Characterizing Mechanisms in Obnoxious Facility Game

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Abstract: In this paper, we study the (group) strategy-proofness of deterministic mechanisms in the obnoxious facility game. In this game, given a set of strategic agents in a metric, we design a mechanism that outputs the location of a facility in the metric based on the locations of the agents reported by themselves. The benefit of an agent is the distance between her location and the facility and the social benefit is the total benefits of all agents. An agent may try to manipulate outputs by the mechanism by misreporting strategically her location. We wish to design a mechanism that is *strategy-proof* (i.e., no agent can gain her benefit by misreporting) or *group strategy-proof* (i.e., there is no coalition of agents such that each member in the coalition can simultaneously gain benefit by misreporting), while the social benefit will be maximized. In this paper, we first prove that, in the line metric, there is no strategy-proof mechanism such that the number of candidates (locations output by the mechanism for some reported locations) is more than two. We next completely characterize (group) strategy-proof mechanisms with exactly two candidates in the general metric and show that there exists a 4-approximation group strategy-proof mechanism in any metric.

1 Introduction

In the *facility game*, given a set of "strategic" agents in a metric, we design a procedure, called a *mechanism*, that outputs the location of a facility in the metric based on reported locations of the agents so that the social cost (or benefit), which is defined to be the sum of individual costs (or benefits) such as the distance from the facility, is minimized (or maximized). We assume that the mechanism is known to all the agents before they report their locations and that an agent may try to manipulate outputs by the mechanism by misreporting strategically her location so that an output location of the facility will be beneficial to her (we also assume that there is no way of testing whether a reported location is a misreported one or not). A mechanism is called *strategy-proof* if no single agent can gain her benefit by misreporting her location. Moreover, a mechanism is called *group strategy-proof* if no coalition of agents can gain benefit of each member in the coalition simultaneously by misreporting the locations of the coalition. Then a (group) strategy-proof

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mechanism may deliver a location of the facility which is not an optimal solution in terms of the social cost (or benefit). Our game-theoretical goal is to design a (group) strategy-proof mechanism with a good approximation ratio between locations output by the mechanism and optimal locations.

In mechanism design, we can consider mechanisms that are allowed to make payments. However, in many settings, money is not used as a medium of compensation due to ethical or legal considerations [12]. For example, in the *social choice* literature, mechanisms without payments are commonly studied. Since the facility game is rather deeply concerned in the social choice, we also design mechanisms without payments in the facility game.

Moulin [9] and Border and Jordan [3] studied a problem of the social choice in economics wherein each agent's preference is a function with a single peak at the most preferred point in a given space but no objective function to be optimized is given. Based on the median voter theorem [2], they characterized the strategy-proof mechanisms in the line and a space of a multidimensional version of the line, respectively. The traditional facility game is a problem of social choice together with a social cost that is to be minimized as an objective function. Schummer and Vohra [11] extended the mechanism [3] to the facility game on tree networks, and characterized the strategy-proof mechanisms to metrics on arbitrary networks containing at least one cycle. Recently strategy-proof *approximation* mechanisms for optimization problem have been studied extensively [1, 7, 8, 10]. Alon et al. [1] gave a complete analysis on the approximation ratio of strategy-proof mechanisms for the facility game in metrics on arbitrary networks. Currently group strategy-proof mechanisms that attain the optimal social cost are known up to tree networks.

In this paper, we study the (group) strategy-proofness of deterministic mechanisms in the obnoxious facility game. In contrast with the traditional game, we regard the distance from each agent to the facility as the benefit of the agent in this game, and the sum of the benefits of all agents will be maximized as the social benefit. Thus each agent's preference is no longer represented as a single-peaked function. This problem setting can be interpreted as a social scenario such that the mayor of a town plans to build a garbage dump in the town according to a set of reported home addresses of the local residents, wishing to maximize the sum of the distances of all residents. Cheng et al. [4] first studied group strategy-proof mechanisms for the obnoxious facility game in the line metric. They demonstrated that a mechanism that simply outputs a socially optimal location is not strategy-proof. They designed a group strategy-proof mechanism which chooses one of two predetermined locations as an output according to the distribution of reported locations, and showed that the mechanism is a 3-approximation. They suggested that the mechanism can be extended to a 3-approximation group strategy-proof mechanism in tree networks. In this paper, we first prove that there is no strategy-proof mechanism in the line metric such that the number of *candidates* (locations output by the mechanism for some reported locations) is more than two. This suggests that we need to know the specific structure of a given metric if we wish to design a strategy-proof mechanism with more than two candidates. We next derive a complete characterization of (group) strategy-proof mechanisms with exactly two candidates in the general metric.

The paper is organized as follows. Section 2 formulates the obnoxious facility game and reviews the definition of (group) strategy-proofness. Section 3 proves that the line metric admits no strategy-proof mechanism with more than two candidates. Section 4 proposes *a valid threshold mechanism* and proves that a mechanism with exactly two candidates in the general metric is (group) strategy-proof if and only if it is a valid threshold mechanism. Section 5 then shows that there always exists a 4-approximation valid threshold mechanism in any metric. Finally Section 6 makes some concluding remarks.

2 Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}_+$ be the sets of nonnegative integers and nonnegative real numbers, respectively. Let (Ω, d) be a metric such that Ω is a set of points (possibly an infinite set) and $d : \Omega \times \Omega \to \mathbb{R}_+$ is a symmetric distance function, i.e., $d(x, y) = d(y, x)$ for every two points $x, y \in \Omega$ and $d(x, y) + d(y, z) \geq d(x, z)$ for every three points $x, y, z \in \Omega$.

