Characterizing GSP Mechanisms to Obnoxious Facility Game in Trees via Output Locations

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Abstract: In the obnoxious facility game with a set of agents in a space, we wish to design a mechanism, a decision-making procedure that determines a location of an undesirable facility based on locations reported by the agents, where we do not know whether the location reported by an agent is where exactly the agent exists in the space. For a location of the facility, the benefit of each agent is defined to be the distance from the location of the facility to where the agent exists. Given a mechanism, all agents are informed of how the mechanism utilizes locations reported by the agents to determine a location of the facility before they report their locations. Some agent may try to manipulate the decision of the facility location by strategically misreporting her location. As a fair decision-making, mechanisms should be designed so that no particular group of agents can get a larger benefit by misreporting their locations. A mechanism is called *group strategy-proof* if no subset of agents can form a group such that every member of the group can increase her benefit by misreporting her location jointly with the rest of the group. For a given mechanism, a point in the space is called a candidate if it can be output as the location of the facility by the mechanism for some set of locations reported by agents.

In this paper, we consider the case where a given space is a tree metric, and characterize the group strategy-proof mechanisms in terms of distribution of all candidates in the tree metric. We prove that there exists a group strategy-proof mechanism in the tree metric if and only if every two candidates have the distance.

Keywords: mechanisms; group strategy-proof; facility location games; trees

1 Introduction

1.1 Social choice theory

In social choice theory, a *mechanisms* is a procedure that determines a social decision based on a vote. More formally, for a set Ω of voting alternatives and a set $N = \{1, 2, \ldots, n\}$ of selfish voters with various utilities, a mechanism is a function $f : \Omega^n \to \Omega$ as a collective

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decision making system. Voters know the exact detail of the operation of the mechanisms before they actually vote, and each voter can find out her expected benefit of her utility in the case when every voter votes truthfully. Each voter may try to manipulate the decision of a mechanism by changing her voting to increase benefit from her personal utility. A voting which aims to manipulate the decision of a mechanism is called a *strategic-voting*. To the effect of making a fair decision, we are interested in mechanisms in which no voter can get a larger benefit by a single-handed strategic-voting. Such a mechanism is called a *strategy-proof* mechanism. Moreover, a mechanism is called a *group strategy-proof* mechanism, if there is no coalition of voters such that each member in the coalition can simultaneously get a larger benefit by their cooperative strategic-voting.

Moulin [8] studied social choice theory under the condition that the set of alternatives is the one-dimensional Euclidean space and each utility function is a single-peaked concave function. Moulin [8] characterized a necessary and sufficient condition of strategy-proofness on single-peaked preferences in the one-dimensional Euclidean space. After that, Border and Jordan [2] extended the result to characterize strategy-proof mechanisms in the multi-dimensional Euclidean space. Schummer and Vohra [10] applied the result of Border and Jordan [2] to obtain characterization of strategy-proof mechanisms when Ω is the set of all points in a tree metric or the set of all points in a graph metric which has at least one cycle.

1.2 Facility game

The *facility game* is a problem in social choice theory where a location of the facility in a metric space will be decided based on locations of agents (votes by voters) and each agent tries to maximize benefit from her utility function defined based on the distance from her location to the location of the facility.

In previous studies of facility games [1, 2, 3, 4, 6, 7, 9, 10], mechanisms are allowed to distinguish agents. In other words, the input of mechanisms is not only location information (i.e., where is reported) but also agents' information (i.e., who reports the location). On the other hand, there is a category of mechanisms which are called *anonymous*, that is, which do not use agents' information.

Another important aspect of mechanisms of facility games is how we can maximize the sum of the utilities over all agents, called the *social utility*, over all strategy-proof (or group strategy-proof) mechanisms. In general, the maximum value of the social utility attained by a strategy-proof (or group strategy-proof) mechanism is smaller than that attained just by choosing the best location of the facility. A possible measurement of the performance for a mechanism is a *benefit-ratio*, the ratio of the social utility attained by the mechanism and such a best possible value to the social utility.

1.2.1 Typical facility game

In a typical facility game setting, it is assumed that the facility is desirable, such as a library. Several studies have been extensively made on the typical facility game, such as designing mechanisms [1, 2, 6, 7, 9, 10]. Procaccia and Tennecholtz [9] proposed a group strategy-proof mechanism which returns the location of the median agent as the facility location when all agents are located on a path. Moreover, they designed a *randomized mechanism*, that is, a mechanism does not output a single facility location but outputs a probability distribution of the facility location over a metric space. In randomized mechanisms, the utility of agents is defined to be the expected value by the probability distribution. On the contrary, a mechanism which outputs a facility location is called *deterministic*.

Alon *et al.* [1] gave a complete analysis on benefit-ratios of group strategy-proof mechanisms for the typical facility game in general graph metrics.

1.2.2 Obnoxious facility game

The *obnoxious facility game* is a facility game such that the facility is undesirable, such as a garbage dump. We call mechanism a *p-candidate mechanism* if the number of distinct outputs is p. In the obnoxious facility game, previous studies [3, 4, 5] focused on mechanisms which output the location of the facility from restricted locations in a metric space. More accurately, those studies first choose p candidates in a set of alternatives, and then they design p-candidate mechanisms which output a location of the facility from these p candidates.

Cheng *et al.* [4] first studied group strategy-proof mechanisms for the obnoxious facility game in the line metric. They have designed a 2-candidate group strategy-proof mechanism and shown that for any set of locations reported by agents, a benefit-ratio of the mechanism is at most 3. Ibara and Nagamochi [5] gave a complete characterization of 2-candidate strategy-proof mechanisms and 2-candidate group strategy-proof mechanisms in general metrics and they proved that in arbitrary metrics, a 2-candidate group strategy-proof mechanism with a benefit-ratio 4 can be designed. Moreover, they have shown that in the line metric, there exists no p-candidate strategy-proof mechanism for any integer $p \geq 3$.

In this paper, we consider group strategy-proof mechanisms in tree metrics. In Section 2, we formulate a model of the obnoxious facility game, and describe out main theorem that characterizes strategy-proof mechanisms in tree metrics in such a way that there exists a p-candidate group strategy-proof mechanism if and only if every two in a set of p candidates have the distance. Section 3 gives a proof to the necessary condition in the theorem, and Section 4 provides a proof to the sufficient condition in the theorem. In Section 5, we make a concluding remarks.

