(u, v) = \emptyset$ for two end-vertices u and v of the three pairwise crossing edges which consecutively appear along $\partial\gamma$, indicating that $V = \{u_1, u_2, u_3, v_1, v_2, v_3\}$.

Vertex u_2 is of degree at least 3, and it has at least two incident edges e_2' and e_2'' other than edge $u_2 v_2$, where neither of edges e_2' and e_2'' can cross e_1 or e_3. This implies that vertex u_2 has exactly three incident edges, $u_2 v_2$, $u_2 u_1$ and $u_2 u_3$. Analogously with other vertices in V, we see that each vertex in V is of degree 3 and $C_{\partial\gamma} = \emptyset$, indicating that G is a complete bipartite graph $K_{3,3}$ between vertex sets $\{u_1, u_3, v_2\}$ and $\{u_2, v_1, v_3\}$. □

Lemma 3. *No triconnected graph G with a vertex of degree ≥ 5 admits an O2PE.*

Proof. Let v be a vertex of degree $d \geq 5$ in G, and γ be an O2PE of G. Since G contains a vertex of degree ≥ 5, G is not $K_{3,3}$ and γ is quasi-planar by Lemma 2. Without loss of generality, the neighbors $u_1, u_2, \ldots, u_d$ of v appear in this order along $\partial\gamma[u_1, u_d]$.

Since $\{v, u_3\}$ is not a cut-pair, there is an edge $e = ab$ that joins a vertex $a \in V_{\partial\gamma}(v, u_3)$ and a vertex $b \in V_{\partial\gamma}(u_3, v)$, where $e = ab$ crosses edge vu_3 and can cross at most one of vu_2 and vu_4.

First assume that $e = ab$ crosses vu_2 or vu_4, say vu_4, where $a \in V_{\partial\gamma}[u_2, u_3)$ holds, and we choose $e = ab$ so that vertex a is closest to u_2 among all choices of such edges ab. Since $\{a, v\}$ is not a cut-pair, there is an edge $e^* = a^* b^*$ that joins a vertex $a^* \in V_{\partial\gamma}(v, a)$ and a vertex $b^* \in V_{\partial\gamma}(a, v)$. Since γ is quasi-planar and e^* cannot cross e, it holds $a^* \in V_{\partial\gamma}[u_2, a)$ and $b^* \in V_{\partial\gamma}(b, v)$, where e^* crosses vu_4 and vu_3 but cannot cross vu_2. This, however, contradicts the choice of edge $e = ab$.

Next assume that no edge ab with $a \in V_{\partial\gamma}(v, u_3)$ and $b \in V_{\partial\gamma}(u_3, v)$ crosses vu_2 or vu_4. Hence $a \in V_{\partial\gamma}[u_2, u_3)$ and $b \in V_{\partial\gamma}(u_3, u_4]$. Since $\{b, v\}$ is not a cut-pair, there is an edge $e' = a'b'$ that joins a vertex $a' \in V_{\partial\gamma}(v, b)$ and a vertex $b' \in V_{\partial\gamma}(b, v)$. Since γ is quasi-planar, edge e' does not cross vu_3 and it holds $a' \in V_{\partial\gamma}[u_3, b)$. Analogously with pair $\{b, v\}$, there must be an edge $e^* = a^* b^*$ that joins a vertex $a^* \in V_{\partial\gamma}(v, a)$ and a vertex $b^* \in V_{\partial\gamma}(a, u_3]$. However, in this case, edge $e = ab$ has three crossings on it, a contradiction.

This proves that no graph with a vertex of degree ≥ 5 admits an O2PE. □

Lemma 4. *Let $G = (V, E)$ be a triconnected graph which contains K_4 as a subgraph. If G admits an O2PE, then $n \leq 6$.*

Proof. Let H be a subgraph of G isomorphic to K_4, and Let γ be an O2PE of G, where the four vertices u_1, u_2, u_3 and u_4 in H appear in this order along γ. To derive a contradiction, assume that $n \geq 7$. Without loss of generality, let $V_{\partial\gamma}(u_1, u_2) \neq \emptyset$. Since $\{u_1, u_2\}$ is not a cut-pair in a triconnected graph, there is an edge $e = ab$ that joins a vertex $a \in V_{\partial\gamma}(u_1, u_2)$ and a vertex $b \in V_{\partial\gamma}(u_2, u_1)$. Note that b can be vertex u_3 or vertex u_4, say $b = u_4$, since otherwise edge ab would cross three edges in H. Now edge $u_2 u_4$ has two crossings on it. Then for each ordered pair $(u, v) \in \{(a, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$, we see that no edge $e = ab$ can join a vertex $a \in V_{\partial\gamma}(u, v)$ and a vertex $b \in V_{\partial\gamma}(v, u)$, and that $V_{\partial\gamma}(u, v) = \emptyset$ holds, since otherwise $\{u, v\}$ would be a cut-pair.

By $n \geq 7$, we have $V_{\partial\gamma}(u_1, a) \neq \emptyset$. Since $\{u_1, a\}$ is not a cut-pair in a triconnected graph, there is an edge $e' = a'b'$ that joins a vertex $a' \in V_{\partial\gamma}(u_1, a)$ and a vertex $b' \in V_{\partial\gamma}(a, u_1)$. In this case, it holds $b' = u_4$, since otherwise e' would cross three edges. Now for each ordered pair $(u, v) \in \{(u_1, a'), (a', u_2)\}$, we see that no edge $e = ab$ can join a vertex $a \in V_{\partial\gamma}(u, v)$ and a vertex $b \in V_{\partial\gamma}(v, u)$, and that $V_{\partial\gamma}(u, v) = \emptyset$ holds, since otherwise $\{u, v\}$ would be a cut-pair. This, however, contradicts that $n \geq 7$. □

By Lemmas 2 and 4, we have the next.

Lemma 5. *Let G be a triconnected graph with at least seven vertices. If G admits an O2PE γ, then G contains no subgraph isomorphic to K_4 and γ is quasi-planar.*

For an O2PE γ of a triconnected graph $G = (V, E)$ with $n \geq 7$, the cyclic order $[v_1, v_2, \ldots, v_n]$ of the vertices in $\partial\gamma$ completely determines the embedding γ by Lemma 5. In what follows, an O2PE γ of a graph G is simply denoted by the cyclic order of the vertices in $\partial\gamma$.

For an inner edge uv in an FO2PE γ of a triconnected graph G, there is an edge ab that crosses uv; i.e., ab joins a vertex $a \in V_{\partial\gamma}(u, v)$ and a vertex $b \in V_{\partial\gamma}(v, u)$, since otherwise $\{u, v\}$ would be a cut-pair. We call an edge ab (u, v)-*hooked* if ab crosses uv and some edge $a'a''$ $(\neq uv)$ with $a', a'' \in V_{\partial\gamma}[u, v]$. We frequently use the following technical lemma.

Lemma 6. *Let γ be an FO2PE of a triconnected graph G, and let u and v be two vertices such that $uv \in E - E_{\partial\gamma}$.*

(i) *If $|V_{\partial\gamma}(u, v)| \geq 3$, then there is a (u, v)-hooked edge ab.*
(ii) *If $|V_{\partial\gamma}(u, v)| = 2$ and there is no (u, v)-hooked edge, then each of the two vertices in $V_{\partial\gamma}(u, v)$ is of degree 3 and the inner edge incident to it crosses uv.*

Proof. Assume that $|V_{\partial\gamma}(u, v)| \geq 2$ and there is no (u, v)-hooked edge in γ. To prove the lemma, it suffices to show that $|V_{\partial\gamma}(u, v)| = 2$ holds and each of the two vertices in $V_{\partial\gamma}(u, v)$ is of degree 3 and has an incident edge crossing uv.

Since $\{u, v\}$ is not a cut-pair, there is an edge ab that joins a vertex $a \in V_{\partial\gamma}(u, v)$ and a vertex $b \in V_{\partial\gamma}(v, u)$. We choose an edge ab so that a is closest to u among all edges ab crossing $e = uv$. If $V_{\partial\gamma}(u, a) \neq \emptyset$, then $\{u, a\}$ would be a cut-pair, since each inner edge incident to a vertex in $V_{\partial\gamma}(u, a)$ cannot cross a non (u, v)-hooked edge ab or edge uv by the choice of a. Hence we have $V_{\partial\gamma}(u, a) = \emptyset$.

Similarly we choose an edge $a'b'$ so that $a' \in V_{\partial\gamma}(u, v)$ is closest to v among all edges $a'b'$ crossing $e = uv$, and we see that $V_{\partial\gamma}(a', v) = \emptyset$.

Now no edge incident to a vertex in $V_{\partial\gamma}(a, a')$ other than ab or $a'b'$ can cross any of edges uv, ab and $a'b'$. This means that $V_{\partial\gamma}(u, v) = \{a, a'\}$ (otherwise $\{a, a'\}$ would be a cut-pair) and $\deg(a) = \deg(a') = 3$, as required. □

5.2 Identifying a constant number of candidate partial embeddings

Let γ be an O2PE of a triconnected graph G. A triangle uvw is called a $(3, 3)$-*rim* (resp., $(3, 4)$-rim) of γ if uv and vw are outer edges in γ and v is a degree-3 (resp., degree-4) vertex. A 4-cycle $uvv'w$ is a 4-*rim* of γ if v and v' are degree-3 vertices and uv, vv' and vw are outer edges in γ. A $(3, 3)$-, $(3, 4)$- or 4-rim is called a *rim*. For example, see Fig. 2.

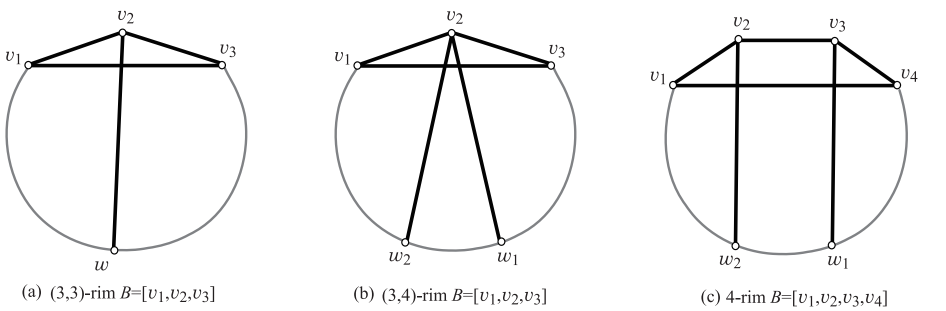

Fig. 2. Illustration for rims: (a) a (3,3)-rim for a triangle $v_1v_2v_3$ with a degree-3 vertex v_2; (b) a (3,4)-rim for a triangle $v_1v_2v_3$ with a degree-4 vertex v_2; (c) a 4-rim for 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3.

We show that any FO2PE of a triconnected graph G contains a rim.

Lemma 7. *Any FO2PE γ of a triconnected graph G has a rim.*

Proof. By Lemma 3, each vertex in G is of degree 3 or 4. Consider an inner edge uv such that $\partial\gamma[u, v]$ is the shortest. Hence there is no (u, v)-hooked edge, since otherwise there exists an inner edge $a'a''$ with $a', a'' \in V_{\partial\gamma}(u, v)$, where $\partial\gamma[a', a'']$ would be shorter than $\partial\gamma[u, v]$. By Lemma 6(i), we have $|V_{\partial\gamma}(u, v)| \leq 2$.

If $|V_{\partial\gamma}(u, v)| = 1$, then for the vertex $w \in V_{\partial\gamma}(u, v)$, triangle uwv is a 3-rim of γ.

Assume that $|V_{\partial\gamma}(u, v)| = 2$. By Lemma 6(ii), each of the two vertices in $V_{\partial\gamma}(u, v)$ is of degree 3, indicating that the 4-cycle with the four vertices in $V_{\partial\gamma}[u, v]$ is a 4-rim of γ. □

Our algorithm for constructing an FO2PE of a given triconnected graph G first generates triangles/4-cycles as rims of possible FO2PEs and tries to extend each of the triangles/4-cycles into an FO2PE. By Lemma 3, we can assume that a given triconnected graph G has a maximum degree at most 4. Then there are $O(n)$ triangles and 4-cycles for candidates of rims in an FO2PE of G. The next lemma reduces the number of triangles/4-cycles to be generated as rims of possible FO2PEs to a constant number.

