Re-embedding a 1-Plane Graph into a Straight-line Drawing in Linear Time

Seok-Hee Hong[1] Hiroshi Nagamochi[2]

[1]School of Information Technologies, University of Sydney, seokhee.hong@sydney.edu.au
[2]Department of Applied Mathematics and Physics, Kyoto University, nag@amp.i.kyoto-u.ac.jp

Abstract: Thomassen characterized some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph is drawable in straight lines if and only if it does not contain the configuration [C. Thomassen, Rectilinear drawings of graphs, J. Graph Theory, 10(3), 335-341, 1988]. In this paper, we characterize some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph can be re-embedded into a straight-line drawable 1-plane embedding of the same graph if and only if it does not contain the configuration. Re-embedding of a 1-plane embedding preserves the same set of pairs of crossing edges. We give a linear-time algorithm for finding a straight-line drawable 1-plane re-embedding or the forbidden configuration.

1 Introduction

Since the 1930s, a number of researchers have investigated *planar* graphs. In particular, a beautiful and classical result, known as *Fáry's Theorem*, asserts that every plane graph admits a *straight-line drawing* [9]. Indeed, a straight-line drawing is the most popular drawing convention in Graph Drawing.

More recently, researchers have investigated *1-planar graphs* (i.e., graphs that can be embedded in the plane with at most one crossing per edge), introduced by Ringel [16]. Subsequently, the structure of 1-planar graphs has been investigated [4, 5, 6, 15, 17]. In particular, Pach and Toth [15] proved that a 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight upper bound. Unfortunately, testing the 1-planarity of a graph is NP-complete [10, 14], but fixed parameter tractable [3]. Linear-time algorithms are available for special subclasses of 1-planar graphs [2, 8, 11].

Thomassen [18] proved that every 1-plane graph (i.e., a 1-planar graph embedded with a given *1-plane embedding*) admits a straight-line drawing if and only if it does not contain any of two special 1-plane graphs, called the *B-configuration* or *W-configuration*, see Fig. 1. Recently, Hong et al. [12] gave an alternative constructive proof, with a linear-time testing algorithm and a drawing algorithm. They also showed that some 1-planar graphs need an

[1]Technical report 2016-002, July 14, 2016.

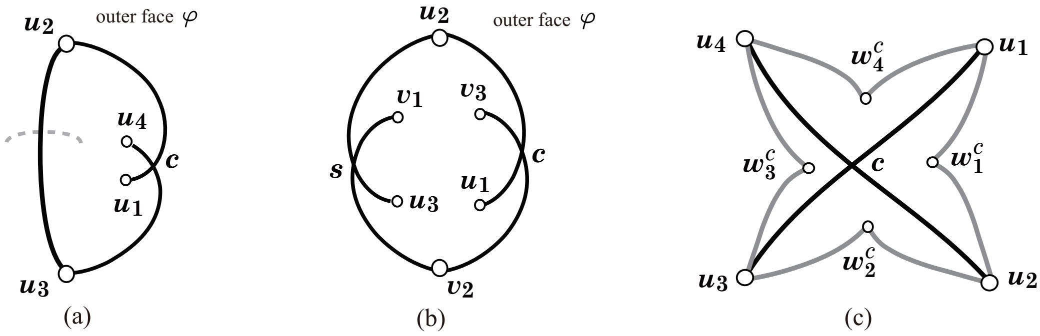

Figure 1: (a) B-configuration with three edges u_1u_2, u_2u_3 and u_3u_4 and one crossing c made by an edge pair $\{u_1u_2, u_3u_4\}$, where edge u_2u_3 may have a crossing when the configuration is part of a 1-plane embedding; (b) W-configuration with four edges u_1u_2, u_2u_3, v_1v_2 and v_2v_3 and two crossings c and s made by edge pairs $\{u_1u_2, v_2v_3\}$ and $\{u_2u_3, v_1v_2\}$, where possibly $u_1 = v_1$ and $u_3 = v_3$; (a) Augmenting a crossing $c \in \chi$ made by edges u_1u_3 and u_2u_4 with a new cycle $Q_c = (u_1, w_1^c, u_2, w_2^c, u_3, w_3^c, u_4, w_4^c)$ depicted by gray lines.

exponential area with straight-line drawing. Grid drawings of triconnected 1-planar graphs was also studied [1].

We call a 1-plane embedding *straight-line drawable* (*SLD* for short) if it admits a straight-line drawing, i.e., it does not contain a B- or W-configuration by Thomassen [18]. In this paper, we investigate a problem of "re-embedding" a given non-SLD 1-plane embedding γ into an SLD 1-plane embedding γ'. For a given 1-plane embedding γ of a graph G, we call another 1-plane embedding γ' of G a *cross-preserving embedding* of γ if exactly the same set of edge pairs make the same crossings in γ'.

More specifically, we first characterize the *forbidden configuration* of 1-plane embeddings that cannot admit an SLD cross-preserving 1-plane embedding. Based on the characterization, we present a linear-time algorithm either detects the forbidden configuration in γ or computes an SLD cross-preserving 1-plane embedding γ'.

Formally, the main problem considered in this paper is defined as follows.

Re-embedding a 1-Plane Graph into a Straight-line Drawing
Input: A 1-planar graph G and a 1-plane embedding γ of G.
Output: Test whether γ admits an SLD cross-preserving 1-plane embedding γ', and construct such an embedding γ' if one exists, or report the forbidden configuration.

To design a linear-time implementation of our algorithm in this paper, we introduce a *rooted-forest representation of non-intersecting cycles* and an efficient procedure of flipping subgraphs in a plane graph. Since these data structure and procedure can be easily implemented, it has advantage over the complicated decomposition of biconnected graphs into triconnected components [13] or the SPQR tree [7].

The paper is organized as follows. Section 2 makes a technical preparation to attain the linear time complexity of our algorithm by introducing a forest representation of non-intersecting cycles and an efficient implementation of a flip operation in a plane graph. Section 3 investigates the structure of cycles that can induce a B- or W-configuration and defines a forbidden configuration to our problem. Section 4 and Section 5 treat the case where the planarization of a 1-plane embedding γ is biconnected and connected, respectively, where a linear-time algorithm is designed to detect the forbidden configuration in γ or to construct an SLD cross-preserving 1-plane embedding of γ. Finally Section 6 makes some concluding remarks.

2 Plane Embeddings and Inclusion Forests

Let U be a set of n elements, and let $\mathcal{S}$ be a family of subsets $S \subseteq U$. We say that two subsets $S, S' \subseteq U$ are *intersecting* if none of $S \cap S'$, $S - S'$ and $S' - S$ is empty. We call $\mathcal{S}$ a *laminar* if no two subsets in $\mathcal{S}$ are intersecting. For a laminar $\mathcal{S}$, the *inclusion-forest* of $\mathcal{S}$ is defined to be a forest $\mathcal{I} = (\mathcal{S}, \mathcal{E})$ of a disjoint union of rooted trees such that (i) the sets in $\mathcal{S}$ are regarded as the vertices of $\mathcal{I}$, and (ii) a set S is an ancestor of a set S' in $\mathcal{I}$ if and only if $S' \subseteq S$.

Lemma 1 *For a cyclic sequence $(u_1, u_2, \ldots, u_\delta)$ of $\delta \geq 2$ elements, define an interval (i, j) to be the set of elements u_k with $i \leq k \leq j$ if $i \leq j$ and $(i, j) = (i, \delta) \cup (1, j)$ if $i > j$. Let $\mathcal{S}$ be a set of intervals. A pair of two intersecting intervals in $\mathcal{S}$ (when $\mathcal{S}$ is not a laminar) or the inclusion-forest of $\mathcal{S}$ (when $\mathcal{S}$ is a laminar) can be obtained in $O(\delta + |\mathcal{S}|)$ time.*

Proof. The inclusion-forest of a laminar $\mathcal{R}$ is denoted by $\mathcal{I}(\mathcal{R})$. Let $\mathcal{S} = \{S_i = (a_i, b_i) \mid i = 1, 2, \ldots, q\}$, and let Δ_j denote the number of elements that appear from u_{a_j+1} to u_{b_j}, where $\Delta_j = b_j - a_j$ if $b_j > a_j$ and $\Delta_j = b_j - a_j + \delta$ if $b_j < a_j$. To make the presentation simpler, we assume without loss of generality that $\Delta_1 = \max_j \Delta_j$ and $u_1 = a_1$, and introduce a fictitious interval S_0 with $a_0 = 1$, $b_0 = \delta + 1$ and $\Delta_0 = \delta + 1$, where S_0 will be the single root of the inclusion-forest $\mathcal{I}(\mathcal{S} \cup \{S_0\})$ if $\mathcal{S}$ is a laminar.

We sort all intervals $S_j \in \mathcal{S} \cup \{S_0\}$ in an increasing order of a_j and a decreasing order of Δ_j, i.e., according to the lexicographic order of $(a_j, -\Delta_j)$, and assume without loss of generality that the intervals $S_0, S_1, S_2, \ldots, S_q$ are indexed so that $(a_0, -\Delta_0)$, $(a_1, -\Delta_1), (a_2, -\Delta_2), \ldots, (a_q, -\Delta_q)$ is the resulting sorted list. For each $i = 0, 1, \ldots, q$, we let $\mathcal{S}_i = \{S_0, S_1, \ldots, S_i\}$, and give a procedure for constructing an inclusion-forest $\mathcal{I}(\mathcal{S}_i)$ or finding a pair of two intersecting intervals in $\mathcal{S}_i$.

We see that $\mathcal{I}(\mathcal{S}_0) = (\{S_0\}, \emptyset)$ is a rooted tree with a single node S_0. Fix an index $i = 0, 1, \ldots, q-1$, assuming that $\mathcal{S}_i$ is a laminar and $\mathcal{I}(\mathcal{S}_i)$ has been obtained. We test whether S_{i+1} is intersecting with some interval S_j with $j < i$, and construct the inclusion-forest $\mathcal{I}(\mathcal{S}_{i+1})$ if $\mathcal{S}_{i+1}$ is a laminar. Note that "$a_i = a_{i+1}$ and $\Delta_i > \Delta_{i+1}$" or "$a_i < a_{i+1}$" by the lexicographic ordering.

Case 1. $a_{i+1} < a_i + \Delta_i$: If $\Delta_i < \Delta_{i+1}$, then $a_i < a_{i+1}$ holds and S_{i+1} is intersecting with S_i, and halt. Otherwise, i.e., when $\Delta_{i+1} \leq \Delta_i$, interval S_{i+1} is contained in S_i and is not intersecting with any interval $S \in \mathcal{S}_i$, since $S_i \subseteq S$ or $S_i \cap S = \emptyset$, and we add S_{i+1} to $\mathcal{I}(\mathcal{S}_i)$ as a child of S_i to obtain $\mathcal{I}(\mathcal{S}_{i+1})$.

Case 2. $a_i + \Delta_i \leq a_{i+1}$: In the rooted tree $\mathcal{I}(\mathcal{S}_i)$, find the ancestors S_x and S_y of S_i such that S_y is the current last child of S_x and $a_y + \Delta_y \leq a_{i+1} < a_x + \Delta_x$, where such ancestors exist since the root S_0 satisfies $a_0 = 1$ and $\Delta_0 = \delta + 1$. If $\Delta_x < \Delta_{i+1}$, then S_{i+1} is intersecting with S_x, and halt. Otherwise, i.e., when $\Delta_{i+1} \leq \Delta_x$, interval S_{i+1} is not intersecting with any interval in $\mathcal{S}_i$, since S_{i+1} is contained in S_x and disjoint with any children of S_x, and we add S_{i+1} to $\mathcal{I}(\mathcal{S}_i)$ as a child of S_x to obtain $\mathcal{I}(\mathcal{S}_{i+1})$.

We repeat the above step until $\mathcal{I}(\mathcal{S}_p)$ is successfully constructed or two intersecting intervals are detected. In the former case, $\mathcal{S}$ is a laminar and we can obtain the inclusion-forest $\mathcal{I}(\mathcal{S})$ from $\mathcal{I}(\mathcal{S}_q)$ by removing the fictitious root S_0.