2 Preliminaries

2.1 Mechanisms

Let $\mathbb{R}_+$ be the set of nonnegative real numbers. Let Ω be a set of points, possibly infinite. A symmetric distance function $d : \Omega \times \Omega \to \mathbb{R}_+$ holds the following conditions, for every point $x \in \Omega$, it holds that $d(x, x) = 0$; for every two points $x, y \in \Omega$, it holds that $d(x, y) = d(y, x)$; and for every three points $x, y, z \in \Omega$, it holds that $d(x, y) + d(y, z) \geq d(x, z)$. Throughout this paper, we use the notation d as a symmetric distance function. Let (Ω, d) denote a metric.

Let $N = \{1, 2, \dots, n\}$ be a set of agents, and assume that exactly one location of an undesirable facility needs to be decided. Let $\Omega_{\text{agents}} \subseteq \Omega$ denote a set of points to which any location that can be reported by an agent in N belong, and let $\Omega_{\text{facility}} \subseteq \Omega$ denote a set of points such that the facility can be located. A set of locations reported by agents in N is denoted by a *location function* $\chi : N \to \Omega_{\text{agents}}$, where $\chi(i)$ denotes the location reported by an agent $i \in N$.

Let χ be a location function. For a set $\Omega' \subseteq \Omega$ of points, let $N(\chi, \Omega')$ denote the set of all agents $i \in N$ with $\chi(i) \in \Omega'$. For a location $y \in \Omega_{\text{facility}}$ of the facility, the benefit $\beta(y, \chi(i))$ of an agent $i \in N$ is defined to be the distance from her location to the facility, i.e.,

$$\beta(y, \chi(i)) = d(y, \chi(i)).$$

For simplicity, for a set $S \subseteq N$ of agents, we write by $\chi(S)$ the multiset $\{\chi(i) \mid i \in S\}$ of locations reported by agents in S, and we denote by $\overline{S}$ the set $N \setminus S$. The multiset $\chi(N)$ is called a *profile* of N. Given a profile $\chi(N)$, a *mechanism* f outputs a facility location based on the profile $\chi(N)$, that is, $f : \Omega^n_{\text{agents}} \to \Omega_{\text{facility}}$.

In the literature on the study of facility games, the following mechanism model appears [1, 2, 3, 4, 6, 7, 9, 10]. The input to mechanisms is a location function χ and mechanisms distinguish each agent's report. For instance, for location functions χ and χ' of a set $N = \{1, 2\}$ of agents and locations $x, y \in \Omega_{\text{agents}}$ such that $\chi(1) = x$, $\chi(2) = y$, $\chi'(1) = y$ and $\chi'(2) = x$, a mechanism f can output different facility locations, that is, $f(\chi(N)) \neq f(\chi'(N))$. *Anonymity* is an important property of mechanisms. A mechanism f is called *anonymous* if it holds that $f(\chi(N)) = f(\chi'(N))$ for any two location functions χ and χ' of a set N of agents that admits a bijection σ on N such that $\chi(i) = \chi'(\sigma(i))$, for every agent $i \in N$ (i.e., $\chi(N) = \chi'(N)$ holds as multisets). In our model, every mechanism is anonymous, that is, the input is a multiset as a set of locations which all agent report.

In this paper, we consider that an intersection of multisets retains the highest multiplicity of elements in the sets. For example, for points $a, b \in \Omega$ and a multiset $A = \{a, b, b\}$, it holds that $A \cap \Omega = \{a, b, b\}$.

Next we review the definition of strategy-proofness and group strategy-proofness of mechanisms [1, 4, 5].

Definition 1 *A mechanism f is strategy-proof (SP for short) if and only if no agent can benefit from misreporting her location. Formally, given a set N of agents and a location function χ, for any agent $i \in N$ and any location function χ' such that $\chi(\overline{\{i\}}) = \chi'(\overline{\{i\}})$, it holds that*

$$\beta(f(\chi(N)), \chi(i)) \geq \beta(f(\chi'(N)), \chi(i)).$$

Definition 2 *A mechanism f is group strategy-proof (GSP for short) if and only if for any group of agents, at least one agent in the group cannot benefit from misreporting her location simultaneously with the rest of the group. Formally, given a set N of agents and a location function χ, for any non-empty set $S \subseteq N$ of agents and for any location function χ' such that $\chi(\overline{S}) = \chi'(\overline{S})$, there exists an agent $i \in S$ satisfying*

$$\beta(f(\chi(N)), \chi(i)) \geq \beta(f(\chi'(N)), \chi(i)).$$

For a mechanism $f : \Omega^n_{\text{agents}} \to \Omega_{\text{facility}}$, a location $y \in \Omega_{\text{facility}}$ is called a *candidate* if there is a profile $\chi(N) \in \Omega^n_{\text{agents}}$ such that $f(\chi(N)) = y$ and the set of all candidates of f is denoted by $C(f) \subseteq \Omega_{\text{facility}}$. A mechanism with $|C(f)| = p$ is called a *p-candidate mechanism.*

2.2 Tree Metric

In this paper, we define a tree metric based on the graph model due to Schummer and Vohra [10]. We define a *graph* G to be a closed, connected subset of Euclidean space. The graph is composed of a finite number of closed curves of finite length, which are called *edges*. The extremities and branch points of the curves are called *vertices*. A *path* is a minimal connected subset of G that contains two points x and y as its endpoints. A *cycle* in G is defined to be the union of two paths whose intersection is equal to the set of both their endpoints.

A *tree* T is defined to be a graph without cycles. A path with two endpoints in a tree is uniquely determined. For two points x and y in a tree, let $P(x, y)$ denote the path with two endpoints x and y, and the distance $d(x, y)$ between x and y is defined to be the length of path $P(x, y)$, and there is a unique point z such that $d(x, z) = d(z, y)$. We call such a point the *middle point* of x and y, and denote it by $m(x, y)$. Note that $d(x, y) = 0$ if and only if $x = y$. In this paper, we consider the tree metric (T, d). Let $T_{\text{agents}} = T$ be a set of points where agents can exist and $T_{\text{facility}} \subseteq T$ be a set of points where the facility can be located. Given a mechanism f and a profile χ, the benefit $\beta(f(\chi(N)), \chi(i))$ of an agent $i \in N$ is defined to be the distance $d(f(\chi(N)), \chi(i))$ from $\chi(i)$ to $f(\chi(N))$.

A *rooted tree* is a tree such that one vertex of the tree is designated as a *root*. Let T be a rooted tree with rooted at a point μ. The *parent* y of a vertex x is the vertex one step closer to root r and lying on the same edge and x is called a *child* of the vertex y. For a vertex u and a child v of u, let (u, v) denote the edge joining u and v. A vertex x is called a *descendant* of a vertex v if v is in path $P(\mu, x)$ between the root μ and x. We define subtrees $T[u]$ and $T(e)$ specified by a vertex u and an edge e as follows. For each vertex u in T, let $T[u]$ be the set of points z in the subtrees induced from T by v and the descendants of v, i.e., z is a point on $P(v, x)$ for some descendant x of v in T. For each edge $e = (u, v)$ in T, let $T(e) \subseteq T$ be the set of points in e and $T[v]$.