Lemma 8. *Let γ be an FO2PE of a triconnected graph $G = (V, E)$ with $n \geq 10$.*

(i) *Assume that G has a triangle, and let t_1 be a triangle in G. Then $\partial\gamma$ contains a sequence $[u, v, w]$ for the set of vertices u, v and w of some triangle $t' = uvw$ sharing an edge with t_1 (possibly $t' = t_1$) as its subsequence.*

(ii) *Assume that G has no triangle, and let $u_1u_2u_3u_4$ be a 4-cycle with degree-3 vertices u_2 and u_3 in G. Then $\partial\gamma$ (or its reversal) contains $[u_1, u_2, u_3, u_4]$ (or $[u_3, u_4, u_1, u_2]$ if $\deg(u_4) = \deg(u_1) = 3$) as its subsequence.*

Proof. Since $n \geq 7$, embedding γ is quasi-planar by Lemma 2. (i) Let $t_1 = u_1u_2u_3$ be a triangle in G. Assume that t_1 is not a 3-rim of γ; i.e., at least two edges in triangle t_1 are inner edges in γ. Since $n \geq 10$, we can assume without loss of generality that u_1, u_2 and u_3 appear in this order along γ, and let $|V_{\partial\gamma}(u_1, u_2)| \geq 3$ and $V_{\partial\gamma}(u_2, u_3) \neq \emptyset$. By Lemma 6(i), there is a (u_1, u_2)-hooked edge a_1b_1 joining a vertex $a_1 \in V_{\partial\gamma}(u_1, u_2)$ and a vertex $b_1 \in V_{\partial\gamma}(u_2, u_1)$. Note that $b_1 = u_3$ holds because edge a_1b_1 already has two crossings.

If $|V_{\partial\gamma}(u_2, u_3)| = 1$, then triangle u_2xu_3 for the vertex $x \in V_{\partial\gamma}(u_2, u_3)$ satisfies the lemma. Assume that $|V_{\partial\gamma}(u_2, u_3)| \geq 2$. If there is a (u_2, u_3)-hooked edge a_2b_2 that joins a vertex $a_2 \in V_{\partial\gamma}(u_2, u_3)$ and a vertex $b_2 \in V_{\partial\gamma}(u_3, u_2)$, then edge a_2b_2 would have the third crossing with edge u_1u_2 or edge a_1b_1, a contradiction. Hence there is no (u_2, u_3)-hooked edge. By Lemma 6(ii), there are two edges that cross u_2u_3. However, these edges cannot cross edge a_1b_1, and must cross u_1u_2, creating three crossings on edge u_1u_2, a contradiction.

(ii) Assume that G has no triangle. Let $u_1u_2u_3u_4$ be a 4-cycle with degree-3 vertices u_2 and u_3 in G. We distinguish three cases.

(a) The vertices in the 4-cycle appear in the order of u_1, u_2, u_3, u_4 or u_4, u_3, u_2, u_1 along $\partial\gamma$: Let u_1, u_2, u_3, u_4 appear in this order along $\partial\gamma$. It suffices to show that "$V_{\partial\gamma}(u_2, u_3) = \emptyset$" or "$V_{\partial\gamma}(u_4, u_1) = \emptyset$ and $\deg(u_4) = \deg(u_1) = 3$." Assume that $V_{\partial\gamma}(u_2, u_3) \neq \emptyset$, where u_2u_3 is an inner edge in γ and it holds $V_{\partial\gamma}(u_1, u_2) = V_{\partial\gamma}(u_3, u_4) = \emptyset$. Then if $V_{\partial\gamma}(u_4, u_1) = \emptyset$, then we see that $\deg(u_4) = \deg(u_1) = 3$ holds, as required.

To derive a contradiction, we consider the case of $V_{\partial\gamma}(u_4, u_1) \neq \emptyset$, where it holds $|V_{\partial\gamma}(u_4, u_1)| \geq 2$ since G has no triangle.

If $|V_{\partial\gamma}(u_4, u_1)| \geq 3$, then there is a (u_4, u_1)-hooked edge ab by Lemma 6(i), which crosses edges u_4u_1 and u_2u_3, since no other inner edge is incident to u_2 or u_3. This is a contradiction, because edge ab has at least three crossings.

Hence $|V_{\partial\gamma}(u_4, u_1)| = 2$ and $|V_{\partial\gamma}(u_2, u_3)| \geq 3$ by $n \geq 10$. By Lemma 6(i), there is a (u_2, u_3)-hooked edge $e' = a'b'$ where e' is incident to u_1 or u_4 since it cannot cross u_1u_4 any more.

Since G has no triangle and $|V_{\partial\gamma}(u_4, u_1)| = 2$, each of the two vertices in $V_{\partial\gamma}(u_4, u_1)$ has an incident edge that crosses edge u_3u_4 and edge u_2u_3. Hence edge u_2u_3 crosses these two edges incident to vertices in $V_{\partial\gamma}(u_4, u_1)$ and edge $e' = a'b'$, creating three crossings, a contradiction.

(b) The vertices in the 4-cycle appear in the order of u_1, u_4, u_2, u_3 or u_3, u_2, u_4, u_1 along $\partial\gamma$: Let u_1, u_4, u_2, u_3 appear in this order along $\partial\gamma$. Since $\deg(u_2) = \deg(u_3) = 3$, we have $V_{\partial\gamma}(u_2, u_3) = \emptyset$. Since $n \geq 10$, it holds one of $|V_{\partial\gamma}(u_3, u_1)| \geq 2$, $|V_{\partial\gamma}(u_4, u_2)| \geq 2$ and $|V_{\partial\gamma}(u_1, u_4)| \geq 3$. If $|V_{\partial\gamma}(u_1, u_4)| \geq 3$, then there is a (u_1, u_4)-hooked edge $e = ab$ by Lemma 6(i), where $e = ab$ must cross edge u_1u_2 or edge u_4u_3 creating three crossings on it. Hence $|V_{\partial\gamma}(u_3, u_1)| \geq 2$ or $|V_{\partial\gamma}(u_4, u_2)| \geq 2$.

Without loss of generality assume that $|V_{\partial\gamma}(u_3, u_1)| \geq 2$. Then $|V_{\partial\gamma}(u_2, u_1)| \geq 3$, and there is a (u_2, u_1)-hooked edge $e = ab$ by Lemma 6(i), where $ab \neq u_3u_4$ since u_3u_4 does not cross any edge incident to u_2. However, in this case, edge $e = ab$ crosses u_4u_1 creating three crossings on it in the quasi-planar embedding γ.

(c) The vertices in the 4-cycle appear in the order of u_1, u_2, u_4, u_3 or u_3, u_4, u_2, u_1 along $\partial\gamma$: Let u_1, u_2, u_4, u_3 appear in this order along $\partial\gamma$. Since $\deg(u_2) = \deg(u_3) = 3$, we have $V_{\partial\gamma}(u_1, u_2) = V_{\partial\gamma}(u_4, u_3) = \emptyset$. Since G has no triangle, $V_{\partial\gamma}(u_2, u_4) \neq \emptyset \neq V_{\partial\gamma}(u_3, u_1)$. Hence there is an edge $e = ab$ that joins a vertex $a \in V_{\partial\gamma}(u_3, u_1)$ and $b \in V_{\partial\gamma}(u_1, u_3)$, where $b = u_4$ holds since γ is quasi-planar. Symmetrically there is an edge $e' = a'u_1$ that joins a vertex $a \in V_{\partial\gamma}(u_3, u_1)$ and vertex u_1. However, edge u_2u_3 crosses three edges in this case. □

In an FO2PE γ of a triconnected graph G, an outer edge e joining a degree-3 vertex u and a degree-4 vertex v is called a *frill* if γ contains a subsequence $[s_1, s_2, s_3, s_4]$ with $\{s_2, s_3\} = \{u, v\}$ such that $s_1s_2s_3$ and $s_2s_3s_4$ are triangles, where the degree-4 vertex v (resp., degree-3 vertex u) is called the *head* (resp., *tail*) of the frill e. We call $[s_1, s_2, s_3, s_4]$ the *span* of frill e. An operation of exchanging the positions of s_2 and s_3 in the cyclic order γ is called *flipping* frill e. It is easy to observe that the cyclic order γ' obtained from γ by flipping a frill is also an FO2PE of G.

Lemma 9. *Let γ be an FO2PE of a triconnected graph $G = (V, E)$ with $n \geq 7$. Then there are at most two frills in γ, and if there are two frills, then their spans share at most one vertex. Moreover flipping a frill in γ never introduces a new frill in the resulting cyclic order γ'.*

Proof. Assume that there are two frills $e = xy$ and $e' = x'y'$ in γ. Denote their spans by $[a, x, y, b]$ and $[a', x', y', b']$. Without loss of generality assume that vertices a, x, y, b (resp., a', x', y', b') appear in this order along $\partial\gamma$ and $\{a, x, y, b\} \cup \{a', x', y', b'\} \subseteq V_{\partial\gamma}(a, b')$. If the spans share at least two vertices, then we see that "$x = a'$, $y = x'$ ($\deg(y) = 4$) and $b = y'$" or "$y = a'$ and $b = x'$ ($\deg(y) = \deg(x') = 4$)" holds. Hence there is no edge between $V_{\partial\gamma}(a, b')$ and $V_{\partial\gamma}(b', a)$, where $V_{\partial\gamma}(b', a) \neq \emptyset$ by $n \geq 7$. This means that $\{a, b'\}$ is a cut-pair, contradicting the triconnectivity of G. Hence their spans share at most one vertex. This also implies that flipping a frill in γ cannot create a triangle for a new frill and thereby never introduces a new frill.

To derive a contradiction, assume that there are three frills e_1, e_2 and e_3 in γ. Let V_i and E_i, $i = 1, 2, 3$ be the set of vertices in the span of e_i and the set of edges in the two triangles sharing frill e_i. For each frill $e_i = x_iy_i$, there is exactly one edge f_i between the head vertex $x_i \in V_i$ of e_i and a vertex $y_i \in V - V_i$. Note that $f_i = x_iy_i$ already crosses an edge in E_i and no other edge than f_i crosses any edge in E_i.

We now define a set E^* of edges as follows. If $f_i = f_j$, then assume that $f_1 = f_2$ and let $E^* = E_1 \cup E_2 \cup \{f_1 = f_2\}$. If $f_i \neq f_j$ for any $1 \leq i < j \leq 3$ but f_i crosses f_j for some $1 \leq i < j \leq 3$, then assume that f_1 crosses f_2 and let $E^* = E_1 \cup E_2 \cup \{f_1, f_2\}$.

Assume that $f_i \neq f_j$ and f_i does not cross f_j for any $1 \leq i < j \leq 3$. Without loss of generality that $V_2 \subseteq V_{\partial\gamma}(x_1, y_1)$. Consider frill e_2, where $V_2 \cup \{y_2\} \subseteq V_{\partial\gamma}(x_1, y_1)$ since f_2 does not cross f_1. In fact, $V_2 \cup \{y_2\}$ is contained in $V_{\partial\gamma}(a, y_2)$ or $V_{\partial\gamma}(y_2, a)$ for an end-vertex $a \in V_2$ of the span of e_2, and an edge h_1 crosses f_2. Similarly if h_1 does not cross f_1, then we can find a sequence of edges h_2, $h_3, \ldots, h_p$ such that h_i crosses edges h_{i-1} and h_{i+1} for each $i = 2, 3, \ldots, p-1$ and h_p crosses f_1. Let $E^* = E_1 \cup E_2 \cup \{f_1, f_2\} \cup \{h_1, h_2, \ldots, h_p\}$.

In any of the above three cases, no edge in E^* crosses any edge in E_3 since only edge f_3 can cross an edge in E_3 and $f_1 \neq f_3 \neq f_2$ by the choice of f_1 and f_2. We denote the set of all end-vertices of edges in E^* by $z_1, z_2, \ldots, z_q$ in the order they appear along $\partial\gamma$. Then for each $i = 1, 2, \ldots, q$, set $V_{\partial\gamma}(z_i, z_{i+1})$ (where $z_{p+1} = z_1$) must be empty, since otherwise no edge in E^* can cross any other edge and $\{z, z_{i+1}\}$ would be a cut-pair. This means that frill e_3 cannot exist anywhere along $\partial\gamma$, a contradiction. □

We start with a triangle or 4-cycle fixed in Lemma 8 as a rim of a possible FO2PE of G, where the rim is a "partial embedding" of G. For a triangle uvw (resp., a 4-cycle $uvv'w$) in a graph G, the instance where edges uv and vw (resp., uv, vv' and $v'w$) are required to appear as outer edges is given by (G, B) with $B = \{uv, vw\}$ (resp., $B = \{uv, vv', v'w\}$). In what follows, we denote the constraint B simply by a vertex sequence $B = [u, v, w]$ (resp., $B = [u, v, v', w]$).