During an execution of the procedure, once a path from S_{i+1} to S_x is backtracked in the current rooted tree $\mathcal{I}(\mathcal{S}_i)$ in Case 2, it will never be backtracked again later. Hence the above procedure can be implemented in $O(\delta + |\mathcal{S}|)$ time. $\square$

Throughout the paper, a graph $G = (V, E)$ stands for a simple undirected graph. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex v, let $E(v)$ be the set of edges incident to v, $N(v)$ be the set of neighbors of v, and $\deg(v)$ denote the degree $|N(v)|$ of v. A simple path with end vertices u and v is called a *u, v-path*. For a subset $X \subseteq V$, let $G - X$ denote the graph obtained from G by removing the vertices in X together with the edges in $\cup_{v \in X} E(v)$.

A *drawing* D of a graph G is a geometric representation of the graph in the plane, such that each vertex of G is mapped to a point in the plane, and each edge of G is drawn as a curve. A drawing D of a graph $G = (V, E)$ is called *planar* if there is no edge crossing. A planar drawing D of a graph G divides the plane into several connected regions, called *faces*, where a face enclosed by a closed walk of the graph is called an *inner face* and the face not enclosed by any closed walk is called the *outer face*.

A planar drawing D *induces* a plane embedding γ of G, which is defined to be a pair (ρ, φ) of the *rotation system* (i.e., the circular ordering of edges for each vertex) ρ, and the facial cycle φ along the outer boundary of D. Let $\gamma = (\rho, \varphi)$ be a plane embedding of a graph $G = (V, E)$. We denote by $F(\gamma)$ the set of faces in γ, and by C_f the facial cycle determined by a face $f \in F$, where we call a subpath of C_f a *boundary path* of f. For a simple cycle C of G, the plane is divided by C in two regions, one containing only inner faces and the other containing the outer area, where we say that the former is *enclosed* by C or the *interior* of C, while the latter is called the *exterior* of C. We denote by $F_{\rm in}(C)$ the set of inner faces in the interior of C, by $E_{\rm in}(C)$ the set of edges in $E(C_f)$ with $f \in F_{\rm in}(C)$, and by $V_{\rm in}(C)$ the set of end-vertices of edges in $E_{\rm in}(C)$. Analogously define $F_{\rm ex}(C)$, $E_{\rm ex}(C)$ and $V_{\rm ex}(C)$ in the exterior of C. Note that $E(C) = E_{\rm in}(C) \cap E_{\rm ex}(C)$ and $V(C) = V_{\rm in}(C) \cap V_{\rm ex}(C)$.

For a subgraph H of G, we define the embedding $\gamma|_H$ of γ induced by H to be a sub-embedding of γ obtained by removing the vertices/edges not in H keeping the same rotation system around each of the remaining vertices/crossings and the same outer face.

2.1 Inclusion Forests of Inclusive Set of Cycles

In this and next subsections, let (G, γ) stand for a plane embedding of $\gamma = (\rho, \varphi)$ of a biconnected simple graph $G = (V, E)$ with $n = |V| \geq 3$.

Let C be a simple cycle in G. We define the *direction* of C to be an ordered pair (u, v) with $uv \in E(C)$ such that the inner faces in $F_{\rm in}(C)$ appear on the right hand side when we traverse C in the order that we start u and next visit v. For simplicity, we say that two simple cycles C and C' are *intersecting* if $F_{\rm in}(C)$ and $F_{\rm in}(C')$ are intersecting.

Let $\mathcal{C}$ be a set of simple cycles in G. We call $\mathcal{C}$ *inclusive* if no two cycles in $\mathcal{C}$ are intersecting, i.e., $\{F_{\rm in}(C) \mid C \in \mathcal{C}\}$ is a laminar. When $\mathcal{C}$ is inclusive, the *inclusion-forest* of $\mathcal{C}$ is defined to be a forest $\mathcal{I} = (\mathcal{C}, \mathcal{E})$ of a disjoint union of rooted trees such that (i) the cycles in $\mathcal{C}$ are regarded as the vertices of $\mathcal{I}$, and (ii) a cycle C is an ancestor of a cycle C'

in $\mathcal{I}$ if and only if $F_{\rm in}(C') \subseteq F_{\rm in}(C)$. Let $\mathcal{I}(\mathcal{C})$ denote the inclusion-forest of $\mathcal{C}$. For a vertex subset $X \subseteq V$, let $\mathcal{C}(X)$ denote the set of cycles $C \in \mathcal{C}$ such that $x \in V(C)$ for some vertex $x \in X$, where we denote $\mathcal{C}(\{v\})$ by $\mathcal{C}(v)$ for short.

Lemma 2 *For (G, γ), let $\mathcal{C}(v)$, $v \in V$ denote the set of cycles $C \in \mathcal{C}$ such that $v \in V(C)$. For a set $\mathcal{C}$ of simple cycles of G, any of the following tasks can be executed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time.*

(i) *Decision of the directions of all cycles in $\mathcal{C}$;*

(ii) *Detection of a pair of two intersecting cycles in $\mathcal{C}$ when $\mathcal{C}$ is not inclusive, and construction of the inclusion-forests $\mathcal{I}(\mathcal{C}(v))$ for all vertices $v \in V$ when $\mathcal{C}$ is inclusive; and*

(iii) *Construction of the inclusion-forest $\mathcal{I}(\mathcal{C})$ when $\mathcal{C}$ is inclusive.*

Proof. (i) Choose an edge $ss' \in E(C_\varphi)$, where s appears immediately after s' in the clockwise order along C_φ, and construct a spanning tree T of G, which we regard as a tree rooted at vertex s, and denote by $\mathrm{p}(v)$ the parent of a non-root vertex v, where we let $\mathrm{p}(s) = s'$. Let the vertices in V be indexed as $v_1\ (= s), v_2, \ldots, v_n$ so that $i < j$ holds if v_i is an ancestor of v_j in T, and let $V_0 = \emptyset$ and $V_i = \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots, n$. In the order of $i = 1, 2, \ldots, n$, we determine the direction of each cycle $C \in \mathcal{C}(v_i) - \mathcal{C}(V_{i-1})$, (i.e., cycle C with $v_i \in V(C) \subseteq V - V_{i-1}$) as follows. Let $u_1\ (= \mathrm{p}(v_i)), u_2, \ldots, u_\delta$, where $\delta = \deg(v_i)$, be the neighbors of v_i in the clockwise order of $\rho(v_i)$. Denote $\mathcal{C}(v_i) - \mathcal{C}(V_{i-1})$ by $\{C_1, C_2, \ldots, C_q\}$, and let a_j and b_j denote the indices of the end-vertices of edges in cycle C_j incident to v_i, i.e., $v_i u_{a_j}, v_i u_{b_j} \in E(C_j)$.

As the base case, let $i = 1$, where edge $v_1 p(v_1) = ss'$ is in the outer facial cycle C_φ. Assume without loss of generality that $1 < a_j < b_j \leq \delta$. Then we see that the direction of each cycle C_j is (v_1, a_j), because the outer face φ appears on the left hand side when we traverse C_j starting from v_1 and next visiting a_j with $j \geq 2$.

Next let $i > 1$, and suppose that we have determined the direction of each cycle in $\mathcal{C}(V_{i-1})$. For each cycle $C_j \in \mathcal{C}(v_i) - \mathcal{C}(V_{i-1})$, where $u_1 = \mathrm{p}(v_i) \notin V(C_j)$ holds, we see that neither of a_j and b_j is 1, and assume without loss of generality that $1 < a_j < b_j \leq \delta$. Since $C_j \in \mathcal{C}(v_i) - \mathcal{C}(V_{i-1})$ has no vertex in V_{i-1}, we see that if we traverse C_j starting from v_1 and next visiting b_j then V_{i-1} would appear on the right hand side. Hence the direction of each cycle C_j is (v_1, a_j). Deciding the directions of cycles in $\mathcal{C}(v_i) - \mathcal{C}(V_{i-1})$ can be done in the time of tracing all edges in these cycles and the edges incident to v_i, i.e., $O(\deg(v_i) + \sum\{|E(C)| \mid C \in \mathcal{C}(v_i) - \mathcal{C}(V_{i-1})\})$ time. Hence the total time for deciding the directions of all cycles in $\mathcal{C}$ is $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$.

(ii) By the result in (i), we first compute the directions of all cycles in $\mathcal{C}$ in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time. We then traverse each cycle $C \in \mathcal{C}$ in its direction to decide for each $uv \in V(C)$ whether (u, v) or (v, u) is the direction of C.

We easily see that $\mathcal{C}$ is inclusive if and only if the set $\mathcal{S}(v) = \{N(v) \cap V_{\rm in}(C) \mid C \in \mathcal{C}(v)\}$ of intervals over the circular sequence $\rho(v)$ is a laminar for all vertices $v \in V$.

For fixed vertex $v \in V$, denote $\mathcal{S}(v) = \{S_i = (a_i, b_i) = N(v) \cap V_{\rm in}(C_i) \mid i = 1, 2, \ldots, q\}$, where a_i and b_i with $(a_i, b_i) = N(v) \cap V_{\rm in}(C_i)$ can be found in $O(1)$ time for each i since we know the direction of C_i. By Lemma 1, a pair of two intersecting intervals in $\mathcal{S}(v)$ (when $\mathcal{S}(v)$ is not inclusive) or the inclusion-forest of $\mathcal{S}(v)$ (when $\mathcal{S}(v)$ is inclusive) can be obtained in $O(\deg(v) + |\mathcal{C}(v)|)$ time. We run the procedure for all vertices in V in $O(\sum_{v\in V}[\deg(v) + |\mathcal{C}(v)|]) = O(n + \sum_{C\in\mathcal{C}} |E(C)|)$ time to conclude that $\mathcal{C}$ is inclusive only when $\mathcal{S}(v)$ is a laminar for all vertices or to detect a pair of intersecting cycles in $\mathcal{C}$.

(iii) For notational convenience, we set $\mathcal{C} := \mathcal{C} \cup \{C_f \mid f \in F(\gamma)\}$, where $n+\sum_{C\in\mathcal{C}} |E(C)| = O(n + \sum_{C\in\mathcal{C}'} |E(C)|)$ still holds since $\sum_{f\in F(\gamma)} |E(C_f)| = O(n)$. Note that the inclusion-forest of the original $\mathcal{C}$ can be obtained from the inclusion-forest of the updated $\mathcal{C}$ by removing the leaves corresponding to newly added cycles $C_f \in \{C_f \mid f \in F(\gamma)\}$ or the root C_φ when C_φ is not in the original $\mathcal{C}$. By the result in (ii), we first construct $\mathcal{I}(\mathcal{C}(v))$ for all vertices $v \in V$ in $O(n+\sum_{C\in\mathcal{C}} |E(C)|)$ time. We show how to construct the inclusion-forest $\mathcal{I}(\mathcal{C})$ from $\{\mathcal{I}(\mathcal{C}(v)) \mid v \in V\}$. Define $\mathrm{Ch}(C)$ to be the set of children of a cycle C in the inclusion-forest $\mathcal{I}(\mathcal{C})$. Our goal is now to construct $\mathrm{Ch}(C)$ for all cycles $C \in \mathcal{C}$, where clearly $\mathrm{Ch}(C) = \emptyset$ if $C = C_f$ for some inner face $f \in F(\gamma)$.

For each cycle $C \in \mathcal{C} - \{C_f \mid f \in F(\gamma)\}$, let $N(C)$ be the set of cycles $C' \in \mathcal{C}$ such that C is the parent of C' in $\mathcal{I}(\mathcal{C}(v))$ for some vertex $v \in V$, where clearly $N(C) \subseteq \mathrm{Ch}(C)$. The difference $\mathrm{Ch}(C) - N(C)$ can be obtained as follows.

For each vertex $v \in V - V(C_\varphi)$, let $R(v)$ denote the set of cycles that are the roots of the inclusion-forest $\mathcal{I}(\mathcal{C}(v))$. Since the facial cycles of all inner faces are included in $\mathcal{C}$, it holds $|R(v)| \geq 2$ for each vertex $v \in V - V(C_\varphi)$. We introduce an equivalence relation $\sim$ over $\mathcal{C}$ such that for two cycles $C, C' \in \mathcal{C}$, it holds $C \sim C'$ if (a) $C = C'$ or $C, C' \in R(v)$ for some vertex $v \in V - V(C_\varphi)$; or (b) $C \sim C''$ and $C'' \sim C'$ for some cycle $C'' \in \mathcal{C}$. We observe the next property.

