Parameterization of Strategy-Proof Mechanisms in the Obnoxious Facility Game

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Abstract

In the obnoxious facility game, a location for an undesirable facility is to be determined based on the voting of selfish agents. Design of group strategy proof mechanisms has been extensively studied, but it is known that there is a gap between the social benefit (i.e., the sum of individual benefits) by a facility location determined by any group strategy proof mechanism and the maximum social benefit over all choices of facility locations; their ratio, called the *benefit ratio* can be 3 in the line metric space. In this paper, we investigate a trade-off between the benefit ratio and a possible relaxation of group strategy proofness, taking 2-candidate mechanisms for the obnoxious facility game in the line metric as an example. Given a real $\lambda \geq 1$ as a parameter, we introduce a new strategy proofness, called "λ-group strategy-proofness," so that each coalition of agents has no incentive to lie unless every agent in the group can increase her benefit by strictly more than λ times by doing so, where the 1-group strategy-proofness is the previously known group strategy-proofness. We next introduce "masking zone mechanisms," a new notion on structure of mechanisms, and prove that every λ-group strategy-proof (λ-GSP) mechanism is a masking zone mechanism. We then show that, for any $\lambda \geq 1$, there exists a λ-GSP mechanism whose benefit ratio is at most $1 + \frac{2}{\lambda}$, which converges to 1 as λ becomes infinitely large. Finally we prove that the bound is nearly tight: given $n \geq 1$ selfish agents, the benefit ratio of λ-GSP mechanisms cannot be better than $1 + \frac{2}{\lambda}$ when n is even, and $1 + \frac{2n-2}{\lambda n+1}$ when n is odd.
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1 Introduction

1.1 Social choice theory

In social choice theory, we design *mechanisms* that determine a social decision based on a vote. That is, for a set Ω of voting alternatives and a set N of selfish voters with various utilities, we design a mechanism $f : \Omega^n \to \Omega$ as a collective decision making system. Voters know the exact detail of the operation of the mechanisms before they actually vote, and each voter can find out her expected benefit in the case when every voter votes truthfully. We call this benefit for each voter her *primary benefit*. Each voter may try to manipulate

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the decision of a mechanism by changing her voting to increase her personal utility. A voting which aims to manipulate the decision of a mechanism is called a *strategic-voting*. To the effect of making a fair decision, we are interested in mechanisms in which no voter can benefit by a single-handed strategic-voting. Such a mechanism is called a *strategy-proof* mechanism. Moreover, a mechanism is called a *group strategy-proof* mechanism, if there is no coalition of voters such that each member in the coalition can simultaneously benefit by their cooperative strategic-voting. Moulin [9] studied social choice theory under the condition that the set of alternatives is the one-dimensional Euclidean space and each utility function is a single peaked concave function, and gave necessary and sufficient conditions of strategy-proofness under such conditions. After that, Border and Jordan [2] extended the characterization into the multi-dimensional Euclidean space, and characterized strategy-proof mechanisms in those metrics. Schummer and Vohra [11] applied the result of Border and Jordan [2] to the case when Ω is the set of all points in a tree metric and characterized strategy-proof mechanisms in those metrics. Moreover, they characterized strategy-proof mechanisms in the case when Ω is the set of all points in a graph metric which has at least one cycle.

1.2 Facility game

The *facility game* can be regarded as a problem in social choice theory where a location of the facility in a metric space will be decided based on locations of agents (votes by voters) and each agent tries to maximize the benefit from her utility function defined based on the distance from her location to the location of the facility.

In a facility game with a set N of agents in a space Ω, each agent reports a point in the space, and a location of a facility will be determined by a procedure called a *mechanism*, where how the set of points reported by the agents is used by the procedure to decide a location of the facility is known to all the agents in advance. Each agent is selfish in the sense that she may misreport her point so that the output by the mechanism becomes more beneficial to her. The facility can be classified as either one of two types, one is *desirable* to agents (or each agent wants the facility to be located near the point reported by the agent), and the other is *obnoxious* to agents (or each agent wants the facility to be located far from the point reported by the agent).

Several studies have been extensively made on the desirable facility game, such as designing mechanisms [1, 2, 7, 8, 10, 11]. Procaccia and Tennenholtz [10] proposed a group strategy-proof mechanism which returns the location of the median agent as the facility location when all agents are located on a path. Moreover, they designed a *randomized mechanism*, that is, a mechanism that does not output a single facility location but outputs a probability distribution of the facility location over a metric space. In randomized mechanisms, the utility of agents is defined to be the expected value by the probability distribution. On the contrary, a mechanism which outputs a facility location is called *deterministic*.

Cheng *et al.* [4] studied the *obnoxious facility game*. For a given mechanism, the benefit for each agent obtained under the assumption that all agents have reported their true locations is called *primary benefit*. Mechanisms which only output one of a predetermined set of k candidates for a facility location are called *k-candidate mechanisms*.

In previous studies of facility games [1, 2, 3, 4, 7, 8, 10, 11], mechanisms are allowed to distinguish agents. In other words, the input of mechanisms is not only location information (i.e., where is reported) but also agents' information (i.e., who reports the location). On the other hand, there is a category of mechanisms which are called *anonymous*, that is, which do not use agents' information. Anonymous mechanisms would be a fair decision-making in the sense that no indication of a particular agent can reflect to outputs by such mechanisms.