Our next aim is to design a procedure for constructing a possible FO2PE of G as an extension of the fixed rim. Suppose that **Algorithm EXTEND**(G, B) is a procedure that returns all FO2PE extensions of (G,B). By executing such a procedure to each candidate of rims, we can enumerate all FO2PE of a triconnected graph G, as described in **Algorithm TRICONNECTED FO2PE** below.

Algorithm TRICONNECTED FO2PE
Input: A triconnected simple graph G with maximum degree at most 4 and $n \geq 10$.
Output: The set Γ of all FO2PEs of G.
1: $\Gamma := \mathcal{B} := \emptyset$;
2: **if** G contains a triangle **then**
3: Choose a triangle t_1 in G;
4: **for** each triangle t' sharing an edge with t_1 (possibly $t' = t_1$) **do**
5: $\mathcal{B} := \mathcal{B} \cup \{[u,v,w],[v,w,u],[w,u,v]\}$ for the vertices u,v,w in triangle t'
6: **end for**;
7: **for** each $[u,v,w] \in \mathcal{B}$ for a triangle uvw **do**
8: $\Gamma := \Gamma \cup \{$**EXTEND**$(G,[u,v,v',w])\}$
9: **end for**
10: **else** /* G has no triangles */
11: **if** G contains a 4-cycle with two adjacent degree-3 vertices **then**
12: Choose a 4-cycle $u_1u_2u_3u_4$ with degree-3 vertices u_2 and u_3;
13: $\mathcal{B} := \{[u_1,u_2,u_3,u_4]\}$;
14: **if** u_1 and u_4 are degree-3 vertices **then**
15: $\mathcal{B} := \mathcal{B} \cup \{[u_3,u_4,u_1,u_2]\}$
16: **end if**;
17: **for** each $[u,v,v',w] \in \mathcal{B}$ for a 4-cycle $uvv'w$ **do**
18: $\Gamma := \Gamma \cup \{$**EXTEND**$(G,[u,v,v',w])\}$
19: **end for**
20: **end if**
21: **end if**; /* $|\Gamma| = O(1)$ */
22: Output Γ after discarding duplications in Γ.

Supposing Lemma 10, we show that the above algorithm correctly runs in $O(n)$ time. By Lemma 8, the set $\mathcal{B}$ of sequences of triangles/4-cycles is a candidate of a rim of some FO2PE extension of (G,B) if any. Hence the set $\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}$ contains all FO2PE extensions of (G,B). Clearly $|\mathcal{B}| = O(1)$ in each of lines 7 and 15. Then $\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}$ can be obtained in $O(n)$ time, where $|\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}| = O(|\mathcal{B}|) = O(1)$ by Lemma 10. We can test if two sequences in $\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}$ are the same cyclic order or not in $O(n)$ time. Since $|\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}| = O(1)$, we can output Γ after discarding duplications from $\{$**EXTEND**$(G,B) \mid B \in \mathcal{B}\}$ in $O(n)$ time. Now to prove Theorem 2, it suffices to show Lemma 10. In the next section, we show how to design **EXTEND**(G,B).

5.3 Reducing instances with fixed rims

In this section, we prove the following result by designing **EXTEND**(G,B).

Lemma 10. *For a triconnected instance (G,B) with a fixed rim, the maximum number of FO2PE extensions of (G,B) is constant, and all FO2PE extensions of (G,B) can be generated in $O(n)$ time.*

To prove Theorem 2, it suffices to show Lemma 10. We call an instance (G,B) *triconnected* if G is triconnected. To prove the lemma, we establish a reduction over triconnected instances (G,B) with fixed rims. We try to extend a given partial embedding (G,B) by fixing some other vertices, and simplify the instance with the newly fixed vertices into a triconnected instance (G',B') so that the new instance (G',B') admits an FO2PE extension if and only if so does the original instance.

For an instance (G,B), a sequence $[s_1,s_2,\ldots,s_k]$ is called *inevitable* if any FO2PE extension $\gamma = [v_1,v_2,\ldots,v_n]$ of (G,B) contains the sequence as its subsequence. Given an instance (G,B) with a fixed rim, we identify an inevitable sequence or a frill contained in any FO2PE extension of (G,B) without generating all possible permutations of the vertices in G. Based on the identified local structure of inevitable sequences or frills, we reduce (G,B) into a smaller new instance (G',B') with a new fixed rim B' such that (G,B) is extendible if and only if so is (G',B').

When we construct a new instance $(G' = G/X, B')$ by contracting a vertex subset X in G into a single vertex v^* and setting B' to be the set V' of a new triangle or 4-cycle, we call a vertex $v \in X$ an *attaching point* of (G', B') if each edge $e = uv^* \in E(v^*; G')$ corresponds to an edge $e \in E(v; G)$. We introduce how to reduce an instance with a fixed $(3, 3)$-rim.

Before we give proofs of Lemmas 12, 13 and 14, we introduce the following technical lemma.

Lemma 11. *Let G be a triconnected graph with $n \geq 8$, and let $B = [v_1, v_2, \ldots, v_p]$ ($p = 3$ or 4), where $B = [v_1, v_2, v_3]$ for a triangle $v_1v_2v_3$ with a degree-3 vertex v_2 and $N(v_2) = \{v_1, v_2, w\}$ (or a degree-4 vertex v_2 and $N(v_2) = \{v_1, v_2, w, w'\}$) or $B = [v_1, v_2, v_3, v_4]$ for a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 with $N(v_2) = \{v_1, v_3, w\}$ and $N(v_3) = \{v_2, v_4, w'\}$. Let γ be an FO2PE extension of (G, B), where we assume that $w \in V_{\partial\gamma}(v_1, w')$ when $p = 3$ and $\deg(v_2) = 4$. Assume that $V_{\partial\gamma}(w, v_1) \neq \emptyset$.*

(i) *If some edge $e = ab$ between a vertex $a \in V_{\partial\gamma}(w, v_1)$ and a vertex $b \in V_{\partial\gamma}[v_p, w)$ has no crossing with any edge $a'a''$ ($\neq v_1w$) with $a', a'' \in V_{\partial\gamma}[w, v_1]$, then it holds $V_{\partial\gamma}(w, v_1) = \{a\}$.*

(ii) *If $|V_{\partial\gamma}(w, v_1)| \geq 2$, then $v_1w \notin E$ holds, there is exactly one edge $e = ab$ between a vertex $a \in V_{\partial\gamma}(w, v_1)$ and a vertex $b \in V_{\partial\gamma}[v_p, w)$, and edge e crosses some edge $a'a''$ ($\neq v_1w$) with $a', a'' \in V_{\partial\gamma}[w, v_1]$.*

Proof. Since $n \geq 8$, embedding γ is quasi-planar by Lemma 2, and $w' \in V_{\partial\gamma}(v_4, w)$ holds for $p = 4$.

(i) Let $e = ab$ be an edge between a vertex $a \in V_{\partial\gamma}(w, v_1)$ and a vertex $b \in V_{\partial\gamma}[v_p, w)$ such that no edge $a'a''$ ($\neq v_1w$) with $a', a'' \in V_{\partial\gamma}[w, v_1]$ crosses e. Note that edge v_2w has two crossings on it and edge v_1v_p crosses only edge v_2w for $p = 3$ (edges v_2w and v_3w' for $p = 4$). Also now no edge $a'a''$ ($\neq v_1w$) with $a', a'' \in V_{\partial\gamma}[w, v_1]$ crosses e. Hence if $V_{\partial\gamma}(u, v) \neq \emptyset$ for a pair $(u, v) \in \{(w, a), (a, v_1)\}$, then (u, v) would be a cut-pair since any possible edge between $V_{\partial\gamma}(u, v)$ and $V_{\partial\gamma}(v, u)$ would create another crossing on edge v_2w or v_1v_p.

(ii) Now $|V_{\partial\gamma}(w, v_1)| \geq 2$. Since $\{v_1, w\}$ is not a cut-pair, there is an edge $e = ab$ between a vertex $a \in V_{\partial\gamma}(w, v_1)$ and a vertex $b \in V_{\partial\gamma}[v_p, w)$. By (i), edge e has a crossing with some edge $a'a''$ ($\neq v_1w$) with $a', a'' \in V_{\partial\gamma}[w, v_1]$. However, in this case, e would cross three edges v_2w, $a'a''$ and v_1w if $v_1w \in E$. Hence $v_1w \notin E$ also holds. □

Lemma 12. **($(3, 3)$-rim reduction)** *Let (G, B) be a triconnected extendible instance with $n \geq 7$ for a fixed $(3, 3)$-rim $B = [v_1, v_2, v_3]$ with $N(v_2) = \{v_1, v_2, w\}$. Then one of the following conditions* (i) *and* (ii) *holds, and the instance (G', B') defined in each condition is triconnected and extendible.*

(i) *Assume that v_1 or v_3, say v_1 is a degree-4 vertex adjacent to w. (See Fig. 3.) Then $[w, v_1, v_2, v_3]$ is inevitable to (G, B). Let $G' = G/\{v_2, v_3\}$ and $B' = [w, v_1, v^*]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = w, u_2 = v_1, u_3 = v^*, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [w, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$.*

(ii) *Assume that v_1 or v_3, say v_1 is a degree-3 vertex not adjacent to w. (See Fig. 4.) Then $[z, v_1, v_2, v_3]$ is inevitable to (G, B). Let $G' = G/\{z, v_1\}$ and $B' = [v^*, v_2, v_3]$. Any FO2PE extension of (G, B) is obtained by modifying any FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$.*

Proof. Let $\gamma = [v_1, v_2, \ldots, v_n]$ be an arbitrary FO2PE extension of (G, B). By Lemma 3, each vertex in G is of degree 3 or 4. If $|V_{\partial\gamma}(w, v_1)| \geq 2$ and $|V_{\partial\gamma}(v_3, w)| \geq 2$, then γ has a (w, v_2)-hooked edge ab ($\neq v_1v_3$) and a (v_2, w)-hooked edge $a'b'$ ($\neq v_1v_3$) by Lemma 11(ii). However, if $ab = a'b'$ then the edge would have three crossings; otherwise ($ab \neq a'b'$) edge v_2w would get three crossings, a contradiction in any way. Hence we have $|V_{\partial\gamma}(w, v_1)| \leq 1$ or $|V_{\partial\gamma}(v_3, w)| \leq 1$. First consider the case where v_1 or v_3, say v_1 is adjacent to w. If $\deg(v_1) = 3$, then $V_{\partial\gamma}(v_3, w) \neq \emptyset$ by $n \geq 7$ and $\{v_3, w\}$ would be a cut-pair. Hence $\deg(v_1) = 4$, satisfying condition (i).

Next assume that neither of v_1 and v_3 is adjacent to w. Recall that $|V_{\partial\gamma}(w, v_1)| \leq 1$ or $|V_{\partial\gamma}(v_3, w)| \leq 1$. Assume without loss of generality that $|V_{\partial\gamma}(w, v_1)| \leq 1$. Since v_1 is not adjacent to w, we see that $\deg(v_1) = 3$, satisfying condition (ii).

(i) Assume that v_1 or v_3, say v_1 is a degree-4 vertex adjacent to w. To prove that $[w, v_1, v_2, v_3]$ is inevitable to (G, B), it suffices to show that $V_{\partial\gamma}(w, v_1) = \emptyset$ in γ. Since $v_1w \in E$, it holds $|V_{\partial\gamma}(w, v_1)| \leq 1$ by Lemma 11(ii). To derive a contradiction, assume that $|V_{\partial\gamma}(w, v_1)| = 1$, where an edge $e = ab$ joins

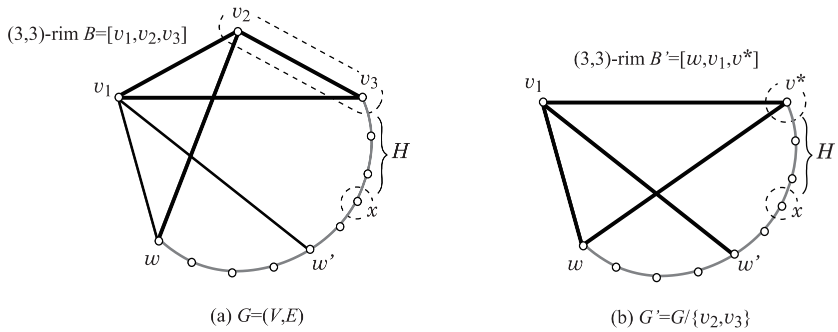

Fig. 3. Illustration for the reduction in Lemma 12(i) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 3)$-rim of a triangle $v_1v_2v_3$ with a degree-3 vertex v_2 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-3 vertex adjacent to w; (b) a new instance $(G' = G/\{v_2, v_3\}, B' = [w, v_1, v^*])$ with a new $(3, 3)$-rim of triangle wv_1v^* with a degree-3 vertex v_1.