Claim *For each cycle $C \in \mathcal{C} - \{C_f \mid f \in F(\gamma)\}$, a cycle $C' \notin \mathcal{C} - N(C) - \{C\}$ belongs to* $\mathrm{Ch}(C)$ *if and only if $C' \sim C''$ for some cycle $C'' \in N(C)$.*

Proof. For two simple cycles C and C' in G, we let $C \sqsubseteq C'$ mean $F_{\mathrm{in}}(C) \subseteq F_{\mathrm{in}}(C')$.

If part: Let $C' = C_1, C_2, \ldots, C_k = C''$ be a shortest sequence such that $C_i, C_{i+1} \in R(v_i)$, $i = 1, 2, \ldots, k-1$ for some vertex $v_i \in V - V(C_\varphi)$, where it holds $\{C_1, C_2, \ldots, C_k\} \cap N(C) = \{C''\}$ by the minimality of the sequence. We first show that $C' \sqsubseteq C$. Since $C_k = C'' \in N(C)$, it holds $C_k \sqsubseteq C$. If $C' \not\sqsubseteq C$, then there is an index j such that $C_j \sqsubseteq C$ and $F_{\mathrm{in}}(C_{j-1}) \cap F_{\mathrm{in}}(C) = \emptyset$, which, however, contradicts that C_j and C_{j-1} are roots of $\mathcal{I}(\mathcal{C}(v_{j-1}))$. Hence $C' \sqsubseteq C$. There is no other cycle $C^\dagger$ such that $C' \sqsubseteq C^\dagger \sqsubseteq C$, since otherwise $C'' \sqsubseteq C^\dagger$ would hold by a similar augment, contradicting that $C'' \in N(C) \subseteq \mathrm{Ch}(C)$. Therefore $C' \in \mathrm{Ch}(C)$.

Only if part: Let a cycle $C' \notin \mathcal{C} - N(C) - \{C\}$ belong to $\mathrm{Ch}(C)$. If $C' \in N(C)$ then $C' \sim C''$ for $C'' = C'$ and we are done. Assume that $C' \notin N(C)$, where $V(C') \cap V(C) = \emptyset$. Let $\mathcal{C}_C$ be the set of cycles $\widetilde{C} \in \mathcal{C} - \{C\}$ such that $\widetilde{C} \sqsubseteq C$, and there is no other cycle $C^\dagger \in \mathcal{C} - \{C\}$ with $\widetilde{C} \sqsubseteq C^\dagger \sqsubseteq C$. Since $\mathcal{C}$ contains all the facial cycles, there is a sequence $\widetilde{C}_1 = C', \widetilde{C}_2, \ldots, \widetilde{C}_k$ of cycles in $\mathcal{C}_C$ such that, for each $i = 1, 2, \ldots, k-1$, $\widetilde{C}_i$ and $\widetilde{C}_{i+1}$ share a vertex $v_i \in V - V(C)$ and $\widetilde{C}_k \in N(C)$. By definition of $\mathcal{C}_C$, for each $i = 1, 2, \ldots, k-1$, there is no other cycle $C^\dagger \in \mathcal{C}(v_i)$ such that $\widetilde{C}_i \sqsubseteq C^\dagger$ or $\widetilde{C}_{i+1} \sqsubseteq C^\dagger$; i.e., it holds $\widetilde{C}_i, \widetilde{C}_{i+1} \in R(v_i)$. This implies that $C' = \widetilde{C}_1 \sim \widetilde{C}_k \in N(C)$, as required. □

Let $B = (V \cup \mathcal{C}, E_R)$ be a bipartite graph between two vertex sets V and $\mathcal{C}$ such that B has an edge uC between a vertex $v \in V$ and a cycle $C \in \mathcal{C}$ if and only if $C \in R(v)$ for some vertex $v \in V - V(C_\varphi)$. Note that the size of B is bounded by that of the union of $\mathcal{I}(\mathcal{C}(v))$ over all vertices $v \in V$. By the above claim, the set $\mathrm{Ch}(C) - N(C)$ of each cycle $C \in \mathcal{C} - \{C_f \mid f \in F(\gamma)\}$ is given by the set of cycles in the components of B which contain some cycle $C'' \in N(C)$. Hence $\mathrm{Ch}(C)$ for all cycles $C \in \mathcal{C}$ and hence $\mathcal{I}(\mathcal{C})$ can be constructed in time linear to the size of B, i.e., $O(n + \sum_{C \in \mathcal{C}'} |E(C)|)$ time. $\square$

2.2 Flipping Spindles

A simple cycle C of G is called a *spindle* (or a *u,v-spindle*) of γ if there are two vertices $u, v \in V(C)$ such that no vertex in $V(C) - \{u, v\}$ is adjacent to any vertex in the exterior of C, where we call vertices u and v the *junctions* of C. Note that each of the two subpaths of C between u and v is a boundary path of some face in $F(\gamma)$.

Given (G, γ), we denote the rotation system around a vertex $v \in V$ by $\rho_\gamma(v)$. For a spindle C in γ, let $J(C)$ denote the set of the two junctions of C. *Flipping a u,v-spindle* C means to modify the rotation system of vertices in $V_{\mathrm{in}}(C)$ as follows:
(i) For each vertex $w \in V_{\mathrm{in}}(C) - J(C)$, reverse the cyclic order of $\rho_\gamma(w)$; and
(ii) For each vertex $u \in J(C)$, reverse the order of subsequence of $\rho_\gamma(u)$ that consists of vertices $N(u) \cap V_{\mathrm{in}}(C)$.

Every two distinct spindles C and C' in γ are non-intersecting, and they always satisfy one of $E_{\mathrm{in}}(C) \cap E_{\mathrm{in}}(C') = \emptyset$, $E_{\mathrm{in}}(C) \subseteq E_{\mathrm{in}}(C')$, and $E_{\mathrm{in}}(C') \subseteq E_{\mathrm{in}}(C)$. Let $\mathcal{C}$ be a set of spindles in γ, which is always inclusive, and let $\mathcal{I}(\mathcal{C})$ denote the inclusion-forest of $\mathcal{C}$.

When we modify the current embedding γ by flipping each spindle in $\mathcal{C}$, the resulting embedding $\gamma_{\mathcal{C}}$ is the same, independent from the ordering of the flipping operation to the spindles, since for two spindles C and C' which share a common junction vertex $u \in J(C) \cap J(C')$, the sets $N(u) \cap V_{\mathrm{in}}(C)$ and $N(u) \cap V_{\mathrm{in}}(C')$ do not intersect, i.e., they are disjoint or one is contained in the other.

Define the *depth* of a vertex $v \in V$ in $\mathcal{I}$ to be the number of spindles $C \in \mathcal{C}$ such that $v \in V_{\mathrm{in}}(C) - J(C)$, and denote by $\mathrm{p}(v)$ the parity of depth of vertex v, i.e., $\mathrm{p}(v) = 1$ if the depth is odd and $\mathrm{p}(v) = -1$ otherwise.

For a vertex $v \in V$, let $\mathcal{C}[v]$ denote the set of spindles $C \in \mathcal{C}$ such that $v \in J(C)$, and let $\gamma_{\mathcal{C}[v]}$ be the embedding obtained from γ by flipping all spindles in $\mathcal{C}[v]$. Let $\mathrm{rev}\langle\sigma\rangle$ mean the reverse of a sequence σ. Then we see that $\rho_{\gamma_{\mathcal{C}}}(v) = \rho_{\gamma_{\mathcal{C}[v]}}(v)$ if $\mathrm{p}(v) = 1$; and $\rho_{\gamma_{\mathcal{C}}}(v) = \mathrm{rev}\langle\rho_{\gamma_{\mathcal{C}[v]}}(v)\rangle$ otherwise. To obtain the embedding $\gamma_{\mathcal{C}}$ from the current embedding γ by flipping each spindle in $\mathcal{C}$, it suffices to show how to compute each of $\mathrm{p}(v)$ and $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.

Lemma 3 *Given (G, γ), let $\mathcal{C}$ be a set of spindles of γ. Then any of the following tasks can be executed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time.*

(i) *Decision of parity* $\mathrm{p}(v)$ *of all vertices $v \in V$; and*

(ii) *Computation of $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.*

Proof. (i) Let $\mathcal{C}' = \mathcal{C} \cup \{C_f \mid f \in F(\gamma)\}$, and $\mathcal{I}$ be the inclusion-forest of $\mathcal{C}'$, where the cycle C_φ for the outer face φ is the root with depth 0 in $\mathcal{I}$. Since $\mathcal{C}'$ is inclusive, the tree $\mathcal{I}$ can be constructed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time by Lemma 2. We index all the cycles in $\mathcal{C}'$ so that for any two cycles $C_i, C_j \in \mathcal{C}'$, it holds $i < j$ when C_i is an ancestor of C_j in $\mathcal{I}$.

For each inner face $f \in F(\gamma)$, let π_f denote the index i of the cycle $C_i \in \mathcal{C} \cup \{C_\varphi\}$ that is the parent of C_f in $\mathcal{I}$. For each vertex $v \in V$, we denote by $\mathrm{i}(v)$ the minimum of π_f over all inner faces $f \in F(\gamma)$ with $v \in V(C_f)$. Then we see that $C_{\mathrm{i}(v)}$ is the cycle $C \in \mathcal{C} \cup \{C_\varphi\}$

with minimum $|F_{\rm in}(C)|$ that contains v in the interior of C, where $v \notin V(C)$, and $\mathrm{p}(v)$ is given by the parity of the depth of the cycle $C_{\mathrm{i}(v)}$. The above step after constructing $\mathcal{I}$ can be executed by traversing all inner faces, taking $O(n)$ time.

(ii) Let $\mathcal{C}(v)$, $v \in V$ denote the set of cycles $C \in \mathcal{C}$ such that $v \in V(C)$, where always $\mathcal{C}[v] \subseteq \mathcal{C}(v)$. We first construct $\mathcal{I}(\mathcal{C}(v))$ for all vertices $v \in V$, from which we obtain the inclusion-forest $\mathcal{I}(\mathcal{C}[v])$ for all vertices $v \in V$. By Lemma 2, this takes $O(n+\sum_{C\in\mathcal{C}}|E(C)|)$ time.

Fix a vertex $v \in V$. We show that sequence $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ can be computed recursively. Let the neighbors of v be indexed as $1, 2, \ldots, \deg(v)$ in the order of $\rho_\gamma(v)$. Without loss of generality that, for each spindle $C \in \mathcal{C}[v]$, the set of vertices in $N(v) \cap V_{\rm in}(C)$ is denoted by $\{i, i+1, \ldots, j\}$ with $1 \le i < j \le \deg(v)$, which we denote by list $L_C = [i, j]$.

Let $[x, y]^*$ denote the list obtained from list $L_C = [x, y]$ of a spindle $C \in \mathcal{C}[v]$ by flipping C and all descendants C' of C in $\mathcal{I}(\mathcal{C}[v])$. Let C_i, $i = 1, 2, \ldots, k$ denote the children of C in $\mathcal{I}(\mathcal{C}[v])$ and let $L_{C'} = [a_i, b_i]$, where $x \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le y$. Then we see that the following recursion holds:

$$\begin{aligned}[x, y]^* = [x, x+1, \ldots, a_1 - 1, \mathrm{rev}\langle[a_1, b_1]^*\rangle, b_1 + 1, \ldots,&\\ a_2 - 1, \mathrm{rev}\langle[a_2, b_2]^*\rangle, b_2 + 1, \ldots,&\\ \cdots&\\ a_k - 1, \mathrm{rev}\langle[a_k, b_k]^*\rangle, b_k + 1, \ldots, y-1, y].&\end{aligned}$$

Based on this, we obtain a recursive procedure for computing $[x, y]^*$ as follows.

Recursive Procedure $\textsc{List}(x, y, \tau)$
Input: The list $L_C = [x, y]$ of a spindle $C \in \mathcal{C}[v]$ and $\tau \in \{-1, 1\}$.
Output: $[x, y]^*$ if $\tau = 1$; and the reverse of $[x, y]^*$ if $\tau = -1$.