Another important aspect of mechanisms of facility games is a measure of the quality of mechanisms. In general, a location of a facility that maximizes some social benefit, such as the sum of all individual benefits, is different from a location output by a strategy-proof (or group strategy-proof) mechanism. In other words, the maximum value of the social utility attained by a strategy-proof (or group strategy-proof) mechanism is smaller than that attained just by choosing the best location of the facility. This raises a problem of designing a strategy-proof (or group strategy-proof) mechanism that outputs a location of a facility that maximizes a social benefit among all strategy-proof (or group strategy-proof) mechanisms. A possible measurement of the performance for a mechanism is a *benefit-ratio*, the ratio of the social utility attained by the mechanism to that attained by a theoretically maximum possible social benefit. For example, Alon *et al.* [1] gave a complete analysis on benefit-ratios of group strategy-proof mechanisms for the desirable facility game in general graph metrics.

We review some recent results on the obnoxious facility game. Ibara and Nagamochi [5, 6] presented a complete characterization of 2-candidate (group) strategy-proof mechanisms in any metric space, giving necessary and sufficient conditions for the existence of such a mechanism in a given metric, and proved that in any metric, a 2-candidate mechanism with a benefit ratio of 4 can always be designed.

For the obnoxious facility game in the line metric, Ibara and Nagamochi [5, 6] showed that there exists no k-candidate group strategy-proof mechanism for any $k \geq 3$. Cheng *et al.* [4] gave a 2-candidate group strategy-proof mechanism in the line metric with a benefit ratio of 3, and showed that this is the best possible over all 2-candidate group strategy-proof mechanisms in the line metric.

1.3 Our Contribution

Since it has been shown that the best benefit ratio is 3 over all 2-candidate group strategy-proof mechanisms for the obnoxious facility game in the line metric, we propound the following questions on the game:

1. Is there any way of relaxing the definition of group strategy-proofness so that the benefit ratio 3 is improved over such relaxed group strategy-proof mechanisms?

2. With an adequate parameter $\lambda \geq 1$, is there any trade-off between a λ-group strategy-proofness (a group strategy-proofness relaxed with λ) and the benefit ratio ρ for λ such that ρ approaches 1 as λ becomes infinitely large; and

3. For each fixed $\lambda \geq 1$, what is the benefit ratio ρ of a λ-group strategy-proof (or can upper and lower bounds on ρ which are tight be derived)?

This paper answers all of these questions affirmatively. First we introduce a relaxed version of (group) strategy-proofness via a parameter $\lambda \geq 1$ by assuming that an agent has no incentive to misreport her own location unless she can increase her benefit by strictly more than λ times her primary benefit. Respectively, in every group of agents, at least one agent cannot get an increase of strictly more than λ times from her primary benefit by strategically changing her report in coalition with the rest of the group. This parameterization serves as a relaxation of the notion of group strategy-proofness. Mechanisms which guarantee the above property are termed "λ-strategy-proof (λ-SP) mechanisms" and "λ-group strategy-proof (λ-GSP) mechanisms," where 1-group strategy-proofness is equivalent to the previously known group strategy-proofness.

Second, we design a λ-GSP 2-candidate mechanism whose benefit ratio ρ is at most $1 + 2/\lambda$, which approaches 1 as the parameter λ tends to ∞. This answers the first and second question.

Finally, we show that there is no λ-SP 2-candidate mechanism whose benefit ratio ρ is smaller than $1+2/\lambda$ for an even n and $1+(2n-2)/(\lambda n+1)$ for an odd n, where n (≥ 1) is the number of agents. This is an answer to the third and second question, since our upper and lower bounds on the benefit ratio are almost tight.

The above results are obtained by introducing a new concept of mechanism design that follows naturally from the introduction of the parameter λ, called "masking zone mechanisms," which in their own right might lend interesting directions for future research.

The paper is organized as follows. In Section 2, we give necessary preliminaries and introduce as a general ideas such concepts as (parameterized) strategy-proofness in Section 2.2, masking zone mechanisms in Section 2.3, social benefit in Section 2.4 and the obnoxious facility game, Section 2.5. Section 3 expands on the topic of masking zone mechanisms, and shows that being a masking zone mechanism is a necessary condition for a mechanism to be λ-SP. Following, Section 4 gives an upper bound on the benefit ratio of λ-GSP mechanisms, by constructing a mechanism f with a benefit ratio of at most $1 + \frac{2}{\lambda}$, which also extends known results in the line metric [4]. Immediately, in Section 5, it is shown that this bound is the best obtainable, by giving lower bounds on the benefit ratio of masking zone mechanisms. Finally, the paper is concluded in Section 6.

2 Preliminaries

2.1 Notation

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of real and nonnegative real numbers, respectively.

Let Ω be a universal set of points. For a positive integer $n \geq 1$, let N be a set of n agents. For a set $S \subseteq N$ of agents (resp., an agent $i \in N$), let $\overline{S} = N \setminus S$ (resp., $\overline{i} = N \setminus \{i\}$). Each agent $i \in N$ chooses a point $p \in \Omega$ as a reported value $\chi_i = p$. Let $\Omega_{\text{agents}} \subseteq \Omega$ denote a set of points that can be chosen by an agent. A vector $\chi \in \Omega^n_{\text{agents}}$ with reported values χ_i, $i \in N$ is called a profile of N.