We here observe a property on GSP mechanisms in the next lemma.

Lemma 1 *Let f be a mechanism in T. Let χ be a location function and $c \in C(f)$ be a candidate such that $c = f(\chi(N))$. If there is a candidate $c' \in C(f)$ such that*

$$d(c, \chi(i)) < d(c', \chi(i)) \text{ for every agent } i \in N,$$

then f is not GSP.

Proof. There is a location function χ' such that $f(\chi'(N)) = c'$. For the set $S = N$, any agent $i \in S$ satisfies

$$\beta(f(\chi(N)), \chi(i)) = d(c, \chi(i)) < d(c', \chi(i)) = \beta(f(\chi'(N)), \chi(i)).$$

Therefore when the agents in S misreport their locations, all agents in S can benefit, that is, the mechanism f is not GSP by Definition 2. □

Definition 3 *We call a set C of locations in a tree T a perimetric distribution if $|C| = 1$ or there is a point $\mu \in T$ such that $d(\mu, c) = d(\mu, c')$ for every two $c, c' \in C$.*

The main result in this paper is the following theorem.

Theorem 1 *Let $C \subseteq T_{facility}$ be a set of $p \geq 1$ points in a tree T. There is a p-candidate GSP mechanism such that $C(f) = C$ if and only if C is a perimetric distribution.*

In the following two sections, we prove the necessity and sufficiency of Theorem 1, respectively.

3 Necessity of Theorem 1

This section proves the necessity of Theorem 1. Thus we prove the next.

Lemma 2 *Let f be a p-candidate mechanism in a tree metric (T, d) such that $C(f)$ is not a perimetric distribution. Then f is not GSP.*

Let f be a p-candidate mechanism such that $C(f)$ is not a perimetric distribution. Hence $p = |C(f)| \geq 3$ since $C(f)$ with $p = |C(f)| \leq 2$ is always a perimetric distribution. Let c_a and c_b be a pair of two most distant candidates in $C(f)$. We define point $\mu = m(c_a, c_b)$ and regard T as a rooted tree by designating μ as the root. We denote $r = d(c_a, \mu) = d(c_b, \mu)$. Define $C_r(f)$ to be the set of candidates which are at distance r from the root μ, i.e.,

$$C_r(f) = \{c \in C(f) \mid d(c, \mu) = r\},$$

where $|C_r(f)| \geq |\{c_a, c_b\}| = 2$. Since $C(f)$ is not a perimetric distribution, it holds $C(f) \setminus C_r(f) \neq \emptyset$. Let c_1 be a candidate such that $c_1 \in C(f) \setminus C_r(f)$, where it holds $d(\mu, c_1) < d(c, \mu)$ for any $c \in C_r(f)$. Fig. 1 illustrates how root μ, points in $C_r(f)$ and c_1 appear on a tree T.

For each vertex u in T, let $\mathrm{Ch}(u)$ be the set of edges $e = (u, v)$ such that $T(e)$ contains at least one candidate $c \in C_r(f)$, where $|\mathrm{Ch}(\mu)| \geq 2$ by $|C_r(f)| \geq 2$. For each edge $e \in \mathrm{Ch}(\mu)$ at root μ, we introduce a partition $T(e)$ into $A(e)$ and $B(e) = T(e) \setminus A(e)$ such that

$$A(e) = \{u \in T(e) \mid d(u, c_1) < d(u, c) \text{ for all } c \in C_r(f) \cap T(e)\},$$

where $B(e) = \{u \in T(e) \mid d(u, c_1) \geq d(u, c) \text{ for some } c \in C_r(f) \cap T(e)\}$. See Fig. 1 for an illustration of subsets $A(e)$ and $B(e)$ of $T(e)$ for an edge $e \in \mathrm{Ch}(\mu)$. Note that $\mu \in A(e)$ and $A(e) \setminus \{\mu\} \neq \emptyset$ for each edge $e \in \mathrm{Ch}(\mu)$, since $d(\mu, c_1) < d(\mu, c)$ and $m(c_1, c) \in T(e)$ hold for all candidates $c \in C_r(f) \cap T(e)$ with $e \in \mathrm{Ch}(\mu)$.

We here observe a property on the structure of set $B(e)$ of an edge $e \in \mathrm{Ch}(\mu)$.

Lemma 3 *Let f be a p-candidate mechanism in a tree metric (T, d) such that $p \geq 3$ and $C(f)$ is not a perimetric distribution, and let $\mu \in T$, $c_1 \in C(f)$ and $C_r(f)$ be defined in the above. Let χ_1 be a location function such that $f(\chi_1(N)) = c_1 \in C(f) \setminus C_r(f)$. If $N(\chi_1, B(e)) = \emptyset$ for some edge $e \in Ch(\mu)$, then f is not GSP.*

Proof. Assume that there is an edge $e \in \mathrm{Ch}(\mu)$ such that $N(\chi_1, B(e)) = \emptyset$. By definition of $T(e)$, there is a candidate c in $C_r(f) \cap T(e)$. To prove that f is not GSP by Lemma 1, it suffices to show that

$$d(c_1, \chi_1(i)) < d(c, \chi_1(i)) \text{ for all agents } i \in N.$$

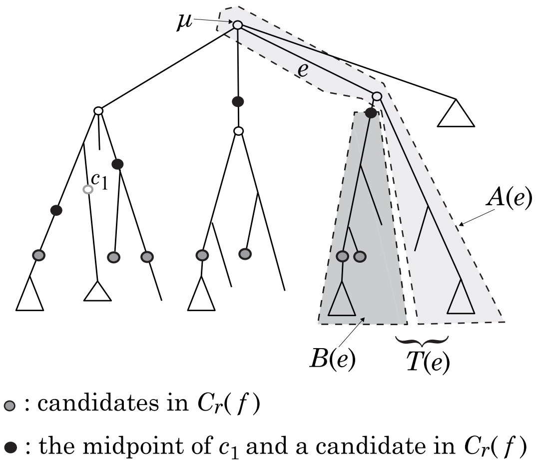

Figure 1: An illustration of root μ, candidate c_1 and sets $C_r(f)$, $T(e)$, $B(e)$ and $A(e)$ for an edge $e \in \mathrm{Ch}(\mu)$ in a tree T.