For a set $N = \{1, 2, \ldots, n\}$ of agents, $x_i \in \Omega$ denotes the location reported by agent $i \in N$ and the multiset $X = \{x_1, x_2, \ldots, x_n\}$ of the locations is called a *profile* of N. For a location $y \in \Omega$ of an obnoxious facility, the benefit of agent i is defined to be the distance between her location and the facility, i.e.,

$$\beta(y, x_i) = d(y, x_i).$$

The *social benefit* of a location $y \in \Omega$ of an obnoxious facility over a profile X is defined to be the total benefit of n agents

$$\text{SB}(y, X) = \sum_{i=1}^{n} \beta(y, x_i).$$

For a profile X, let $\text{OPT}(X)$ denote the optimal obnoxious social benefit, i.e., $\text{OPT}(X) = \max_{y \in \Omega} \text{SB}(y, X)$.

In the obnoxious facility game, a deterministic mechanism outputs a facility location based on a given profile X, where we do not distinguish two profiles $X = \{x_1, x_2, \ldots, x_n\}$ and $X' = \{x'_1, x'_2, \ldots, x'_n\}$ of N if there is a bijection $\sigma : N \to N$ such that $x_i = x'_{\sigma(i)}$ for all $i \in N$. We write $X = X'$ if there is such a bijection σ. A *mechanism* is defined to be a function $f : \Omega^n \to \Omega$ such that $f(X) = f(X')$ for two profiles X and X' of N with $X = X'$. We say that a mechanism f has an approximation ratio γ if

$$\text{OPT}(X) \leq \gamma \text{SB}(f(X), X) \quad \text{for all profiles } X \in \Omega^n \text{ of } N.$$

In the following we define the strategy-proofness and the group strategy-proofness of mechanisms. For a profile $X = \{x_1, x_2, \ldots, x_n\}$ of N and an agent set $S \subseteq N$, let X_S denote the profile of S obtained from X by eliminating locations x_i such that $i \in N - S$. We denote X_{N-S} simply by X_{-S}. In particular, for $S = \{i\}$, X_{-S} is denoted by $X_{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. Location profile X may be written by $\langle x_i, X_{-i} \rangle$ or $\langle X_S, X_{-S} \rangle$. For simplicity, we write $f(x_i, X_{-i}) = f(\langle x_i, X_{-i} \rangle)$ and $f(X_S, X_{-S}) = f(\langle X_S, X_{-S} \rangle)$.

Definition 1 *A mechanism f is* strategy-proof *(SP for short) if no agent can benefit from misreporting her location. Formally, given an agent i, a profile $X = \langle x_i, X_{-i} \rangle \in \Omega^n$ and a misreported location $x'_i \in \Omega$, it holds that*

$$\beta(f(x_i, X_{-i}), x_i) \geq \beta(f(x'_i, X_{-i}), x_i).$$

Definition 2 *A mechanism f is* group strategy-proof *(GSP for short) if for any group of agents, at least one of them cannot benefit from misreporting their locations simultaneously.*

Formally, given a non-empty set $S \subseteq N$, a profile $X = \langle X_S, X_{-S} \rangle \in \Omega^n$ and a misreported profile $X'_S \in \Omega^{|S|}$ of S, there exists $i \in S$ satisfying

$$\beta(f(X_S, X_{-S}), x_i) \geq \beta(f(X'_S, X_{-S}), x_i).$$

We remark that a stronger notation of group strategy-proofness requires that any set of misreporting agents with a strict gain contains at least one agent who strictly loses [6]. Our GSP results in this paper do not hold under the stronger definition. However, the above weaker definition of group strategy-proofness is rather common in the social choice, since in the settings without payments an agent has no incentive to misreport unless it strictly benefits.

For a mechanism $f : \Omega^n \to \Omega$, a point $y \in \Omega$ is called a *candidate* if there is a profile $X \in \Omega^n$ such that $f(X) = y$, and the set of all candidates of f is denoted by C_f. A mechanism with $|C_f| = p$ is called by a *p-candidate mechanism.* Any 1-candidate mechanism is group strategy-proof, but its approximation ratio γ can be infinitely large.

3 Mechanisms in the line metric

This section proves that there is a metric that admits no p-candidate SP mechanism for any $p \geq 3$. Let (I, d) be the line metric, where I denotes the 1-dimensional Euclidean space.

Theorem 3 *There is no p-candidate SP mechanism for any $p \geq 3$ in the line metric.*

We prove Theorem 3 via the next lemma.

Lemma 4 *Let f be a p-candidate SP mechanism. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a profile of N, and X'_S be a misreported profile of a coalition $S \subseteq N$. Then:*

(i) *If there is a candidate $c \in C_f$ such that $f(X) < c$ and $\max\{x_i \mid i \in S\} < \frac{f(X)+c}{2}$, then it holds $f(X'_S, X_{-S}) < c$; and*

(ii) *If there is a candidate $c \in C_f$ such that $f(X) > c$ and $\min\{x_i \mid i \in S\} > \frac{f(X)+c}{2}$, then it holds $f(X'_S, X_{-S}) > c$.*

Proof. We prove (i) ((ii) can be treated symmetrically). To derive a contradiction, we assume that there is a candidate $c \in C_f$ such that $f(X) < c \leq f(X'_S, X_{-S})$ and $\max\{x_i \mid i \in S\} < \frac{f(X)+c}{2}$. We assume that S is minimal subject to the condition $c \leq f(X'_S, X_{-S})$, from which any proper subset $T \subset S$ satisfies $f(X'_T, X_{-T}) < c$.