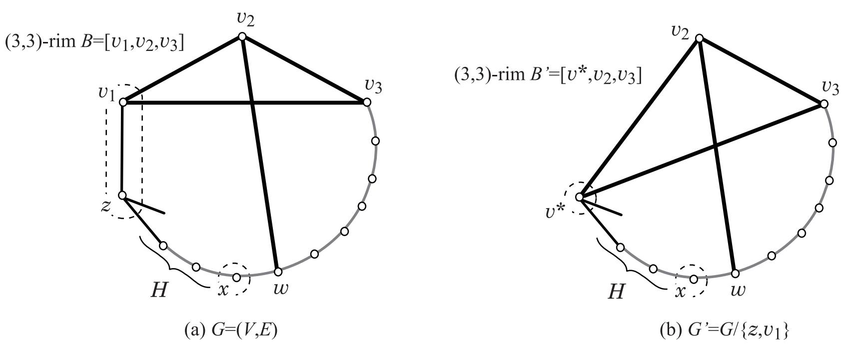

Fig. 4. Illustration for the reduction in Lemma 12(ii) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 3)$-rim of a triangle $v_1v_2v_3$ with a degree-3 vertex v_2 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-3 vertex not adjacent to w; (b) a new instance $(G' = G/\{z, v_1\}, B' = [v^*, v_2, v_3])$ with a new $(3, 3)$-rim of triangle $v^*v_2v_3$ with a degree-3 vertex v_2.

the vertex $a \in V_{\partial\gamma}(w, v_1)$ and a vertex $b \in V_{\partial\gamma}[v_3, w)$ since $\{w, v_1\}$ is not a cut-pair. Also by $n \geq 7$, it holds $|V_{\partial\gamma}(v_2, w)| \geq 3$, and there is a (v_2, w)-hooked edge $e' = a'b'$ that joins a vertex $a' \in \{a, v_1\}$ and a vertex $b' \in V_{\partial\gamma}[v_3, w)$. However, if $e' = e$ then the edge would have three crossings; otherwise ($e \neq e'$) edge v_2w would get three crossings, a contradiction in any way. Therefore $V_{\partial\gamma}(w, v_1) = \emptyset$, as required.

We show that $G' = G/\{v_2, v_3\}$ is triconnected, If G' is not triconnected, i.e., there is a pair of vertices u and u' such that $G' - \{u, u'\}$ is disconnected, then $v^* \in \{u, u'\}$ holds, since otherwise a component H in $G' - \{u, u'\}$ not containing v^* still can be separated in $G - \{u, u'\}$, contradicting the triconnectivity of G. Therefore, to show that $G' = G/\{v_2, v_3\}$ is triconnected, it suffices to show that $G - \{v_2, v_3, x\}$ remains connected for any vertex x in G. Let $N(v_1; G) = \{v_2, v_3, w, w'\}$. Note that v_3 is an attaching point of (G', B') (i.e., any edge $e = uv^* \in E(v^*; G') - \{v_1v^*, wv^*\}$ corresponds to an edge $e \in E(v_3; G)$). Since γ is a Hamiltonian cycle where v_1 and v_2 appear consecutively, if $G-\{v_2, v_3, x\}$ is not connected then one of the components, say H in the graph is given by $G[V_{\partial\gamma}(x, v_1]]$, where $w' \in V_{\partial\gamma}(x, w)$ by $v_1w' \in E$. Since v_3 is an attaching point of (G', B'), H is still a component of $G - \{v_3, x\}$, contradicting the triconnectivity of G.

We next show that $(G', B' = [w, v_1, v^*])$ is extendible. Let γ'' be the cyclic order obtained from $\gamma = [v_1, v_2, \ldots, v_n]$ by replacing v_2 and v_3 with v^*. Then $C_{\partial\gamma''} = \emptyset$ holds, $B' = [w, v_1, v^*]$ is a $(3, 3)$-rim with degree-3 vertex v_1 in γ'', and each edge not in the new triangle wv_1v^* has the same number of crossing on it, implying that γ'' is a FO2PE extension of (G', B').

Conversely for any FO2PE extension $\gamma' = [u_1 = w, u_2 = z, u_3 = v^*, u_4, \ldots, u_{n'}]$ of (G', B'), let $\gamma = [w, z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$ be the cyclic order obtained from γ' by replacing subsequence $[z, v^*, u_4]$ with subsequence $[z, v_1, v_2, v_3, u_4]$. In γ, no new crossing is introduced by the expansion of v^* into $\{v_2, v_3\}$ because v_3 is an attaching point of (G', B'). Hence γ is an FO2PE extension of (G, B). The way of constructing γ from γ' is the reverse operation of the way of constructing the above FO2PE extension γ'' of (G', B') from an FO2PE extension γ of (G, B). Hence if $\gamma' = \gamma''$, then the original FO2PE extension γ can be obtained from γ''. This means that any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension γ' of (G', B').

(ii) Assume that v_1 or v_3, say v_1 is a degree-3 vertex not adjacent to w. Since $\deg(v_1) = 3$, the remaining incident edge zv_1 must be an outer edge in any FO2PE extension of (G, B), and $[z, v_1, v_2, v_3]$ is inevitable to (G, B). Note that vertex z is an attaching point of $(G' = G/\{z, v_1\}, B' = [v^*, v_2, v_3])$.

To prove that $G' = G/\{z, v_1\}$ is triconnected, it suffices to show that $G-\{z, v_1, x\}$ remains connected for any vertex x in G. Since γ is a Hamiltonian cycle where z and v_1 appear consecutively, if $G-\{z, v_1, x\}$ is not connected then one of the components, say H in the graph is given by $G[V_{\partial\gamma}(x, v_1]]$, and $w \notin V_{\partial\gamma}(x, z) \supseteq N(z) - \{v_1\}$ holds. Since z is an attaching point of (G', B'), H is still a component of $G - \{z, x\}$, contradicting the triconnectivity of G.

Analogously with (i), we can show that (G', B') is extendible and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$. □

The next lemma provides how to reduce an instance with a fixed $(3, 4)$-rim. Note that for an instance $(G, B = [v_1, v_2, v_3])$ with $N(v_2) = \{v_1, v_2, w_1, w_2\}$ for a $(3, 4)$-rim, we do not know the order of vertices w_1 and w_2 along the boundary of an FO2PE extension of (G, B).

Lemma 13. **($(3, 4)$-rim reduction)** *Let (G, B) be a triconnected extendible instance with $n \geq 7$ for a fixed $(3, 4)$-rim $B = [v_1, v_2, v_3]$ with $N(v_2) = \{v_1, v_2, w_1, w_2\}$. Then one of the following conditions* (i)-(iv) *holds, and the instance (G', B') defined in each condition is triconnected and extendible.*

(i) *Assume that v_1 or v_4, say v_1 is a degree-3 vertex, where $N(v_1) = \{v_2, v_3, z\}$. (See Fig. 5.) Then $[z, v_1, v_2, v_3]$ is inevitable to (G, B). Let $G' = G/\{v_1, z\}$ and $B' = [v^*, v_2, v_3]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$.*

(ii) *Assume that v_1 or v_4, say v is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w, and there is a pair of a degree-3 vertex z and a vertex y such that vwz and wzy are triangles. Let $v = v_1$ without loss of generality. (See Fig. 6.) Then any FO2PE extension of (G, B) has zw as a frill. Let $G' = G/\{y, z, w, v_1\}$ and $B' = [v^*, v_2, v_3]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [y, z, w, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$ and $[y, w, z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$.*

(iii) *Assume that v_1 or v_4, say v is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w, but there is no pair of a degree-3 vertex z and a vertex y such that vwz and wzy are triangles. Let $(v, w) = (v_1, w_2)$ without loss of generality.* (*See Fig. 7.*) *Then $[w_2, v_1, v_2, v_3]$ is inevitable to (G, B). Let G' be the graph obtained from G by replacing edges v_1v_3 and v_2w_2 with a new edge w_1v_3, and $B' = [w_2, v_1, v_2, v_3]$. Any FO2PE extension of (G, B) is obtained as an FO2PE extension $\gamma' = [u_1, u_2, u_3, u_4, \ldots, u_{n'}]$ of (G', B').*

(iv) *Assume that none of the above conditions* (i)-(iii) *holds and there is an edge $z_1z_2 \in E$ between two degree-3 vertices $z_1 \in N(w)$ and $z_2 \in N(w')$ for $\{w, w'\} = \{w_1, w_2\}$ or a degree-4 vertex $z \in N(w_1) \cap N(w_2)$.* (*See Fig. 8.*) *Then any FO2PE extension of (G, B) contains exactly one of $[w, z_1, z_2, w']$ and $[w', z_2, z_1, w]$* (*or exactly one of $[w, z, w']$ and $[w', z, w]$*) *as a sequence. Let G' be the graph obtained from G by removing vertex v_2 and adding a new edge w_1w_2, and $B' = [w, z_1, z_2, w']$* (*or $B' = [w_1, z, w_2]$*)*. Vertices v_1 and v_3 appear consecutively in any FO2PE extension γ' of (G', B'). Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v_1, u_2 = v_3, u_3, \ldots, u_{n'}]$ of (G', B') into $\gamma = [v_1, v_2, v_3, u_3, \ldots, u_{n'}]$.*

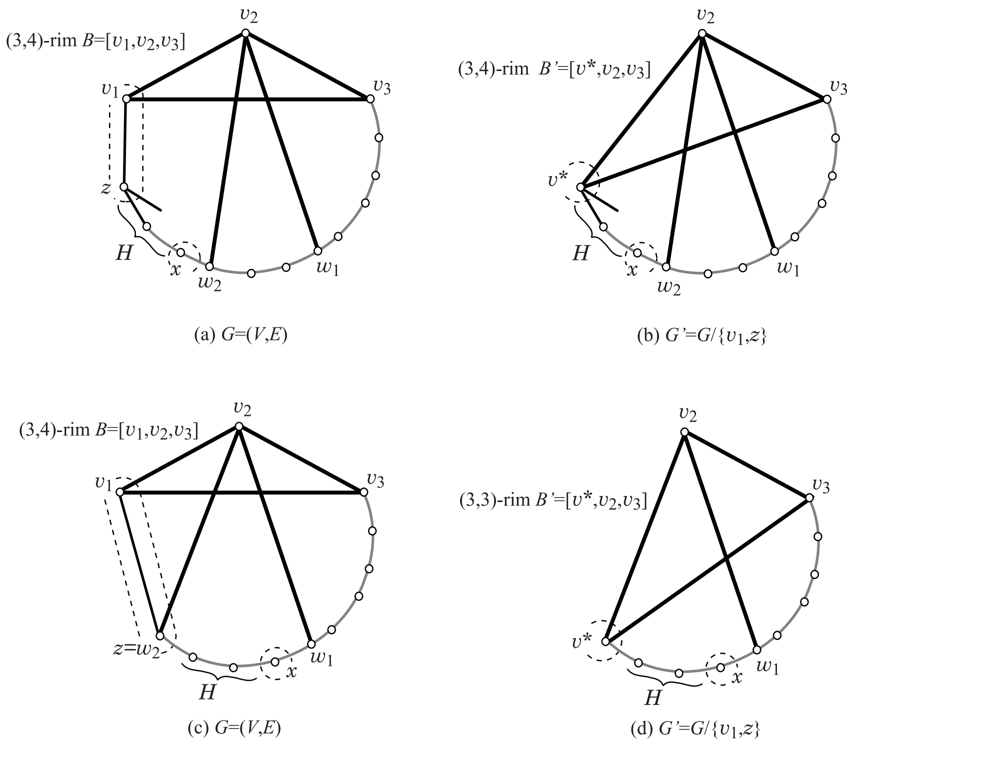

Fig. 5. Illustration for the reduction in Lemma 13(i) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 4)$-rim of a triangle $v_1v_2v_3$ with a degree-4 vertex v_2 and a degree-3 vertex v_1 ($N(v_1) = \{v_2, v_3, z\}$) to a new instance (G', B'): (a) a graph G such that $z \notin \{w_1, w_2\}$; (b) a new instance $(G' = G/\{z, v_1\}, B' = [v^*, v_2, v_3])$ with a new $(3, 4)$-rim of triangle $v^*v_2v_3$ with a degree-4 vertex v_2; (c) a graph G such that $z \in \{w_1, w_2\}$; (d) a new instance $(G' = G/\{z, v_1\}, B' = [v^*, v_2, v_3])$ with a new $(3, 3)$-rim of triangle $v^*v_2v_3$ with a degree-3 vertex v_2.