A mechanism f is a function that given a profile χ of N outputs a point $t \in \Omega$. Let $\Omega_{\text{facility}} \subseteq \Omega$ denote a set of points that can be output by a mechanism, where a mechanism f is a mapping $f(\chi) : \Omega_{\text{agents}}^n \to \Omega_{\text{facility}}$. It is common in the literature, e.g., [3, 4], to represent the locations reported by agents as an n-dimensional vector $\vec{x}$, where $\vec{x}_i$ is the point reported by an agent $i \in N$. Under these circumstances, the notion of anonymity plays an important role. A mechanism f is *anonymous* if $f(\vec{x}) = f(\vec{x'})$ holds for any two vectors $\vec{x}$ and $\vec{x'}$ that admit a bijection σ on N such that $\vec{x}'_i = \vec{x}_{\sigma(i)}$ for all $i \in N$.

In what follows, we treat a profile χ of N as a multiset $\{\chi_i \mid i \in N\}$ of n elements. For convenience, given a profile χ and a set $S \subseteq N$ of agents, let χ_S denote the multiset $\{\chi_i \mid i \in S\}$ of $|S|$ elements. For a subspace $\Omega' \subseteq \Omega$, the restriction $\chi|_{\Omega'}$ of a profile χ on Ω' is defined to be the multiset

$$\chi|_{\Omega'} = \{\chi_i \mid i \in N, \chi_i \in \Omega'\},$$

where $|\chi|_{\Omega'}|$ means the number of elements in $\chi|_{\Omega'}$, i.e., the number of agents in $\chi|_{\Omega'}$.

The *benefit* of an agent $i \in N$ with respect to a point $p \in \Omega_{\text{agents}}$ and a point $t \in \Omega_{\text{facility}}$ is specified by a function $\beta_i : \Omega_{\text{facility}} \times \Omega_{\text{agents}} \to \mathbb{R}$. We assume that a larger value in β_i is preferable to the agent $i \in N$. For a mechanism $f : \Omega_{\text{agents}}^n \to \Omega_{\text{facility}}$, a point $t \in \Omega_{\text{facility}}$ is called a candidate if there is a profile $\chi \in \Omega_{\text{agents}}^n$ such that $f(\chi) = t$, and the set of all candidates of f is denoted by $C(f) \subseteq \Omega_{\text{facility}}$. A mechanism with $|C(f)| = k$ is called a *k-candidate mechanism*.

2.2 Strategy Proofness

In this paper, we assume that a larger value of β_i is preferable to the agent $i \in N$.

First we review the definitions of strategy-proofness and group strategy-proofness of mechanisms [1, 4, 5, 6]. Following, we introduce a new notion of "λ-strategy-proofness," an extension of strategy-proofness. Henceforth, let χ be a profile of a set N of $n \geq 1$ agents.

A mechanism f is called *strategy-proof* (SP for short) if no agent can strictly benefit from changing her report. Formally, for any agent $i \in N$ who changes her report from χ_i to $\chi'_i \in \Omega_{\text{agents}}$, for the profile $\chi' = \chi_{\bar{i}} \cup \chi'_i$ it holds that

$$\beta_i(f(\chi), \chi_i) \geq \beta_i(f(\chi'), \chi_i)$$

A mechanism f is called *group strategy-proof* (GSP for short) if for every group of agents, at least one agent in the group cannot benefit from changing her report in coalition with the rest of the group. Formally, for any non-empty set $S \subseteq N$ of agents and for any profile χ' such that $\chi'_{\overline{S}} = \chi_{\overline{S}}$, there exists an agent $i \in S$ for whom

$$\beta_i(f(\chi), \chi_i) \geq \beta_i(f(\chi'), \chi_i).$$

Now we extend the (group) strategy-proofness by introducing a parameter $\lambda \geq 1$.

A mechanism f is called *λ-strategy-proof* (λ-SP for short) if no agent can gain strictly more than λ times her primary benefit by changing her report. Formally, for any agent $i \in N$ and any profile χ' such that $\chi'_{\bar{i}} = \chi_{\bar{i}}$, it holds that

$$\lambda\beta_i(f(\chi), \chi_i) \geq \beta_i(f(\chi'), \chi_i). \tag{1}$$

A mechanism f is called λ-*group strategy-proof* (λ-GSP for short) if for every group of agents, at least one agent in the group cannot gain strictly more than λ times her primary benefit by changing her report in coalition with the rest of the group. Formally, for any non-empty set $S \subseteq N$ of agents and for any profile χ' such that $\chi'_{\overline{S}} = \chi_{\overline{S}}$, there exists an agent $i \in S$ for whom

$$\lambda\beta_i(f(\chi), \chi_i) \geq \beta_i(f(\chi'), \chi_i). \tag{2}$$

By definition, any λ-GSP mechanism is λ-SP, and the 1-strategy-proofness (resp., 1-group strategy-proofness) is equivalent to the strategy-proofness (resp., group strategy-proofness).

2.3 Masking Zone Mechanisms

In this paper, we introduce "masking zone mechanisms," another new concept on the structure of mechanisms.

Definition 1 *Let $\mathcal{S}$ be a family of nonempty disjoint subsets of Ω, and $\overline{S} = \Omega \setminus \bigcup_{S \in \mathcal{S}}$. A mechanism f is a* masking zone mechanism *with set of masking zones $\mathcal{S}$ if it delivers the same output $f(\chi) = f(\chi')$ for any two profiles χ and χ' such that*

$$\chi|_{\overline{S}} = \chi'|_{\overline{S}} \text{ and } |\chi|_S| = |\chi'|_S| \text{ for all } S \in \mathcal{S}.$$

In other words, $f(\chi)$ of a profile χ never changes as long as a point $\chi_i \in S \in \mathcal{S}$ changes to a point in the same subset S.