Without loss of generality assume that $x_k \leq \cdots \leq x_2 \leq x_1$ and denote $X'_S = \{x'_1, x'_2, \ldots, x'_k\}$. We define profiles X^i, $i = 0, 1, \ldots, k$ as follows.

$$X^0 = X \quad \text{and} \quad X^i = \langle x'_i, (X^{i-1})_{-i} \rangle,\ i = 1, 2, \ldots, k.$$

Thus X^i is obtained from X by replacing $x_1, x_2, \ldots, x_i$ with $x'_1, x'_2, \ldots, x'_i$. By the assumption on $X = X^0$ and the minimality of S, we observe

$$f(X^i) < c \leq f(X^k) \text{ for } i = 0, 1, \ldots, k-1. \tag{1}$$

Since f is SP, we see from Definition 1 that for each agent $i \in S$ and profile X^{i-1}, it holds $\beta(f(x_i, (X^{i-1})_{-i}), x_i) \geq \beta(f(x'_i, (X^{i-1})_{-i}), x_i)$; i.e.,

$$|f(X^{i-1}) - x_i| \geq |f(X^i) - x_i| \text{ for } i = 1, 2, \ldots, k. \tag{2}$$

Since $x_1 < \frac{f(X^0)+c_\ell}{2}$ and $f(X^0) < c_\ell \leq f(X^k)$ by inequality (1) with $i = 0$, we obtain $x_1 < f(X^k)$. Then $x_k \leq x_1 < f(X^k)$ and inequality (2) with $i = k$ imply that

$$|f(X^{k-1}) - x_k| \geq f(X^k) - x_k. \tag{3}$$

If $x_k \leq f(X^{k-1})$ then $f(X^{k-1}) \geq f(X^k)$ would hold by inequality (3), contradicting that inequality (1) with $i = k-1$, $f(X^{k-1}) < f(X^k)$. Hence it must hold

$$f(X^{k-1}) < x_k. \tag{4}$$

From this and inequality (3), we have $\frac{f(X^{k-1})+f(X^k)}{2} \leq x_k$. By recalling that $x_k < \frac{f(X^0)+c}{2} \leq \frac{f(X^0)+f(X^k)}{2}$, we obtain $f(X^{k-1}) < f(X^0)$, where we see that $k \geq 2$ must hold.

Let j be the minimum index with $1 \leq j \leq k-1$ such that $f(X^j) \leq f(X^{k-1})$. Then it holds $f(X^j) \leq f(X^{k-1}) < f(X^{j-1})$ because $f(X^{k-1}) < f(X^0)$. This and inequality (1) mean that

$$f(X^j) \leq f(X^{k-1}) < f(X^{j-1}) < f(X^k). \tag{5}$$

Now by $f(X^j) \leq f(X^{k-1})$, inequality (4) and $x_k \leq x_j$, we obtain $f(X^j) < x_j$. By $f(X^j) < x_j$, inequality (2) with $i = j$ is written as

$$|f(X^{j-1}) - x_j| \geq x_j - f(X^j). \tag{6}$$

If $f(X^{j-1}) \leq x_j$ then $f(X^{j-1}) \leq f(X^j)$ would hold by inequality (6), contradicting that $f(X^j) < f(X^{j-1})$. Hence $x_j < f(X^{j-1})$ holds, and inequality (6) implies $x_j \leq \frac{f(X^{j-1})+f(X^j)}{2}$.

Since $\frac{f(X^{k-1})+f(X^k)}{2} \leq x_k$, $x_j \leq \frac{f(X^{j-1})+f(X^j)}{2}$ and $x_k \leq x_j$ hold, we have $f(X^{k-1}) + f(X^k) \leq f(X^{j-1}) + f(X^j)$. This, however, is a contradiction to inequality (5). □

We assume that f is a p-candidate SP mechanism with $C_f = \{c_1, c_2, \ldots, c_p\} \subset I$ in (I, d), where $c_1 < c_2 < \cdots < c_p$. Fix a candidate $c_t \in C_f - \{c_1, c_p\}$, and let $I_t = \{x \in I \mid \frac{c_1+c_t}{2} < x < \frac{c_t+c_p}{2}\}$ (where $\frac{c_1+c_p}{2} \in I_t$), $I_{t,1} = \{x \in I \mid \frac{c_1+c_t}{2} < x < \frac{c_1+c_p}{2}\}$ and $I_{t,p} = \{x \in I \mid \frac{c_1+c_p}{2} < x < \frac{c_t+c_p}{2}\}$. For a profile X of N, let $S^a(X) = \{i \in N \mid x_i \leq \frac{c_1+c_t}{2}\}$ and $S^b(X) = \{i \in N \mid \frac{c_t+c_p}{2} \leq x_i\}$. We prove Theorem 3 by deriving a contradiction.

Proof of Theorem 3. Let X be a profile such that $f(X) = c_t \in C_f - \{c_1, c_p\}$.

Let X^a be a profile obtained from X by replacing x_i for each $i \in S^a(X)$ with a new location $x'_i \in I_{t,1}$. Since $f(X) = c_t < c_{t+1}$ and every $i \in S = S^a(X)$ satisfies $x_i \leq \frac{c_1+c_t}{2} < \frac{f(X)+c_{t+1}}{2}$, we see that $f(X^a) < c_{t+1}$ holds by Lemma 4(i) with $c = c_{t+1}$.