Proof. (i) Since $\deg(v_1) = 3$, clearly $[z, v_1, v_2, v_3]$ is inevitable to (G, B). Note that vertex z is an attaching point of $(G' = G/\{z, v_1\}, B' = [v^*, v_2, v_3])$. Analogously with the proof of Lemma 12(i)-(ii), we can show that (G', B') is triconnected and extendible and that for any FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B'), $\gamma = [z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$ is an FO2PE extension of (G, B).

(ii) Assume that v_1 is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w, and there is a pair of a degree-3 vertex z and a vertex y such that v_1wz and wzy are triangles. Let $w' \in N(v_1) - \{v_2, v_3, w\}$.

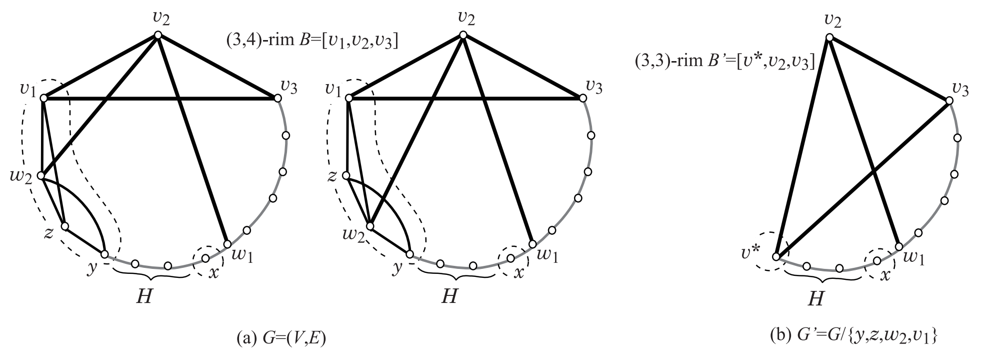

Fig. 6. Illustration for the reduction in Lemma 13(ii) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 4)$-rim of a triangle $v_1v_2v_3$ with a degree-4 vertex v_2 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w_2, and there is a pair of a degree-3 vertex z and a vertex y such that v_1w_2z and w_2zy are triangles; (b) a new instance $(G' = G/\{y, z, w_2, v_1\}, B' = [v^*, v_2, v_3])$ with a new $(3, 3)$-rim of triangle $v^*v_2v_3$ with a degree-3 vertex v_2.

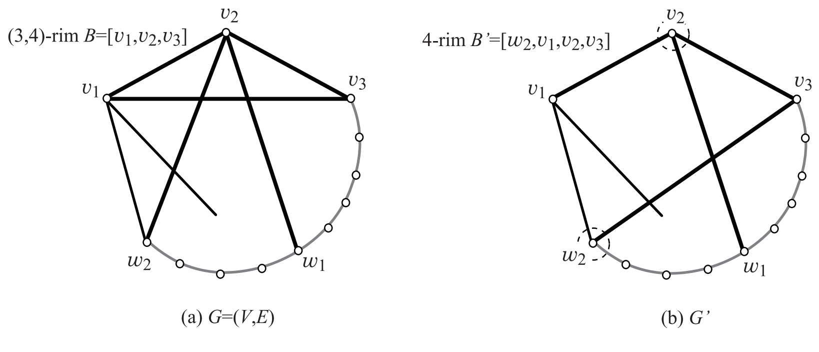

Fig. 7. Illustration for the reduction in Lemma 13(iii) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 4)$-rim of a triangle $v_1v_2v_3$ with a degree-4 vertex v_2 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w_2, but there is no pair of a degree-3 vertex z and a vertex y such that v_1w_2z and w_2zy are triangles; (b) a new instance $(G', B' = [w_2, v_1, v_2, v_3])$ with a new 4-rim of 4-cycle $w_2v_1v_2v_3$ with degree-3 vertices v_1 and v_2, where G' is obtained from G by replacing edges v_1v_3 and v_2w_2 with a new edge w_1v_3.

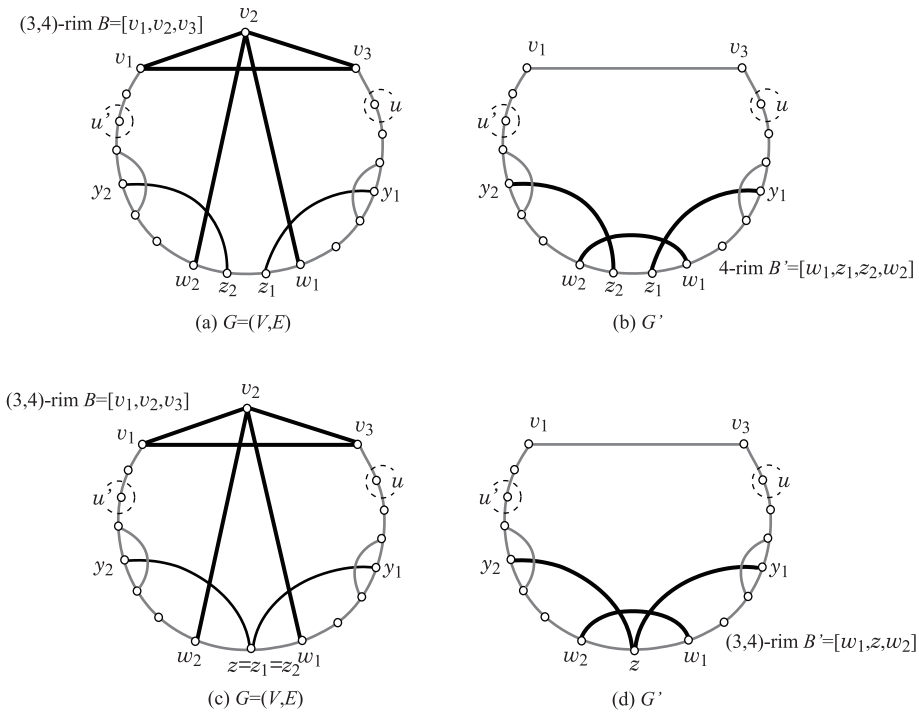

Fig. 8. Illustration for the reduction in Lemma 13(iv) from an instance $(G, B = [v_1, v_2, v_3])$ with a fixed $(3, 4)$-rim of a triangle $v_1v_2v_3$ with a degree-4 vertex v_2 to a new instance (G', B'): (a) a graph G such that none of conditions (i)-(iii) in Lemma 13 holds and there is an edge $z_1z_2 \in E$ between two degree-3 vertices $z_1 \in N(w)$ and $z_2 \in N(w')$ for $\{w, w'\} = \{w_1, w_2\}$; (b) a new instance $(G', B' = [w, z_1, z_2, w'])$ with a new 4-rim of 4-cycle wz_1z_2w' with degree-3 vertices z_1 and z_2, where G' is obtained from G by removing vertex v_2 and adding a new edge w_1w_2; (c) a graph G such that none of conditions (i)-(iii) in Lemma 13 holds and there is a degree-4 vertex $z \in N(w_1) \cap N(w_2)$; (d) a new instance $(G', B' = [w_1, z, w_2])$ with a new $(3, 4)$-rim of 3-cycle wzw' with a degree-3 vertex z, where G' is obtained from G by removing vertex v_2 and adding a new edge w_1w_2.

Let $\gamma = [v_1, v_2, \ldots, v_n]$ be an arbitrary FO2PE extension of (G, B), where $w_2 \in V_{\partial\gamma}(w_1, v_1)$ without loss of generality. Note that vertex y is an attaching point of $(G' = G/\{y, z, w, v_1\}, B' = [v^*, v_2, v_3])$.

We first show that γ contains $[y, z, w, v_1, v_2]$ or $[y, w, z, v_1, v_2]$ as a subsequence. Clearly γ contains $[w, v_1, v_2]$ or $[z, v_1, v_2]$, since it is Hamiltonian. Consider the case where γ contains $[w, v_1, v_2]$. Since $N(z) = \{v_1, w, y\}$, vertices y, z and w must appear consecutively in this order (otherwise z would have only one outer edge incident to it in γ). Similarly when γ contains $[z, v_1, v_2]$ we see from $N(w) = \{v_1, v_2, z, y\}$ that vertices y, w and z appear consecutively in this order (otherwise w would have only one outer edge incident to it in γ). Therefore γ contains frill zw, and $w = w_2$ for the vertex $w_2 \in V_{\partial\gamma}(w_1, v_1)$.

We next show that $G' = G/\{y, z, w, v_1\}$ is triconnected. For this, it suffices to show that $G - \{y, z, w = w_2, v_1, x\}$ remains connected for any vertex x in G. Since γ is a Hamiltonian cycle where the vertices y, $\{z, w_2\}$ and v_1 appear consecutively in this order, if $G - \{y, z, w = w_2, v_1, x\}$ is not connected then one of the components, say H in the graph is given by $G[V_{\partial\gamma}(x, y)]$, where $w_1 \in V_{\partial\gamma}(v_3, x)$ holds by $v_2w_1 \in E$. Since y is an attaching point of (G', B'), H is still a component of $G - \{y, x\}$, contradicting the triconnectivity of G.

Analogously with the proof of Lemma 12(i)-(ii), we can show that $(G', B' = [v^*, v_2, v_3])$ is extendible, and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $[y, z, w, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$ and $[y, w, z, v_1, v_2, v_3, u_4, \ldots, u_{n'}]$.

(iii) Assume that v_1 is a degree-4 vertex adjacent to exactly one of w_1 and w_2, say w, but there is no pair of a degree-3 vertex z and a vertex y such that vwz and wzy are triangles. Let $(v, w) = (v_1, w_2)$ without loss of generality. Let $z \in N(v_1) - \{v_2, v_3, w\}$, and $\gamma = [v_1, v_2, \ldots, v_n]$ be an arbitrary FO2PE extension of (G, B).

We first claim that edge v_1w is an outer edge in γ. Otherwise by applying Lemma 11(ii) to inner edge v_2w, we see that $V_{\partial\gamma}(w, v_1) = \{z\}$ and z has exactly one inner edge za incident to it. Now $\deg(z) = 3$ and v_1zw is a triangle, but we see that a is not adjacent to w, since wz is not in two triangles by the assumption on (iii). Hence $V_{\partial\gamma}(a, w) \neq \emptyset$, which, however, implies that $\{a, w\}$ is a cut-pair since each of edges wv_2 and za has already two crossings. This proves the claim, and $[w, v_1, v_2, v_3]$ is inevitable to (G, B). Then $w = w_2$ without loss of generality, where $w_2 \in V_{\partial\gamma}(w_1, v_1)$ holds.

Let G' be the graph obtained from G by replacing edges v_1v_3 and v_2w_2 with a new edge w_1v_3. To show that G' remains triconnected, we assume that G' has a cut-pair $\{u, u'\}$. We remove edges v_1v_3 and v_2w_2 and add a new edge w_1v_3 in the FO2PE extension $\gamma = [v_1, v_2, \ldots, v_n]$ of (G, B). Since the same set of outer edges still forms a Hamiltonian cycle in the resulting embedding, we see that u and u' are not consecutive along the cycle and both of them must be contained in $V_{\partial\gamma}[v_3, w_2]$ or $V_{\partial\gamma}[w_2, v_3]$ in G'. In the former case, the component H in $G' - \{u, u'\}$ with $V(H) \subseteq V_{\partial\gamma}[v_3, w_2]$ would be separated in $G - \{u, u'\}$, contradicting the triconnectivity of G. In the latter, $\{u, u'\}$ is given by $\{w_2, v_2\}$ or $\{v_1, v_3\}$, which, however cannot be a cut-pair in G' due to edges v_1w' and v_2w_1. This proves that G' is triconnected.

Any edge e incident to a vertex in $V(G) - \{w_2, v_1, v_2, v_3\} = V(G') - \{w_2, v_1, v_2, v_3\}$ has the same number crossings in γ even for G', implying that γ is also an FO2PE extension of $(G', B' = [w_2, v_1, v_2, v_3])$. Hence (G', B') is extendible. Similarly for any FO2PE extension $\gamma' = [u_1, u_2, \ldots, u_n]$ of (G', B'), any edge e incident to a vertex in $V(G) - \{w_2, v_1, v_2, v_3\}$ has the same number crossings in γ' even for G, γ' is also an FO2PE extension of (G, B).

(iv) Assume that in G, each of v_1 and v_3 is a degree-4 vertex which is adjacent to both of w_1 and w_2 or neither of them in G. Let $\gamma = [v_1, v_2, \ldots, v_n]$ be an arbitrary FO2PE extension of (G, B), where $w_2 \in V_{\partial\gamma}(w_1, v_1)$ without loss of generality. By Lemma 3, each vertex in G is of degree 3 or 4.

