A Polynomial-Space Exact Algorithm for TSP in Degree-8 Graphs

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Abstract: Previously, the authors of this work have presented in a series of papers polynomial-space algorithms for the TSP in graphs with degree at most five, six and seven. Each of these algorithms is the first algorithm specialized for the TSP in graphs of limited degree five, six, and seven respectively, and the running time bound of these algorithms outperforms Gurevich and Shelah's $O^*(4^n n^{\log n})$ algorithm for the TSP in n-vertex graphs (SIAM Journal of Computation, 16(3), pp. 486–502, 1987). Now we ask what is the highest degree i until which a specialized polynomial-space algorithm for the TSP in graphs with maximum degree i outperforms Gurevich and Shelah's $O^*(4^n n^{\log n})$ algorithm? As an answer to this question, this paper presents the first polynomial-space exact algorithm specialized for the TSP in graphs with degree at most eight. We develop a set of branching rules to aid the analysis of the branching algorithm, and we use the measure-and-conquer method to effectively analyze our branching algorithm. We obtain a running time of $O^*(4.1485^n)$, and this running time bound does not give an advantageous algorithm for the TSP in degree-8 graphs over Gurevich and Shelah's algorithm for the TSP in general, but it gives a limit as to the applicability of our choice of branching rules and analysis method for designing a polynomial-space exact algorithm for the TSP in graphs of limited degree.

Keywords: Traveling Salesman Problem, Exact Exponential Algorithm, Branch-and-Reduce, Measure-and-Conquer.

1 Introduction

The Traveling Salesman Problem, TSP, is one of the most well-known combinatorial optimization problems. In the multitude of investigated algorithms for the TSP, we confine our exposition to those who use polynomial execution space. Gurevich and Shelah [5] have shown that the TSP in a general n-vertex graph is solvable in time $O^*(4^n n^{\log n})$. This had remained the only result for nearly two decades until Eppstein [2] started the exploration into polynomial-space TSP algorithms specialized for graphs of bounded degree. From this viewpoint, let degree-i graph stand for a graph in which each vertex has at most i incident edges.

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Eppstein [2] designed an algorithm for degree-3 graphs that runs in $O^*(1.260^n)$ time. Iwama and Nakashima [6] have claimed an improvement of Eppstein's time bound to $O^*(1.251^n)$ time for the TSP in degree-3 graphs. Later, Liskiewicz and Schuster [7] have discovered some oversights made in Iwama and Nakashima's analysis, and proved that their algorithm actually runs in $O^*(1.257^n)$ time. Liskiewicz and Schuster then made some minor modifications of Eppstein's algorithm and showed that this modified algorithm runs in $O^*(1.2553^n)$ time. Recently, Xiao and Nagamochi [12] have presented an $O^*(1.2312^n)$-time algorithm for the TSP in degree-3 graphs, and this improved all previous time bounds for polynomial-space algorithms.

For the TSP in degree-4 graphs, Eppstein [2] designed an algorithm that runs in $O^*(1.890^n)$ time. Later, Xiao and Nagamochi [13] showed an improved value for the upper bound of the running time and showed that their algorithm runs in $O^*(1.692^n)$ time. Currently, this is the fastest algorithm for the TSP in degree-4 graphs. To the best of our knowledge, presently the only investigation on the TSP in graphs of degree five and up to seven has been done by Md Yunos et al. [8, 9, 10]. Md Yunos et al. [8] gave an $O^*(2.4723^n)$-time algorithm for the TSP in degree-5 graphs, followed by an $O^*(3.0335^n)$-time algorithm for the TSP in degree-6 graphs [9], and an $O^*(3.5939^n)$-time algorithm for the TSP in degree-7 graphs [10].

Above all, there exist no reports in the literature of exact algorithms specialized to the TSP in graphs of degree higher than seven. Furthermore, the following question arises; until which value i of a maximum degree does a specialized polynomial-space algorithm for degree-i graphs outperform Gurevich and Shelah's $O^*(4^n n^{\log n})$-time algorithm? Therefore, in this paper, not only do we present the first polynomial-space branching algorithm for the TSP in degree-8 graphs, but also breach the time bound of $O^*(4^n)$. This result does not give an advantageous algorithm for the TSP in degree-8 graphs over Gurevich and Shelah, but gives a limit as to the applicability of our choice of branching rules and analysis method for designing a polynomial-space exact algorithm for the TSP in graphs of limited degree. This means that in the quest of designing polynomial-space exact algorithms for the TSP in graphs of limited degree, possibly different and improved branching rules and analysis method should be sought for in order to achieve better results.

2 Preliminaries

For a graph G, let $V(G)$ denote the set of vertices in G, and let $E(G)$ denote the set of edges in G. A vertex u is a neighbor of a vertex v if u and v are adjacent by an edge uv. We denote the set of all neighbors of a vertex v by $N(v)$, and denote by $d(v)$ the cardinality $|N(v)|$ of $N(v)$, also called the *degree* of v. For a subset of vertices $W \subseteq V(G)$, let $N(v; W) = N(v) \cap W$. For a subset of edges $E' \subseteq E(G)$, let $N_{E'}(v) = N(v) \cap \{u \mid uv \in E'\}$, and let $d_{E'}(v) = |N_{E'}(v)|$. Analogously, let $N_{E'}(v; W) = N_{E'}(v) \cap W$, and $d_{E'}(v, W) = |N_{E'}(v, W)|$. Also, for a subset E' of $E(G)$, we denote by $G - E'$ the graph $(V, E \setminus E')$ obtained from G by removing the edges in E'.

We employ a known generalization of the TSP proposed by Rubin [11], and named the *forced* Traveling Salesman Problem by Eppstein [2]. We define an instance $I = (G, F)$ that consists of a simple, edge weighted, undirected graph G, and a subset F of edges in G, called *forced.* For brevity, throughout this paper let U denote $E(G) \setminus F$. A vertex is called *forced* if exactly one of its incident edges is forced. Similarly, it is called *unforced* if no forced edge is incident to it. A Hamiltonian cycle in G is called a *tour* if it passes through all the forced edges in F. Under these circumstances, the forced TSP requests to find a minimum cost tour of an instance (G, F).

Throughout this paper, we assume that the maximum degree of a vertex in G is at most eight. We denote a forced (resp., unforced) vertex of degree i as a type fi vertex (resp., ui vertex). We are interested in 12 types of vertices in an instance of (G, F), namely, ui and fi for $i = 3, 4, \ldots, 8$. As shall be seen in Subsection 3.1, forced and unforced vertices of degree two and one are treated as special cases. Let $V_{\mathrm{f}i}$ (resp., $V_{\mathrm{u}i}$), $i = 3, 4, \ldots, 8$ denote the set of fi-vertices (resp., ui-vertices) in (G, F).

3 A Polynomial-Space Branching Algorithm

Our algorithm consists of two major steps which are repeated iteratively. In the first step, the algorithm applies reduction rules until no further reduction is possible. In the second step, the algorithm applies branching rules in a reduced instance to search for a solution.

3.1 Reduction Rules

Reduction is a process of transforming an instance to a smaller instance optimality. It takes polynomial time to generate a solution of an original instance from a solution to a smaller instance obtained through reduction.

If an instance admits no tour, we call it *infeasible*. Observation 1 gives two sufficient conditions for an instance to be infeasible as observed by Rubin [11]. These two sufficient conditions will be checked when executing the reduction rules.

Observation 1 *If one of the following conditions holds, then the instance (G, F) is infeasible.*

(i) $d(v) \leq 1$ *for some vertex* $v \in V(G)$.

(ii) $d_F(v) \geq 3$ *for some vertex* $v \in V(G)$.

An instance (G, F) is called *semi-feasible* if it does not satisfy any of the conditions in Observation 1. If the instance is semi-feasible, then the reduction rules will be executed. In this paper, we apply two reduction rules as stated in Md Yunos et al. [8]. The reduction rules as stated in Observation 2 preserve the minimum cost tour of an instance, and they are applied in each of the branching operations.

Observation 2 *Each of the following reductions preserves the feasibility and a minimum cost tour of an instance (G, F).*

(i) *If $d(v) = 2$ for a vertex v, then add to F any unforced edge incident to the vertex v; and*

(ii) *If $d(v) > 2$ and $d_F(v) = 2$ for a vertex v, then remove from G any unforced edge incident to vertex v.*

Our reduction algorithm is described as Algorithm 1. An instance (G, F) is called *reduced* if it does not satisfy any of the conditions in Observation 1 and Observation 2.

3.2 Branching Rules

Our algorithm iteratively branches on an unforced edge e in a reduced instance $I = (G, F)$ by either including e into F, **force**(e), or excluding it from G, **delete**(e). By applying a branching operation, the algorithm generates two new instances, called branches.

To describe our branching algorithm, let (G, F) be a reduced instance. Recall that we assume that an input graph has degree at most eight. Due to our reduction and branching operations, the degree in sub-instance will never increase.

In (G, F), an unforced edge $e = vt$ incident to a vertex v of degree eight is called *optimal*, if it satisfies a condition c-i with minimum index i, over all unforced edges vt in (G, F). We refer to the following conditions for choosing an optimal edge to branch on, c-1 to c-29, as the *branching rules*. The set of branching rules for conditions c-1 to c-18 is illustrated in Figure 1, and the set of branching rules for conditions c-19 to c-29 is illustrated in Figure 2. Details of our branching algorithm are described in Algorithm 2.