Let $X^{ab} = \{x'_1, \ldots, x'_n\}$ be a profile obtained from X^a by replacing x_j for each $j \in S^b(X)$ with a new location $x'_j \in I_{t,p}$. Assume that $f(X^{ab}) \neq c_1$ (the case of $f(X^{ab}) \neq c_p$ can be treated symmetrically).

Note that X^a can be obtained from X^{ab} by changing the locations of agents in $S^b(X)$. For $S' = S^b(X)$, every $i \in S'$ satisfies $\frac{c_1+f(X^{ab})}{2} \leq \frac{c_1+c_p}{2} < x'_i$ and it holds $f(X^{ab}) > c_1$. Hence by Lemma 4(ii) with $c = c_1$, we obtain $f(X^a) > c_1$.

Let $X^a = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$ and Y be a profile such that $f(Y) = c_1 \in C_f$. Note that Y can be obtained from X^a by by changing the locations of agents in N. Since $S^a(X^a) = \emptyset$ and $f(X^a) \leq c_t$, every $i \in N$ satisfies $\frac{c_1+f(X^a)}{2} \leq \frac{c_1+c_t}{2} < \tilde{x}_i$. From this and $f(X^a) > c_1$, we see from Lemma 4(ii) with $c = c_1$ that it holds $f(Y) > c_1$. This, however, is a contradiction to $f(Y) = c_1$. □

4 2-candidate SP/GSP mechanisms in the general metric

This section gives a complete characterization of 2-candidate SP/GSP mechanisms in the general metric (not necessarily on the basis of particular graphs). In the following part, we propose *valid threshold mechanisms.*

For fixed two points $a, b \in \Omega$, we partition Ω into three subspaces $\Omega_a = \{x \in \Omega \mid d(a,x) < d(b,x)\}$, $\Omega_m = \{x \in \Omega \mid d(a,x) = d(b,x)\}$ and $\Omega_b = \{x \in \Omega \mid d(a,x) > d(b,x)\}$. For a profile X of N, the set of agents and the number of agents in Ω_a are denoted by S_a and n_a, respectively, i.e., $S_a = \{i \mid x_i \in \Omega_a\}$ and $n_a = |S_a|$. Analogously, we denote $S_m = \{i \mid x_i \in \Omega_m\}$, $n_m = |S_m|$, $S_b = \{i \mid x_i \in \Omega_b\}$ and $n_b = |S_b|$.

For each integer $\ell = 0, 1, \ldots, n$, let θ_ℓ be a function that maps a profile $M \in \Omega_m^\ell$ of ℓ agents with locations in Ω_m to an integer. A mechanism f on N is called a *threshold mechanism* if there are two points $a, b \in \Omega$ and a set $\{\theta_0, \ldots, \theta_n\}$ of functions such that f returns a for all profiles X with $n_a < \theta_{n_m}(X_{S_m})$ and returns b for the other profiles X, i.e., f is given by

$$f(X) = \begin{cases} a & \text{if } n_a < \theta_{n_m}(X_{S_m}) \\ b & \text{if } \theta_{n_m}(X_{S_m}) \leq n_a. \end{cases}$$

A threshold mechanism f is symmetric in terms of a and b in the sense that $f(X) = b$ if $n_b < \overline{\theta}_{n_m}(X_{S_m})$ and $f(X) = a$ otherwise for the set of complement functions $\overline{\theta}_\ell$ on Ω_m^ℓ, $\ell = 0, 1, \ldots, n$ such that

$$\overline{\theta}_\ell(M) = n + 1 - \ell - \theta_\ell(M) \text{ for } M \in \Omega^\ell \text{ and } \ell = 0, 1, \ldots, n.$$

Furthermore a threshold mechanism f is called *valid* if the set $\{\theta_0, \ldots, \theta_n\}$ of functions satisfies the two conditions: (i) $\theta_0(\emptyset) \notin \{0, n+1\}$ and $0 \leq \theta_\ell(M) \leq n + 1 - \ell$ for $0 \leq \ell \leq n$ and $M \in \Omega_m^\ell$; and (ii) $\theta_\ell(M) - 1 \leq \theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ for $0 \leq \ell \leq n-1$, $M \in \Omega_m^\ell$ and $x \in \Omega_m$. Note that the set of complement functions also satisfies the above two conditions.

We show that a 2-candidate mechanism is SP (or GSP) if and only if it is a valid threshold mechanism via the next two theorems.

Theorem 5 *Every valid threshold mechanism is a 2-candidate GSP mechanism.*

Theorem 6 *Every 2-candidate SP mechanism is a valid threshold mechanism.*

For another profile $X' = \{x'_1, \ldots, x'_n\}$ of N, we use the following notation. The set of agents and the number of agents in Ω_a are denoted by S'_a and n'_a, respectively, i.e., $S'_a = \{i \mid x'_i \in \Omega_a\}$ and $n'_a = |S'_a|$. Analogously, we denote $S'_m = \{i \mid x'_i \in \Omega_m\}$, $n'_m = |S'_m|$, $S'_b = \{i \mid x'_i \in \Omega_b\}$ and $n'_b = |S'_b|$. We first prove Theorem 5.