2.4 Social Benefit

We introduce an objective function $\text{sb}(t, \chi)$ that evaluates the quality of a point t determined based on a given profile χ. For a point $t \in \Omega_{\text{facility}}$ and a profile χ, we define the *social benefit* $\text{sb}(t, \chi)$ to be the sum of individual benefits over all agents, i.e.,

$$\text{sb}(t, \chi) = \sum_{i \in N} \beta_i(t, \chi_i).$$

Given a profile χ, let $\text{opt}(\chi)$ denote the maximum social benefit over all choices of points $t \in \Omega_{\text{facility}}$; i.e.,

$$\text{opt}(\chi) = \max_{t \in \Omega_{\text{facility}}} \{\text{sb}(t, \chi)\}.$$

The *benefit ratio* $\rho_f \geq 1$ of a mechanism f is defined to be

$$\rho_f = \sup_{\chi \in \Omega^n_{\text{agents}}} \frac{\text{opt}(\chi)}{\text{sb}(f(\chi), \chi)}.$$

When λ becomes infinitely large, the constraints of Eqs. (1) and (2) are no longer effective. If there is no such constraint as in Eq. (1) or (2), then a λ-SP or λ-GSP mechanism can deliver a point $t \in \Omega_{\text{facility}}$ that maximizes $\text{sb}(t, \chi)$ and $\rho_f = 1$ always holds in this case.

2.5 Obnoxious Facility Game in the Line Metric

In this paper, we consider the obnoxious facility game in the line metric [4]. Let (Ω, d) be a metric with a space $\Omega \subseteq \mathbb{R}$ in the line and the distance function $d : \Omega^2 \to \mathbb{R}_+$ such that

$$d(x,y) = |x-y| = \begin{cases} x-y & \text{if } x \geq y, \\ y-x & \text{otherwise.} \end{cases}$$

We assume that, for any agent $i \in N$, the benefit β_i is given by

$$\beta_i(t,p) = d(t,p) \quad t \in \Omega_{\text{facility}},\ p \in \Omega_{\text{agents}}.$$

In interest of space and clarity, given a profile χ and a candidate location $t \in \Omega_{\text{facility}}$, henceforth we omit referring to the benefit $\beta_i(t, \chi_i)$ of agent i, and directly write $d(t, \chi_i)$. Also we let $d(t, P)$ denote $\sum_{p \in P} d(t, p)$ for a point $t \in \mathbb{R}$ and a multiset P of points in $\mathbb{R}$, where $\mathrm{sb}(c, \chi) = d(c, \chi)$ for a profile χ of N and a location $c \in \mathbb{R}$.

Recall that there is no k-candidate SP mechanism in the line metric, for any $k \geq 3$ [5, 6] and that $\rho_f = 3$ for a GSP mechanism f and no GSP mechanism f attains $\rho_f < 3$ [4].

In this paper, we examine the benefit ratio of a 2-candidate λ-GSP mechanism f, and assume without loss of generality that $C(f) = \{0, 1\} = \Omega_{\text{facility}} \subseteq \Omega \subseteq \mathbb{R}$.

Given a real $\lambda \geq 1$, we define subsets I_0 and I_1 of $\mathbb{R}$ as follows:

$$I_0 = \left(\frac{-1}{\lambda-1}, \frac{1}{\lambda+1}\right),\ I_1 = \left(\frac{\lambda}{\lambda+1}, \frac{\lambda}{\lambda-1}\right) \quad \text{for } \lambda > 1,$$

and $I_0 = \{p \in \mathbb{R} \mid p < \frac{1}{2}\}$ and $I_1 = \{p \in \mathbb{R} \mid p > \frac{1}{2}\}$ for $\lambda = 1$. Let $I = I_0 \cup I_1$ and $\overline{I} = \mathbb{R} \setminus I$. Then we observe the next property.

Proposition 1 *Given a real $\lambda \geq 1$, a point $p \in \mathbb{R}$ satisfies $\lambda d(0,p) < d(1,p)$ (resp., $\lambda d(1,p) < d(0,p)$) if and only if $p \in I_0$ (resp., $p \in I_1$).*

Proof. We have that $\{p \in \mathbb{R} \mid \lambda d(0,p) - d(1,p) < 0\} = \{p \in \mathbb{R} \mid \lambda^2 p^2 - (p-1)^2 = (\lambda p + (p-1))(\lambda p - (p-1)) < 0\} = (\frac{-1}{\lambda-1}, \frac{1}{\lambda+1}) = I_0$. Analogously we see that $\{p \in \mathbb{R} \mid \lambda d(1,p) - d(0,p) < 0\} = (\frac{\lambda}{\lambda+1}, \frac{\lambda}{\lambda-1}) = I_1$. In the case of $\lambda = 1$, we have that $d(0,p) - d(1,p) < 0 \Leftrightarrow p < \frac{1}{2}$, and $d(0,p) - d(1,p) > 0 \Leftrightarrow p > \frac{1}{2}$, as required. □

In this paper, we first show that all 2-candidate λ-SP mechanisms are masking zone mechanisms.

Theorem 1 *Let $\lambda \geq 1$. Every 2-candidate λ-SP mechanism f with candidate set $C(f) = \{0, 1\}$ in the line metric is a masking zone mechanism with set of masking zones $\{I_0, I_1\}$.*

Based on this, we next design a masking zone λ-GSP mechanism whose benefit ratio is at most $1 + 2/\lambda$.