For convenience of the analysis of the algorithm, cases c-5, c-8, c-11, c-14 and c-17 have been divided into sub-cases according to the cardinality of the neighborhood intersection for vertex v of degree eight and vertex t of degree four, five, six, seven and eight, respectively. Vertex pairs with intersections of lower cardinality take precedence over higher ones.

BRANCHING RULES

- (c-1) $v \in V_{f8}$ and $t \in N_U(v; V_{f3})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-2) $v \in V_{f8}$ and $t \in N_U(v; V_{f3})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
- (c-3) $v \in V_{f8}$ and $t \in N_U(v; V_{u3})$;
- (c-4) $v \in V_{f8}$ and $t \in N_U(v; V_{f4})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-5) $v \in V_{f8}$ and $t \in N_U(v; V_{f4})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
  - (I) $|N_U(v) \cap N_U(t)| = 1$; and
  - (II) $|N_U(v) \cap N_U(t)| = 2$;
- (c-6) $v \in V_{f8}$ and $t \in N_U(v; V_{u4})$;
- (c-7) $v \in V_{f8}$ and $t \in N_U(v; V_{f5})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-8) $v \in V_{f8}$ and $t \in N_U(v; V_{f5})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
  - (I) $|N_U(v) \cap N_U(t)| = 1$;
  - (II) $|N_U(v) \cap N_U(t)| = 2$; and
  - (III) $|N_U(v) \cap N_U(t)| = 3$;
- (c-9) $v \in V_{f8}$ and $t \in N_U(v; V_{u5})$;
- (c-10) $v \in V_{f8}$ and $t \in N_U(v; V_{f6})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-11) $v \in V_{f8}$ and $t \in N_U(v; V_{f6})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
  - (I) $|N_U(v) \cap N_U(t)| = 1$;
  - (II) $|N_U(v) \cap N_U(t)| = 2$;
  - (III) $|N_U(v) \cap N_U(t)| = 3$; and
  - (IV) $|N_U(v) \cap N_U(t)| = 4$;
- (c-12) $v \in V_{f8}$ and $t \in N_U(v; V_{u6})$;
- (c-13) $v \in V_{f8}$ and $t \in N_U(v; V_{f7})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-14) $v \in V_{f8}$ and $t \in N_U(v; V_{f7})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
  - (I) $|N_U(v) \cap N_U(t)| = 1$;
  - (II) $|N_U(v) \cap N_U(t)| = 2$;
  - (III) $|N_U(v) \cap N_U(t)| = 3$;
  - (IV) $|N_U(v) \cap N_U(t)| = 4$; and
  - (V) $|N_U(v) \cap N_U(t)| = 5$;
- (c-15) $v \in V_{f8}$ and $t \in N_U(v; V_{u7})$;
- (c-16) $v \in V_{f8}$ and $t \in N_U(v; V_{f8})$ such that $N_U(v) \cap N_U(t) = \emptyset$;
- (c-17) $v \in V_{f8}$ and $t \in N_U(v; V_{f8})$ such that $N_U(v) \cap N_U(t) \neq \emptyset$;
  - (I) $|N_U(v) \cap N_U(t)| = 1$;
  - (II) $|N_U(v) \cap N_U(t)| = 2$;
  - (III) $|N_U(v) \cap N_U(t)| = 3$;
  - (IV) $|N_U(v) \cap N_U(t)| = 4$;
  - (V) $|N_U(v) \cap N_U(t)| = 5$; and
  - (VI) $|N_U(v) \cap N_U(t)| = 6$;
- (c-18) $v \in V_{f8}$ and $t \in N_U(v; V_{u8})$;
- (c-19) $v \in V_{u8}$ and $t \in N_U(v; V_{f3})$;
- (c-20) $v \in V_{u8}$ and $t \in N_U(v; V_{u3})$;
- (c-21) $v \in V_{u8}$ and $t \in N_U(v; V_{f4})$;
- (c-22) $v \in V_{u8}$ and $t \in N_U(v; V_{u4})$;
- (c-23) $v \in V_{u8}$ and $t \in N_U(v; V_{f5})$;
- (c-24) $v \in V_{u8}$ and $t \in N_U(v; V_{u5})$;
- (c-25) $v \in V_{u8}$ and $t \in N_U(v; V_{f6})$.
- (c-26) $v \in V_{u8}$ and $t \in N_U(v; V_{u6})$;
- (c-27) $v \in V_{u7}$ and $t \in N_U(v; V_{f7})$;
- (c-28) $v \in V_{u8}$ and $t \in N_U(v; V_{u7})$; and
- (c-29) $v \in V_{u8}$ and $t \in N_U(v; V_{u8})$.

Algorithm 1 Red(G, F)

Input: An instance (G, F).

Output: A reduced instance (G', F') of (G, F); or a message for the infeasibility of (G, F), which evaluates to ∞.

Initialize $(G', F') := (G, F)$;
while (G', F') is not a reduced instance **do**
    if there is a vertex v in (G', F') such that $d(v) \leq 1$ or $d_{F'}(v) \geq 3$ **then**
        return message "Infeasible"
    else if there is a vertex v in (G', F') such that $2 = d(v) > d_{F'}(v)$ **then**
        Let $E^\dagger$ be the set of unforced edges incident to all such vertices;
        set $F' := F' \cup E^\dagger$
    else if there is a vertex v in (G', F') such that $d(v) > d_{F'}(v) = 2$ **then**
        Let $E^\dagger$ be the set of unforced edges incident to all such vertices;
        set $G' := G' - E^\dagger$
    end if
end while;
return (G', F').

Algorithm 2 tsp8(G, F)

Input: An instance (G, F) such that the maximum degree of G is at most 8.

Output: The minimum cost of a tour of (G, F); or a message for the infeasibility of (G, F), which evaluates to ∞.

Run Red(G, F);
if Red(G, F) returns message "Infeasible" **then**
    return message "Infeasible"
else
    Let $(G', F') :=$ Red(G, F);
    if $V_{u8} \cup V_{f8} \neq \emptyset$ **then**
        Choose an optimal unforced edge e;
        if both tsp8$(G', F' \cup \{e\})$ and tsp8$(G' - \{e\},\ F')$ return message "Infeasible" **then**
            return message "Infeasible"
        else
            return min{tsp8$(G', F' \cup \{e\})$, tsp8$(G' - \{e\},\ F')$}
        end if
    else /* the maximum degree of any vertex in (G', F') is at most 7 */
        return tsp7(G', F')
    end if
end if.

Note: The input and output of algorithm tsp7(G, F) are as follows:
Input: An instance (G, F) such that the maximum degree of G is at most 7.
Output: The minimum cost of a tour of (G, F); or a message for the infeasibility of (G, F), which evaluates to ∞.

4 Analysis

4.1 Analysis Framework

To effectively analyze the running time of our branching algorithm, we use the measure-and-conquer method as introduced by Fomin et al. [3]. Given an instance $I = (G, F)$ of the forced TSP, we assign a nonnegative weight $\omega(v)$ to each vertex $v \in V(G)$ according to its type. To this effect, we set a non-negative vertex weight function $\omega : V \to \mathbb{R}_+$ in the graph G, and we