Proof of Theorem 5. Let f be a valid threshold mechanism. First we show that $C_f = \{a, b\}$. Clearly $C_f \subseteq \{a, b\}$. For a profile X of N with $n_a = 0$ and $n_m = 0$, it holds that $f(X) = a$ since $n_a < \theta_0(\emptyset) \in \{1, 2, \ldots, n\}$. Similarly, for a profile X with $n_a = n$ and $n_m = 0$, we have $f(X) = b$ since $n_a \geq \theta_0(\emptyset) \in \{1, 2, \ldots, n\}$. Hence $C_f = \{a, b\}$ holds and it means that f is a 2-candidate mechanism.

Let us show the group strategy-proofness of f, i.e., not all agents in any coalition S can gain simultaneously by misreporting their locations. Fix a profile $X = \{x_1, \ldots, x_n\}$ of N and a coalition $S \subseteq N$ wherein x'_i denotes the misreported location of each agent $i \in S$. A misreported profile of S is denoted by $X'_S = \{x'_i \mid i \in S\}$ and we denote $X' = \langle X'_S, X_{-S}\rangle$.

We prove that there is an agent $i \in S$ such that $\beta(f(X), x_i) \geq \beta(f(X'), x_i)$. We consider the case of $f(X) = a$, i.e., $n_a < \theta_{n_m}(X_{S_m})$ (the other case can be treated symmetrically by considering the complement functions $\overline{\theta}_\ell$).

If $f(X') = a$, then $\beta(f(X), x_i) = \beta(f(X'), x_i)$ for any $i \in S$ and we are done. Assume that $f(X') = b$. If there is an agent $i \in S - S_a$, then for such an agent i it holds $d(a, x_i) \geq d(b, x_i)$, i.e., $\beta(f(X), x_i) \geq \beta(f(X'), x_i)$, and we are done. Hence it suffices to prove that $S \subseteq S_a$ implies $f(X') = a$. From $S \subseteq S_a$, we have $n'_a \leq n_a$ and $n'_m \geq n_m$. Let $k = n'_m - n_m$ ($\leq n_a - n'_a$) and we denote $S'_m - S_m = \{1, 2, \ldots, k\}$ and $M'_i = \{x'_1, \ldots, x'_i\} \in \Omega^i_m$. By repeatedly applying the second property of functions $\theta_0, \ldots, \theta_n$, we obtain $\theta_{n_m}(X_{S_m}) - k \leq \theta_{n_m+1}(\langle M'_1, X_{S_m}\rangle) - (k-1) \leq \cdots \leq \theta_{n_m+k}(\langle M'_k, X_{S_m}\rangle) = \theta_{n'_m}(X'_{S'_m})$. Then by $k \leq n_a - n'_a$, it holds that $\theta_{n_m}(X_{S_m}) - n_a \leq \theta_{n'_m}(X'_{S'_m}) - n'_a$, which implies $n'_a < \theta_{n'_m}(X'_{S'_m})$ and $f(X') = a$ since $n_a < \theta_{n_m}(X_{S_m})$ now holds by $f(X) = a$ and the assumption on f. □

We next prove Theorem 6 via the next lemma.

Lemma 7 *Let f be a 2-candidate SP mechanism with $C_f = \{a, b\}$, and X and X' be two profiles of N with $X_{S_m} = X'_{S'_m}$. Then $f(X) = f(X')$ if $f(X) = a$ and $n_a \geq n'_a$; or $f(X) = b$ and $n_b \geq n'_b$.*

Proof. For a profile X with $f(X) = a$, if $f(x'_i, X_{-i}) = b$ hold for a misreported location x'_i of some agent $i \in S_a$, then we would have $\beta(f(x_i, X_{-i}), x_i) = d(a, x_i) < d(b, x_i) = \beta(f(x'_i, X_{-i}), x_i)$, contradicting that $\beta(f(x_i, X_{-i}), x_i) \geq \beta(f(x'_i, X_{-i}), x_i)$ holds for any profile X and agent $i \in N$ in an SP f. This means that if $f(X) = a$ then $f(X'_S, X_{-S}) = a$ holds no matter how a subset $S \subseteq S_a$ misreports $X'_S \in \Omega^{|S|}$. Symmetrically if $f(X) = b$ then $f(X'_S, X_{-S}) = b$ for any $X'_S \in \Omega^{|S|}$ with a subset $S \subseteq S_b$.

To prove the lemma, we consider the case where $f(X) = a$ and $n_a \geq n'_a$ (the other case can be treated symmetrically). We construct a new profile X'' of N with $X''_{S_m} = X_{S_m}$ by changing the locations of agents i with $x_i \in \Omega_a$ as follows. We choose a set $T_a \subseteq S_a$ of n'_a agents, and let $x''_i = x'_i$ for each $i \in T_a$, x''_i be any location in Ω_b for each $i \in S_a - T_a$, and $x''_i = x_i$ for each $i \in N - S_a$. Since $f(X) = a$ and X'' is obtained from X by changing the locations of agents only in S_a, it holds $f(X'') = a$. Since X'' is obtained from X' by changing the locations of agents only in S'_b, it holds that if $f(X') = b$ then $f(X'') = b$, i.e., if $f(X'') = a$ then $f(X') = a$. Therefore we have $f(X') = a$. □ □

Now we give a proof of Theorem 6.