Theorem 2 *Let $\lambda \geq 1$. In the line metric, there is a 2-candidate λ-GSP mechanism f such that $\rho_f \leq 1 + \frac{2}{\lambda}$.*

Finally we examine the converse, showing that no masking zone λ-SP mechanism f attains a benefit ratio smaller than $1 + 2/\lambda$ for an even $n = |N|$ or $1 + (2n-2)/(\lambda n+1)$ for an odd $n = |N|$.

Theorem 3 *Let $\lambda \geq 1$ and $n = |N| \geq 1$. In the line metric, there is no 2-candidate λ-SP mechanism f such that*

$$\rho_f < \begin{cases} 1 + \frac{2}{\lambda} & \text{if } n \text{ is even,} \\ 1 + \frac{2n-2}{\lambda n+1} & \text{otherwise.} \end{cases}$$

3 Masking Zone Mechanisms

This section shows that any λ-SP mechanism is a masking zone mechanism. By the following lemma, we derive a necessary condition for a mechanism in the line metric to be λ-SP.

Lemma 1 *Given a real $\lambda \geq 1$, let f be a λ-SP mechanism with candidate set $C(f) = \{0, 1\}$. Let χ be a profile of N such that $f(\chi) = c \in \{0, 1\}$. If there is an agent i with $\chi_i \in I_c$, then the profile $\widehat{\chi}$ obtained from χ by changing χ_i to a point in I_c still satisfies $f(\widehat{\chi}) = c$, where $\widehat{\chi}_{\bar{i}} = \chi_{\bar{i}}$ and $\widehat{\chi}_i \in I_c$.*

Proof. To derive a contradiction, we assume that $f(\widehat{\chi}) = 1 - c$. Since $\chi_i \in I_c$, we know $\lambda d(c, \chi_i) < d(1-c, \chi_i)$ by Proposition 1, i.e., $\lambda d(f(\chi), \chi_i) = \lambda d(c, \chi_i) < d(1-c, \chi_i) = d(f(\widehat{\chi}), \chi_i)$. Since $\widehat{\chi}_{\bar{i}} = \chi_{\bar{i}}$, this contradicts that f is λ-SP. □

We are now ready to prove Theorem 1.

Proof of Theorem 1 Let f be a λ-SP mechanism with candidate set $C(f) = \{0, 1\}$. We say that two profiles χ and χ' of N are *zone-equivalent* if $\chi|_{\overline{I}} = \chi'|_{\overline{I}}$ and $|\chi|_{I_c}| = |\chi'|_{I_c}|$ for each $c \in C(f) = \{0, 1\}$. It suffices to show that, for any zone-equivalent profiles χ and χ', it holds that $f(\chi) = f(\chi')$. To derive a contradiction, assume that there are zone-equivalent profiles χ and χ' with $f(\chi) \neq f(\chi')$, and let χ and χ' minimize the number $|\chi|_{\overline{I}} \setminus \chi'|_{\overline{I}}| + |\chi'|_{\overline{I}} \setminus \chi|_{\overline{I}}|$ of different locations between them among all such pairs.

Since χ and χ' are zone-equivalent, there are two distinct locations $\chi_i \in I_c$ and $\chi'_j \in I_c$ for some agents $i, j \in N$ and some $c \in \{0, 1\}$. Without loss of generality $f(\chi) = c$ and $f(\chi') = 1 - c$ (if necessary we exchange the role of χ and χ'). Let $\widehat{\chi}$ be the profile obtained from χ by changing the location $\chi_i \in I_c$ of agent i to the point $\chi'_j \in I_c$. By Lemma 1 and $f(\chi) = c$, we see that $f(\widehat{\chi}) = c$.

Notice that $\widehat{\chi}$ and χ' remain zone-equivalent, and now they have fewer different locations than χ and χ' have. Then by the choice of χ and χ', it must hold $f(\widehat{\chi}) = f(\chi') = 1 - c$, which contradicts that $f(\widehat{\chi}) = c$, proving Theorem 1. □

4 Upper Bounds on the Benefit Ratio

In this section, given a real $\lambda \geq 1$, we prove Theorem 2 by constructing a 2-candidate λ-GSP mechanism f whose benefit ratio ρ_f is at most $1 + 2/\lambda$.

Having in mind that for a given profile χ, the condition for λ-group strategy-proofness of Eq. (2) concerns exactly the agents $i \in N$ with $\chi_i \in I$, we define a distorted distance between a point $c \in \{0,1\}$ and a point $p \in \Omega$ of to be

$$\mu(c,p) = \begin{cases} d(c,p) & \text{if } p \in \overline{I}, \\ 0 & \text{if } p \in I_c, \\ 1 & \text{if } p \in I_{1-c}, \end{cases}$$

where clearly $-1 \leq \mu(c,p) - \mu(1-c,p) \leq 1$ always holds. Also we let $\mu(c,P)$ denote $\sum_{p \in P} \mu(c,p)$ for a point $c \in \{0,1\}$ and a multiset P of points in $\mathbb{R}$. Then for a profile χ of N and a location $c \in \{0,1\}$, we have

$$\mu(c,\chi) = \sum_{i \in N} \mu(c,\chi_i) = d(c, \chi|_{\overline{I}}) + |\chi|_{I_{1-c}}|.$$

Based on this, we propose the following masking zone mechanism f with candidate set $C(f) = \{0,1\}$.