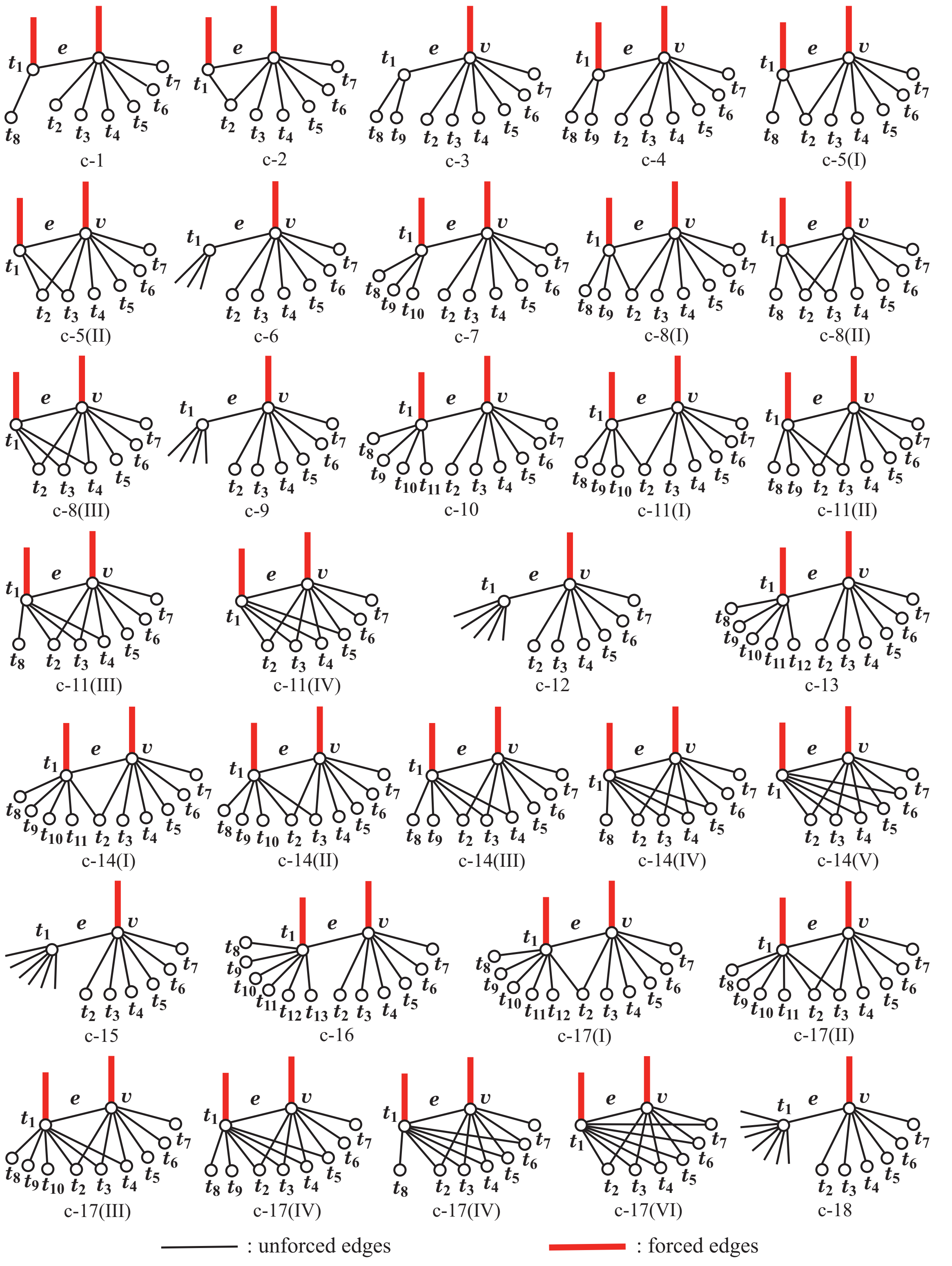


Figure 1: Illustration of the branching rules c-1 to c-18.

use the sum of weights of all vertices in the graph as the measure $\mu(I)$ of instance I, that is,

$$\mu(I) = \sum_{v \in V(G)} \omega(v). \tag{1}$$

It is important for the analysis to find a measure which satisfies the following properties

(i) $\mu(I) = 0$ if and only if I can be solved in polynomial time;

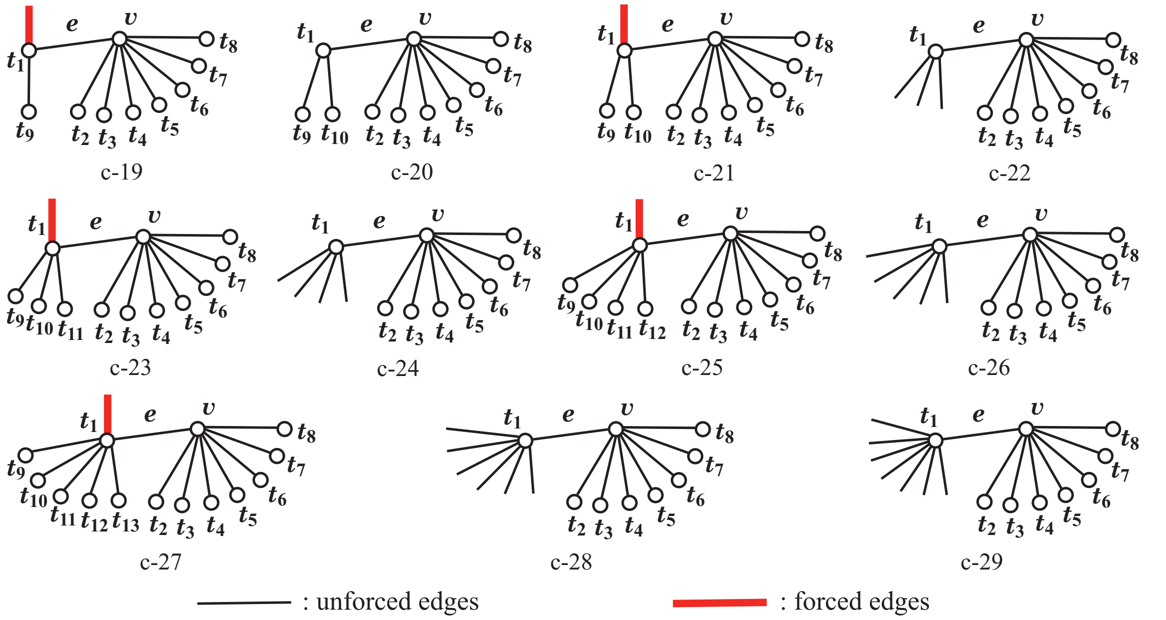


Figure 2: Illustration of the branching rules c-19 to c-29.

(ii) If I' is a sub-instance of I obtained through a reduction or a branching operation, then $\mu(I') \leq \mu(I)$.

We call a measure μ satisfying conditions (i) and (ii) above a *proper measure.*

We perform the time analysis of the branching algorithm via appropriately constructed recurrences over the measure $\mu = \mu(I)$ of an instance $I = (G, F)$, for each branching rule of the algorithm. Let $T(\mu)$ denote the number of nodes in the search tree generated by our algorithm when invoked on the instance I with measure μ. Let I' and I'' be instances obtained from I by a branching operation, and let $a \leq \mu(I) - \mu(I')$ and $b \leq \mu(I) - \mu(I'')$ be lower bounds on the amounts of decrease in the measure. We call (a, b) the *branching vector* of the branching operation, and this implies the linear recurrence

$$T(\mu) \leq T(\mu - a) + T(\mu - b). \tag{2}$$

To evaluate the performance of this branching vector, we can use any standard method for linear recurrence relations. In fact, it is known that $T(\mu)$ is of the form $O(\tau^{\mu})$, where τ is the unique positive real root of the function $f(x) = 1 - \left(x^{-a} + x^{-b}\right)$. The value τ is called the *branching factor* of the branching vector (a, b). The running time of the algorithm is determined by considering the worst branching factor over all branching vectors generated by the branching rules. For further details justifying this approach, as well as a solid introduction to branching algorithms, the reader is referred to the book of Fomin and Kratsch [4].

4.2 Weight Constraints

In order to obtain a measure which will naturally give a running time bound as a function of the size of a TSP instance, we require that the weight of each vertex to be not greater than one. In what follows, we examine some necessary constraints which the vertex weights should satisfy in order for us to obtain a proper measure.

For each $i = 3, 4, \ldots, 8$, we denote by w_i the weight of a ui-vertex, and by w'_i the weight of an fi-vertex. The conditions for a proper measure require that the measure of an instance obtained through a branching or a reduction operation will not be greater than the measure of the original instance. Thus, the vertex weights should satisfy the following relations:

$$w_8 \le 1, \tag{3}$$
$$w'_i \le w_i, \;\; 3 \le i \le 8 \tag{4}$$
$$w_i \le w_j, \;\; 3 \le i < j \le 8, \text{ and} \tag{5}$$
$$w'_i \le w'_j, \;\; 3 \le i < j \le 8. \tag{6}$$

The vertex weight for vertices of degree less than three is set to be zero.

Lemma 1 states that given Algorithms 1 and 2, setting vertex weights which satisfy the conditions of Eqs. (4) to (6) is sufficient to obtain a proper measure. We can prove Lemma 1 in a similar way as Lemma 3 by Md Yunos et al. [8, Lemma 3].

Lemma 1 *If the weights of vertices are chosen as in Eqs.* (4) *to* (6)*, then the measure* $\mu(I)$ *never increases as a result of the reduction or the branching operations of Algorithm 1 and Algorithm 2.*

To simplify some arguments and the list of the branching vectors we are about to derive, we introduce the following notation:

$$\Delta_i = w_i - w'_i, \;\; 3 \le i \le 8$$
$$\Delta_{i,j} = w_i - w_j, \;\; 3 \le j < i \le 8, \text{ and}$$
$$\Delta'_{i,j} = w'_i - w'_j, \;\; 3 \le j < i \le 8$$

further,

$$m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{7}$$
$$m_2 = \min\{w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{8}$$
$$m_3 = \min\{w'_3, \Delta_3, w'_4, \Delta_4, w'_5, \Delta_5, w'_6, \Delta_6, w'_7, \Delta_7, w'_8, \Delta_8\}, \tag{9}$$
$$m_4 = \min\{\Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{10}$$
$$m_5 = \min\{w'_4, w_4, \Delta'_{5,3}, \Delta_{5,3}, \Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}, \tag{11}$$
$$m_6 = \min\{\Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{12}$$
$$m_7 = \min\{\Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{13}$$
$$m_8 = \min\{\Delta'_{5,3}, \Delta_{5,3}, \Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}, \tag{14}$$
$$m_9 = \min\{\Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{15}$$
$$m_{10} = \min\{\Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{16}$$
$$m_{11} = \min\{\Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}, \tag{17}$$
$$m_{12} = \min\{\Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{18}$$
$$m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{19}$$
$$m_{14} = \min\{\Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}, \tag{20}$$
$$m_{15} = \min\{\Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}, \tag{21}$$
$$m_{16} = \min\{\Delta'_{8,7}, \Delta_{8,7}\}, \tag{22}$$
$$m_{17} = \min\{\Delta'_{8,6}, \Delta_{8,6}\}, \tag{23}$$
$$m_{18} = \min\{w'_3, \Delta_3, w'_4, \Delta_4, w'_5, \Delta_5, w'_6, \Delta_6, w'_7, \Delta_7, \Delta_8\}, \tag{24}$$
$$m_{19} = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta_{8,7}\}. \tag{25}$$

4.3 Main Result

Let a vertex weight function $\omega(v)$ be chosen as follows:

$$\omega(v) = \begin{cases} w_8 = 1 & \text{for a u8-vertex } v \\ w'_8 = 0.511412 & \text{for an f8-vertex } v \\ w_7 = 0.899136 & \text{for a u7-vertex } v \\ w'_7 = 0.464212 & \text{for an f7-vertex } v \\ w_6 = 0.779985 & \text{for a u6-vertex } v \\ w'_6 = 0.406731 & \text{for an f6-vertex } v \\ w_5 = 0.636671 & \text{for a u5-vertex } v \\ w'_5 = 0.349250 & \text{for an f5-vertex } v \\ w_4 = 0.454189 & \text{for a u4-vertex } v \\ w'_4 = 0.259479 & \text{for an f4-vertex } v \\ w_3 = 0.208484 & \text{for a u3-vertex } v \\ w'_3 = 0.116721 & \text{for an f3-vertex } v \\ \quad 0 & \text{otherwise} \end{cases} \tag{26}$$

The vertex weight function $\omega(v)$ given in Eq. (26) is obtained as a solution to a quasiconvex program, according to the method introduced by Eppstein [1]. All the branching vectors are in fact constraints in the quasiconvex program.