Let C_i, $i = 1, 2, \ldots, k$ denote the children of C in $\mathcal{I}(\mathcal{C}[v])$ and let $L_{C'} = [a_i, b_i]$, where $x \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le y$;
if $\tau = 1$ **then**

Return $[x, x+1, \ldots, a_1 - 1,\ \textsc{List}(a_1, b_1, -1), b_1 + 1, \ldots,$
$a_2 - 1,\ \textsc{List}(a_2, b_2, -1), b_2 + 1, \ldots,$
$\cdots$
$a_k - 1,\ \textsc{List}(a_k, b_k, -1), b_k + 1, \ldots, y-1, y]$

else /* $\tau = -1$ */

Return $[y, y-1, \ldots, a_k + 1,\ \textsc{List}(a_k, b_k, 1), b_k - 1, \ldots,$
$\cdots$
$a_2 + 1,\ \textsc{List}(a_2, b_2, 1), b_2 - 1, \ldots,$
$a_1 + 1,\ \textsc{List}(a_1, b_1, 1), b_1 - 1, \ldots, x+1, x]$

end if.

Hence $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ can be obtained by executing $\textsc{List}(1, \deg(v), 1)$, taking $O(\deg(v)+|\mathcal{C}[v]|)$ time. Since $|\mathcal{C}| = O(n)$ holds for any inclusive set $\mathcal{C}$, the total time for computing $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$ is $O(\sum_{v\in V}[\deg(v) + |\mathcal{C}[v]|]) = O(n + \sum_{C\in\mathcal{C}}|E(C)|)$ time. $\square$

3 Re-embedding 1-plane Graph and Forbidden Configuration

A drawing D of a graph $G = (V, E)$ is called a *1-planar drawing* if each edge has at most one crossing. A 1-planar drawing D of graph G induces a *1-plane embedding* γ of G, which is defined to be a tuple (χ, ρ, φ) of the *crossing system* χ of E, the rotation system ρ of V, and the outer face φ of D. The *planarization* $\mathcal{G}(G, \gamma)$ of a 1-plane embedding γ of graph G is the plane embedding obtained from γ by regarding crossings also as graph vertices, called crossing-vertex. The set of vertices in $\mathcal{G}(G, \gamma)$ is given by $V \cup \chi$. For a notational convenience, we may say a subgraph/face of $\mathcal{G}(G, \gamma)$ as a subgraph/face in γ. We denote by $F(\gamma)$ the set of faces in the plane graph $\mathcal{G}(G, \gamma)$.

Let $\gamma = (\chi, \rho, \varphi)$ be a 1-plane embedding of graph G. We call another 1-plane embedding $\gamma' = (\chi', \rho', \varphi')$ of graph G a *cross-preserving* 1-plane embedding of γ when the same set of edge pairs makes crossings, i.e., $\chi = \chi'$. In other words, the planarization $\mathcal{G}(G, \gamma')$ is another plane embedding of $\mathcal{G}(G, \gamma)$ such that the alternating order of edges incident to each crossing-vertex $c \in \chi$ is preserved.

To eliminate the additional constraint on the rotation system on each crossing-vertex $c \in \chi$, we *augment* the end-vertices of each pair of crossing edges as follows. In the plane graph, $\mathcal{G}(G, \gamma)$, for each crossing-vertex $c \in \chi$ and its neighbors u_1, u_2, u_3 and u_4 that appear in the clockwise order around c, we add four new vertices w_i^c, $i = 1, 2, 3, 4$ and eight new edges $u_i w_i^c$ and $w_i^c u_{i+1}$, $i = 1, 2, 3, 4$ to form a cycle Q_c of length 8 whose interior contains no other vertex than c, as shown in Fig. 1(c).

Let H be the resulting graph augmented from G and Γ be the resulting 1-plane embedding of H augmented from γ, where $|V(H)| \leq |V(G)| + 4|\chi|$ holds. We easily see that if γ admits an SLD cross-preserving embedding γ' then Γ admits an SLD cross-preserving embedding Γ'. This is because a straight-line drawing $D_{\gamma'}$ of γ' can be changed into a straight-line drawing $D_{\Gamma'}$ of some cross-preserving embedding Γ' of Γ by placing the newly introduced vertices w_i^c within the region sufficiently close to the position of c. We here see that cycle Q_c can be drawn by straight-line segments without intersecting with other straight-line segments in $D_{\gamma'}$.

We call an instance (G, γ) of 1-plane embedding *circular* when for each crossing $c \in \chi$, the four end-vertices of the two crossing edges that create c are contained in a cycle Q_c of eight crossing-free edges as described in the above, where c is not necessarily enclosed and the instance (G, γ') remains circular for any cross-preserving embedding γ' of γ. In the rest of paper, let (G, γ) stand for a circular instance $(G = (V, E), \gamma = (\chi, \rho, \varphi))$ with $n \geq 3$ vertices and let $\mathcal{G}$ denote its planarization $\mathcal{G}(G, \gamma)$. Fig. 2 shows examples of circular instances (G, γ), where $\mathcal{G}$ is oneconnected.

Just to test whether the current 1-plane embedding contains a B- or W-configuration or not, we can check each block in $\mathcal{G}$ separately, namely, a given instance $\mathcal{G}$ can be assumed to be biconnected. As will be discussed in Section 5, the biconnectivity cannot be assumed without loss of generality to finding an SLD re-embedding, where we have to examine how the blocks in $\mathcal{G}$ are connected via cut-vertices.

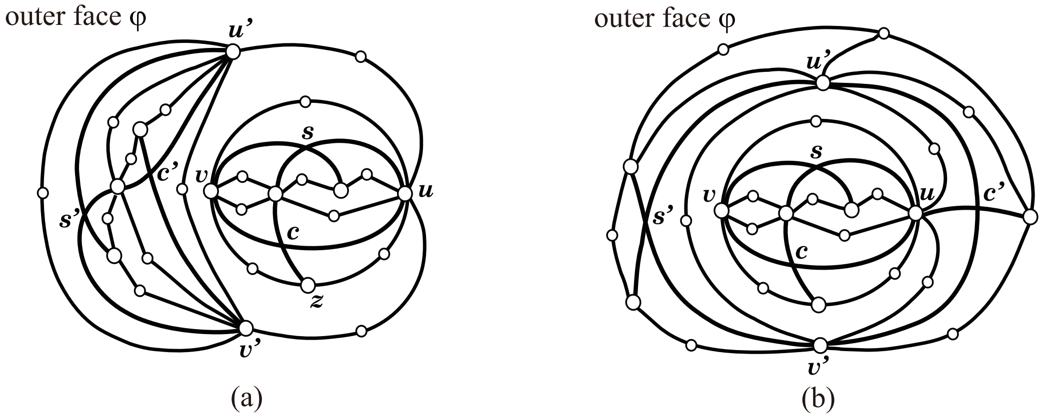

Figure 2: Circular instances (G, γ) with a cut-vertex u of $\mathcal{G}$, where the crossing edges are depicted by slightly thicker lines: (a) hard B-cycles $C = (u, c, v, s)$ and $C' = (u', c', v', s')$, (b) hard B-cycle $C = (u, c, v, s)$ and a nega-cycle $C' = (u', c', v', s')$ whose reversal is a hard B-cycle, where vertices $u, v, u', v' \in V$ and crossings $c, s, c', s' \in \chi$.

3.1 Candidate Cycles, B/W Cycle, Posi/Nega Cycle, Hard/Soft Cycle

For a circular instance (G, γ), finding a cross-preserving embedding of γ is effectively equivalent to finding another plane embedding of $\mathcal{G}$ so that all the current B- and W-configurations are eliminated and no new B- or W-configurations are introduced. To detect the cycles that can be the boundary of a B- or W-configuration in changing the plane embedding of $\mathcal{G}$, we categorize cycles containing crossing vertices in $\mathcal{G}$.

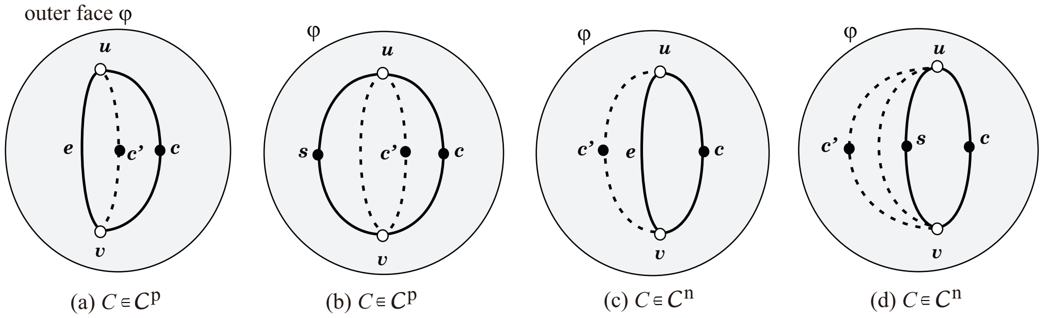

Figure 3: Candidate posi- and nega-cycles $C = (u, c, v)$ and $C = (u, c, v, s)$ in $\mathcal{G}$, where white circles represent vertices in V while black ones represent crossings in χ: (a) candidate posi-cycle of length 3, (b) candidate posi-cycle of length 4, (c) candidate nega-cycle of length 3, and (d) candidate nega-cycle of length 4.

A *candidate posi-cycle* (resp., *candidate nega-cycle*) in $\mathcal{G}$ is defined to be a cycle $C = (u, c, v)$ or $C = (u, c, v, s)$ in $\mathcal{G}$ with $u, v \in V$ and $c, s \in \chi$ such that

> the interior (resp., exterior) of C does not contain a crossing-free edge $uv \in E$ and any other crossing vertex c' adjacent to both u and v.

Fig. 3(a)-(b) and (c)-(d) illustrate candidate posi-cycles and candidate nega-cycles, respectively. Let $\mathcal{C}^{\mathrm{p}}$ and $\mathcal{C}^{\mathrm{n}}$ be the sets of candidate posi-cycles and candidate nega-cycles, respectively. By definition we see that the set $\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{n}} \cup \{C_f \mid f \in F(\gamma)\}$ is inclusive, and hence $|\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{n}} \cup \{C_f \mid f \in F(\gamma)\}| = O(n)$.

A candidate posi-cycle C with $C = (u, c, v)$ (resp., $C = (u, c, v, s)$) is called a *B-cycle* if

(a)-(B) the exterior of C contains no vertices in $V - \{u, v\}$ adjacent to c (resp., contains exactly one vertex in $V - \{u, v\}$ adjacent to c or s).

Note that $uv \in E$ when $C = (u, c, v, s)$ is a B-cycle, as shown in Fig. 4(a). Fig. 4(b) and (d) illustrate the other types of B-cycles.

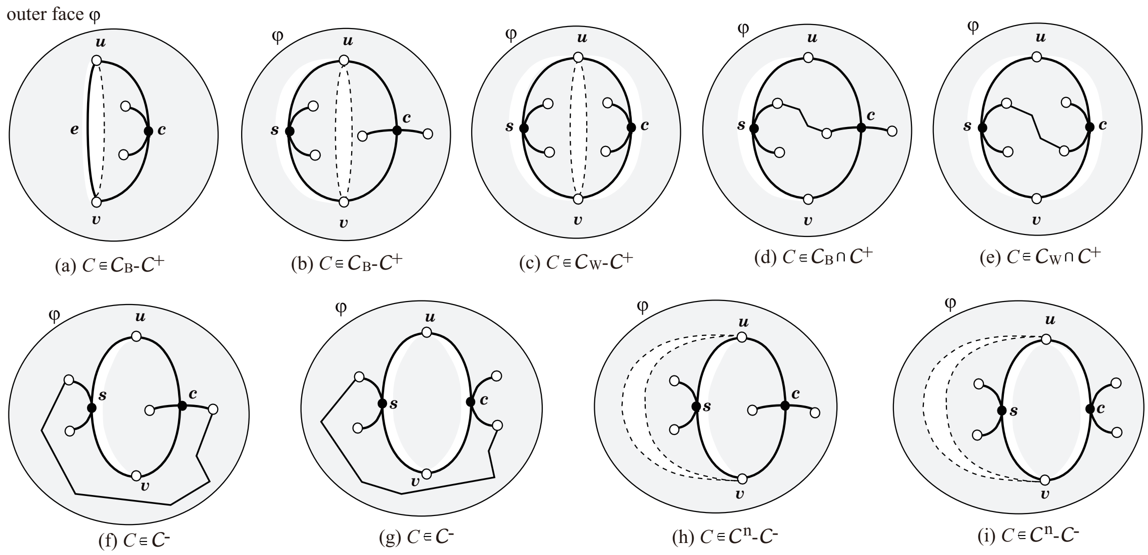

Figure 4: Illustration of types of cycles $C = (u, c, v)$ and $C = (u, c, v, s)$ in $\mathcal{G}$, where white circles represent vertices in V while black ones represent crossings in χ: (a) B-cycle of length 3, which is always soft, (b) soft B-cycle of length 4, (c) soft W-cycle, (d) hard B-cycle of length 4, (e) hard W-cycle, (f) nega-cycle whose reversal is a hard B-cycle, (g) nega-cycle whose reversal is a hard W-cycle, (h) candidate nega-cycle of length 4 that is not a nega-cycle whose reversal is a hard B-cycle, and (i) candidate nega-cycle of length 4 that is not a nega-cycle whose reversal a hard W-cycle.