For each agent $i \in N(\chi_1, T(e))$, where $N(\chi_1, T(e)) = N(\chi_1, A(e))$ by $N(\chi_1, B(e)) = \emptyset$, we have

$$d(c_1, \chi_1(i)) < d(c, \chi_1(i)).$$

On the other hand, for each agent $i \in N \setminus N(\chi_1, T(e))$, it holds that

$$\begin{aligned} d(c_1, \chi_1(i)) &\leq d(c_1, \mu) + d(\mu, \chi_1(i)) \quad \text{(by triangle inequality)} \\ &< d(c, \mu) + d(\mu, \chi_1(i)) \quad \text{(by } c_1 \in C(f) \setminus C_r(f)\text{)} \\ &= d(c, \chi_1(i)), \end{aligned}$$

as required. □

Now we are ready to prove Lemma 2.

Proof of Lemma 2. Let χ_1 be a location function such that $f(\chi_1(N)) = c_1$. See Fig. 2(a) for an illustration of profile $\chi_1(N)$ and point c_1 in tree T rooted at μ. We can assume that

$$N(\chi_1, B(e)) \neq \emptyset \text{ for each edge } e \in \mathrm{Ch}(\mu),$$

since otherwise f is not GSP by Lemma 3, and we are done. For each edge $e \in \mathrm{Ch}(\mu)$, we select an arbitrary point $t_e \in A(e) \setminus \{\mu\}$. To prove Lemma 2, we introduce two location functions $\chi_k(N)$, $k = 2, 3$ by modifying $\chi_1(N)$.

Let $\chi_2(N)$ be the profile obtained from $\chi_1(N)$ by changing the locations of all agents $i \in N(\chi_1, B(e))$ to t_e for each edge $e \in \mathrm{Ch}(\mu)$; i.e.,

> For each edge $e \in \mathrm{Ch}(\mu)$ and all agents $i \in N(\chi_1, B(e))$, let $\chi_2(i) = t_e \in A(e) \setminus \{\mu\}$; and
>
> For all agents $i \in N \setminus \bigcup\{N(\chi_1, B(e)) \mid e \in \mathrm{Ch}(\mu)\}$, let $\chi_2(i) = \chi_1(i)$.

See Fig. 2(b) for an illustration of the new profile $\chi_2(N)$. Note that $\chi_2(N) \neq \chi_1(N)$, because $\mathrm{Ch}(\mu) \neq \emptyset$ and $N(\chi_1, B(e)) \neq \emptyset$ for each edge $e \in \mathrm{Ch}(\mu)$ by assumption.

Let $c_2 = f(\chi_2(N))$, and let $e' = (\mu, v)$ be the edge incident to root μ such that $c_2 \in T(e')$. If $e' \in \mathrm{Ch}(\mu)$ (i.e., $T(e') \cap C_r(f) \neq \emptyset$) and $c_2 \neq \mu$, then we define e_2 to be e'. Otherwise ($e' \notin \mathrm{Ch}(\mu)$ or $c_2 = \mu$), we choose an arbitrary edge in $\mathrm{Ch}(\mu)$ as e_2.

Let $\chi_3(N)$ be the profile obtained from $\chi_1(N)$ by changing the locations of all agents $i \in N(\chi_1, B(e))$ to t_e for each edge $e \in \mathrm{Ch}(\mu)$ except $e = e_2$; i.e.,

For each edge $e \in \mathrm{Ch}(\mu) \setminus \{e_2\}$ and all agents $i \in N(\chi_1, B(e))$, let $\chi_3(i) = \chi_2(i) = t_e \in A(e) \setminus \{\mu\}$; and

For all agents $i \in N \setminus \bigcup\{N(\chi_1, B(e)) \mid e \in \mathrm{Ch}(\mu) \setminus \{e_2\}\}$, let $\chi_3(i) = \chi_1(\mathrm{i})$.

See Fig. 2(c) for an illustration of the new profile $\chi_3(N)$. Observe that profile $\chi_2(N)$ is obtained from $\chi_3(N)$ by changing the locations of all agents $i \in N(\chi_1, B(e_2))$ from $\chi_3(i) = \chi_1(i) \in B(e_2)$ to $\chi_2(i) = t_{e_2} \in A(e_2) \setminus \{\mu\}$. Note that $\chi_3(N) \neq \chi_1(N)$, because $|\mathrm{Ch}(\mu)| \geq 2$, $\mathrm{Ch}(\mu) \setminus \{e_2\} \neq \emptyset$ and $N(\chi_1, B(e)) \neq \emptyset$ for each edge $e \in \mathrm{Ch}(\mu) \setminus \{e_2\} \neq \emptyset$ by assumption. Also $\chi_2(N) \neq \chi_3(N)$, since $N(\chi_1, B(e_2)) \neq \emptyset$ by $e_2 \in \mathrm{Ch}(\mu)$, as we have assumed that $N(\chi_1, B(e)) \neq \emptyset$ for all edges $e \in \mathrm{Ch}(\mu)$. Let $c_3 = f(\chi_3(N))$.

To know how location function χ_k changes into χ_3, $k = 1, 2$, we define the set $N_{k,3}$ of agents whose locations change, i.e., $N_{k,3} = \{i \in N \mid \chi_k(i) \neq \chi_3(i)\}$, and choose a special agent in $N_{k,3}$ as follows.

For $k = 1$, we see that

$$N_{1,3} = \{i \in N \mid \chi_1(i) \neq \chi_3(i)\} = \{N(\chi_1, B(e)) \mid e \in \mathrm{Ch}(\mu) \setminus \{e_2\}\} \neq \emptyset.$$

We can assume that there is at least one agent $i_{1,3} \in N_{1,3}$ such that

$$d(c_3, \chi_1(i_{1,3})) \leq d(c_1, \chi_1(i_{1,3})), \tag{1}$$

since otherwise f is not GSP with respect to a group $S = N_{1,3}$ by Definition 2 and we are done. Let $e_{1,3} \in \mathrm{Ch}(\mu)$ be the edge such that $\chi_1(i_{1,3}) \in T(e_{1,3})$ for agent $i_{1,3} \in N_{1,3}$. We show that $e_{1,3} \neq e_2$. Since $N_{1,3} = \bigcup\{N(\chi_1, B(e)) \mid e \in \mathrm{Ch}(\mu) \setminus \{e_2\}\}$, we obtain $N_{1,3} \cap N(\chi_1, T(e_2)) = \emptyset$ and in particular agent $i_{1,3} \in N_{1,3}$ satisfies $\chi_1(i_{1,3}) \notin T(e_2)$. From this and $\chi_1(i_{1,3}) \in T(e_{1,3})$, we have $e_{1,3} \neq e_2$.