Lemma 2 *If the vertex weight function $\omega(v)$ is set as in Eq.* (26)*, then each of the branching operations in Algorithm 2 has a branching factor not greater than 4.148449.*

A proof of Lemma 2 can be derived analytically by analyzing the branching vectors which result by applying the branching and reduction operations. From Lemma 2, we get our main result as stated in Theorem 1.

Theorem 1 *The TSP in an n-vertex graph G with maximum degree eight can be solved in $O^*(4.1485^n)$ time and polynomial space.*

In the remainder of the analysis, for an optimal edge $e = vt_1$, we refer to $N_U(v)$ by $\{t_1, t_2, \ldots, t_a\}$, $a = d_U(v)$, and to $N_U(t_1) \setminus \{v\}$ by $\{t_{a+1}, t_{a+2}, \ldots, t_{a+b}\}$, $b = d_U(t_1) - 1$. We assume without loss of generality that $t_{1+i} = t_{a+i}$ for $i = 1, 2, \ldots, c$, where $c = |N_U(v) \cap N_U(t_1)|$ is the number of common neighbors of v and t_1.

If there exists an f3-vertex t_{a+i} in $N_U(t_1) \setminus \{v\}$, let $x \in N_U(t_{a+i}) \setminus \{v, t_1\}$. We see that the choice of vertex x is unique, because t_{a+i} is of type f3 and $|N_U(t_{a+i}) \setminus \{v, t_1\}| = 1$. This vertex x will plays a key role in our analysis, as shown in Fig. 3.

4.4 Branching on Edges around f8-vertices (c-1 to c-18)

This subsection will show how we derive the branching vectors for the branching operations on an optimal edge $e = vt_1$, incident to a forced vertex v of degree eight, distinguishing the 18 cases for conditions c-1 to c-18. We analyze the branching vectors in a similar manner with the analysis of the algorithm for the TSP in degree-5 graphs by Md Yunos et al. [8].

Case c-1. There exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f3}})$ such that $N_U(v) \cap N_U(t_1) = \emptyset$ (see Figure 4): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.

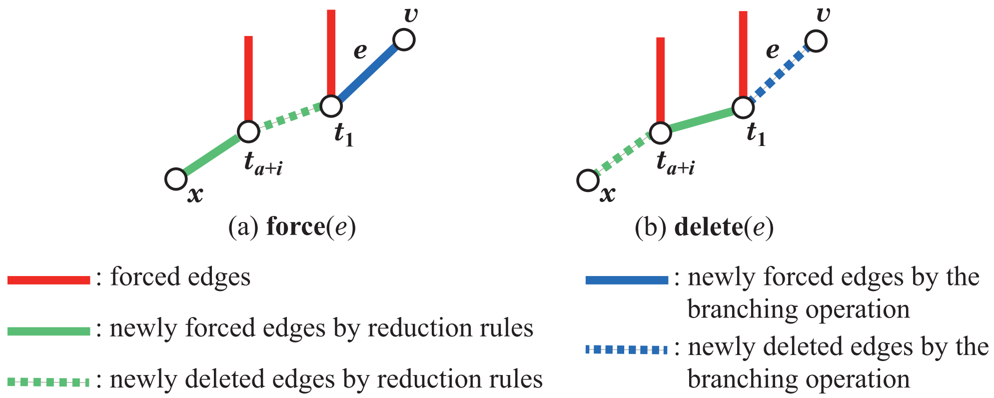


Figure 3: Illustration of (a) newly forced and (b) deleted edge by a branching operation and reduction rules for an f3 vertex t_{a+i}.

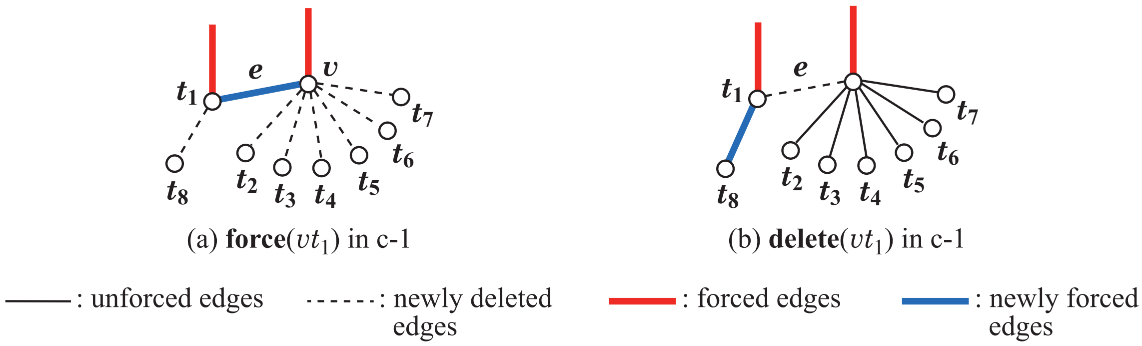


Figure 4: Illustration of branching rule c-1, where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f3}})$.

In the branch of $\mathbf{force}(vt_1)$, the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, and t_1t_8 will be deleted from G' by the reduction rules. Both v and t_1 will become vertices of degree two. From Eq. (26), the weight of vertices of degree two is zero. So the weight of vertex v decreases by w'_8 and the weight of vertex t_1 decreases by w'_3. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$.

In the branch of $\mathbf{delete}(vt_1)$, the edge vt_1 will be deleted from G' by the branching operation, and the edge t_1t_8 will be added to F' by the reduction rules. The weight of vertex v decreases by $\Delta'_{8,7}$ and the weight of vertex t_1 decreases by w'_3.

There are two cases for vertex t_8; 1) vertex t_8 is of type f3, and 2) otherwise. We will analyze these two cases separately for each of branches $\mathbf{force}(vt_1)$ and $\mathbf{delete}(vt_1)$.

First, we will analyze the case where vertex t_8 is an f3-vertex (see Figure 3). Recall that in this case, we denote by x the unique vertex in $N_U(t_8) \setminus \{t_1\}$. In the branch of $\mathbf{force}(vt_1)$, edge xt_8 will be added to F' by the reduction rules. Hence the weight of vertex t_8 decreases by w'_3. If vertex x is an f3-vertex (resp., u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease α_1 of vertex x will be w'_3 (resp., Δ_3, w'_4, Δ_4, w'_5, Δ_5, w'_6, Δ_6, w'_7, Δ_7, w'_8, and Δ_8). Thus the total weight decrease for this case in the branch of $\mathbf{force}(vt_1)$ is at least $w'_8 + w'_3 + w'_3 + 6m_1 + \alpha_1$.

In the branch of $\mathbf{delete}(vt_1)$, edge xt_8 will be deleted from G' be the reduction rules. Hence the weight of vertex t_8 decreases by w'_3. If vertex x is an f3-vertex (resp., u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease β_1 of vertex x will be w'_3 (resp., w_3, $\Delta'_{4,3}$, $\Delta_{4,3}$, $\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, and $\Delta_{8,7}$). Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_3 + w'_3 + \beta_1$.

As a result, for the ordered pair (α_1, β_1) taking values in $\{(w'_3, w'_3), (\Delta_3, w_3), (w'_4, \Delta'_{4,3}), (\Delta_4, \Delta_{4,3}), (w'_5, \Delta'_{5,4}), (\Delta_5, \Delta_{5,4}), (w'_6, \Delta'_{6,5}), (\Delta_6, \Delta_{6,5}), (w'_7, \Delta'_{7,6}), (\Delta_7, \Delta_{7,6}), (w'_8, \Delta'_{8,7}), (\Delta_8, \Delta_{8,7})\}$, we get the following 12 branching vectors:

$$(w'_8 + 2w'_3 + 6m_1 + \alpha_1,\ w'_8 - w'_7 + 2w'_3 + \beta_1). \tag{27}$$

Next, we examine the case where vertex t_8 is not an f3-vertex. In the branch of **force**(vt_1), if vertex t_8 is a u3-vertex (resp., f4, u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease α_2 of vertex t_8 will be w_3 (resp., $\Delta'_{4,3}$, $\Delta_{4,3}$, $\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, and $\Delta_{8,7}$). Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_3 + 6m_1 + \alpha_2$.