A candidate posi-cycle $C = (u, c, v, s)$ is called a *W-cycle* if

(a)-(W) the exterior of C contains no vertices in $V - \{u, v\}$ adjacent to c or s.

Fig. 4(c) and (e) illustrate W-cycles.

Let $\mathcal{C}_{\mathrm{W}}$ (resp., $\mathcal{C}_{\mathrm{B}}$) be the set of W-cycles (resp., B-cycles) in γ. Clearly a W-cycle (resp., B-cycle) gives rise to a W-configuration (resp., B-configuration). Conversely, by choosing a W-configuration (resp., B-configuration) so that the interior is minimal, we obtain a W-cycle (resp., B-cycle). Hence we observe that the current embedding γ admits a straight-line drawing if and only if $\mathcal{C}_{\mathrm{W}} = \mathcal{C}_{\mathrm{B}} = \emptyset$.

A W- or B-cycle C is called *hard* if

(b) length of C is 4, and the interior of $C = (u, c, v, s)$ contains no inner face f whose facial cycle C_f contains both vertices u and v, i.e., some path connects c and s without passing through u or v.

On the other hand, a W- or B-cycle $C = (u, c, v, s)$ of length 4 that does not satisfy condition (b) or a B-cycle of length 3 is called *soft*. We also call a hard B- or W-cycle a *posi-cycle*. Fig. 4(d) and (e) illustrate a hard B-cycle and a hard W-cycles, respectively, whereas Fig. 4(a) and (b) (resp., (c)) illustrate soft B-cycles (resp., a soft W-cycle).

A cycle $C = (u, c, v, s)$ is called a *nega-cycle* if it becomes a posi-cycle when an inner face in the interior of C is chosen as the outer face. In other words, a nega-cycle is a candidate nega-cycle $C = (u, c, v, s)$ of length 4 that satisfies the following conditions (a') and (b'), where (a') (resp., (b')) is obtained from the above conditions (a)-(B) and (a)-(W) (resp., (b)) by exchanging the roles of "interior" and "exterior":

(a') the interior of C contains at most one vertex in $V - \{u, v\}$ adjacent to c or s; and

(b') the exterior of C contains no face f whose facial cycle C_f contains both vertices u and v.

Fig. 2(a)-(b) and Fig. 5(a)-(b) show examples of a hard B-cycle C and a nega-cycle C'. Fig. 4(f) and (g) illustrate nega-cycles, whereas Fig. 4(h) and (i) illustrate candidate nega-cycles that are not nega-cycles.

Let $\mathcal{C}^+$ (resp., $\mathcal{C}^-$) denote the set of posi-cycles (resp., nega-cycles) in γ. By definition, it holds that $\mathcal{C}^+ \subseteq \mathcal{C}_{\mathrm{W}} \cup \mathcal{C}_{\mathrm{B}} \subseteq \mathcal{C}^{\mathrm{p}}$ and $\mathcal{C}^- \subseteq \mathcal{C}^{\mathrm{n}}$.

3.2 Forbidden Cycle Pairs

We define a forbidden configuration that characterizes 1-plane embeddings, which cannot be re-embedded into SLD ones. A *forbidden cycle pair* is defined to be a pair $\{C, C'\}$ of a posi-cycle $C = (u, c, v, s)$ and a posi- or nega-cycle $C' = (u', c', v', s')$ in $\mathcal{G}$ with $u, v, u', v' \in V$ and $c, s, c', s' \in \chi$ to which $\mathcal{G}$ has a u, u'-path P_1 and a v, v'-path P_2 such that:

(i) when $C' \in \mathcal{C}^+$, paths P_1 and P_2 are in the exterior of C and C', i.e., $V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\mathrm{ex}}(C) \cap V_{\mathrm{ex}}(C')$, where C and C' cannot have any common inner face; and

(ii) when $C' \in \mathcal{C}^-$, paths P_1 and P_2 are in the exterior of C and the interior of C', i.e., $V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\mathrm{ex}}(C) \cap V_{\mathrm{in}}(C')$, where C is enclosed by C'.

In (i) and (ii), P_1 and P_2 are not necessary disjoint, and possibly one of them consists of a single vertex, i.e., $u = u'$ or $v = v'$.

The pair of cycles C and C' in Fig. 5(a) (resp., Fig. 5(b)) is a forbidden cycle pair, because there is a pair of a u, u'-path $P_1 = (u, x, z, y, u')$ and a v, v'-path $P_2 = (v, x', z, y', v')$ that satisfy the above conditions (i) (resp., (ii)). Note that the pair of cycles C and C' in Fig. 2(a)-(b) is not forbidden cycle pair, because there are no such paths.

Our main result of this paper is as follows.

Theorem 4 *A circular instance (G, γ) admits an SLD cross-preserving embedding if and only if it has no forbidden cycle pair. Finding an SLD cross-preserving embedding of γ or a forbidden cycle pair in $\mathcal{G}$ can be computed in linear time.*

Proof of necessity: The necessity of the theorem follows from the next lemma.

For a cycle $C = (u, c, v, s) \in \mathcal{C}^+$ (resp., $\mathcal{C}^-$) with $u, v \in V$ and $c, s \in \chi$ in $\mathcal{G}$, we call a vertex $z \in V$ an *in-factor* of C if the exterior of $C \in \mathcal{C}^+$ (resp., the interior of $C \in \mathcal{C}^-$) has a z, u-path $P_{z,u}$ and a z, v-path $P_{z,v}$, i.e., $V(P_{z,u} - \{u\}) \cup V(P_{z,v} - \{v\})$ is in $V_{\mathrm{ex}}(C)$ (resp., $V_{\mathrm{in}}(C)$). Paths $P_{z,u}$ and $P_{z,v}$ are not necessarily disjoint.

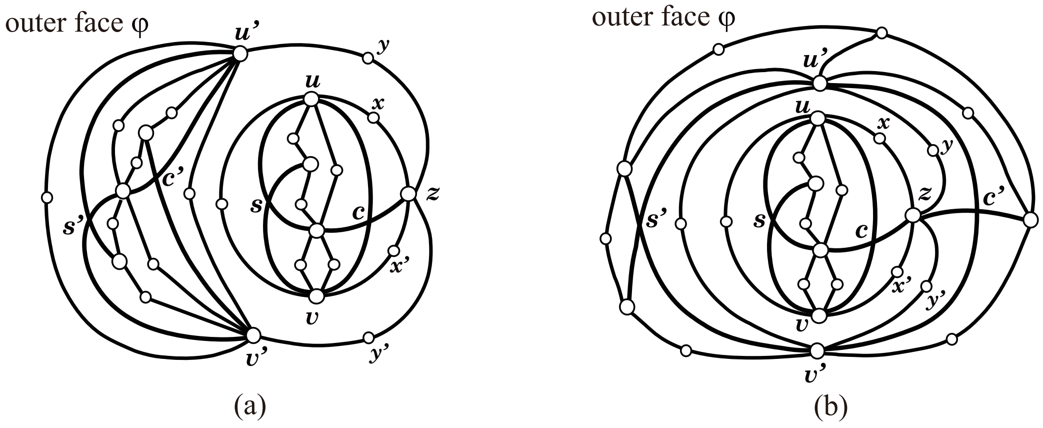

Figure 5: Illustration of circular instances (G, γ) with a cut-vertex z of $\mathcal{G}$, where the crossing edges are depicted by slightly thicker lines: (a) forbidden cycle pair with hard B-cycles $C = (u, c, v, s)$ and $C' = (u', c', v', s')$ (b) forbidden cycle pair with a hard B-cycle $C = (u, c, v, s)$ and a nega-cycle $C' = (u', c', v', s')$ whose reversal is a hard B-cycle, where vertices $u, v, u', v' \in V$ and crossings $c, s, c', s' \in \chi$.

Lemma 5 *Given $\mathcal{G} = \mathcal{G}(G, \gamma)$, let γ' be a cross-preserving embedding of γ. Then:*

(i) *Let $z \in V$ be an in-factor of a cycle $C \in \mathcal{C}^+ \cup \mathcal{C}^-$ in $\mathcal{G}$. Then cycle C is a posi-cycle (resp., a nega-cycle) in $\mathcal{G}(G, \gamma')$ if and only if z is in the exterior (resp., interior) of C in γ';*

(ii) *For a forbidden cycle pair $\{C, C'\}$, one of C and C' is a posi-cycle in $\mathcal{G}(G, \gamma')$.*

Proof. (i) Let $C = (u, c, v, s)$ be a posi-cycle with $u, v \in V$ and $c, s \in \chi$ in $\mathcal{G}$, where the case where C is a nega-cycle can be treated analogously. By definition, the exterior of C together with vertices u and v contains a z, u-path $P_{z,u}$ and a z, v-path $P_{z,v}$, not necessarily disjoint. These paths contain a u, v-path $P^{\mathrm{out}}_{u,v}$ with vertices in $V_{\mathrm{ex}}(C) \cup \{u, v\}$. Since the instance is circular, the interior of C together with vertices u and v contains a u, v-path $P_{u,v}$ consisting of crossing-free edges. Since C is a hard B- or W-cycle, the interior of C together with crossing-vertices c and s contains a c, s-path $P_{c,s}$. We denote by x (resp., y) the first (resp., last) vertex in $V(P_{c,s}) \cap V(P_{u,v})$ that appear along $P_{c,s}$ from c to s, where possibly $x = y$. Let $P_{c,x}$ and $P_{y,s}$ be the c, x-path and y, s-path of $P_{c,s}$, respectively. Let $H_{C,z}$ denote the subgraph of $\mathcal{G}(G, \gamma)$ that consists of cycle C and paths $P^{\mathrm{out}}_{u,v}$, $P_{u,v}$, $P_{c,x}$ and $P_{y,s}$. Note that $H_{C,z}$ is a pseudo-triconnected graph, a graph obtained from a triconnected graph by replacing edges with paths, and its plane embedding with a specified outer face is unique up to reversal. In particular, paths $P_{z,u}$ and $P_{z,v}$ are enclosed by C in a new cross-preserving embedding γ' if and only if so is z in γ'. Therefore C remains a posi-cycle if z is in the exterior of C in γ' or becomes a nega-cycle otherwise.

(ii) Let $C = (u, c, v, s)$ be a posi-cycle and $C' = (u', c', v', s')$ be a posi- or nega-cycle with $u, v, u', v' \in V$ and $c, s, c', s' \in \chi$ in $\mathcal{G}$. First consider the case where C' is a posi-cycle. To derive a contradiction, assume that both C and C' are nega-cycles in a new cross-preserving embedding γ', where C does not enclose C' without loss of generality. In the original embedding γ, the interior of C' contains a u', v'-path P' passing through a vertex $z \in V_{\mathrm{in}}(C') - \{u', v'\}$, where z has a z, u-path and a z, v-path, since γ has a u, u'-path and a v, v'-path in the exterior of C and C' by definition. By (i), vertex z is enclosed by C. Since C does not enclose C', the path P' connecting z and C' would make a new crossing with C, a contradiction.

Next consider the case where C' is a nega-cycle. By definition, $E(C) \neq E(C')$, and $\mathcal{G}$ has an inner face f that is in the interior of C' and in the exterior of C. We change the outer face from φ to f to obtain a cross-preserving embedding γ_f, where $\{C, C'\}$ becomes a forbidden cycle pair such that both C and C' are posi-cycles. Hence by applying the above argument to the pair $\{C, C'\}$, we see that both C and C' cannot be nega-cycles at the same time in any cross-preserving embedding of γ_f or γ. □

Proof of sufficiency: In the rest of paper, we prove the sufficiency of Theorem 4 by designing a linear-time algorithm.