We first claim that neither of v_1 and v_4 is adjacent to both of w_1 and w_2. To derive a contradiction, let $N(v_1) = \{v_2, v_4, w_1, w_2\}$. Then $V_{\partial\gamma}(w_2, v_1) = \emptyset$. If $|V_{\partial\gamma}(w_1, v_1)| \geq 3$ (resp., $|V_{\partial\gamma}(v_2, w_1)| \geq 3$), then γ would have a (w_2, v_1)-hooked edge (resp., a (v_2, w_1)-hooked edge) e, which, however crosses edges w_1v_1 and w_1v_2 too, a contradiction. Hence $|V_{\partial\gamma}(w_1, v_1)| \leq 2$ and $|V_{\partial\gamma}(v_2, w_1)| \leq 2$, where $|V_{\partial\gamma}(w_1, w_2)| = |V_{\partial\gamma}(v_3, w_1)| = 1$ holds by $n \geq 7$ and an edge ab joins the vertex $a \in V_{\partial\gamma}(w_1, w_2)$ and the vertex $b \in V_{\partial\gamma}(v_3, w_1)$. However, the edge v_3x with $x \in N(v_3) - \{v_1, v_2, b\}$ crosses edge ab or edge w_1v_2, creating the third crossing there, a contradiction. This proves the claim. Now each of v_1 and v_3 is a degree-4 vertex which is adjacent to neither of w_1 and w_2 in G.

Let the two neighbors x_1 and x_2 in $N(v_1) - \{v_2, v_3\}$ appear in this order along $\partial\gamma(v_3, v_1)$. We show that $x_1, x_2 \in V_{\partial\gamma}(w_2, v_1)$. Since v_1 is not adjacent to w_2, we have $x_2 \in V_{\partial\gamma}(w_2, v_1)$. If $x_1 \in V_{\partial\gamma}(v_3, w_2)$, then an edge ab joins a vertex $a \in V_{\partial\gamma}(w_2, v_1)$ and a vertex $b \in V_{\partial\gamma}(v_1, w_2)$ since $\{w_2, v_1\}$

is not a cut-pair. However, edge ab creates the third crossing on edge w_2v_2. Hence we have $\{x_1, x_2\} \subseteq V_{\partial\gamma}(w_2, v_1)$. Symmetrically we have $N(v_3) - \{v_2, v_3\} \subseteq V_{\partial\gamma}(v_3, w_1)$. Since $|V_{\partial\gamma}(w_2, v_2)| \geq 3$ (resp., $|V_{\partial\gamma}(v_2, w_1)| \geq 3$), there is a (w_2, v_2)-hooked edge y_2z_2 between $y_2 \in V_{\partial\gamma}(w_2, v_1)$ and $z_2 \in V_{\partial\gamma}(v_3, w_2)$ (resp., a (v_2, w_1)-hooked edge y_1z_1 between $y_1 \in V_{\partial\gamma}(v_3, w_1)$ and $z_1 \in V_{\partial\gamma}(w_1, v_1)$). In fact, it must hold that $z_1 \in V_{\partial\gamma}(w_1, z_2]$ and $z_2 \in V_{\partial\gamma}[z_1, w_2)$ since otherwise one of edges v_2w_1, v_2w_2, y_1z_1 and y_2z_2 would get three crossings. Note that possibly $z_1 = z_2$. Since each of these four edges already has two crossings, we see that $V_{\partial\gamma}(w_1, w_2) = \{z_1, z_2\}$ (otherwise one of $\{w_1, z_1\}$, $\{z_1, z_2\}$ and $\{z_2, w_2\}$ would be a cut-pair), and that $\deg(z_1) = \deg(z_2) = 3$ when $z_1 \neq z_2$. We easily see that there is no other pair $\{z_1', z_2'\}$ than $\{z_1, z_2\}$ which satisfies condition (iv), since otherwise edge y_1z_1 would further cross some edge in the cycle $w_1z_1z_2w_2z_2'z_1'$ (in other words, if a vertex pair $\{z_1', z_2'\}$ satisfies condition (iv) then $\{z_1', z_2'\} = \{z_1, z_2\}$). Therefore for $\{w, w'\} = \{w_1, w_2\}$, any FO2PE extension of (G, B) contains exactly one of $[w, z_1, z_2, w']$ and $[w', z_2, z_1, w]$ (when $z_1 \neq z_2$) or exactly one of $[w_1, z, w_2]$ and $[w_2, z, w_1]$ (when $z = z_1 = z_2$) as a sequence.

Let G' be the graph obtained from $G - v_2$ by adding a new edge w_1w_2, and $B' = [w, z_1, z_2, w']$ (or $B' = [w, z, w']$). We show that (G', B') is triconnected and extendible. Given any FO2PE extension $\gamma = [v_1, v_2, v_3, \ldots, v_n]$ of (G, B), we easily see that $\gamma'' = [v_1, v_3, \ldots, v_n]$ is an FO2PE extension of (G', B'), since the added edge w_1w_2 has two crossings with edges y_1z_1 and y_2z_2. Hence (G', B') is extendible.

To prove the triconnectivity of G', we assume that G' has a cut-pair $\{u, u'\}$. In γ'', only a vertex pair $\{u, u'\}$ such that $|\{u, u'\} \cap \{a, b\}| \in \{0, 2\}$ for any inner edge $ab \in \{y_1z_1, y_2z_2, w_1w_2\}$ can be a cut-pair in G'. Thus, $\{u, u'\}$ is contained in one of $V_{\partial\gamma''}[y_1, w_1]$, $V_{\partial\gamma''}[w_2, w_2]$ and $V_{\partial\gamma''}[y_2, y_1]$. Also if $\{u, u'\} \subseteq V_{\partial\gamma''}[v_3, w_1]$ or $\{u, u'\} \subseteq V_{\partial\gamma''}[w_2, v_1]$, then clearly $\{u, u'\}$ is also a cut-pair in G. Hence it must hold that $u \in V_{\partial\gamma''}[v_3, y_1]$, $u' \in V_{\partial\gamma''}[y_2, v_1]$ and $\{u, u'\} \neq \{v_1, v_3\}$. Let H be the component in $G' - \{u, u'\}$ containing vertex v_1 or v_3, say v_3. Note that no vertex in $V_{\partial\gamma''}[v_3, u)$ has a neighbor in $V_{\partial\gamma''}[y_2, v_1)$ since edge v_2w_2 has two crossings in γ. Consider the vertex set $V_{\partial\gamma''}(v_3, u) \subseteq V(H)$, where $V_{\partial\gamma''}(v_3, u) \neq \emptyset$ since $\deg(v_3; G') = 3$ and v_3 has no neighbor in $V_{\partial\gamma''}[y_2, v_1)$. This means that the vertex set $V_{\partial\gamma''}(v_3, u)$ will be separated in $G - \{v_3, u\}$, contradicting the triconnectivity of G.

Finally we show how to construct an FO2PE extension of (G, B) from an FO2PE extension γ' of (G', B') after deriving an important property on γ'. We first examine the graph structure of (G, B) which admits an FO2PE extension γ. Let $A_1 = V_{\partial\gamma}(v_3, z_1)$ and $A_2 = V_{\partial\gamma}(z_2, v_1)$ when $z_1 \neq z_2$, and $A_1 = V_{\partial\gamma}(v_3, z)$ and $A_2 = V_{\partial\gamma}(z, v_1)$ when $z_1 \neq z_2$. Consider the case of $z_1 \neq z_2$ (the case of $z = z_1 = z_2$ can be treated analogously). Without loss of generality denote $w \in N(z_1)$ by w_1 and $w' \in N(z_1)$ by w_2. Then $B' = [w, z_1, z_2, w'] = [w_1, z_1, z_2, w_2]$, and any FO2PE extension γ' of (G', B') contains $[w_1, z_1, z_2, w_2]$ as a subsequence by definition. Then G' has only two edges between A_1 and A_2, i.e., edges z_1z_2 and v_1v_3. This means that the vertices in A_1 appear in some order consecutively along $\partial\gamma'$ of any FO2PE extension γ' of (G', B'), since otherwise a crossing would be generated on the boundary $\partial\gamma'$. Thus any FO2PE extension $\gamma' = [u_1 = v_1, u_2 = v_3, u_3, \ldots, u_{n'}]$ of (G', B') satisfies $A_1 \subseteq V_{\partial\gamma'}(u_3, u_i)$ for $u_i = z_1$ and $A_2 \subseteq V_{\partial\gamma'}(u_{i+1} = z_2, u_1)$. In particular $v_1, v_3, w_1, z_1, z_2, w_2$ appear in this order and v_1 and v_3 appear consecutively along $\partial\gamma'$ (recall that vertices w_1, z_1, z_2, w_2 appear in this order in an FO2PE extension γ of (G, B)). Note that there is no edge between $V_{\partial\gamma'}(u_3, u_i = z_1)$ and $V_{\partial\gamma'}(u_{i+1} = z_2, u_1)$. Therefore the cyclic order $\gamma = [v_1, v_2, v_3, u_3, \ldots, u_{n'}]$ obtained from γ' by inserting v_2 between $u_1 = v_1$ and $u_2 = v_2$ is an FO2PE extension of (G, B), since the edge w_1w_2 is replaced with edges w_1v_2 and w_2v_2 in G without creating any new crossings on the other edges in G. The way of constructing γ from γ' is the reverse operation of the way of constructing the above FO2PE extension γ'' of (G', B') from an FO2PE extension γ of (G, B). Hence any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension γ' of (G', B').

We here remark that computing the sets A_1 and A_2 would take $\Omega(n)$ time. However, without knowing $\{A_1, A_2\}$, in particular for the case of $z = z_1 = z_2$, we can reduce (G, B) into (G', B') only by identifying z_1 and z_2 (or $z = z_1 = z_2$), which can be done in $O(1)$ time. $\square$

The next lemma provides how to reduce an instance with a fixed 4-rim. Note that for an instance $(G, B = [v_1, v_2, v_3, v_4])$ with $N(v_2) = \{v_1, v_3, w_2\}$ and $N(v_3) = \{v_2, v_4, w_1\}$ for a 4-rim, we see that w_1 and w_2 appear always in this order after vertices v_1, v_2, v_3, v_4 appear along the boundary of any "quasi-planar" FO2PE extension of (G, B).

Lemma 14. **(4-rim reduction)** *Let (G, B) be a triconnected extendible instance with $n \geq 8$ for a fixed 4-rim $B = [v_1, v_2, v_3, v_4]$ with $N(v_2) = \{v_1, v_3, w_2\}$ and $N(v_3) = \{v_2, v_4, w_1\}$ (possibly $w_1 = w_2$). Then one of the following conditions* (i)-(v) *holds, and the instance (G', B') defined in each condition is triconnected and extendible.*

(i) *Assume that v_1 or v_4, say v_1 is a degree-3 vertex adjacent to neither of w_1 and w_2. (See Fig. 9.) Then for $z \in N(v_1) - \{v_2, v_3\}$, $[z, v_1, v_2, v_3, v_4]$ is inevitable to (G, B). Let $G' = G/\{v_1, z\}$ and $B' = [v^*, v_2, v_3, v_4]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4 = v_4, u_5, \ldots, u_{n'}]$ of (G', B') into $\gamma = [z, v_1, v_2, v_3, v_4, u_5, \ldots, u_{n'}]$.*

(ii) *Assume that for $(v, w) = (v_1, w_2)$ or (v_4, w_1), v is a degree-3 vertex adjacent to w. Let $(v, w) = (v_1, w_2)$ without loss of generality. (See Fig. 10.) Then $[w_2, v_1, v_2, v_3, v_4]$ is inevitable to (G, B). Let $G' = G/\{w_2, v_1, v_2\}$ and $B' = [v^*, v_3, v_4]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_3, u_3 = v_4, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [w_2, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$.*

(iii) *Assume that for $(v, w) = (v_1, w_2)$ or (v_4, w_1), v is a degree-4 vertex adjacent to w, and there is a pair of a degree-4 vertex z and a vertex y such that vwz and wzy are triangles. Let $(v, w) = (v_1, w_2)$ without loss of generality. (See Fig. 11.) Then any FO2PE extension $\gamma = [v_1, v_2, \ldots, v_n]$ of (G, B) has zw_2 as a frill. Let $G' = G/\{y, z, w_2, v_1, v_2\}$ and $B' = [v^*, v_3, v_4]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_3, u_3 = v_4, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [y, z, w_2, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$ and $[y, w_2, z, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$.*