Let $c_{1,3}$ be a candidate in $T(e_{1,3}) \cap C_r(f)$ and $m_{1,3}$ be the middle point of $c_{1,3}$ and c_1, i.e., $m_{1,3} = m(c_{1,3}, c_1)$, where $m_{1,3} \in T(e_{1,3})$ since $d(\mu, c_1) < d(\mu, c_{1,3})$. See Figs. 2(c) and 3 for illustrations of $c_{1,3}$, $m_{1,3}$ and $B(e_{1,3})$. In profile $\chi_3(N)$, it holds $N(\chi_3, B(e_{1,3})) = \emptyset$ by construction of $\chi_3(N)$ and $e_{1,3} \neq e_2$, and we see that point $m_{1,3}$ is always on path $P(c_{1,3}, \chi_3(i))$ for any agent $i \in N$. Hence

$$d(c_{1,3}, \chi_3(i)) = d(c_{1,3}, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \text{ for all } i \in N. \tag{2}$$

For $k = 2$, we see that

$$N_{2,3} = \{i \in N \mid \chi_2(i) \neq \chi_3(i)\} = N(\chi_1, B(e_2)) \neq \emptyset.$$

We can assume that there is at least one agent $i_{2,3} \in N_{2,3}$ such that

$$d(c_3, \chi_2(i_{2,3})) \leq d(c_2, \chi_2(i_{2,3})), \tag{3}$$

since otherwise f is not GSP with respect to a group $S = N_{2,3}$ by Definition 2 and we are done.

To prove Lemma 2, we derive the next inequality

$$d(c_{1,3}, \chi_3(i)) > d(c_3, \chi_3(i)) \text{ for all agents } i \in N, \tag{4}$$

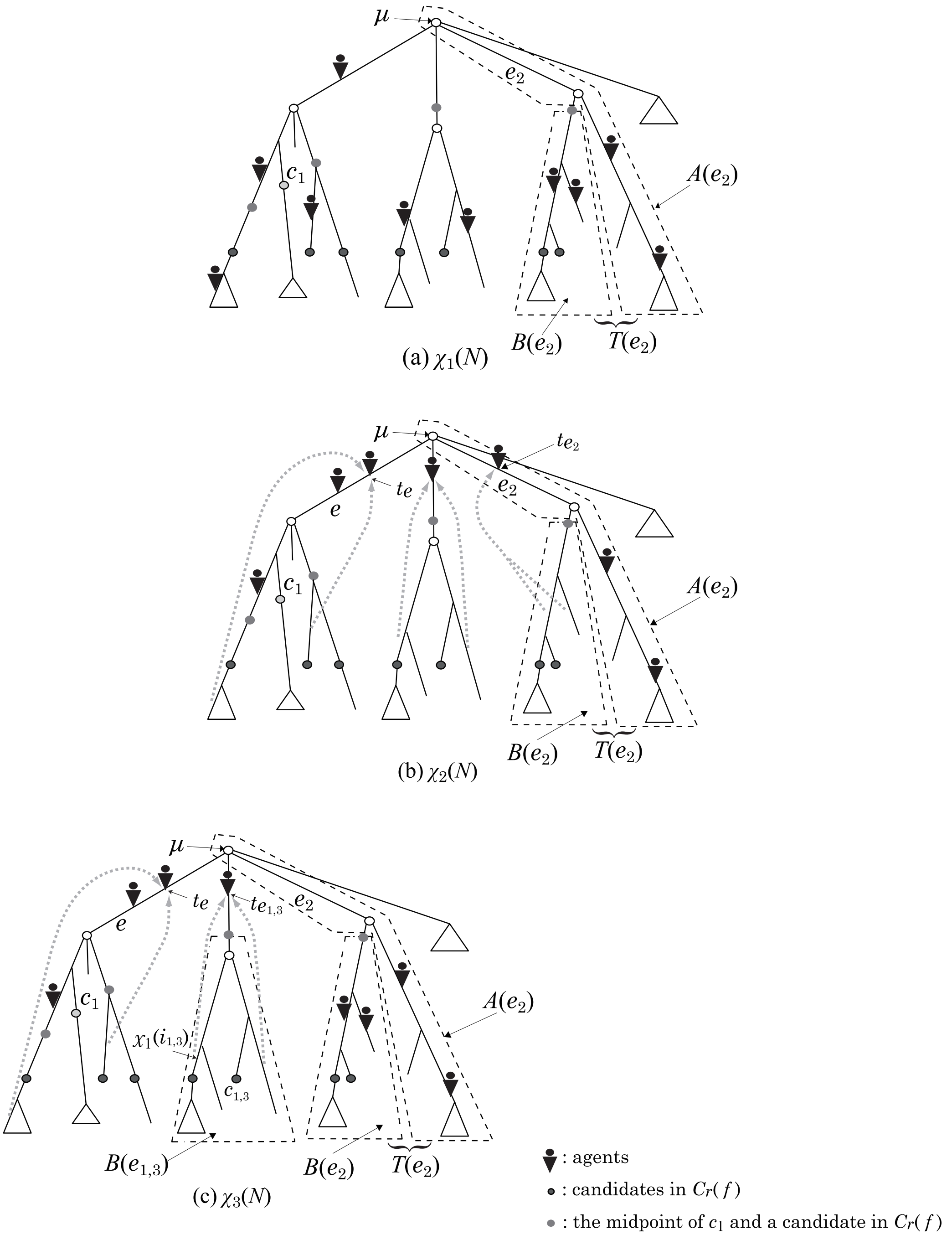

Figure 2: An illustration of profiles: (a) the original profile $\chi_1(N)$ such that $f(\chi_1(N)) = c_1 \notin C_r(f)$, (b) profile $\chi_2(N)$ obtained from $\chi_1(N)$ by changing the locations of all agents $i \in N(\chi_1, B(e))$ to t_e for each edge $e \in \mathrm{Ch}(\mu)$, and (c) profile $\chi_3(N)$ obtained from $\chi_1(N)$ by changing the locations of all agents $i \in N(\chi_1, B(e))$ to t_e for each edge $e \in \mathrm{Ch}(\mu) \setminus \{e_2\}$.

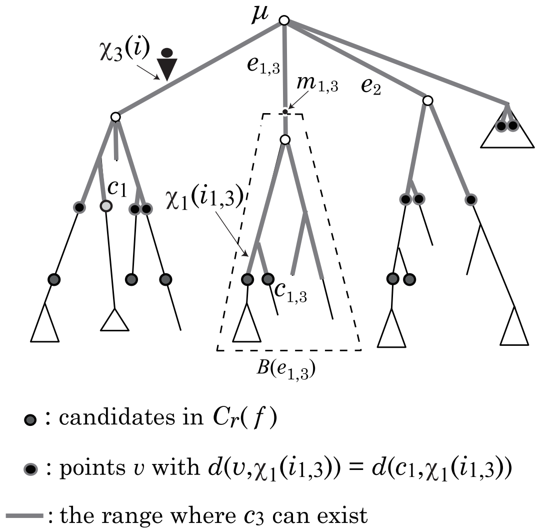

Figure 3: An illustration of points $c_{1,3}$, $m_{1,3}$ and the range that c_3 exists. The candidate c_3 can exist in the gray thick lines.

which implies that f is not GSP by Lemma 1.