Proof of Theorem 6. Let f be a 2-candidate SP mechanism with $C_f = \{a, b\}$. For an integer $0 \leq \ell \leq n$ and a set $M \in \Omega^\ell_m$ of locations, let $\theta_\ell(M)$ be the minimum integer n_a such that there is a profile X of N such that $X_{S_m} = M$ satisfying $f(X) = b$, where if $f(X) = a$ (resp., $f(X) = b$) for all such X then we define $\theta_\ell(M) = n + 1 - \ell$ (resp., $\theta_\ell(M) = 0$). Then by Lemma 7, $f(X) = a$ holds for all profiles X such that $\theta_\ell(M) > n_a$ and $X_{S_m} = M$. Thus, f is a threshold mechanism. Now it suffices to show that f is valid, i.e., the set $\{\theta_0, \ldots, \theta_n\}$ of the above functions satisfies $\theta_0(\emptyset) \notin \{0, n+1\}$ and $0 \leq \theta_\ell(M) \leq n + 1 - \ell$ for $0 \leq \ell \leq n$ and $M \in \Omega^\ell_m$; and $\theta_\ell(M) - 1 \leq \theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ for $1 \leq \ell \leq n-1$, $M \in \Omega^\ell_m$ and $x \in \Omega_m$. Note that we have shown that $\theta_\ell(M) \in \{0, \ldots, n+1-\ell\}$ for $0 \leq \ell \leq n$.

We first show inequality $\theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ (inequality $\theta_\ell(M) - 1 \leq \theta_{\ell+1}(\langle x, M\rangle)$ follows from the inequality $\overline{\theta}_{\ell+1}(\langle x, M\rangle) \leq \overline{\theta}_\ell(M)$ on the complement functions). If $\theta_\ell(M) \geq n - \ell$, then $\theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ is immediate, since $\theta_{\ell+1}(\langle x, M\rangle) \leq n + 1 - (\ell+1) = n - \ell$ by definition. Consider the case of $\theta_\ell(M) < n - \ell$. Then there is a profile X of N such that

$X_{S_m} = M$, $n_a = \theta_\ell(M)$, $f(X) = b$ and $n_b \geq 1$. We choose an agent $t \in S_b$ and change its location from x_t to an arbitrary location $x'_t \in \Omega_m$ to obtain a new profile $X' = \langle x'_t, X_{-t}\rangle$ of N. By Definition 1, it holds that $\beta(f(X), x_t) \geq \beta(f(X'), x_t)$, i.e., $d(b, x_t) \geq d(f(X'), x_t)$. Since $d(a, x_t) > d(b, x_t)$, $f(X') = b$ holds for the profile X' such that $n'_a = \theta_\ell(M)$ and $X'_{S'_m} = \langle x'_t, M\rangle \in \Omega_m^{\ell+1}$. Recall that $\theta_{\ell+1}(\langle x'_t, M\rangle)$ is the minimum integer n_a such that there is a profile X of N such that $X_{S_m} = \langle x'_t, M\rangle$ satisfying $f(X) = b$. Hence we have $\theta_{\ell+1}(\langle x'_t, M\rangle) \leq n'_a = \theta_\ell(M)$.

Finally, we prove that $\theta_0(\emptyset) \neq 0$ (property $\theta_0(\emptyset) \neq n+1$ follows from $\overline{\theta}_0(\emptyset) \neq 0$ on the complement function). If $\theta_0(\emptyset) = 0$, then inequality $\theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ $(0 \leq \ell \leq n-1$, $M \in \Omega_m^\ell$, $x \in \Omega_m)$ inductively implies that $\theta_\ell(M) = 0$ for any $M \in \Omega_m^\ell$ and $0 \leq \ell \leq n$. This, however, means that $f(X) = b$ for all profiles X of N, contradicting that $C_f = \{a, b\}$. □

5 Approximation ratio of 2-candidate mechanisms

This section analyzes the approximate ratio $\gamma = \max_{X \in \Omega^n} \frac{\mathrm{OPT}(X)}{\mathrm{SB}(f(X), X)}$ of 2-candidate SP mechanisms in the general metric.

Upper bound We first derive an upper bound on the approximate ratio γ. Let f be a 2-candidate SP mechanism on a set N of n agents in a metric (Ω, d), where f can be given by choosing two points $a, b \in C_f$ and functions θ_i, $i = 0, 1, \ldots, n$ so that f becomes a valid threshold mechanism by Theorem 6.

Theorem 8 *Let f be a 2-candidate mechanism for a set N of n agents. If $C_f = \{a, b\}$ is a pair of most distant points in Ω and f is a valid threshold mechanism by a set $\{\theta_0, \ldots, \theta_n\}$ of functions, then the approximate ratio γ of f is less than* $\max\{\frac{2n}{\theta_0(\emptyset)}, \frac{2n}{n+1-\theta_0(\emptyset)}\}$.

Proof. Let $d(a, b) = 2r$. For a profile X of N with $f(X) = a$, we derive an upper bound on γ. We have $\mathrm{SB}(f(X), X) = \sum_{i \in N} d(a, x_i) > n_m r + n_b r = (n - n_a) r$, since $d(a, x_i) > r$ for n_b agents $i \in S_b$, and $d(a, x_i) = r$ for n_m agents $i \in S_m$. Let $c_X \in \Omega$ denote an optimal facility location, i.e., $\mathrm{OPT}(X) = \mathrm{SB}(c_X, X)$. On the other hand, we see that $\mathrm{OPT}(X) \leq 2nr$, since $d(c_X, x_i) \leq 2r$ for any location $x_i \in \Omega$ by the choice of a and b. Hence we have $\gamma < \frac{2nr}{(n-n_a)r} = \frac{2n}{n-n_a}$. Since $f(X) = a$, it holds $n_a < \theta_{n_m}(X_{S_m}) \leq \theta_0(\emptyset)$, where we use the property $\theta_{\ell+1}(\langle x, M\rangle) \leq \theta_\ell(M)$ of functions to get the second inequality. Hence $\gamma < \frac{2n}{n+1-\theta_0(\emptyset)}$. When $f(X) = b$, we apply the same argument to the complement function to obtain $\gamma < \frac{2n}{n+1-\overline{\theta}_0(\emptyset)} = \frac{2n}{\theta_0(\emptyset)}$. This proves the theorem. □ □