In the branch of **delete**(vt_1), if vertex t_8 is a u3-vertex (resp., f4, u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease β_2 of vertex t_8 will be Δ_3 (resp., w'_4, Δ_4, w'_5, Δ_5, w'_6, Δ_6, w'_7, Δ_7, w'_8, and Δ_8). Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_3 + \beta_2$.

As a result, for the ordered pair (α_2, β_2) taking values in $\{(w_3, \Delta_3), (\Delta'_{4,3}, w'_4), (\Delta_{4,3}, \Delta_4), (\Delta'_{5,4}, w'_5), (\Delta_{5,4}, \Delta_5), (\Delta'_{6,5}, w'_6), (\Delta_{6,5}, \Delta_6), (\Delta'_{7,6}, w'_7), (\Delta_{7,6}, \Delta_7), (\Delta'_{8,7}, w'_8), (\Delta_{8,7}, \Delta_8)\}$, we get the following 11 branching vectors:

$$(w'_8 + w'_3 + 6m_1 + \alpha_2,\ w'_8 - w'_7 + w'_3 + \beta_2). \tag{28}$$

Case c-2. Case c-1 is not applicable, and there exist vertices $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{f3}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 5). We branch on the edge vt_1.

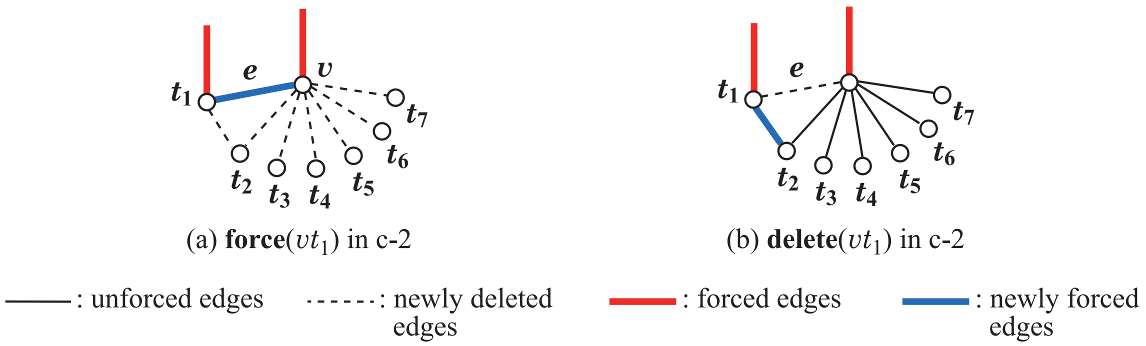


Figure 5: Illustration of branching rule c-2, where vertex $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{f3}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7 and t_1t_2 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_3. Each of the vertices t_3, t_4, t_5, t_6 and t_7 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$.

If vertex t_2 is an f3 or a u3-vertex, after applying the branching operation, t_2 would become a vertex of degree one. From Observation 1, case (i), this is infeasible, and the algorithm will return a message of infeasibility and terminate. Otherwise, if vertex t_2 is an f4-vertex (resp., u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease α_3 of vertex t_2 will be w'_4 (resp., w_4, $\Delta'_{5,3}$, $\Delta_{5,3}$, $\Delta'_{6,4}$, $\Delta_{6,4}$, $\Delta'_{7,5}$, $\Delta_{7,5}$, $\Delta'_{8,6}$, and $\Delta_{8,6}$). Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_3 + 5m_1 + \alpha_3$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation, and the edge t_1t_2 will be added to F' by the reduction rules. So the weights of vertices v and t_1 decrease by $\Delta'_{8,7}$ and w'_3, respectively. If vertex t_2 is an f4-vertex (resp., u4, f5, u5, f6, u6, f7, u7, f8, or a u8-vertex), then the weight decrease β_3 of vertex t_2 will be w'_4 (resp., Δ_4, w'_5, Δ_5, w'_6, Δ_6, w'_7, Δ_7, w'_8, and Δ_8). Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_3 + \beta_3$.

As a result, for the ordered pair (α_3, β_3) taking values in $\{(w'_4, w'_4)$, (w_4, Δ_4), $(\Delta'_{5,3}, w'_5)$, $(\Delta_{5,3}, \Delta_5)$, $(\Delta'_{6,4}, w'_6)$, $(\Delta_{6,4}, \Delta_6)$, $(\Delta'_{7,5}, w'_7)$, $(\Delta_{7,5}, \Delta_7)$, $(\Delta'_{8,6}, w'_8)$, $(\Delta_{8,6}, \Delta_8)\}$, we get the following 10 branching vectors:

$$(w'_8 + w'_3 + 5m_1 + \alpha_3,\ w'_8 - w'_7 + w'_3 + \beta_3). \tag{29}$$

Case c-3. Case c-1 and case c-2 are not applicable, and there exist vertices $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{u3}})$ (see Figure 6): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

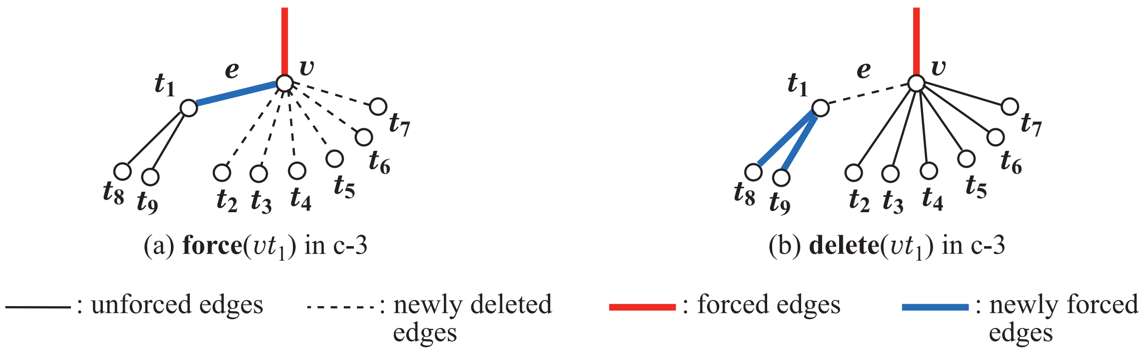


Figure 6: Illustration of branching rule c-3, where vertex $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{u3}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by Δ_3. None of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can be an f3-vertex because it would have been chosen as an optimal edge in some previous case. Hence, each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_2 = \min\{w_3$, $\Delta'_{4,3}$, $\Delta_{4,3}$, $\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, $\Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w_3 - w'_3 + 6m_2$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation, and edges t_1t_8 and t_1t_9 will be added to F' by the reduction rules. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by w_3. Each of vertices t_8 and t_9 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_3 = \min\{w'_3$, Δ_3, w'_4, Δ_4, w'_5, Δ_5, w'_6, Δ_6, w'_7, Δ_7, w'_8, $\Delta_8\}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w_3 + 2m_3$.

As a result, we get the following branching vector:

$$(w'_8 + w_3 - w'_3 + 6m_2,\ w'_8 - w'_7 + w_3 + 2m_3). \tag{30}$$

Case c-4. None of the previous cases are applicable, and there exist vertices $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f4})$ (see Figure 7): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

(a) **force**(vt_1) in c-4 (b) **delete**(vt_1) in c-4

: unforced edges : newly deleted edges : forced edges : newly forced edges

Figure 7: Illustration of branching rule c-4, where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f4})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_8, and t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_4. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_4 = \min\{\Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8 and t_9 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_4 + 6m_4 + 2m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{4,3}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_4 - w'_3$.

As a result, we get the following branching vector:

$$(w'_8 + w'_4 + 6m_4 + 2m_1,\ w'_8 - w'_7 + w'_4 - w'_3). \tag{31}$$

Case c-5. None of the previous cases are applicable, and there exist vertices $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f4})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: We distinguish two sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$,

(c-5(I)) $|N_U(v) \cap N_U(t_1)| = 1$; and

(c-5(II)) $|N_U(v) \cap N_U(t_1)| = 2$.

Case c-5(I). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 8): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2 and t_1t_8 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_4. Vertex t_2 can only be one of types f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least $m_5 = \min\{w'_4, w_4, \Delta'_{5,3}, \Delta_{5,3}, \Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_3, t_4, t_5, t_6, and t_7 can only be one of types f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_4 = \min\{\Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Vertex t_8 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least

(a) **force**(vt_1) in c-5(I) (b) **delete**(vt_1) in c-5(I)

——: unforced edges ------: newly deleted edges ▬: forced edges ▬: newly forced edges

Figure 8: Illustration of branching rule c-5(I), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f4})$.

$m_1 = \min\{w_3', w_3, \Delta_{4,3}', \Delta_{4,3}, \Delta_{5,4}', \Delta_{5,4}, \Delta_{6,5}', \Delta_{6,5}, \Delta_{7,6}', \Delta_{7,6}, \Delta_{8,7}', \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8' + w_4' + m_5 + 5m_4 + m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}'$, and the weight of vertex t_1 decreases by $\Delta_{4,3}'$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8' - w_7' + w_4' - w_3'$.

As a result, we get the following branching vector:

$$(w_8' + w_4' + m_5 + 5m_4 + m_1,\ w_8' - w_7' + w_4' - w_3'). \tag{32}$$

Case c-5(II). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 9): We branch on the edge vt_1.