4 Biconnected Case

In this section, (G, γ) stands for a circular instance such that the connectivity of the plane graph $\mathcal{G}$ is at least 2. In a biconnected graph $\mathcal{G}$, any two posi-cycles $C = (u, c, v, s)$, $C' = (u', c', v', s') \in \mathcal{C}^+$with $u, v, u', v' \in V$ give a forbidden cycle pair if they do not share an inner face, because there is a pair of u, u'-path and v, v'-path in the exterior of C and C'. Analogously any pair of a posi-cycle C and a nega-cycle C' such that C' encloses C is also a forbidden cycle pair in a biconnected graph $\mathcal{G}$.

To detect such a forbidden pair in $\mathcal{G}$ in linear time, we first compute the sets $\mathcal{C}_{\rm p}$, $\mathcal{C}_{\rm n}$, $\mathcal{C}_{\rm W}$, $\mathcal{C}_{\rm B}$, $\mathcal{C}^+$ and $\mathcal{C}^-$ in γ in linear time by using the inclusion-forest from Lemma 2.

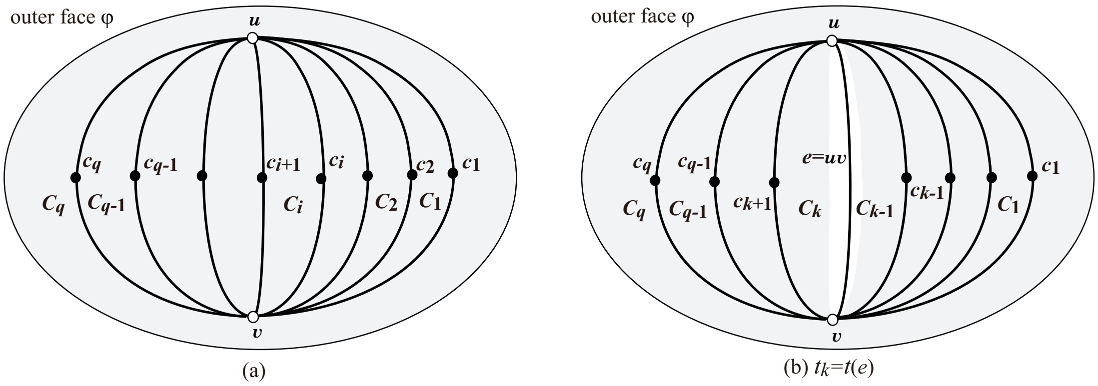

Figure 6: Illustration of u, v-paths of length 1 or 2 generated by an interval $I = [t_1, t_2, \ldots, t_q]$, where white circles represent vertices in V while black ones crossings in χ: (a) The case where uv is not a crossing-free edge, (b) The case where uv is a crossing-free edge, where I contains a tuple $t_k = t(e)$.

Lemma 6 *Given (G, γ), the following in (i)-(iv) can be computed in $O(n)$ time.*

(i) *The sets $\mathcal{C}_{\rm p}$, $\mathcal{C}_{\rm n}$ and the inclusion-forest $\mathcal{I}$ of $\mathcal{C}_{\rm p} \cup \mathcal{C}_{\rm n} \cup \{C_f \mid f \in F(\gamma)\}$;*

(ii) *The sets $\mathcal{C}_{\rm W}$ and $\mathcal{C}_{\rm B}$;*

(iii) *The sets $\mathcal{C}^+$, $\mathcal{C}^-$ and the inclusion-forest $\mathcal{I}^*$ of $\mathcal{C}^+ \cup \mathcal{C}^-$; and*

(iv) *A set* $\{f_C \mid C \in (\mathcal{C}_{\rm W} \cup \mathcal{C}_{\rm B}) - \mathcal{C}^+\}$ *such that* f_C *is an inner face in the interior of a soft B- or W-cycle* C *with* $V(C_f) \supseteq V(C)$.

Proof. We first introduce a total order $\prec$ over the vertex set V.

The rotation system $\rho(v)$ of each vertex $v \in V$ represents a cyclic order $N(v) \to \{1, 2, \ldots, \deg(v)\}$ around v, where for each neighbor $u \in N(v)$, we let $\rho(c; v)$ denote the rank i of u such that u appears as the i-th vertex in the order.

(i) To find all cycles in $\mathcal{C}_{\rm n}$ and $\mathcal{C}_{\rm p}$, we prepare a tuple $t(e)$ of each crossing-free edge $e \in E$ and six tuples $t_i(c)$, $i = 1, 2, \ldots, 6$ for each crossing $c \in \chi$ as follows.

> For each crossing-free edge $e = uv \in E$ with $u \prec v$, let $t(e) = (u, v, \rho(u; v), e)$; and
> For each crossing $c \in \chi$ and the two crossing edges $ab, a'b' \in E$ that create c, let $t_i(c)$, $i = 1, 2, \ldots, 6$ be six tuples $(u, v, \rho(u; u_c), c)$ for all pairs $u \prec v$ with $u, v \in \{a, b, a', b'\}$ and the vertex u_c with edge $uu_c \in \{ab, a'b'\}$.

Note that each of the above tuples represents a u, v-path of length 1 or 2 in $\mathcal{G}$. We compute a lexicographically sorted list L of all tuples in $(\cup_{e \in E} t(e)) \cup (\cup_{c \in \chi}\{t_i(c) \mid i = 1, 2, \ldots, 6\})$, which takes $O(n)$ time using the bucket sorting. Then tuples that have the same pair of the first and second entries appear consecutively in L, and we call such a maximal subsequence of L an *interval*.

Let $I = [t_1, t_2, \ldots, t_q]$ be an interval of L such that $q \geq 2$, where there are vertices $u \prec v$ and each tuple t_j in I is given as $t(e) = (u, v, \rho(u, v), e)$ for a crossing-free edge $e = uv \in E$ or $t(c_i)_k = (u, v, \rho(u; u_{c_i}), c_i)$ for a crossing $c_i \in \chi$ and some $k = 1, 2, \ldots, 6$. Then the plane is divided into q regions, each of which is enclosed by cycle $C_i = (u, c_i, v, c_{i+1})$, $i = 1, 2, \ldots, q$, where we interpret $c_{q+1} = c_1$ and regard c_k in C_k and C_{k+1} as nulls for the index k with $t_k = t(e)$ and $e = uv$ when $e = uv \in E$.

We compute the direction of cycle C_i by using Lemma 2, and assume without loss of generality that the outer face is not in the interior of each C_i with $i = 1, 2, \ldots, q-1$. Since any crossing c' adjacent to both u and v in $\mathcal{G}$ appears in this interval I, we see that C_p is a candidate nega-cycle, and C_i with $i = 1, 2, \ldots, q-1$ is a candidate posi-cycle. Then $\mathcal{C}_{\rm n}$ and $\mathcal{C}_{\rm p}$ are given as the sets of the candidate nega-cycles and posi-cycles constructed over all intervals. Finally the inclusion-forest $\mathcal{I}$ of $\mathcal{C}_{\rm p} \cup \mathcal{C}_{\rm n} \cup \{C_f \mid f \in F(\gamma)\}$ can be obtained in $O(n)$ time by Lemma 2.

(ii) Let $\mathcal{I}$ be the inclusion-forest obtained in (i). By definition, a cycle $C = (u, c, v)$ of $\mathcal{G}$ such that $c \in \chi$ is a B-cycle if and only if the exterior of C contains no vertex in $V - \{u, v\}$ adjacent to c; and a cycle $C = (u, c, v, s)$ of $\mathcal{G}$ such that $c, s \in \chi$ is a B-cycle (resp., W-cycle) if and only if the exterior of C contains exactly one vertex (resp., no vertex) in $V - \{u, v\}$ adjacent to c or s. The above test can be done in $O(1)$ time for each cycle in $\mathcal{C}_{\rm p}$, and we can find the sets $\mathcal{C}_{\rm B}$ and $\mathcal{C}_{\rm W}$ in $O(|\mathcal{C}_{\rm p}|) = O(n)$ time.

(iii) Let $\mathcal{I}$ be the inclusion-forest obtained in (i). By definition, a cycle C of $\mathcal{G}$ is a posi-cycle if and only if C is a cycle $(u, c, v, s) \in \mathcal{C}_{\rm B} \cup \mathcal{C}_{\rm W}$ of length 4 such that $c, s \in \chi$ and $\mathcal{I}$ has no facial cycle C_f with $u, v \in V(C_f)$ as a child of C in $\mathcal{I}$. For each cycle $C = (u, c, v, s) \in \mathcal{C}_{\rm B} \cup \mathcal{C}_{\rm W}$ of length 4, we traverse each of the facial cycles C_f that is a child of C in $\mathcal{I}$, and conclude that $C \in \mathcal{C}^+$ if u and v are not visited by a single facial cycle C_f; and C is a soft B- or W-cycle if u and v are visited by a single facial cycle of some inner face, say f_C.

Since each facial cycle in $\mathcal{G}$ is traversed once over all cycles in $\mathcal{C}_{\rm p}$, the set $\mathcal{C}^+$ can be found in $O(n)$ time. By definition, a cycle C of $\mathcal{G}$ is a nega-cycle if and only if C is a cycle $(u, c, v, s) \in \mathcal{C}_{\rm n}$ such that $c, s \in \chi$, $\{u, v\} \not\subseteq V(C_\varphi)$ and $\mathcal{I}$ has no facial cycle C_f with $u, v \in V(C_f)$ as a sibling of C in $\mathcal{I}$. Note that no two cycles $C, C' \in \mathcal{C}_{\rm n}$ can be siblings in $\mathcal{I}$. We traverse each of the facial cycles C_f that are siblings of a cycle $C = (u, c, v, s) \in \mathcal{C}_{\rm n}$ in $\mathcal{I}$, and conclude that $C \in \mathcal{C}^-$ if $\{u, v\} \not\subseteq V(C_\varphi)$ and u and v are not visited by a single facial cycle C_f. Again, each facial cycle in $\mathcal{G}$ is traversed once over all cycles in $\mathcal{C}_{\rm n}$, the set $\mathcal{C}^-$ can be found in $O(n)$ time.

(iv) The procedure in (iii) actually detects for each soft B- or W-cycle $C = (u, c, v, s) \in \mathcal{C}_{\rm B} \cup \mathcal{C}_{\rm W}$, an inner face f_C whose facial cycle contains both vertices u and v. □

Given (G, γ), a face $f \in F(\gamma)$ is called *admissible* if all posi-cycles enclose f but no nega-cycle encloses f. Let $A(\gamma)$ denote the set of all admissible faces in $F(\gamma)$.

Lemma 7 *Given (G, γ), it holds $A(\gamma) \neq \emptyset$ if and only if no forbidden cycle pair exists in γ. A forbidden cycle pair, if one exists, and $A(\gamma)$ can be obtained in $O(n)$ time.*

Proof. Let $\mathcal{C}^+_{\rm min}$ (resp., $\mathcal{C}^-_{\rm max}$) be the set of cycles C in $\mathcal{C}^+$ (resp., $\mathcal{C}^-$) that does not enclose any other cycle $C' \in \mathcal{C}^+$ (resp., that is not enclosed by any other cycle $C' \in \mathcal{C}^-$). From the inclusion-forest of $\mathcal{C}^+ \cup \mathcal{C}^-$, we can find $\mathcal{C}^+_{\rm min}$ and $\mathcal{C}^-_{\rm max}$ in $O(n)$ time. If $|\mathcal{C}^+_{\rm min}| \geq 2$, then any two cycles $C, C' \in \mathcal{C}^+_{\rm min}$ give a forbidden cycle pair. If $|\mathcal{C}^+_{\rm min}| = 1$ and a cycle $C' \in \mathcal{C}^-_{\rm max}$ encloses the cycle $C \in \mathcal{C}^+_{\rm min}$, then C and C' give a forbidden cycle pair. Otherwise (i.e., "$|\mathcal{C}^+_{\rm min}| = 1$ and no cycle $C' \in \mathcal{C}^-_{\rm max}$ encloses the cycle $C \in \mathcal{C}^+_{\rm min}$" or "$|\mathcal{C}^+_{\rm min}| = 0$") there is no forbidden cycle pair in γ.