(iv) *Assume that for $(v, w) = (v_1, w_2)$ or (v_4, w_1), v is a degree-4 vertex adjacent to w, but there is no pair of a degree-4 vertex z and a vertex y such that vwz and wzy are triangles. Let $(v, w) = (v_1, w_2)$ without loss of generality. (See Fig. 12.) Then $[w_2, v_1, v_2, v_3, v_4]$ is inevitable to (G, B). Let G' be the graph obtained from G by replacing edges v_1v_4 and v_2w_2 with a new edge w_2v_4 and contracting v_1 and v_2 into a single vertex v^*, and $B' = [w_2, v^*, v_3, v_4]$. Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = w_2, u_2 = v^*, u_3 = v, u_4 = v_4, u_5, \ldots, u_{n'}]$ of (G', B') into $\gamma = [w_2, v_1, v_2, v_3, v_4, u_5, \ldots, u_{n'}]$.*

(v) *Assume that none of the above conditions* (i)-(iv) *holds, $w_1 \neq w_2$, and there is an edge $z_1z_2 \in E$ between two degree-3 vertices $z_1 \in N(w_1)$ and $z_2 \in N(w_2)$ (there is a degree-4 vertex $z \in N(w_1) \cap N(w_2)$). (See Fig. 13.) Then $[w_1, z_1, z_2, w_2]$ (or $[w_1, z, w_2]$) is inevitable to (G, B). Let G' be the graph obtained from G by removing vertices v_2 and v_3 and adding a new edge w_1w_2, and $B' = [w_1, z_1, z_2, w_2]$ (or $B' = [w_1, z, w_2]$). Vertices v_1 and v_4 appear consecutively in any FO2PE extension γ' of (G', B'). Any FO2PE extension of (G, B) is obtained by modifying an FO2PE extension $\gamma' = [u_1 = v_1, u_2 = v_4, u_3, \ldots, u_{n'}]$ of (G', B') into $\gamma = [v_1, v_2, v_3, v_4, u_3, \ldots, u_{n'}]$.*

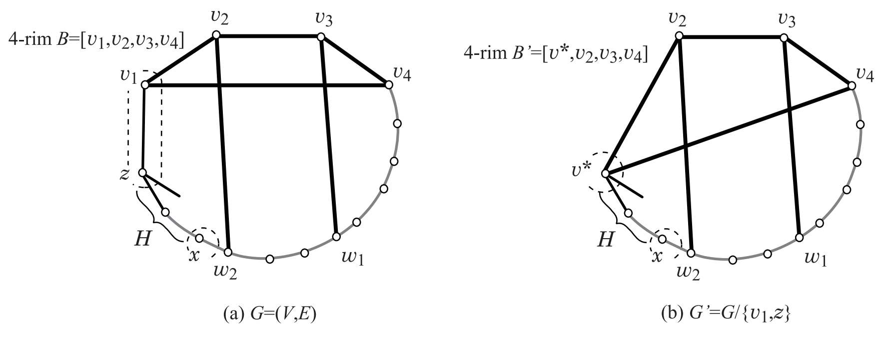

Fig. 9. Illustration for the reduction in Lemma 14(i) from an instance $(G, B = [v_1, v_2, v_3, v_4])$ with a fixed 4-rim of a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-3 vertex adjacent to neither of w_1 and w_2; (b) a new instance $(G' = G/\{v_1, z\}, B' = [v^*, v_2, v_3, v_4])$ with a new 4-rim of 4-cycle $v^*v_2v_3v_4$ with degree-3 vertices v_2 and v_3.

Proof. Let $\gamma = [v_1, v_2, \ldots, v_n]$ be an arbitrary FO2PE extension of (G, B). By Lemma 3, each vertex in G is of degree 3 or 4. Since $n \geq 7$, embedding γ is quasi-planar by Lemma 2, and hence it holds $w_2 \in V_{\partial\gamma}(w_1, v_1)$.

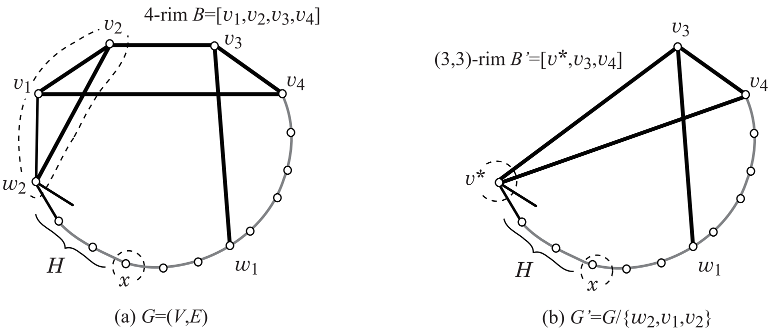

Fig. 10. Illustration for the reduction in Lemma 14(ii) from an instance $(G, B = [v_1, v_2, v_3, v_4])$ with a fixed 4-rim of a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-3 vertex adjacent to w_2; (b) a new instance $(G' = G/\{w_2, v_1, v_2\}, B' = [v^*, v_3, v_4])$ with a new $(3, 3)$-rim of triangle $v^*v_3v_4$ with a degree-3 vertex v_3.

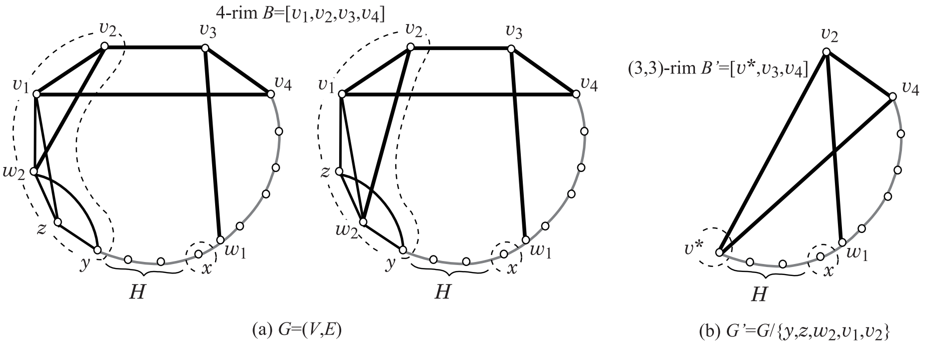

Fig. 11. Illustration for the reduction in Lemma 14(iii) from an instance $(G, B = [v_1, v_2, v_3, v_4])$ with a fixed 4-rim of a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-4 vertex adjacent to w_2, and there is a pair of a degree-4 vertex z and a vertex y such that v_1w_2z and w_2zy are triangles; (b) a new instance $(G' = G/\{y, z, w_2, v_1, v_2\}, B' = [v^*, v_3, v_4])$ with a new $(3, 3)$-rim of triangle $v^*v_3v_4$ with a degree-3 vertex v_3.

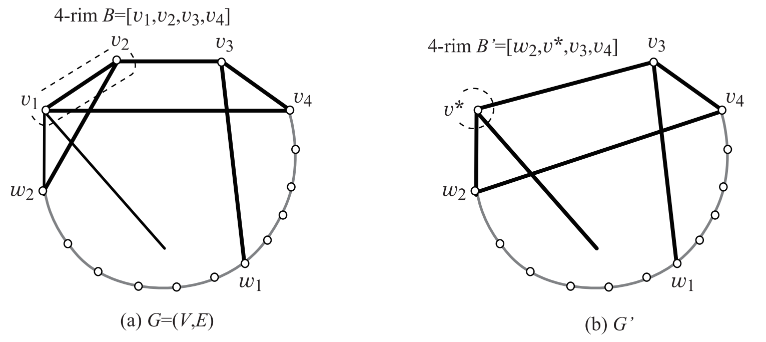

Fig. 12. Illustration for the reduction in Lemma 14(iv) from an instance $(G, B = [v_1, v_2, v_3, v_4])$ with a fixed 4-rim of a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 to a new instance (G', B'): (a) a graph G such that v_1 is a degree-4 vertex adjacent to w_2, but there is no pair of a degree-4 vertex z and a vertex y such that v_1w_2z and w_2zy are triangles; (b) a new instance $(G', B' = [w_2, v^*, v_3, v_4])$ with a new 4-rim of 4-cycle $w_2, v^*v_3v_4$ with degree-3 vertices v^* and v_3, where G' is obtained from G by replacing edges v_1v_4 and v_2w_2 with a new edge w_2v_4 and contracting v_1 and v_2 into a single vertex v^*.

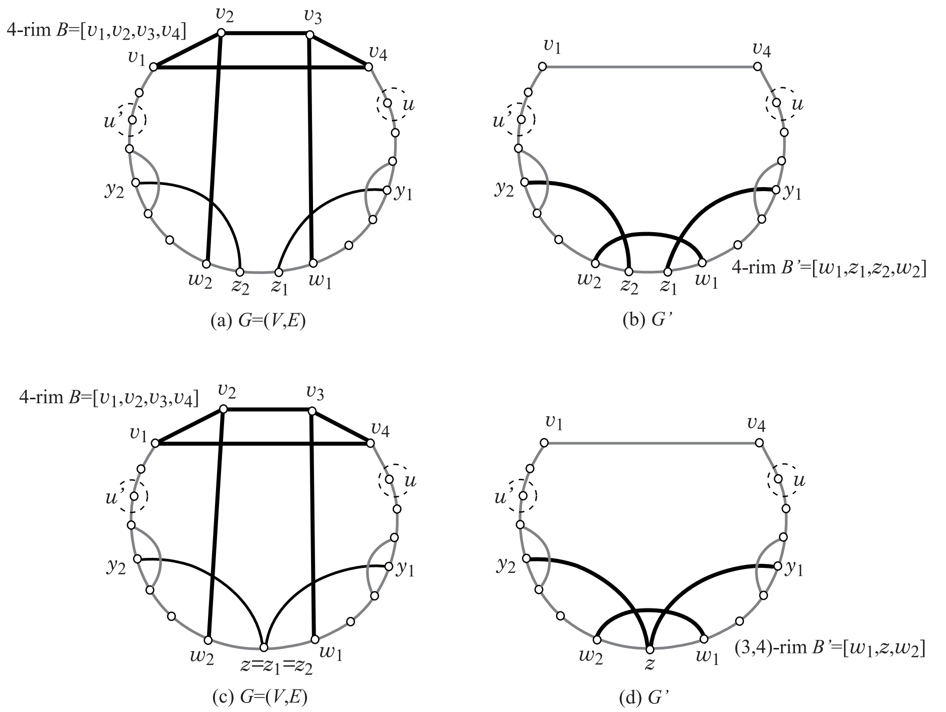

Fig. 13. Illustration for the reduction in Lemma 14(v) from an instance $(G, B = [v_1, v_2, v_3, v_4])$ with a fixed 4-rim of a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3 to a new instance (G', B'): (a) a graph G such that none of conditions (i)-(iv) in Lemma 14 holds, $w_1 \neq w_2$, and there is an edge $z_1z_2 \in E$ between two degree-3 vertices $z_1 \in N(w_1)$ and $z_2 \in N(w_2)$; (b) a new instance $(G', B' = [w_1, z_1, z_2, w_2])$ with a new 4-rim of 4-cycle $w_1z_1z_2w_2$ with degree-3 vertices z_1 and z_2, where G' is obtained from G by removing vertices v_2 and v_3 and adding a new edge w_1w_2; (c) a graph G such that none of conditions (i)-(iv) in Lemma 14 holds, $w_1 \neq w_2$, and there is a degree-4 vertex $z \in N(w_1) \cap N(w_2)$; (d) a new instance $(G', B' = [w_1, z, w_2])$ with a new $(3, 4)$-rim of triangle w_1zw_2 with degree-4 vertex z, where G' is obtained from G by removing vertices v_2 and v_3 and adding a new edge w_1w_2.

Let $z \in N(v_1) = \{v_2, v_4, z\}$, where $z \in V_{\partial\gamma}(w_2, v_1)$. Hence if v_1 cannot be adjacent to w_1, and symmetrically v_4 cannot be adjacent to w_2. This means that when v_1 or v_4 is a degree-3 vertex, condition (i) or (ii) holds. Also when v_1 is a degree-4 vertex adjacent to w_2 or v_4 is a degree-4 vertex adjacent to w_1, condition (iii) or (iv) holds.

We consider the remaining case where v_1 is a degree-4 vertex not adjacent to w_2 and v_4 is a degree-4 vertex not adjacent to w_1.