To derive Eq. (4), we distinguish two cases: Case a. $c_3 \notin B(e_{1,3})$; and Case b. $c_3 \in B(e_{1,3})$.

Case a. $c_3 \notin B(e_{1,3})$: See Fig. 3 for an illustration of location $\chi_1(i_{1,3})$ in $B(e_{1,3}) \subseteq T(e_{1,3})$. Since $\chi_1(i_{1,3}) \in B(e_{1,3})$, we see from $c_3 \notin B(e_{1,3})$ that point $m_{1,3}$ lies on path $P(\chi_1(i_{1,3}), c_3)$. Hence

$$d(c_3, \chi_1(i_{1,3})) = d(c_3, m_{1,3}) + d(m_{1,3}, \chi_1(i_{1,3})).$$

For edge $e_{1,3} \in \mathrm{Ch}(\mu)$, $B(e_{1,3}) = \{u \in T(e_{1,3}) \mid d(u, c_1) \geq d(u, c) \text{ for some } c \in C_r(f) \cap T(e_{1,3})\}$ by definition. Since no $c \in C_r(f)$ satisfies $0 = d(c_1, c_1) \geq d(c_1, c)$, we know $c_1 \notin B(e_{1,3})$. Then $\chi_1(i_{1,3}) \in B(e_{1,3})$ and $c_1 \notin B(e_{1,3})$ imply that point $m_{1,3}$ lies on path $P(\chi_1(i_{1,3}), c_1)$. Hence we have

$$d(c_1, \chi_1(i_{1,3})) = d(c_1, m_{1,3}) + d(m_{1,3}, \chi_1(i_{1,3})).$$

By the above two equations and Eq. (1), we obtain

$$d(c_1, m_{1,3}) \geq d(c_3, m_{1,3}). \tag{5}$$

We claim that $\chi_3(i) \notin B(e_{1,3})$ for all agents $i \in N$. Every agent $i \in N(\chi_1, B(e_{1,3}))$ has the location $\chi_3(i) = t_{e_{1,3}} \in A(e_{1,3}) \setminus \{\mu\}$ in profile $\chi_3(N)$ by the definition of χ_3 and $e_{1,3} \neq e_2$. Also every agent $i \notin N(\chi_1, B(e_{1,3}))$ has a location $\chi_3(i) \notin B(e_{1,3})$ by the definition of χ_3. This proves the claim.

By noting that $c_3 \notin B(c_{1,3})$ in Case a, we see that, for each agent $i \in N$, it holds $\chi_3(i) \notin B(e_{1,3})$ and thereby path $P(\chi_3(i), c_3)$ does not pass though point $m_{1,3} \in B(e_{1,3})$. Hence we have

$$d(c_3, \chi_3(i)) < d(c_3, m_{1,3}) + d(m_{1,3}, \chi_3(i)). \tag{6}$$

Therefore by Eq. (2), for every agent $i \in N$, it holds that

$$\begin{aligned} d(c_{1,3}, \chi_3(i)) &= d(c_{1,3}, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \\ &= d(c_1, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \quad \text{(by definition of } m_{1,3}) \\ &\geq d(c_3, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \quad \text{(by Eq. (5))} \\ &> d(c_3, \chi_3(i)) \quad \text{(by Eq (6))}. \end{aligned}$$

Case b. $c_3 \in B(e_{1,3})$: To handle this case, we use the next claim, where a proof of it will be given later.

Claim 1 $c_3 \notin C_r(f) \setminus T(e_2)$.

Fig. 3 illustrates the range where c_3 can exist in T.

By $e_{1,3} \neq e_2$ and Claim 1, the distance between $m_{1,3}$ and $c_{1,3}$ is larger than that between $m_{1,3}$ and c_3, i.e., it holds that

$$d(c_{1,3}, m_{1,3}) > d(c_3, m_{1,3}). \tag{7}$$

By Eq (2), for every agent $i \in N$, it holds that

$$\begin{aligned} d(c_{1,3}, \chi_3(i)) &= d(c_{1,3}, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \\ &> d(c_3, m_{1,3}) + d(m_{1,3}, \chi_3(i)) \quad \text{(by Eq. (7))} \\ &= d(c_3, \chi_3(i)), \end{aligned}$$

as required.

Finally we give a proof of Claim 1.

Proof of Claim 1. To prove the claim, it suffices to show that

$$d(c_3, \chi_2(i_{2,3})) < d(c, \chi_2(i_{2,3})) \text{ for all } c \in C_r(f) \setminus T(e_2).$$

Since $d(c_3, \chi_2(i_{2,3})) \leq d(c_2, \chi_2(i_{2,3}))$ by Eq. (3), it futher suffices to prove that

$$d(c_2, \chi_2(i_{2,3})) < d(c, \chi_2(i_{2,3})) \text{ for all } c \in C_r(f) \setminus T(e_2).$$

We distinguish two cases: $c_2 \in T(e_2)$ and $c_2 \neq \mu$; and $c_2 \notin T(e_2)$ or $c_2 = \mu$.

Case 1. $c_2 \in T(e_2)$ and $c_2 \neq \mu$: We first show that $\mu \notin P(\chi_2(i_{2,3}), c_2)$. By the definition of χ_2, for any agent $i \in N(\chi_1, B(e_2))$, $\chi_2(i) = t_{e_2}$. Then agent $i_{2,3} \in N_{2,3} = N(\chi_1, B(e_2))$ has location $\chi_2(i_{2,3}) = t_{e_2} \in A(e_2) \setminus \{\mu\}$. By assumption of $\mu \neq c_2 \in T(e_2)$ in Case 1, we see that $P(\chi_2(i_{2,3}), c_2) \subseteq T(e_2) \setminus \{\mu\}$.

Hence $d(c_2, \chi_2(i_{2,3})) < d(c_2, \mu) + d(\mu, \chi_2(i_{2,3}))$. From this, we see that for any candidate $c \in C_r(f) \setminus T(e_2)$, it holds that

$$\begin{aligned} d(c_2, \chi_2(i_{2,3})) &< d(c_2, \mu) + d(\mu, \chi_2(i_{2,3})) \\ &\leq d(c, \mu) + d(\mu, \chi_2(i_{2,3})) \text{ (by } c \in C_r(f)) \\ &= d(c, \chi_2(i_{2,3})), \end{aligned}$$

as required.