Let F^+ be the set of all inner faces that are contained in all posi-cycles, where $F^+ = \emptyset$ if $|\mathcal{C}^+_{\rm min}| \geq 2$; $F^+ = F(\gamma)$ if $\mathcal{C}^+_{\rm min} = \emptyset$; and F^+ is the set of inner faces enclosed by the cycle $C \in \mathcal{C}^+_{\rm min}$ otherwise. Let F^- be the set of all inner faces that are not contained in any nega-cycle, where $F^- = F(\gamma) - \{f \in F(\gamma) \mid f \text{ is enclosed by some cycle } C \in \mathcal{C}^-_{\rm max}\}$. Then it holds $A(\gamma) = F^+ \cap F^-$, and γ has no forbidden cycle pair if and only if $A(\gamma) \neq \emptyset$. From $\mathcal{C}^+_{\rm min}$ and $\mathcal{C}^-_{\rm max}$, we can compute $A(\gamma)$ and a forbidden cycle pair if one exists in $O(n)$ time. □

By the lemma, if (G, γ) has no forbidden cycle pair, i.e., $A(\gamma) \neq \emptyset$, then any new embedding obtained from γ by changing the outer face with a face in $A(\gamma)$ is a cross-preserving embedding of γ which has no hard B- or W-cycle.

4.1 Eliminating Soft B- and W-cycles

Suppose that we are given a circular instance (G, γ) such that $\mathcal{G}$ is biconnected and $\mathcal{C}^+ = \emptyset$. We now show how to eliminate all soft B- and W-cycles in $\mathcal{G}$ in linear time using the inclusion-forest from Lemma 2 and the spindles from Lemma 3.

Lemma 8 *Given (G, γ) with $\mathcal{C}^+ = \emptyset$, there exists an SLD cross-preserving embedding $\gamma' = (\chi, \rho', \varphi')$ of γ such that $V(C_{\varphi'}) \supseteq V(C_\varphi)$, which can be constructed in $O(n)$ time.*

Proof. We show how to specify necessary subgraphs of $\mathcal{G}$ so that flipping the spindles induced by the subgraphs eliminates all soft B- and W-cycles. This results in an SLD cross-preserving embedding, because flipping spindles will not change the outer face of any embedding induced by a pseudo-triconnected subgraph and thereby it will not introduce any new posi-cycle by Lemma 5.

Fix a cut-pair $\{u, v\}$ such that γ has a soft B- or W-cycle $C \in \mathcal{C}_{\mathrm{W}} \cup \mathcal{C}_{\mathrm{B}}$ with $u, v \in V(C)$. We consider the case where uv is a crossing-free edge in γ (the other case can be treated analogously by regarding $k = q$ in the following argument). Let c_i, $1 \le i \le q$, and $i \ne k$ be the crossings such that u, v-path (u, c_i, v) is a subpath of a soft B- or W-cycle, and regard (u, c_k, v) as the crossing-free edge uv for a notational convenience, as shown in Fig. 6(b).

Assume without loss of generality that $c_1, c_2, \ldots, c_p$ is the clockwise order around u, and each cycle (u, c_i, v, c_{i+1}) with $1 \le i \le q-1$ denoted by C_i does not enclose the outer face φ. Let I_{soft} be the index i such that C_i is a soft B- or W-cycle. For each soft B- or W-cycle C_i with $i \in I_{\mathrm{soft}}$, let f^i be an inner face in the interior of C_i whose facial cycle C_{f^i} contains u and v, and let C_{f^i} consist of two u, v-paths P^i_{r} and P^i_{l} where P^i_{r} appears before P^i_{l} around u. We call P^i_{r} and P^i_{l} the *supporting paths* of the soft B- or W-cycle C_i.

Define a spindle S_i for each $i \in I_{\mathrm{soft}}$ as follows: For $i < k$ (resp., $i > k$), let S_i be the cycle that consists of (u, c_i, v) and P^i_{r} (resp., P^i_{l} and (u, c_{i+1}, v)). Let $\mathcal{S}_{u,v}$ be the set of spindles S_i, $i \in I_{\mathrm{soft}}$ for this vertex pair $\{u, v\}$. We easily see that no soft B- or W-cycle appears between u and v any more after flipping all the spindles in $\mathcal{S}_{u,v}$. Note that no spindle in $\mathcal{S}_{u,v}$ contains any vertex in $V(C_\varphi) - \{u, v\}$ since S_1 contains the u, v-path (u, c_1, v) and S_{q-1} contains the u, v-path (u, c_q, v).

Let $\mathcal{S}$ be the union of $\mathcal{S}_{u,v}$ over all vertex pairs $\{u, v\}$ in a soft B- or W-cycle in γ. Note that no vertex along each spindle in $\mathcal{S}$ except for its junctions is in $V(C_\varphi)$. Also for two distinct soft B- or W-cycle C and C' in γ, their supporting paths are edge-disjoint, because otherwise some part of the supporting path of C or C', say C would be enclosed by C'. Hence we see that no two spindles in $\mathcal{S}$ have an edge in their supporting path sides, which implies that the total number of edges in all soft B- or W-cycles and their supporting paths is $O(n)$. Therefore finding the inner faces in all soft B- or W-cycles can be done in $O(n)$ time by Lemma 2(iii), and flipping all spindles in $\mathcal{S}$ to obtain a new embedding γ' can be executed in $O(n)$ time by Lemma 3. The resulting embedding γ' still keeps all the vertices in φ along the new outer boundary, since no non-junction vertex along each spindle in $\mathcal{S}$ is in $V(C_\varphi)$. □

Given an instance (G, γ) with a biconnected graph $\mathcal{G}$, we can test whether it has either a forbidden cycle pair or an admissible face by Lemmas 6 and 7. In the former, it cannot have an SLD cross-preserving embedding by Lemma 5. In the latter, we can eliminate all hard B- and W-cycles by choosing an admissible face as a new outer face, and then eliminate all soft B- and W-cycles by a flipping procedure based on Lemma 8. All the above can be done in linear time.

To treat the case where $\mathcal{G}$ is oneconnected in the next section, we now characterize 1-plane embeddings that can have an SLD cross-preserving embedding such that a specified vertex appears along the outer boundary. For a vertex $z \in V$ in a graph G, we call a 1-plane embedding γ of G *z-exposed* if vertex z appears along the outer boundary of γ. We call (G, γ) *z-feasible* if it admits a z-exposed SLD cross-preserving embedding γ' of γ.

Lemma 9 *Given (G, γ) such that $A(\gamma) \ne \emptyset$, let z be a vertex in V. Then:*

(i) *The following conditions are equivalent:*
  (a) *γ admits no z-exposed SLD cross-preserving embedding;*
  (b) *$A(\gamma)$ contains no face f with $z \in V(C_f)$; and*
  (c) *$\mathcal{G}$ has a posi- or nega-cycle C to which z is an in-factor;*

(ii) *A z-exposed SLD cross-preserving embedding or a posi- or nega-cycle C to which z is an in-factor can be computed in $O(n)$ time.*

Proof. (i) Let z be a vertex in V, and F_z be the set of faces $f \in F(\gamma)$ with $z \in V(C_f)$.

(a)$\Rightarrow$(b): Assume that $A(\gamma) \cap F_z \neq \emptyset$. Then choose a face $f \in A(\gamma) \cap F_v$, change the outer face from φ to face f to obtain a cross-preserving embedding $\gamma' = (\chi, \rho', \varphi')$ of γ that has no posi-cycle in $\mathcal{G}(G, \gamma')$, and then eliminate all soft B- or W-cycles to obtain an embedding $\gamma'' = (\chi, \rho'', \varphi'')$. By Lemma 8, embedding γ'' is an SLD cross-preserving embedding of γ and can be obtained from γ in $O(n)$ time. We observe that γ'' is z-exposed since $V(C_{\gamma''}) \supseteq V(C_{\gamma'})$ by Lemma 8.

(b)$\Rightarrow$(c): On the other hand, assume that $A(\gamma) \cap F_z = \emptyset$. We distinguish two cases.

(α) there is a face $f \in F_z$ not enclosed by a posi-cycle C: Then $|\mathcal{C}^+_{\min}| = 1$ since $A(\gamma) \neq \emptyset$. If z is on the posi-cycle $C \in \mathcal{C}^+_{\min}$, say $C = (z, c, v, s)$ with $z, v \in V$ and $c, s \in \chi$, then there is a face $f' \in F_z$ which is enclosed by C in γ, and $A(\gamma) \cap F_z = \emptyset$ means that this face f' must be enclosed by some nega-cycle C', contradicting that $\{C, C'\}$ is not a forbidden cycle pair. Hence z is not on the posi-cycle $C \in \mathcal{C}^+_{\min}$, i.e., z is properly in the interior of C, as required.

(β) each face $f \in F_v$ is enclosed by a nega-cycle: If there is no nega-cycle which encloses vertex z properly, then each face $f \in F_z$ is contained in a nega-cycle $C' = (z, c, v, s)$, which, however, contradicts that no two nega-cycles sharing a vertex z can share an edge incident to z. Hence there is a nega-cycle C which encloses vertex z properly, as required. In any of (α) and (β), there is a cycle C to which z is an in-factor.

(c)$\Rightarrow$(a): Let $C = (u, c, v, s)$ be a posi-cycle (resp., nega-cycle) to which z is an in-factor, where the biconnected graph $\mathcal{G}(G, \gamma)$ has a z, u-path and a z, v-path in the exterior (resp., interior) of C. By Lemma 5(i), we see that C is a posi-cycle when z appears along the outer boundary of any cross-preserving embedding γ'. Hence if a cycle satisfying condition (i)-(c) of the lemma exists, then no z-exposed cross-preserving embedding can be SLD.

(ii) By Lemma 6, we can find a cycle C to which z is an in-factor, if one exists. On the other hand, a z-exposed SLD cross-preserving embedding of γ constructed in (i) can be computed in linear time, as we have observed after Lemma 8. $\square$

5 One-connected Case

In this section, we prove the sufficiency of Theorem 4 by designing a linear-time algorithm claimed in the theorem. Given a circular instance (G, γ), where $\mathcal{G}$ may be disconnected, obviously we only need to test each connected component of $\mathcal{G}$ separately to find a forbidden cycle pair. Thus we first consider a circular instance (G, γ) such that the connectivity of $\mathcal{G}$ is 1; i.e., $\mathcal{G}$ is connected and has some cut-vertices.

A block B of $\mathcal{G}$ is a maximal biconnected subgraph of $\mathcal{G}$. For a biconnected graph $\mathcal{G}$, we already know how to find a forbidden cycle pair or an SLD cross-preserving embedding from the previous section. For a trivial block B with $|V(B)| = 2$, there is nothing to do. If some block B of $\mathcal{G}$ with $|V(B)| \geq 3$ contains a forbidden cycle pair, then (G, γ) cannot admit any SLD cross-preserving embedding by Lemma 5.

We now observe that $\mathcal{G}$ may contain a forbidden cycle pair even if no single block of $\mathcal{G}$ has a forbidden cycle pair.

Lemma 10 *For a circular instance (G, γ) such that the connectivity of $\mathcal{G}$ is 1, let B_1 and B_2 be blocks of $\mathcal{G}$ and let $z_i \in V(B_i)$ be the closest vertex to B_j with $j \in \{1, 2\} - \{i\}$. If $\gamma|_{B_i}$ has a posi- or nega-cycle C_i to which z_i is an in-factor for each $i = 1, 2$, then $\{C_1, C_2\}$ is a forbidden cycle pair in $\mathcal{G}$.*

Proof. For each $i = 1, 2$, let $C_i = (u_i, c_i, v_i, s_i)$ with $u_i, v_i \in V$ and $c_i, s_i \in \chi$, and we know that each block B_i has a u_i, v_i-path P_i that passes through z_i in the exterior (resp., interior) of C_i if C_i is a posi-cycle (resp., nega-cycle). Let $P_{1,2}$ be a shortest z_1, z_2-path of $\mathcal{G}$, where $V(B_i) \cap V(P_{1,2}) = \{z_i\}$ for each $i = 1, 2$. Denote by H the subgraph of $\mathcal{G}$ consisting of these paths P_i, $i = 1, 2$ and $P_{1,2}$. We distinguish two cases.

(i) Both C_1 and C_2 are posi-cycles: If one of C_1 and C_2, say C_1 encloses the other in the current embedding γ, then z_1, z_2-path $P_{1,2}$ would create a crossing along cycle C_2. Hence C_1 and C_2 share no inner face, and subgraph H contains a u_1, u_2-path and a v_1, v_2-path in the exterior of C_1 and C_2. Therefore $\{C_1, C_2\}$ is a forbidden cycle pair.