Let the two neighbors $x_1, x_2 \in N(v_1) - \{v_2, v_4\}$ appear in this order along $\partial\gamma(v_4, v_1)$. We show that $x_1, x_2 \in V_{\partial\gamma}(w_2, v_1)$. Since v_1 is not adjacent to w_2, we have $x_2 \in V_{\partial\gamma}(w_2, v_1)$. If $x_1 \in V_{\partial\gamma}(v_4, w_2)$, then an edge ab joins a vertex $a \in V_{\partial\gamma}(w_2, v_1)$ and a vertex $b \in V_{\partial\gamma}(v_1, w_2)$ since $\{w_2, v_1\}$ is not a cut-pair. However, edge ab creates the third crossing on edge w_2v_2. Hence we have $\{x_1, x_2\} \subseteq V_{\partial\gamma}(w_2, v_1)$. Symmetrically we have $N(v_4) - \{v_1, v_3\} \subseteq V_{\partial\gamma}(v_4, w_1)$. Since $|V_{\partial\gamma}(w_2, v_2)| \geq 3$ (resp., $|V_{\partial\gamma}(v_3, w_1)| \geq 3$), there is a (w_2, v_2)-hooked edge y_2z_2 between $y_2 \in V_{\partial\gamma}(w_2, v_1)$ and $z_2 \in V_{\partial\gamma}(v_4, w_2)$ (resp., a (v_3, w_1)-hooked edge y_1z_1 between $y_1 \in V_{\partial\gamma}(v_4, w_1)$ and $z_1 \in V_{\partial\gamma}(w_1, v_1)$). In fact, it must hold that $w_1 \neq w_2$, $z_1 \in V_{\partial\gamma}(w_1, z_2)$ and $z_2 \in V_{\partial\gamma}(z_1, w_2)$ since otherwise one of edges v_2w_1, v_2w_2, y_1z_1 and y_2z_2 would get three crossings. Note that possibly $z_1 = z_2$. Since each of these four edges already has two crossings, we see that $V_{\partial\gamma}(w_1, w_2) = \{z_1, z_2\}$ (otherwise one of $\{w_1, z_1\}$, $\{z_1, z_2\}$ and $\{z_2, w_2\}$ would be a cut-pair), and that $\deg(z_1) = \deg(z_2) = 3$ when $z_1 \neq z_2$. This proves that condition (v) holds when none of (i)-(iv) occurs.

(i) Since $\deg(v_1) = 3$, clearly $[z, v_1, v_2, v_3, v_4]$ is inevitable to (G, B). Note that vertex z is an attaching point of $(G' = G/\{v_1, z\}, B' = [v^*, v_2, v_3, v_4])$. Analogously with the proof of Lemma 12(i)-(ii), we can show that (G', B') is triconnected and extendible and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [z, v_1, v_2, v_3, v_4, u_5, \ldots, u_{n'}]$.

(ii) Since $\deg(v_1) = 3$, clearly $[w_2, v_1, v_2, v_3, v_4]$ is inevitable to (G, B). Note that vertex w_2 is an attaching point of $(G' = G/\{w_2, v_1, v_2\}, B' = [v^*, v_3, v_4])$. We can show that (G', B') is triconnected and extendible and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_2, u_3 = v_3, u_4, \ldots, u_{n'}]$ of (G', B') into $\gamma = [w_2, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$.

(iii) Note that vertex y is an attaching point of $(G' = G/\{y, z, w_2, v_1, v_2\}, B' = [v^*, v_3, v_4])$. Analogously with the proof of Lemma 13(ii), we can show that any extension B into an FO2PE $\gamma = [v_1, v_2, \ldots, v_n]$ of G has zw_2 as a frill, and that any FO2PE extension of (G, B) can be obtained by modifying any FO2PE extension $\gamma' = [u_1 = v^*, u_2 = v_3, u_3 = v_4, u_4, \ldots, u_{n'}]$ of (G', B') into $[y, z, w_2, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$ and $[y, w_2, z, v_1, v_2, v_3, v_4, u_4, \ldots, u_{n'}]$.

(iv) Assume that for $(v, w) = (v_1, w_2)$ or (v_4, w_1), v is a degree-4 vertex adjacent to w, but there is no pair of a degree-4 vertex z and a vertex y such that vwz and wzy are triangles. Let $(v, w) = (v_1, w_2)$ without loss of generality. Analogously with the proof of Lemma 13(iii), we can show that $[w_2, v_1, v_2, v_3, v_4]$ is inevitable to (G, B).

Let $G^\dagger$ be the graph obtained from G by replacing edges v_1v_4 and v_2w_2 with a new edge w_2v_4, and $G' = G^\dagger/\{v_1, v_2\}$. Then v_1 is an attaching point to $(G', B' = [w_2, v^*, v_3, v_4])$. Analogously with the proof of Lemma 13(iii), we can prove that $G^\dagger$ is triconnected. Analogously with the proof of Lemma 12(i), we see that $G' = G^\dagger/\{v_1, v_2\}$ remains triconnected.

Analogously with the proof of Lemma 13(iii), we can prove that $(G', B' = [w_2, v^*, v_3, v_4])$ is extendible and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = w_2, u_2 = v^*, u_3 = v, u_4 = v_4, u_5, \ldots, u_{n'}]$ of (G', B') into $\gamma = [w_2, v_1, v_2, v_3, v_4, u_5, \ldots, u_{n'}]$.

(v) Analogously with the proof of Lemma 13(iv), we see that any FO2PE extension γ of (G, B) satisfies the following properties: $N(v_1) - \{v_2, v_4\} \subseteq V_{\partial\gamma}(w_2, v_1)$, $N(v_3) - \{v_2, v_4\} \subseteq V_{\partial\gamma}(v_4, w_1)$, there is a (w_2, v_2)-hooked edge y_2z_2 between $y_2 \in V_{\partial\gamma}(w_2, v_1)$ and $z_2 \in V_{\partial\gamma}[z_1, w_2)$ (resp., a (v_2, w_1)-hooked edge y_1z_1 between $y_1 \in V_{\partial\gamma}(v_4, w_1)$ and $z_1 \in V_{\partial\gamma}(w_1, z_2])$ such that $V_{\partial\gamma}(w_1, w_2) = \{z_1, z_2\}$ (possibly $z_1 = z_2$), and $\deg(z_1) = \deg(z_2) = 3$ when $z_1 \neq z_2$. Hence $w_1 \neq w_2$. Also no other pair $\{z_1', z_2'\}$ than $\{z_1, z_2\}$ satisfies condition (iv). Therefore any FO2PE extension of (G, B) contains $[w_1, z_1, z_2, w_2]$ (when $z_1 \neq z_2$) or $[w_1, z, w_2]$ (when $z = z_1 = z_2$) as a sequence.

Analogously with the proof of Lemma 13(iv), we can prove that (G', B') is triconnected and extendible and that any FO2PE extension of (G, B) can be obtained by modifying an FO2PE extension $\gamma' = [u_1 = v_1, u_2 = v_4, u_3, \ldots, u_{n'}]$ of (G', B') into $\gamma = [v_1, v_2, v_3, v_4, u_3, \ldots, u_{n'}]$. □

Note that in each of Lemmas 12, 13 and 14, constructing a new instance (G', R') and modifying an FO2PE extension γ' of (G', B') into an FO2PE extension γ of (G, B) can be executed in $O(1)$ since G is a degree-bounded graph and γ can be obtained by inserting a subsequence.

The **Algorithm EXTEND**(G, B), which takes a triconnected graph G and a permutation B of vertices in a triangle uvw or a 4-cycle $uvv'w$ with degree-3 vertices v and v', and outputs all FO2PE extensions of (G, B), is described below.

Algorithm EXTEND(G, B)
Input: A triconnected simple graph $G = (V, E)$ with $n \geq 7$ and a permutation B of vertices in a triangle uvw or a 4-cycle $uvv'w$ with degree-3 vertices v and v'.
Output: All FO2PE extensions of (G, B).

1: **if** $n \leq 7$ **then**
2: Return all FO2PE extensions γ of (G, B) (if any), or Return $\emptyset$ (otherwise);
3: **else**
/* Partial embedding B is specified as one of the following:
Case 1: $B = [v_1, v_2, v_3]$ for a triangle $v_1v_2v_3$ with a degree-3 vertex v_2,
where $N(v_2) = \{v_1, v_3, w\}$;
Case 2: $B = [v_1, v_2, v_3]$ for a triangle $v_1v_2v_3$ with a degree-4 vertex v_2,
where $N(v_2) = \{v_1, v_3, w_1, w_2\}$; and
Case 3: $B = [v_1, v_2, v_3, v_4]$ for a 4-cycle $v_1v_2v_3v_4$ with degree-3 vertices v_2 and v_3,
where $N(v_2) = \{v_1, v_3, w_2\}$ and $N(v_2) = \{v_2, v_4, w_1\}$ */
4: **if** Case 1 (resp., Case 2, 3) holds, but none of the conditions (i)- (ii) in Lemma 12
(resp., (i)- (v) in Lemma 13, Lemma 14) holds **then**
5: Return $\emptyset$;
6: **else**
7: Construct (G', B') according to the the conditions (i)- (ii) in Lemma 12
(resp., (i)- (v) in Lemma 13, Lemma 14) currently satisfied by (G, B);
8: **if EXTEND**$(G', B') \neq \emptyset$ **then**
9: Modify each $\gamma' \in$**EXTEND**(G', B') into an FO2PE extension γ of (G, B) according to
the operation in Lemma 12 (resp., Lemma 13, Lemma 14), where two FO2PE extensions
of (G, B) will be constructed from the same γ' for the cases (ii)
in Lemma 13 and (iii) in Lemma 14;
10: Return all the resulting FO2PE extensions γ
11: **else**
12: Return $\emptyset$
13: **end if**
14: **end if**
15: **end if**.

Based on **Algorithm EXTEND**(G, B), we finally prove Lemma 10. We first show that **Algorithm EXTEND**(G, B) correctly delivers all FO2PE extensions of (G, B), if any. In line 9, if **Algorithm EXTEND**(G', B') returns all FO2PE extensions γ' of (G', B'), then all FO2PE extensions of (G, B) can be obtained according to the modifications stated in Lemmas 12, 13 and 14. Since **Algorithm EXTEND**(G', B') returns all FO2PE extensions when $n \leq 7$, we see by induction that **EXTEND**(G, B) correctly delivers all FO2PE extensions of (G, B).

We next show that **Algorithm EXTEND**(G, B) delivers a constant number of solutions. When $n \leq 7$, the graph G has at most $n - |B| \leq 4$ vertices to be arranged along the boundary of a possible FO2PE extension of (G, B), and at most $4!$ FO2PE extensions of (G, B) will be constructed. We construct exactly one FO2PE extension γ of (G, B) from an FO2PE extension γ' of (G', B'), except for the cases (ii) in Lemma 13 and (iii) in Lemma 14 wherein exactly two FO2PE extensions, say γ_1 and γ_2 of (G, B) will be generated from the same FO2PE extension γ' of (G', B'). Note that in this case, γ_1 is obtained from γ_2 by flipping a frill zw in the lemmas, and the frill in γ_i will be preserved in any extensions obtained from γ_i until it is output as a final solution. By Lemma 9, any FO2PE of a graph can contain at most two frills, which means that generating two FO2PE extensions in line 9 can occur at most twice. Therefore, **Algorithm EXTEND**(G, B) delivers a constant number of FO2PE extensions of (G, B).

As we have already observed, constructing a new instance (G', R') and modifying an FO2PE can be done in $O(1)$ time, **Algorithm EXTEND**(G, B) runs in $O(n)$ time. This completes a proof of Lemma 10.

6 Proof of Theorem 3

In this section, we prove Theorem 3.

Proof. Assume that a given connected graph $G = (V, E)$ admits an O2PE γ. When G is not biconnected, we first augment the embedding γ by adding new edges so that it remains to be an O2PE γ' of the resulting "biconnected graph" $G' = (V, E')$. For this, we traverse the boundary $\partial\gamma$ in the clockwise order starting with a vertex v_1. During this, we skip visiting a cut-vertex already traversed to form a permutation $[v_1, v_2, \ldots, v_n]$ of the vertices in V in the order that we first visit. In the outer face of γ, we add new edges between non-adjacent vertices v_i and v_{i+1}, $1 \leq i < n$. Note that we have skipped a vertex v only when it is a cut-vertex already traversed. The resulting embedding γ' remains outer-2-planar, and the boundary $\partial\gamma'$ forms a simple cycle of the augmented graph G', which is now biconnected. Hence it suffices to show the lemma only when a given graph is biconnected, since the added edges can be removed from any straight-line drawing of G' to obtain any straight-line drawing of G.

Let $[v_1, v_2, \ldots, v_n]$ be the cyclic order of an O2PE γ of a biconnected graph. Then fix the positions of vertices as the apices of a convex n-gon P_n, which automatically determines straight-line segments of all edges. Clearly two inner edges v_iv_j and v_kv_h cross only when $i < k < j < h$ on the cyclic order in the topological embedding γ. In the geometric embedding by P_n, the straight-line segments of two inner edges v_iv_j and v_kv_h intersect only when $i < k < j < h$. This implies that P_n gives a straight-line drawing of γ. $\square$

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