Case 2. $c_2 \notin T(e_2)$ or $c_2 = \mu$: We first show that $c_2 \notin C_r(f)$. When $c_2 = \mu$, it holds $c_2 = \mu \notin C_r(f)$ since $|C_r(f)| \geq 2$ and $|\mathrm{Ch}(\mu)| \geq 2$. Assume otherwise (i.e., $c_2 \neq \mu$ and $c_2 \notin T(e_2)$). In this case, by the definition of e_2, $e' \notin \mathrm{Ch}(\mu)$ holds for the edge $e' = (\mu, v)$ with $c_2 \in T(e')$. Clearly $e' \notin \mathrm{Ch}(\mu)$ means that $c_2 \notin C_r(f)$.

Hence for any candidate $c \in C_r(f) \setminus T(e_2)$, we have

$$\begin{aligned} d(c_2, \chi_2(i_{2,3})) &= d(c_2, \mu) + d(\mu, \chi_2(i_{2,3})) \\ &< d(c, \mu) + d(\mu, \chi_2(i_{2,3})) \quad (\text{by } c_2 \in C(f) \setminus C_r(f)) \\ &= d(c, \chi_2(i_{2,3})), \end{aligned}$$

as required. □

This completes a proof of Lemma 2.

4 Sufficiency of Theorem 1

In this section, we prove the sufficiency of Theorem 1. Let C be a set of points in a tree T such that C is a perimetric distribution. When $|C| = 1$, any mechanism f with $C(f) = C$ outputs a unique facility location for all profiles of agents, and thereby for any agent set S, all agents in S cannot benefit by misreporting their location. Therefore the mechanism f is GSP. We consider the case that $|C| \geq 2$.

First we design a voting mechanism f with the set C of candidates. Let μ be the middle point between the most distant two points in C. Since C is a perimetric distribution, for any $c, c' \in C$, we have $d(\mu, c) = d(\mu, c')$. If a point $w \in \{\mu\} \cup C$ is on an edge (u, v) of T, then we regard w as a vertex of T and replace (u, v) with two edges (u, w) and (w, v). We regard T as a rooted tree by designating μ as the root. For each vertex u, let $\mathrm{Ch}(u)$ be the set of edges $e = (u, v)$ such that there is at least one $c \in C$ in $T(e)$.

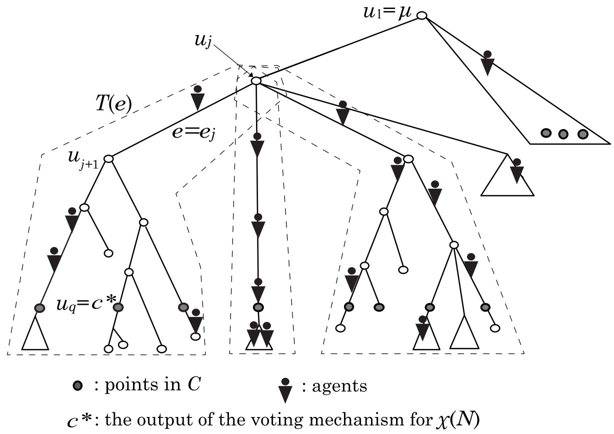

Figure 4: The voting mechanism f outputs a facility location c^* such that path $P(\mu, c^*) = (u_1 = \mu, u_2, \ldots, u_q = c^*)$ satisfies that for each $j = 1, 2, \ldots, q-1$, $(|N(\chi, T((u_j, u_{j+1})))|, \mathrm{id}((u_j, u_{j+1}))) \prec (|N(\chi, T(e))|, \mathrm{id}(e))$ for all edges $e \in \mathrm{Ch}(u_j) \setminus \{(u_j, u_{j+1})\}$.

Let E denote the set of all edges in the tree T. We define a *lexicographical order* between two vectors (a, b) and (a', b') so that

$$(a, b) \prec (a', b')$$

if $a < a'$, or $a = a'$ and $b < b'$. Given a perimetric distribution $C \subseteq T$, we define a voting mechanism f as follows. We introduce an arbitrary total order among all edges in E with an injective function $\mathrm{id} : E \to \mathbb{N}$, i.e., for two edges $e, e' \in E$ if $e \neq e'$, then $\mathrm{id}(e) \neq \mathrm{id}(e')$. Given a profile $\chi(N)$, we let f output $f(\chi(N)) = c^* \in C$ so that path

$$P(\mu, c^*) = (u_1 = \mu, u_2, \ldots, u_q = c^*)$$

is formed by choosing each edge $e_j = (u_j, u_{j+1}),\ j = 1, 2, \ldots, q-1$ that satisfies the lexicographic order:

$$(|N(\chi, T(e_j))|, \mathrm{id}(e_j)) \prec (|N(\chi, T(e))|, \mathrm{id}(e)) \ \text{ for all edges } e \in \mathrm{Ch}(u_j) \setminus \{e_j\}.$$

Fig. 4 illustrates a profile $\chi(N)$ in a rooted tree T and its output by the voting mechanism f.

We show the group strategy-proofness of the voting mechanism f via the next lemma.

Lemma 4 *Given a perimetric distribution $C \subseteq T$ with $|C| \geq 2$, let f be the voting mechanism defined in the above. Let χ be a location function, and $(u_1 = \mu, u_2, \ldots, u_q = f(\chi(N)))$ be the sequence of vertices on path $P(\mu, f(\chi(N)))$. Let $S \subseteq N$ be a subset of agents, and χ' be a location function such that*

$$\begin{array}{l} \chi(\overline{S}) = \chi'(\overline{S}) \text{ and} \\ d(f(\chi(N)), \chi(i)) < d(f(\chi'(N)), \chi(i)) \text{ for all } i \in S. \end{array}$$

If $f(\chi'(N)) \in T[u_j]$ holds for some $j = 1, 2, \ldots, q-1$, then we have $f(\chi'(N)) \in T[u_{j+1}]$.

Proof. We assume that $f(\chi'(N)) \in T[u_j]$ for some $j = 1, 2, \ldots, q-1$. Since C is a perimetric distribution, it holds that $d(f(\chi(N)), u_j) = d(f(\chi'(N)), u_j)$ by $u_q = f(\chi(N)), f(\chi'(N)) \in T[u_j]$. Hence $f(\chi'(N)) \in T((u_j, u_{j+1}))$ implies $f(\chi'(N)) \in T[u_{j+1}]$. So we assume that $f(\chi'(N)) \in T[u_j] \setminus T((u_j, u_{j+1}))$ holds to prove the lemma by deriving a contradiction. Let $e = (u_j, u_{j+1})$, where $f(\chi(N)) \in T(e)$. Let $e' \in \mathrm{Ch}(u_j) \setminus \{e\}$ be the edge such that $f(\chi'(N)) \in T(e')$.