(ii) One of C_1 and C_2, say C_1 is a nega-cycle: In the current embedding γ, vertex z_1 is in the interior of C_1. Hence the subgraph H with $z_1 \in V(H)$ is also in the interior of C_1. This implies that C_2 cannot be a nega-cycle, since otherwise H also need to be in the interior of C_2 and would create a new crossing with C_1 or C_2. Then C_2 is a posi-cycle, which is in the interior of C_1, and H exists in the interior of C_1 and in the exterior of C_2, indicating that $\{C_1, C_2\}$ is a forbidden cycle pair. $\square$

For a linear-time implementation, we do not apply the lemma for all pairs of blocks in $\mathcal{B}$. A block of $\mathcal{G}$ is called a *leaf block* if it contains only one cut-vertex of $\mathcal{G}$, where we denote the cut-vertex in a leaf block B by v_B. Without directly searching for a forbidden cycle pair in $\mathcal{G}$, we use the next lemma to reduce a given embedding by repeatedly removing leaf blocks.

Lemma 11 *For a circular instance (G, γ) such that the connectivity of $\mathcal{G} = \mathcal{G}(G, \gamma)$ is 1 and a leaf block B of $\mathcal{G}$ such that $\gamma|_B$ is v_B-feasible, let $H = G - (V(B) - \{v_B\})$ be the graph obtained by removing the vertices in $V(B) - \{v_B\}$. Then*

(i) *The instance $(H, \gamma|_H)$ is circular; and*

(ii) *If $(H, \gamma|_H)$ admits an SLD cross-preserving embedding γ^*_H, then an SLD cross-preserving embedding γ^* of γ can be obtained by placing a v_B-exposed SLD cross-preserving embedding γ^*_B of $\gamma|_B$ within a space next to the cut-vertex v_B in γ^*_H.*

Proof. (i) The instance $(H, \gamma|_H)$ remains circular, because for any crossing c in $\gamma|_H$, cut-vertex v_B separates no two vertices in the cycle of eight crossing-free edges that surrounds c in G.

(ii) The embedding γ^* obtained from γ^*_H and γ^*_B is an SLD cross-preserving embedding of γ, since neither of γ^*_H and γ^*_B contains a W- or B-cycle. $\square$

Given a circular instance (G, γ) such that $\mathcal{G} = \mathcal{G}(G, \gamma)$ is connected, an algorithm **Algorithm Re-Embed-1-Plane** for Theorem 4 is designed by the following three steps.

The first step test whether $\mathcal{G}$ has a block B such that $\gamma|_B$ has a forbidden cycle pair, based on Lemma 9. If one exists, the algorithm outputs a forbidden cycle pair and halts.

After the first step, no block has a forbidden cycle pair. In the current circular instance (G, γ), one of the following holds:

(i) the number of blocks in $\mathcal{G}$ is at least two and there is at most one leaf block B such that $\gamma|_B$ is not v_B-feasible; and
(ii) $\mathcal{G}$ has two leaf blocks B and B' such that $\gamma|_B$ is not v_B-feasible and $\gamma|_{B'}$ is not $v_{B'}$-feasible; and
(iii) the number of blocks in $\mathcal{G}$ is at most one.

In (ii), v_B is an in-factor of a cycle C in $\gamma|_B$ and $v_{B'}$ is an in-factor of a cycle C' in $\gamma|_{B'}$ by Lemma 9, and we obtain a forbidden cycle pair $\{C, C'\}$ by Lemma 10. Otherwise if (i) holds, then we can remove all leaf blocks B such that $\gamma|_B$ is not v_B-feasible by Lemma 11. The second step keeps removing all leaf blocks B such that $\gamma|_B$ is not v_B-feasible until (ii) or (iii) holds to the resulting embedding. If (i) occurs, then the algorithm outputs a forbidden cycle pair and halts.

When all the blocks of $\mathcal{G}$ can be removed successfully, say in an order of $B^1, B^2, \ldots, B^m$, the third step constructs an embedding with no B- or W-cycles by starting with such an SLD embedding of B^m and by adding an SLD embedding of B^i to the current embedding in the order of $i = m-1, m-2, \ldots, 1$. By Lemma 11, this results in an SLD cross-preserving embedding of the input instance (G, γ).

The entire description of algorithm **Algorithm Re-Embed-1-Plane** is given as follows.

Algorithm Re-Embed-1-Plane
Input: A circular instance (G, γ) such that $\mathcal{G} = \mathcal{G}(G, \gamma)$ is connected and has $m \geq 1$ blocks.
Output: Either an SLD cross-preserving embedding of γ^* of γ or a forbidden cycle pair in $\mathcal{G}$.

STEP 1:
Among the m blocks in $\mathcal{G}$, test whether there is a block B such that $\gamma|_B$ has a forbidden cycle pair, based on Lemma 9;
if $\gamma|_B$ for some block B of $\mathcal{G}$ contains a forbidden cycle pair **then**
  Halt outputting a forbidden cycle pair in $\gamma|_B$
end if;

STEP 2:
Let $\mathcal{G}' := \mathcal{G}$; $\gamma' := \gamma$; $p := 1$;
while the number of blocks of $\mathcal{G}'$ is at least two **do**
  Test whether the embedding $\gamma|_B$ of each leaf block B of $\mathcal{G}'$ is v_B-feasible,
  i.e., v_B is not an in-factor of any cycle C in $\gamma|_B$, based on Lemma 9;
  if for some two leaf blocks B and B', v_B is an in-factor of a cycle C in $\gamma|_B$ and
    $v_{B'}$ is an in-factor of a cycle C' in $\gamma|_{B'}$ **then**
      Halt outputting the forbidden cycle pair $\{C, C'\}$
  else
      Let $B^p, B^{p+1}, \ldots, B^q$ be the leaf blocks of $\mathcal{G}'$ such that $\gamma|_{B^i}$ is v_{B^i}-feasible
      in γ';   /* possibly one leaf block B such that $\gamma|_B$ is v_B-infeasible is left */
      For $H = G - \cup_{p \leq i \leq q}(V(B^i) - \{v_{B^i}\})$, let $\gamma' := \gamma|_H$ and $\mathcal{G}' := \mathcal{G}(H, \gamma|_H)$;
      Let $p := q + 1$;
  end if
end while;
if $\mathcal{G}'$ is not empty; i.e., $\mathcal{G}'$ contains only one block B **then**
  Remove $B^m := B$ from $\mathcal{G}'$

end if;
/* $B^1, B^2, \ldots, B^m$ are the m blocks of $\mathcal{G}$ indexed such that B^i is the i-th block removed from $\mathcal{G}$ during Step 2 */

Step 3:
Denote $H^i = G - \cup_{1 \le j \le i-1}(V(B^j) - \{v_{B^j}\})$ for $i = 1, \ldots, m$;
Construct an SLD cross-preserving embedding $\gamma^*_{H^m} = \gamma^*_{B^m}$ of $\gamma|_{B^m}$ based on Lemma 9;
for $i = m-1, m-2, \ldots, 1$ **do**
Construct a v_{B^i}-exposed SLD cross-preserving embedding $\gamma^*_{B^i}$ of $\gamma|_{B^i}$ based on Lemma 9;
Place the embedding $\gamma^*_{B^i}$ within a region next to the cut-vertex v_{B^i} in $\gamma^*_{H^{i+1}}$ to obtain a cross-preserving embedding $\gamma^*_{H^i}$ of $\gamma|_{H^i}$
end for;
Output embedding $\gamma^*_{H^1}$, which is an SLD cross-preserving embedding of γ.

Note that we can obtain an SLD cross-preserving embedding $\gamma^*_{H^1}$ of γ in the third step when the first and second step did not find any forbidden cycle pair. Thus the algorithm finds either an SLD cross-preserving embedding of γ or a forbidden cycle pair. This proves the sufficiency of Theorem 4.

By the time complexity result from Lemma 9, we see that the algorithm can be implemented in linear time.

6 Concluding Remarks

In this paper, we studied the problem of re-embedding a 1-plane graph so that it is drawn as a straight-line drawing. By Thomassen's forbidden characterization via the B- and W-configurations, this is reduced to a problem of embedding the planarization of a given 1-plane graph so that special types of cycles do not appear. Just to test whether the current embedding contains such a cycle or not, we can check each block separately or can assume that a given instance is biconnected without loss of generality. However, this is not the case to finding an SDL re-embedding, where we have to examine how the blocks are connected in the whole connected graph.

To detect sa special type of cycle and a subgraph to be flipped, it would be natural to use the triconnected component decomposition or the SPQR tree to see the whole structure of cut-pairs in *some* graph. However, a hard B-cycle is not related to any cut-pair of the planarization and is related to a cut-pair of the spanning subgraph induced by the crossing-free edges. This would require us to construct the SPQR tree of the crossing-free spanning graph and establish a method based on some comparison between the SPQR tree and the planarization. On the other hand, all soft B- and W-cycles are related to only cut-pairs in the planarization, which suggests to us use of the SPQR tree of the planarization. As we have observed in this paper, we could avoid such troublesome situation by devising the inclusion-forest of non-intersecting cycles and a fast implementation of flipping procedure. It would be interesting to see how our new approach can be extended to a problem of re-embedding k-plane graphs with $k \geq 2$ into a straight-line drawing.

References

[1] Md. Alam, F. Brandenburg and S. Kobourov, Straight-line grid drawings of 3-connected 1-planar graphs, Proc. of GD 2013, pp. 83-94, 2014.

[2] C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth and J. Reislhuber, Recognizing outer 1-planar graphs in linear time, Proc. of GD 2013, pp. 107-118, 2014.

[3] M. J. Bannister, S. Cabello and D. Eppstein, Parameterized complexity of 1-planarity, Proc. of WADS 2013, pp. 97-108, 2013.

[4] O. V. Borodin, A. V. Kostochka, A. Raspaud and E. Sopena, Acyclic colouring of 1-planar graphs, Discrete Applied Mathematics, 114(1-3), pp. 29-41, 2001.

[5] Z. Chen and M. Kouno, A Linear-time algorithm for 7-coloring 1-plane graphs, Algorithmica, 43(3), pp. 147-177, 2005.

[6] I. Fabrici and T. Madaras, The structure of 1-planar graphs, Discrete Mathematics, 307(7-8), pp. 854-865, 2007.

[7] G. Di Battista and R. Tamassia, On-line planarity testing, SIAM J. on Comput., 25(5), pp. 956-997, 1996.

[8] P. Eades, S-H. Hong, N. Katoh, G. Liotta, P. Schweitzer and Y. Suzuki, A linear time algorithm for testing maximal 1-planarity of graphs with a rotation system, Theor. Comput. Sci., 513, pp. 65-76, 2013.

[9] I. Fáry, On straight line representations of planar graphs, Acta Sci. Math. Szeged, 11, pp. 229-233, 1948.

[10] A. Grigoriev and H. Bodlaender, Algorithms for graphs embeddable with few crossings per edge, Algorithmica, 49(1), pp. 1-11, 2007.

[11] S-H. Hong, P. Eades, N. Katoh, G. Liotta, P. Schweitzer and Y. Suzuki, A linear-time algorithm for testing outer-1-planarity, Algorithmica, 72(4), pp. 1033-1054, 2015.

[12] S. Hong, P. Eades, G. Liotta and S. Poon, Fary's theorem for 1-planar graphs, Proc. of COCOON 2012, pp. 335-346, 2012.

[13] J. E. Hopcroft and R. E. Tarjan, Dividing a graph into triconnected components, SIAM J. on Comput., 2, pp. 135-158, 1973.

[14] V. P. Korzhik and B. Mohar, Minimal obstructions for 1-immersions and hardness of 1-planarity testing, J. Graph Theory, 72(1), pp. 30-71, 2013.

[15] J. Pach and G. Toth, Graphs drawn with few crossings per edge, Combinatorica, 17(3), pp. 427-439, 1997.

[16] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, Abh. Math. Semin. Univ. Hamb., 29, pp. 107-117, 1965.

[17] Y. Suzuki, Re-embeddings of maximum 1-planar graphs, SIAM J. Discrete Math., 24(4), pp. 1527-1540, 2010.

[18] C. Thomassen, Rectilinear drawings of graphs, J. Graph Theory, 10(3), pp. 335-341, 1988.