Mechanism $f(\chi)$: given a multiset χ, return a candidate $c \in C(f) = \{0,1\}$

$$f(\chi) = \begin{cases} 0 & \text{if } \mu(0,\chi) > \mu(1,\chi), \\ 1 & \text{if } \mu(0,\chi) \leq \mu(1,\chi). \end{cases} \tag{3}$$

We claim that the mechanism f is λ-GSP.

Lemma 2 *The mechanism f of Eq. (3) is λ-GSP.*

Proof. To derive a contradiction, we assume that f is not λ-GSP; i.e., by definition, there is a non-empty subset $S \subseteq N$ which influences two profiles χ and χ' with

$$\chi'_{\overline{S}} = \chi_{\overline{S}} \text{ and } f(\chi) \neq f(\chi')$$

such that every agent $i \in S$ satisfies

$$\lambda d(f(\chi), \chi_i) < d(f(\chi'), \chi_i) \text{ or } \chi_i \in I_c \text{ for } c = f(\chi),$$

by Proposition 1, where $f(\chi) \neq f(\chi')$ must hold by $\lambda \geq 1$. Let S be minimal subject to the above condition, and j be an arbitrary agent in S.

We prove that the profile χ'' obtained from χ' just by changing χ'_j to χ_j satisfies $f(\chi') = f(\chi'')$, or equivalently

$$\mu(0,\chi') - \mu(1,\chi') \leq 0 \text{ if and only if } \mu(0,\chi'') - \mu(1,\chi'') \leq 0 \tag{4}$$

by the definition of f of Eq. (3). Observe that $\chi''_{\overline{j}} = \chi'_{\overline{j}}$, $\chi''_j = \chi_j$, and

$$\mu(c,\chi'') = \mu(c,\chi') - \mu(c,\chi'_j) + \mu(c,\chi_j) \text{ for each } c \in \{0,1\}$$

by the definition of μ. From this, we have

$$\mu(0,\chi'') - \mu(1,\chi'') = \mu(0,\chi') - \mu(1,\chi') + [\mu(1,\chi'_j) - \mu(0,\chi'_j)] + [\mu(0,\chi_j) - \mu(1,\chi_j)], \tag{5}$$

where we know $-1 \leq \mu(1, \chi'_j) - \mu(0, \chi'_j) \leq 1$ by the definition of μ. By $\chi_j \in I_c$ for $f(\chi) = c$ (i.e., $f(\chi') = 1 - c$), we also know

$$\mu(0, \chi_j) - \mu(1, \chi_j) = \begin{cases} -1 & \text{if } f(\chi') = 1 \text{ (i.e., } \mu(0, \chi') - \mu(1, \chi') \leq 0 \text{ by Eq. (3)),} \\ 1 & \text{if } f(\chi') = 0 \text{ (i.e., } \mu(0, \chi') - \mu(1, \chi') > 0). \end{cases}$$

Therefore, if $\mu(0, \chi') - \mu(1, \chi')$ is nonnegative (resp., positive), then the right hand side of Eq. (5) is also nonnegative (resp., positive), implying Eq. (4).

Finally we observe that $f(\chi'') = f(\chi')$ contradicts the minimality of S.
(i) $S = \{j\}$: Since $\chi'_{\overline{S}} = \chi_{\overline{S}}$, we see that $\chi'' = \chi$ and $f(\chi'') = f(\chi) \neq f(\chi')$, a contradiction.
(ii) $S - \{j\} \neq \emptyset$: If $f(\chi'') = f(\chi')$ then the subset $T = S - \{j\}$ would satisfy $\chi''_{\overline{S}} = \chi'_{\overline{S}} = \chi_{\overline{S}}$, $\chi''_j = \chi_j$ and $\lambda d(f(\chi), \chi_j) < d(f(\chi'), \chi_j) = d(f(\chi''), \chi_j)$ for all $i \in T$, contradicting the minimality of S. □

Now we derive an upper bound on the benefit ratio of the mechanism f.

Lemma 3 *The benefit ratio of the mechanism f of Eq. (3) is at most $1 + 2/\lambda$ for any real $\lambda \geq 1$.*

Proof. In the following, we use the fact that $f(\chi) = c$ for $c \in \{0, 1\}$ imply $\mu(c, \chi) \geq \mu(1 - c, \chi)$ in Eq. (3), i.e.,

$$d(c, \chi|_{\overline{I}}) \leq d(1 - c, \chi|_{\overline{I}}) + m_c - m_{1-c}, \tag{6}$$

which is symmetric with $c \in \{0, 1\}$. For a notational simplicity, we consider the case of $f(\chi) = 0$, because the other case of $f(\chi) = 1$ can be treated symmetrically.

For each $c \in \{0, 1\}$, define $I_c^- = \{h \in I_c \mid h < c\}$, $I_c^+ = \{h \in I_c \mid h \geq c\}$, $m_c^- = |\chi|_{I_c^-}|$, $m_c^+ = |\chi|_{I_c^+}|$ and $m_c = |\chi|_{I_c}| = m_c^- + m_c^+$. For

$$D = d(0, \chi|_{\overline{I}}) + d(0, \chi|_{I_0^-}) + d(0, \chi|_{I_1^+}) \; (\geq 0),$$