For every agent $i \in S$, we have

$$\begin{aligned} d(f(\chi(N)), \chi(i)) &< d(f(\chi'(N)), \chi(i)) \ \text{(by the lemma assumption)} \\ &\leq d(f(\chi'(N)), u_j) + d(u_j, \chi(i)) \ \text{(by triangle inequality)} \\ &= d(f(\chi(N)), u_j) + d(u_j, \chi(i)) \ \text{(by } d(f(\chi(N)), u_j) = d(f(\chi'(N)), u_j)\text{)}. \end{aligned}$$

This means that u_j is not on $P(\chi(i), f(\chi(N)))$ for any agent $i \in S$. Hence

$$\chi(S) \subseteq T(e) \setminus \{u_j\}. \tag{8}$$

Here we use the next claim, whose correctness will be given later.

Claim 2 *Edges e' and e satisfy $|N(\chi', T(e'))| < |N(\chi, T(e'))|$ or $|N(\chi', T(e))| > |N(\chi, T(e))|$.*

When $|N(\chi', T(e'))| < |N(\chi, T(e'))|$ holds, we see that it implies that there is an agent $s \in S$ such that $\chi'(s) \in T \setminus T(e')$ and $\chi(s) \in T(e')$ since $\chi(\overline{S}) = \chi'(\overline{S})$. This, however, contradicts Eq. (8). On the other hand, $|N(\chi', T(e))| > |N(\chi, T(e))|$ implies that there is an agent $s \in S$ such that $\chi'(s) \in T(e)$ and $\chi(s) \in T \setminus T(e)$. This again contradicts Eq. (8).

Proof of Claim 2. Let $e_1 = e$ and $\chi_1 = \chi$, and let $e_2 = e'$ and $\chi_2 = \chi'$. It suffices to show that for $\{k, k'\} = \{1, 2\}$, edges e_k and $e_{k'}$ satisfy $|N(\chi_{k'}, T(e_{k'}))| < |N(\chi_k, T(e_{k'}))|$ or $|N(\chi_{k'}, T(e_k))| > |N(\chi_k, T(e_k))|$. To derive a contradiction, we assume that

$$\begin{aligned} &|N(\chi_k, T(e_{k'}))| \leq |N(\chi_{k'}, T(e_{k'}))| \text{ and} \\ &|N(\chi_{k'}, T(e_k))| \leq |N(\chi_k, T(e_k))|. \end{aligned} \tag{9}$$

For profile $\chi_k(N)$, mechanism f outputs $f(\chi_k(N)) \in T(e_k)$, which implies that the lexicographic order

$$(|N(\chi_k, T(e_k))|, \mathrm{id}(e_k)) \prec (|N(\chi_k, T(e_{k'}))|, \mathrm{id}(e_{k'})). \tag{10}$$

In particular, it holds $|N(\chi_k, T(e_k))| \leq |N(\chi_k, T(e_{k'}))|$. Symmetrically for profile $\chi_{k'}(N)$, mechanism f outputs $f(\chi_{k'}(N)) \in T(e_{k'})$, which implies that

$$(|N(\chi_{k'}, T(e_{k'}))|, \mathrm{id}(e_{k'})) \prec (|N(\chi_{k'}, T(e_k))|, \mathrm{id}(e_k)). \tag{11}$$

In particular, it holds $|N(\chi_{k'}, T(e_{k'}))| \leq |N(\chi_{k'}, T(e_k))|$. From these inequalities and Eq. (9), we have $|N(\chi_k, T(e_k))| \leq |N(\chi_k, T(e_{k'}))| \leq |N(\chi_{k'}, T(e_{k'}))| \leq |N(\chi_{k'}, T(e_k))| \leq |N(\chi_k, T(e_k))|$, where the four inequalities can hold by equality only. Hence now from Eqs. (10) and (11), it must hold that $\mathrm{id}(e_k) < \mathrm{id}(e_{k'})$ and $\mathrm{id}(e_{k'}) < \mathrm{id}(e_k)$, respectively. This, however, is a contradiction, proving Claim 2.

This completes a proof of Lemma 4. □

Now we are ready to prove that our voting mechanism is always GSP. Let χ be a location function and let $c^* = f(\chi(N))$, and $(u_1 = \mu, u_2, \ldots, u_q = c^*)$ denote the sequence of vertices in path $P(\mu, c^*)$, as shown in Fig. 4. Let $S \subseteq N$ be a subset of agents. To derive a contradiction, we assume that all agents in S benefit by misreporting their locations from $\chi(S)$ to $\chi'(S)$; i.e., let χ' be a location function such that $\chi(\overline{S}) = \chi'(\overline{S})$ and $d(f(\chi(N)), \chi(i)) < d(f(\chi'(N)), \chi(i))$ for all agents $i \in S$. In particular, it holds $f(\chi(N)) \neq d(f(\chi'(N))$. We show that $f(\chi'(N)) \in T[u_q]$ by an induction on u_j, $j = 1, 2, \ldots, q$. Obviously we have $f(\chi'(N)) = c^* \in T = T[\mu] = T[u_1]$. Suppose that $f(\chi'(N)) \in T[u_j]$ for some $j = 1, 2, \ldots, q-1$. Then by Lemma 4, we have $f(\chi'(N)) \in T[u_{j+1}]$. This means that $f(\chi'(N)) \in T[u_q] = T[c^*]$. Since C is a perimetric distribution, we have $T[c^*] \cap C = \{c^*\}$. Hence we have $f(\chi'(N)) = c^* = f(\chi(N))$, a contradiction to $f(\chi(N)) \neq d(f(\chi'(N))$. This completes a proof that the voting mechanism f is GSP.

5 Concluding Remarks

In this paper, we characterized a possible distribution of candidates (locations of the facility that can be output) by GSP mechanisms in a tree metric. That is, for a set C of p points in a tree, there exists a p-candidate GSP mechanism whose output set $C(f)$ is equal to

C if and only if C is a perimetric distribution. This explains the non-existence of $p \geq 3$-candidate GSP mechanisms in a line metric (e.g., [5]), because no set C with at least three points can be a perimetric distribution in a line metric. However, it remains open to show whether the set $C(f)$ of candidates of an SP mechanism f in a tree metric also needs to be a perimetric distribution or not. Also it is left as a future work to examine a possible distribution of candidates of SP or GSP mechanisms in a metric on a more complex graph or in an Euclidean space.

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