The bound $\max\{\frac{2n}{\theta_0(\emptyset)}, \frac{2n}{n+1-\theta_0(\emptyset)}\}$ in Theorem 8 is minimized and $\gamma \leq 4$ holds when $\theta_0(\emptyset) = \lceil n/2 \rceil$. In fact, such a valid threshold mechanism f for a set of n agents can be constructed as follows. For a pair $C_f = \{a, b\}$ of most distant points in Ω, let f return $f(X) = a$ if $n_a + n_m < n_b$; $f(X) = b$ otherwise.

Lower bound We have shown that every valid threshold mechanism f with $\theta_0(\emptyset) = \lceil n/2 \rceil$ has an approximation ratio $\gamma = 4$. Now we give a tight example (Ω, d) such that for every choice of $a, b \in \Omega$, the approximation ratio γ attained by a valid threshold mechanism f with $C_f = \{a, b\}$ and $\theta_0(\emptyset) = \lceil n/2 \rceil$ is at least 4. Such an example (Ω_G, d) is constructed from a graph G as follows. Let $G = (V, E)$ be the graph with a set V of ten vertices, s and u_i, v_i, t_i $(i = 1, 2, 3)$, and a set E of 12 edges, $t_i v_i, v_i s, v_i u_{i-1}, v_i u_{i+1}$ $(i = 1, 2, 3)$, where we interpret $u_4 = u_1$ and $u_0 = u_3$ (see Fig. 1). We regard each edge as a line segment of

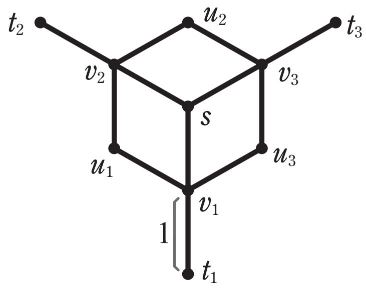

Figure 1: An undirected graph G

length 1, and a point x on an edge e is denoted by $x \in e$. Let Ω_G be the set of points in all edges including the end points. The distance $d(x, x')$ for two points $x, x' \in \Omega_G$ is defined to be the length of a shortest path between x and x'.

Lemma 9 *For any two points $c_1, c_2 \in \Omega_G$, there are a point $c^* \in \Omega_G$ and a shortest path P between c_1 and c_2 in (Ω_G, d) such that*

$$\rho = \frac{\max\{d(c_1, c^*), d(c_2, c^*)\} + d(x, c^*)}{d(c_1, c_2)/2} \geq 4$$

for the middle point x on P.

Proof. Given points c_1 and c_2, we choose P which does not pass through s if any. By symmetry, we only need to consider the case where x is on one of edges v_3t_3, v_3u_2 and v_3s. We show that $c^* = t_1$ suffices the lemma. Let $d(c_2, c^*) \geq d(c_1, c^*)$ without loss of generality. When $d(c_1, c_2) \leq 2$, we easily see that $\rho \leq 4$ since $\max\{d(c_1, c^*), d(c_2, c^*)\} + d(x, c^*) \geq 4$ and $d(c_1, c_2)/2 \leq 1$. In what follows, assume that $d(c_1, c_2) > 2$ and hence c_1 and c_2 are in two nonadjacent edges, respectively. Note that $x \in v_3t_3$ implies $d(c_1, c_2) \leq 2$. We distinguish two cases.

Case 1. $x \in v_3u_2$: We do not need to consider the case where one of c_1 and c_2, say c_i is on edge v_3t_3, since we can move $c_i \in v_3t_3$ to a point on edge v_3u_3 without changing both of the position of x and $d(c_1, c_2)$ or increasing $\max\{d(c_1, c^*), d(c_2, c^*)\}$. In this case, we can assume that $c_1 \in v_3u_3$ and c_2 is on one of edges v_2u_2, v_2u_1 and v_2t_2; or $c_1 \in v_1u_3$ and $c_2 \in v_2u_2$. In any case, we obtain $d(c_2, c^*) + d(c_2, x) + d(x, c^*) = 8 + 2\alpha$ and $d(c_1, c_2)/2 \leq (3 + \alpha)/2$, where $\alpha = d(c_2, v_2) \leq 1$ when $c_2 \in v_2t_2$ and $\alpha = 0$ otherwise. Hence we have $\rho \geq \frac{8+2\alpha-d(c_2,x)}{d(c_1,c_2)/2} \geq -1 + \frac{16+4\alpha}{3+\alpha} = -1 + 4 + \frac{4}{3+\alpha} \geq 4$.