we prove

$$\mathrm{sb}(0, \chi) = d(0, \chi) \geq D + m_1^- \frac{\lambda}{\lambda + 1}$$

and

$$\mathrm{opt}(\chi) \leq D + m_1^- \left(1 + \frac{1}{\lambda + 1}\right),$$

which implies the desired result

$$\frac{\mathrm{opt}(\chi)}{\mathrm{sb}(0, \chi)} \leq 1 + \frac{2}{\lambda}.$$

By noting that χ is a disjoint union of five multisets $\chi|_{\overline{I}}$, $\chi|_{I_0^-}$, $\chi|_{I_0^+}$, $\chi|_{I_1^-}$ and $\chi|_{I_1^+}$, we get

$$\begin{aligned} d(0, \chi) &= d(0, \chi|_{\overline{I}}) + d(0, \chi|_{I_0^-}) + d(0, \chi|_{I_0^+}) + d(0, \chi|_{I_1^-}) + d(0, \chi|_{I_1^+}) \\ &\geq D + d(0, \chi|_{I_1^-}) \geq D + m_1^- \frac{\lambda}{\lambda + 1} \quad (\text{by } I_1^- = (\tfrac{\lambda}{\lambda+1}, 1)). \end{aligned}$$

On the other hand, for $\mathrm{opt}(\chi) = \mathrm{sb}(1, \chi) = d(1, \chi)$, we get

$$\begin{aligned}
\mathrm{opt}(\chi) &= d(1, \chi|_{\overline{I}}) + d(1, \chi|_{I_0}) + d(1, \chi|_{I_1}) \\
&< d(0, \chi|_{\overline{I}}) + m_1 - m_0 + d(1, \chi|_{I_0}) + d(1, \chi|_{I_1}) \ \ \text{(by Eq. (6))} \\
&= d(0, \chi|_{\overline{I}}) + m_1 + d(0, \chi|_{I_0^-}) - d(0, \chi|_{I_0^+}) + d(0, \chi|_{I_1^+}) - m_1^+ + d(1, \chi|_{I_1^-}) \\
&\le D + m_1 - m_1^+ + d(1, \chi|_{I_1^-}) \\
&= D + m_1^- + d(1, \chi|_{I_1^-}) \\
&\le D + m_1^- + m_1^- \frac{1}{\lambda + 1} \quad \text{(by } I_1^- = (\tfrac{\lambda}{\lambda+1}, 1)\text{)},
\end{aligned}$$

as required. □

In conclusion, the results of Lemma 2 and Lemma 3, together give a proof of Theorem 2.

In light of a previous result by Cheng *et al.* [4], who have demonstrated a strategy-proof GSP mechanism with a benefit ratio at most 3 in the line metric, we see that the result of Theorem 2 follows as a natural extension of the introduction of λ-strategy proofness, matching the result of Cheng *et al.* [4] for $\lambda = 1$.

5 Lower Bounds on the Benefit Ratio

This section derives a lower bound on the benefit ratio of all 2-candidate λ-SP mechanisms in the line metric.

By Theorem 1, we only need to handle a masking zone 2-candidate λ-SP mechanism. We show that every such λ-SP mechanisms f admits a profile χ_f such that $\frac{\mathrm{opt}(\chi_f)}{\mathrm{sb}(f(\chi_f), \chi_f)}$ is not smaller than $1 + 2/\lambda$ if n is even; $1 + (2n - 2)/(\lambda n + 1)$ otherwise.

The following lemma establishes a lower bound on the benefit ratio of any 2-candidate masking zone mechanisms in the line.

Lemma 4 *Given a real $\lambda \ge 1$ and a set N of n (≥ 1) agents, let f be a masking zone mechanism f with candidate set $C(f) = \{0, 1\}$ and set $\{I_0, I_1\}$ of masking zones. Then for any real $\delta > 0$, there is a profile χ such that*

$$\frac{\mathrm{opt}(\chi)}{\mathrm{sb}(f(\chi), \chi)} \ge \begin{cases} 1 + \frac{2}{\lambda}(\frac{1-\delta}{1+\delta}), & \textit{if } n \textit{ is even} \\ 1 + \frac{2n-2}{\lambda n+1}(\frac{1-\delta}{1+\delta}), & \textit{otherwise.} \end{cases}$$

Proof. Let f be an arbitrary masking zone mechanism with set $\{I_0, I_1\}$ of intervals and candidate set $C(f) = \{0, 1\}$. Given $\delta > 0$, let ε be a real with $0 < \varepsilon < \min\{\frac{1}{\lambda+1}, \frac{\delta}{\lambda+1}\}$. We distinguish two cases.

Case 1. n (≥ 2) is even: Let χ be an arbitrary profile such that $|\chi|_{I_0}| = |\chi|_{I_1}| = \frac{n}{2}$ and $\chi|_{\overline{I}} = \emptyset$. By symmetry, we can assume without loss of generality that $f(\chi) = 0$ holds. We modify χ into a new profile χ' such that

$$\chi_i' = \begin{cases} 0 & \text{for } \chi_i \in I_0 \\ 1 - \frac{1}{\lambda+1} + \varepsilon & \text{for } \chi_i \in I_1, \end{cases}$$

where $\chi'|_{\overline{I}} = \chi|_{\overline{I}}$, and $|\chi'|_{I_c}| = |\chi|_{I_c}|$ for each $c = 0, 1$. Since f is a masking zone mechanism with set $\mathcal{I} = \{I_0, I_1\}$, it holds that $f(\chi') = f(\chi) = 0$. By definition, we have

$$\mathrm{sb}(0, \chi') = 0 \cdot \frac{n}{2} + \left(1 - \frac{1}{\lambda+1} + \varepsilon\right)\frac{n}{2} = \left(1 - \frac{1}{\lambda+1} + \varepsilon\right)\frac{n}{2}$$

and

$$\begin{aligned}\mathrm{opt}(\chi') &= \max\{\mathrm{sb}(1, \chi'), \mathrm{sb}(0, \chi')\} \geq \mathrm{sb}(1, \chi')\\ &= 1 \cdot \frac{n}{2} + \left(\frac{1}{\lambda+1} - \varepsilon\right)\frac{n}{2} = \left(1 + \frac{1}{\lambda+1} - \varepsilon\right)\frac{n}{2}.\end{aligned}$$