(a) **force**(vt_1) in c-5(II) (b) **delete**(vt_1) in c-5(II)

——: unforced edges ------: newly deleted edges ▬: forced edges ▬: newly forced edges

Figure 9: Illustration of branching rule c-5(II), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f4})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2 and t_1t_3 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w_8', and the weight of vertex t_1 decreases by w_4'. Each of the vertices t_2 and t_3 can only be one of types f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_5 = \min\{w_4', w_4, \Delta_{5,3}', \Delta_{5,3}, \Delta_{6,4}', \Delta_{6,4}, \Delta_{7,5}', \Delta_{7,5}, \Delta_{8,6}', \Delta_{8,6}\}$. Each of the vertices t_4, t_5, t_6, and t_7 can only be one of types f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_4 = \min\{\Delta_{4,3}', \Delta_{4,3}, \Delta_{5,4}', \Delta_{5,4}, \Delta_{6,5}', \Delta_{6,5}, \Delta_{7,6}', \Delta_{7,6}, \Delta_{8,7}', \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8' + w_4' + 2m_5 + 4m_4$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}'$, and the weight of vertex t_1 decreases by $\Delta_{4,3}'$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8' - w_7' + w_4' - w_3'$.

As a result, we get the following branching vector:

$$(w_8' + w_4' + 2m_5 + 4m_4,\ w_8' - w_7' + w_4' - w_3'). \tag{33}$$

Case c-6. None of the previous cases are applicable, and there exist vertices $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{u4}})$ (see Figure 10): We branch on the edge vt_1.

(a) **force**(vt_1) in c-6 (b) **delete**(vt_1) in c-6

——: unforced edges ------: newly deleted edges ——: forced edges ——: newly forced edges

Figure 10: Illustration of branching rule c-6, where vertex $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{u4}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w_8', and the weight of vertex t_1 decreases by Δ_4. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types u4, f5, u5, f6, u6, f7, u7 f8, and a u8-vertex, and each of their weights decreases by at least $m_6 = \min\{\Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8' + w_4 - w_4' + 6m_6$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{4,3}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8' - w_7' + w_4 - w_3$.

As a result, we get the following branching vector:

$$(w_8' + w_4 - w_4' + 6m_6,\ w_8' - w_7' + w_4 - w_3). \tag{34}$$

Case c-7. None of the previous cases are applicable, and there exist vertices $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{f5}})$ (see Figure 11): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}\}$.

(a) **force**(vt_1) in c-7 (b) **delete**(vt_1) in c-7

——: unforced edges ------: newly deleted edges ——: forced edges ——: newly forced edges

Figure 11: Illustration of branching rule c-7, where vertex $v \in V_{\mathrm{f8}}$ and $t_1 \in N_U(v; V_{\mathrm{f5}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_8, t_1t_9, and t_1t_{10} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_5. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types f5, u5, f6, u6, f7, u7 f8, and a u8-vertex, and each of their weights decreases by at least $m_7 = \min\{\Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8, t_9 and t_{10} can be any of the types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_5 + 6m_7 + 3m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{5,4}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_5 - w'_4$.

As a result, we get the following branching vector:

$$(w'_8 + w'_5 + 6m_7 + 3m_1,\ w'_8 - w'_7 + w'_5 - w'_4). \tag{35}$$

Case c-8. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f5}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: We distinguish three sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$,

(c-8(I)) $|N_U(v) \cap N_U(t_1)| = 1$;

(c-8(II)) $|N_U(v) \cap N_U(t_1)| = 2$; and

(c-8(III)) $|N_U(v) \cap N_U(t_1)| = 3$.

Case c-8(I). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 12): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

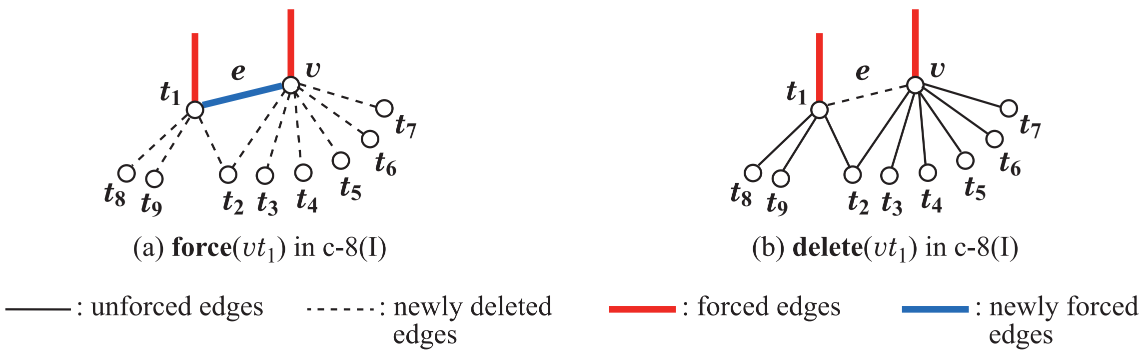


Figure 12: Illustration of branching rule c-8(I), where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f5}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_8 and t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_5. Vertex t_2 can only be one of types f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least $m_8 = \min\{\Delta'_{5,3}, \Delta_{5,3}, \Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_3, t_4, t_5, t_6 and t_7 can only be one of types f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_7 = \min\{\Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8 and t_9 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_5 + m_8 + 5m_7 + 2m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{5,4}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_5 - w'_4$.

As a result, we get the following branching vector:

$$(w'_8 + w'_5 + m_8 + 5m_7 + 2m_1,\ w'_8 - w'_7 + w'_5 - w'_4). \tag{36}$$

Case c-8(II). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 13): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.

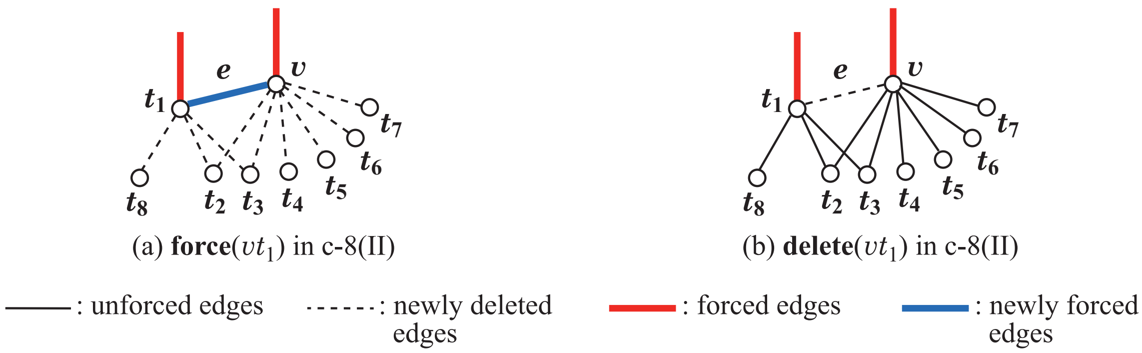


Figure 13: Illustration of branching rule c-8(II), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f5})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3 and t_1t_8 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_5. Each of the vertices t_2 and t_3 can only be one of types f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_8 = \min\{\Delta'_{5,3},\ \Delta_{5,3},\ \Delta'_{6,4},\ \Delta_{6,4},\ \Delta'_{7,5},\ \Delta_{7,5},\ \Delta'_{8,6},\ \Delta_{8,6}\}$. Each of the vertices t_4, t_5, t_6 and t_7 can only be one of types f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_7 = \min\{\Delta'_{5,4},\ \Delta_{5,4},\ \Delta'_{6,5},\ \Delta_{6,5},\ \Delta'_{7,6},\ \Delta_{7,6},\ \Delta'_{8,7},\ \Delta_{8,7}\}$. Vertex t_8 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least $m_1 = \min\{w'_3,\ w_3,\ \Delta'_{4,3},\ \Delta_{4,3},\ \Delta'_{5,4},\ \Delta_{5,4},\ \Delta'_{6,5},\ \Delta_{6,5},\ \Delta'_{7,6},\ \Delta_{7,6},\ \Delta'_{8,7},\ \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_5 + 2m_8 + 4m_7 + m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{5,4}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_5 - w'_4$.

As a result, we get the following branching vector:

$$(w'_8 + w'_5 + 2m_8 + 4m_7 + m_1,\ w'_8 - w'_7 + w'_5 - w'_4). \tag{37}$$

Case c-8(III). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 14): We branch on the edge vt_1.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3 and t_1t_4 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_5. Each of the vertices t_2, t_3 and t_4 can only be one of types f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_8 = \min\{\Delta'_{5,3},\ \Delta_{5,3},\ \Delta'_{6,4},\ \Delta_{6,4},\ \Delta'_{7,5},\ \Delta_{7,5},\ \Delta'_{8,6},\ \Delta_{8,6}\}$. Each of the vertices t_5, t_6 and t_7 can only be one of types f5, u5, f6,

Figure 14: Illustration of branching rule c-8(III), where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f5}})$.

u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_7 = \min\{\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, $\Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_5 + 3m_8 + 3m_7$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{5,4}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_5 - w'_4$.

As a result, we get the following branching vector:

$$(w'_8 + w'_5 + 3m_8 + 3m_7,\ w'_8 - w'_7 + w'_5 - w'_4). \tag{38}$$

Case c-9. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u5}})$ (see Figure 15): We branch on the edge vt_1.