Case 2. $x \in v_3s$: We can assume that c_2 is on v_3u_3, v_3u_2 or v_3t_3 and c_1 is on sv_1 or sv_2, where c_1 is not on v_1t_1 or v_1u_1 by the choice of P. In addition, we can assume $c_1 \notin sv_2$ and $c_2 \notin v_3u_2$, since we can move $c_1 \in sv_2$ (resp., $c_2 \in v_3u_2$) to a point on edge sv_1 (resp., v_3t_3) without changing both of the position of x and $d(c_1, c_2)$ or increasing $\max\{d(c_1, c^*), d(c_2, c^*)\}$. Then we obtain $d(c_2, c^*) + d(c_2, x) + d(x, c^*) = 6 + 2\alpha$ and $d(c_1, c_2)/2 \leq (2+\alpha)/2$, where $\alpha = d(c_2, v_3) \leq 1$ when $c_2 \in v_3t_3$ and $\alpha = 0$ otherwise. Hence we have $\rho \geq \frac{6+2\alpha-d(c_2,x)}{d(c_1,c_2)/2} \geq -1 + \frac{12+4\alpha}{2+\alpha} = -1 + 4 + \frac{4}{2+\alpha} > 4$, as required. □ □

By Lemma 9, we can get the following theorem.

Theorem 10 *For the obnoxious facility game on the above metric (Ω_G, d), let f be a valid threshold mechanism with $\theta_0(\emptyset) = \lceil n/2 \rceil$ of a set N of n agents. Then for any choice of $C_f = \{a, b\}$, the approximation ratio of f is not smaller than $4(1 - \frac{4}{n+2})$.*

Proof. For $C_f = \{a, b\}$, there are points $c^*, m \in \Omega_G$ such that $d(a, m) = d(b, m)$ and $\rho = \frac{\max\{d(a,c^*),d(b,c^*)\}+d(m,c^*)}{d(a,b)/2} \geq 4$ by Lemma 9. We consider the case of $d(a, c^*) > d(b, c^*)$ (the other case can be treated analogously). For a sufficiently small $\epsilon > 0$, let $m_b \in \Omega_G$ be a point that is closer to b than a in a neighbor of m within distance ϵ; $d(a, m_b) \leq d(a, m) + \epsilon$ and $d(m_b, c^*) \geq d(m, c^*) - \epsilon$ hold. Construct a profile X of N such that $\lceil n/2 \rceil - 1$ agents are situated on point a while the other $\lfloor n/2 \rfloor + 1$ agents on point m_b. Since $n_m = 0$ and $n_a = \lceil n/2 \rceil - 1 < \theta_0(\emptyset)$, we have $f(X) = a$, $\mathrm{SB}(f(X), X) = (\lfloor n/2 \rfloor + 1)d(a, m_b) \leq (\lfloor n/2 \rfloor + 1)(d(a, b)/2 + \epsilon)$ and $\mathrm{OPT}(X) \geq \mathrm{SB}(c^*, X) = (\lceil n/2 \rceil - 1)d(a, c^*) + (\lfloor n/2 \rfloor + 1)d(m_b, c^*)$ $\geq (\lceil n/2 \rceil - 1)(d(a, c^*) + d(m, c^*) - \epsilon)$. Hence it holds $\frac{\mathrm{OPT}(X)}{\mathrm{SB}(f(X),X)} \geq \frac{(\lfloor n/2 \rfloor+1)(d(a,c^*)+d(m,c^*)-\epsilon)}{(\lceil n/2 \rceil-1)(d(a,b)/2+\epsilon)}$, which approaches to $4(1 - \frac{4}{n+2})$ when $\epsilon \to 0$. □ □

We remark that it is still open whether there exists an example (Ω, d) such that for every choice of $a, b \in \Omega$, the approximation ratio γ attained by a valid threshold mechanism f with $\theta_0(\emptyset) \neq \lceil n/2 \rceil$ is at least 4.

6 Concluding remarks

In this paper, we studied SP/GSP mechanisms for the obnoxious facility game. We first showed that there is a metric that admits no p-candidate SP mechanism for any $p \geq 3$. We then proved that a valid threshold mechanism is a complete characterization of (group) strategy-proof mechanisms with exactly two candidates in the general metric. We also proved that there always exists a 4-approximation valid threshold mechanism in any metric. Note that for any integer $p \geq 3$, there is a metric (Ω, d) that admits a p-candidate GSP mechanism. For example, let (Ω, d) be a metric on a star network with a center v_c and p leaf edges $v_c v_j$ $j = 1, 2, \ldots, p$ of length 1, and f be a p-candidate mechanism that returns $f(X) = v_k$ for a profile X such that $n_k = \min_{1 \leq j \leq p} n_j$ for $n_j = |\{i \in N \mid x_i \in v_c v_j\}|$. Then we can prove that this p-candidate mechanism is GSP and the approximation ratio is at most $2 + \frac{1}{p-1}$.

There are still several open problems on the obnoxious facility game. First, given a metric (Ω, d), it is important to know the maximum number $p(\Omega, d)$ of candidates such that there exists a $p(\Omega, d)$-candidate SP/GSP mechanism. Also for such a maximum value $p(\Omega, d)$, it is left open whether we can construct a p'-candidate SP/GSP mechanism in the metric for any $p' < p(\Omega, d)$ or not. The problem of placing an obnoxious facility when the locations of agents are fixed is called *the 1-maxian problem* [5, 13]. In the 1-maxian problem, the number of optimal locations of an obnoxious facility in a network metric is known to be finite. It would be interesting to investigate the relationship between solutions of the 1-maxian problem and candidates of SP/GSP mechanisms of the obnoxious facility game. Also it is another interesting issue to derive a counterpart/extension of our arguments in randomized mechanisms in the general metric (see [4] for a randomized 2-candidate GSP mechanism in the line metric).

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