Hence

$$\frac{\mathrm{opt}(\chi')}{\mathrm{sb}(0, \chi')} \geq \frac{1 + \frac{1}{\lambda+1} - \varepsilon}{1 - \frac{1}{\lambda+1} + \varepsilon} = 1 + \frac{2 - 2(\lambda+1)\varepsilon}{\lambda + (\lambda+1)\varepsilon} \geq 1 + \frac{2}{\lambda}\left(\frac{1 - (\lambda+1)\varepsilon}{1 + (\lambda+1)\varepsilon}\right) \geq 1 + \frac{2}{\lambda}\left(\frac{1-\delta}{1+\delta}\right).$$

Case 2. n (≥ 1) is odd: Let χ be an arbitrary profile such that $|\chi|_{I_0}| = |\chi|_{I_1}| = \frac{n-1}{2}$ and $\chi|_{\overline{I}} = \{\frac{1}{2}\}$. By symmetry, we can assume without loss of generality that $f(\chi) = 0$ holds. We modify χ into a new profile χ' such that

$$\chi'_i = \begin{cases} 0 & \text{for } \chi_i \in I_0 \\ 1 - \frac{1}{\lambda+1} + \varepsilon & \text{for } \chi_i \in I_1 \\ \frac{1}{2} & \text{for } \chi_i \in \overline{I}, \end{cases}$$

where $\chi'|_{\overline{I}} = \chi|_{\overline{I}}$, and $|\chi'|_{I_c}| = |\chi|_{I_c}|$ for each $c = 0, 1$. Since f is a masking zone mechanism with set $\mathcal{I} = \{I_0, I_1\}$, it holds that $f(\chi') = f(\chi) = 0$. By definition, we have

$$\mathrm{sb}(0, \chi') = 0 \cdot \frac{n-1}{2} + \left(1 - \frac{1}{\lambda+1} + \varepsilon\right)\frac{n-1}{2} + \frac{1}{2} = \left(1 - \frac{1}{\lambda+1} + \varepsilon\right)\frac{n-1}{2} + \frac{1}{2}$$

and

$$\begin{aligned}\mathrm{opt}(\chi') &= \max\{\mathrm{sb}(1, \chi'), \mathrm{sb}(0, \chi')\} \geq \mathrm{sb}(1, \chi')\\ &= 1 \cdot \frac{n-1}{2} + \left(\frac{1}{\lambda+1} - \varepsilon\right)\frac{n-1}{2} + \frac{1}{2} = \left(1 + \frac{1}{\lambda+1} - \varepsilon\right)\frac{n-1}{2} + \frac{1}{2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\mathrm{opt}(\chi')}{\mathrm{sb}(0, \chi')} &\geq \frac{1 + \frac{1}{\lambda+1} - \varepsilon + \frac{1}{n-1}}{1 - \frac{1}{\lambda+1} + \varepsilon + \frac{1}{n-1}} = 1 + \frac{2(n-1) - 2(n-1)(\lambda+1)\varepsilon}{\lambda(n-1) + (\lambda+1)(n-1)\varepsilon + (\lambda+1)}\\ &= 1 + \frac{2(n-1)}{\lambda n + 1}\left(\frac{1 - (\lambda+1)\varepsilon}{1 + \frac{(\lambda+1)(n-1)}{\lambda n+1}\varepsilon}\right) \geq 1 + \frac{2(n-1)}{\lambda n + 1}\left(\frac{1 - (\lambda+1)\varepsilon}{1 + (\lambda+1)\varepsilon}\right)\\ &\geq 1 + \frac{2(n-1)}{\lambda n + 1}\left(\frac{1-\delta}{1+\delta}\right).\end{aligned}$$

□

By Theorem 1 and Lemma 4, we obtain Theorem 3.

6 Concluding remarks

This paper studied a trade-off between the benefit ratio and a relaxation of group strategy proofs, taking 2-candidate mechanisms for the obnoxious facility game in the line metric as an example. As a result we introduce λ-group strategy-proofness, a parameterized strategy proofness, and demonstrated a mechanism that has a desired property of a benefit ratio of at most $1 + \frac{2}{\lambda}$, which tends to 1, as the parameter λ tends to ∞. This result was obtained via a novel view on mechanism properties, and the introduction of the concept of masking zone mechanisms, which is a necessary condition for λ-GSP mechanisms. On the other hand, we also derived lower bounds on the benefit ratio of for masking zone mechanisms: $1 + \frac{2}{\lambda}$ when $n = |N|$ is even and $1 + \frac{2n-2}{\lambda n+1}$ when $n = |N|$ is odd. The above bounds are tight when $|N|$ is even, meaning that the upper bound on the benefit ratio is the best we can hope for, but it remains an open question to the slight gap between the upper and lower bounds for the case when $|N|$ is odd.

For future work, it remains to investigate the trade-off between the benefit ratio and the λ-GSP mechanisms in other metrics such as trees, circles and Euclidean space.

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