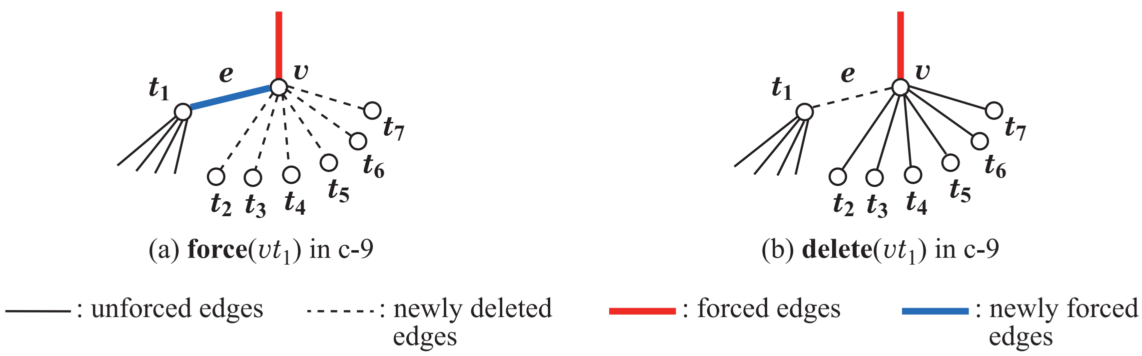


Figure 15: Illustration of branching rule c-9, where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u5}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by Δ_5. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_9 = \min\{\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, $\Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w_5 - w'_5 + 6m_9$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{5,4}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w_5 - w_4$.

As a result, we get the following branching vector:

$$(w'_8 + w_5 - w'_5 + 6m_9,\ w'_8 - w'_7 + w_5 - w_4). \tag{39}$$

Case c-10. None of the previous cases are applicable, and there exist vertices $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f6})$ (see Figure 16): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}\}$.

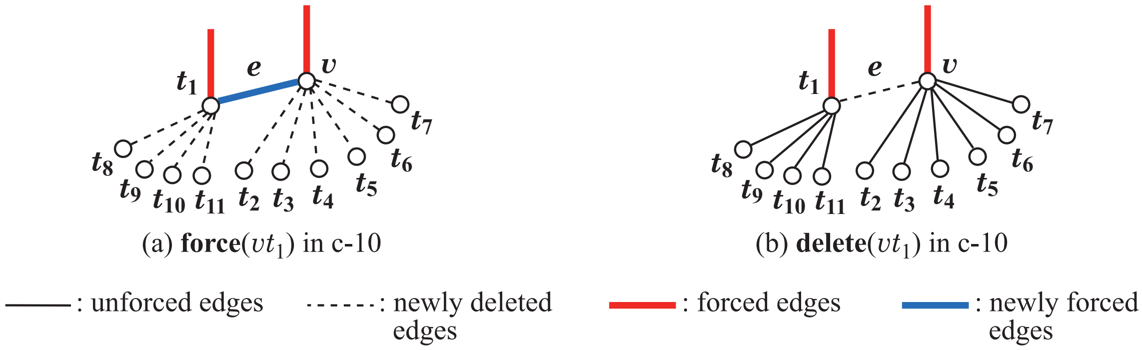


Figure 16: Illustration of branching rule c-10, where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f6})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_8, t_1t_9, t_1t_{10} and t_1t_{11} will be deleted from G by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_6. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types f6, u6, f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{10} = \min\{\Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8, t_9, t_{10} and t_{11} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_6 + 6m_{10} + 4m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_6 - w'_5$.

As a result, we get the following branching vector:

$$(w'_8 + w'_6 + 6m_{10} + 4m_1,\ w'_8 - w'_7 + w'_6 - w'_5). \tag{40}$$

Case c-11. None of the previous cases are applicable, and there exist vertices $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f6})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: We distinguish four sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$,

(c-11(I)) $|N_U(v) \cap N_U(t_1)| = 1$;

(c-11(II)) $|N_U(v) \cap N_U(t_1)| = 2$;

(c-11(III)) $|N_U(v) \cap N_U(t_1)| = 3$; and

(c-11(IV)) $|N_U(v) \cap N_U(t_1)| = 4$.

Case c-11(I). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 17): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_8, t_1t_9, t_1t_{10} will be deleted from G' by the

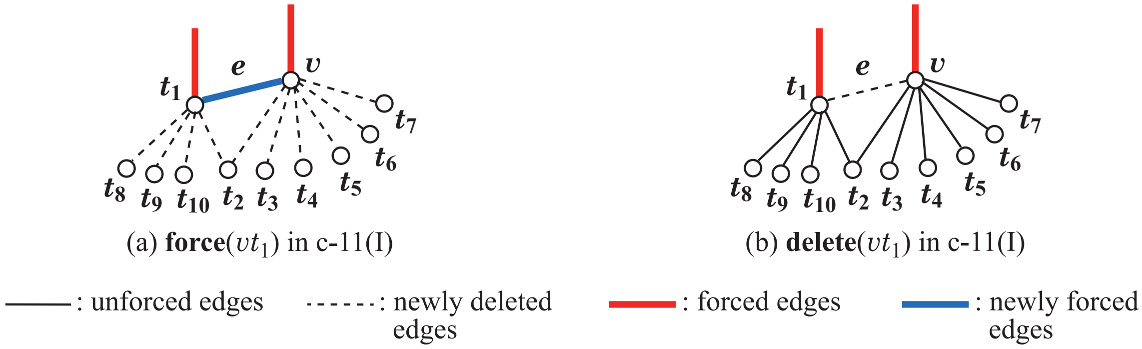


Figure 17: Illustration of branching rule c-11(I), where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f6}})$.

reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_6. Vertex t_2 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least $m_{11} = \min\{\Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_3, t_4, t_5, t_6 and t_7 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{10} = \min\{\Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8, t_9 and t_{10} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_6 + m_{11} + 5m_{10} + 3m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_6 - w'_5$.

As a result, we get the following branching vector:

$$(w'_8 + w'_6 + m_{11} + 5m_{10} + 3m_1,\ w'_8 - w'_7 + w'_6 - w'_5). \tag{41}$$

Case c-11(II). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 18): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

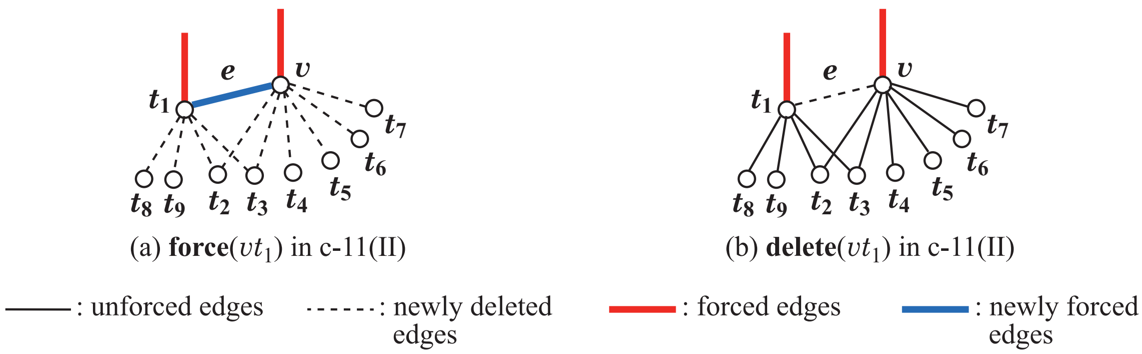


Figure 18: Illustration of branching rule c-11(II), where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f6}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_8 and t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_6. Each of the vertices t_2 and t_3 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{11} = \min\{\Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_4, t_5, t_6 and t_7 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{10} = \min\{\Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8 and t_9 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_6 + 2m_{11} + 4m_{10} + 2m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_6 - w'_5$.

As a result, we get the following branching vector:

$$(w'_8 + w'_6 + 2m_{11} + 4m_{10} + 2m_1,\ w'_8 - w'_7 + w'_6 - w'_5). \tag{42}$$

Case c-11(III). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 19): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.

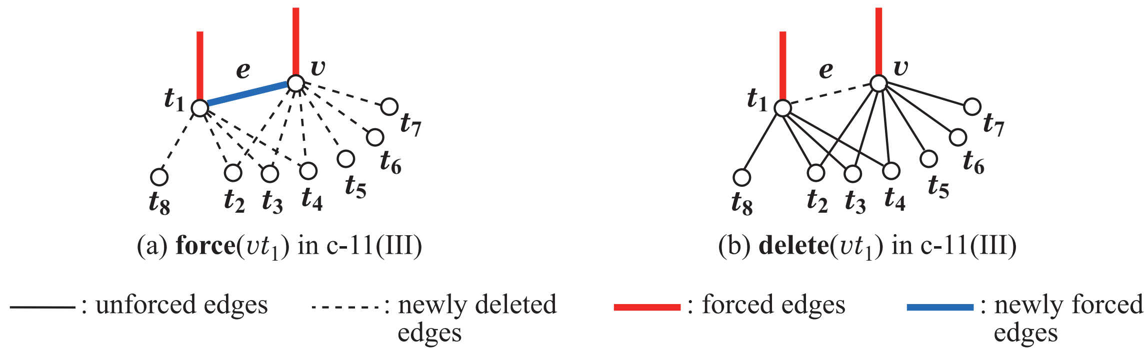


Figure 19: Illustration of branching rule c-11(III), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f6})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4 and t_1t_8 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_6. Each of the vertices t_2, t_3 and t_4 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{11} = \min\{\Delta'_{6,4}, \Delta_{6,4}, \Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_5, t_6 and t_7 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{10} = \min\{\Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Vertex t_8 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and its weight decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_6 + 3m_{11} + 3m_{10} + m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_6 - w'_5$.

As a result, we get the following branching vector:

$$(w'_8 + w'_6 + 3m_{11} + 3m_{10} + m_1,\ w'_8 - w'_7 + w'_6 - w'_5). \tag{43}$$

Case c-11(IV). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5\}$ (see Figure 20): We branch on the edge vt_1.

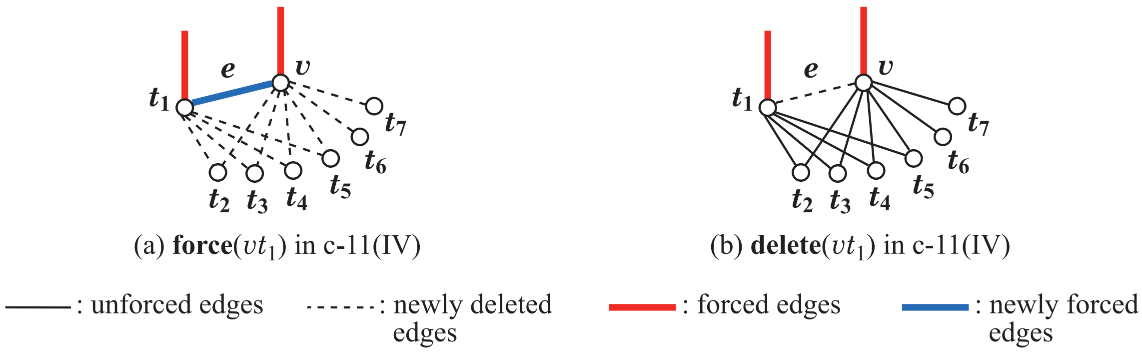


Figure 20: Illustration of branching rule c-11(IV), where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f6}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4 and t_1t_5 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_6. Each of the vertices t_2, t_3, t_4 and t_5 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{11} = \min\{\Delta'_{6,4}$, $\Delta_{6,4}$, $\Delta'_{7,5}$, $\Delta_{7,5}$, $\Delta'_{8,6}$, $\Delta_{8,6}\}$. Each of the vertices t_6 and t_7 can only be one of types f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{10} = \min\{\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, $\Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_6 + 4m_{11} + 2m_{10}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_6 - w'_5$.

As a result, we get the following branching vector:

$$(w'_8 + w'_6 + 4m_{11} + 2m_{10},\ w'_8 - w'_7 + w'_6 - w'_5). \tag{44}$$

Case c-12. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u6}})$ (see Figure 21): We branch on the edge vt_1.

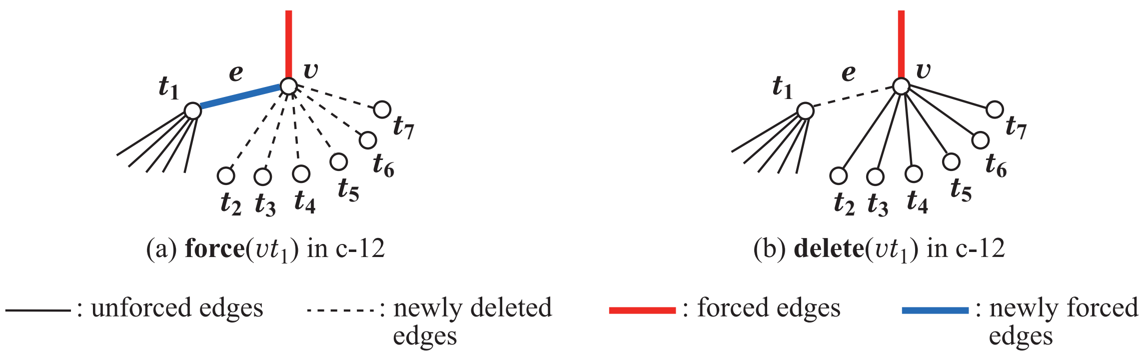


Figure 21: Illustration of branching rule c-12, where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u6}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by Δ_6. Each of vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_{12} = \min\{\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, $\Delta'_{8,7}$, $\Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w_6 - w'_6 + 6m_{12}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{6,5}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w_6 - w_5$.

As a result, we get the following branching vector:

$$(w'_8 + w_6 - w'_6 + 6m_{12},\ w'_8 - w'_7 + w_6 - w_5). \tag{45}$$

Case c-13. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f7}})$ (see Figure 22): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}, t_{12}\}$.

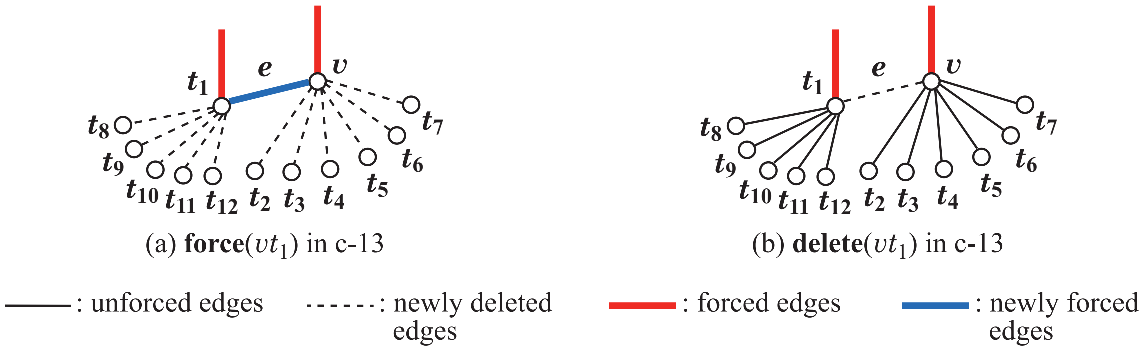


Figure 22: Illustration of branching rule c-13, where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f7}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_8, t_1t_9, t_1t_{10}, t_1t_{11} and t_1t_{12} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8, t_9, t_{10}, t_{11} and t_{12} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + 6m_{13} + 5m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + 6m_{13} + 5m_1,\ w'_8 - w'_6). \tag{46}$$

Case c-14. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f7}})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: We distinguish five sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$,

(c-14(I)) $|N_U(v) \cap N_U(t_1)| = 1$;

(c-14(II)) $|N_U(v) \cap N_U(t_1)| = 2$;

(c-14(III)) $|N_U(v) \cap N_U(t_1)| = 3$;

(c-14(IV)) $|N_U(v) \cap N_U(t_1)| = 4$; and

(c-14(V)) $|N_U(v) \cap N_U(t_1)| = 5$.

Case c-14(I). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 23): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}\}$.

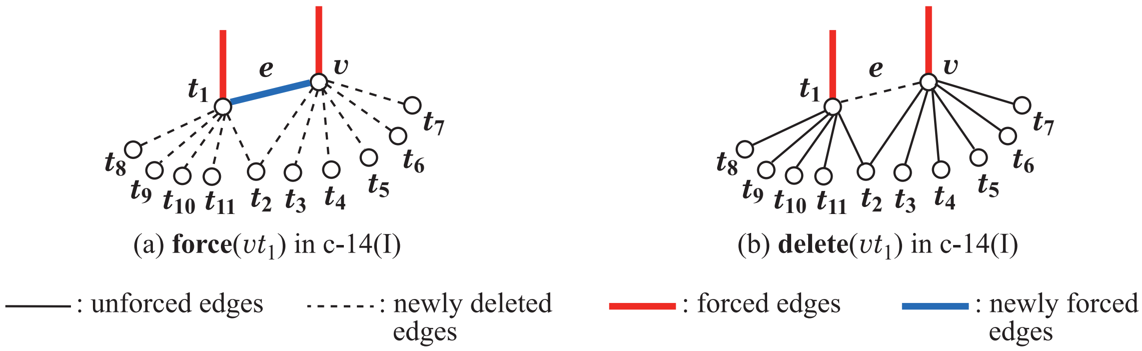


Figure 23: Illustration of branching rule c-14(I), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f7})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_8, t_1t_9, t_1t_{10} and t_1t_{11} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Vertex t_2 can only be one of types f7, u7, f8 and a u8-vertex, and its weight decreases by at least $m_{14} = \min\{\Delta'_{7,5},\ \Delta_{7,5},\ \Delta'_{8,6},\ \Delta_{8,6}\}$. Each of the vertices t_3, t_4, t_5, t_6 and t_7 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{13} = \min\{\Delta'_{7,6},\ \Delta_{7,6},\ \Delta'_{8,7},\ \Delta_{8,7}\}$. Each of the vertices t_8, t_9, t_{10} and t_{11} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8, and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3,\ w_3,\ \Delta'_{4,3},\ \Delta_{4,3},\ \Delta'_{5,4},\ \Delta_{5,4},\ \Delta'_{6,5},\ \Delta_{6,5},\ \Delta'_{7,6},\ \Delta_{7,6},\ \Delta'_{8,7},\ \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + m_{14} + 5m_{13} + 4m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + m_{14} + 5m_{13} + 4m_1,\ w'_8 - w'_6). \tag{47}$$

Case c-14(II). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 24): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}\}$.

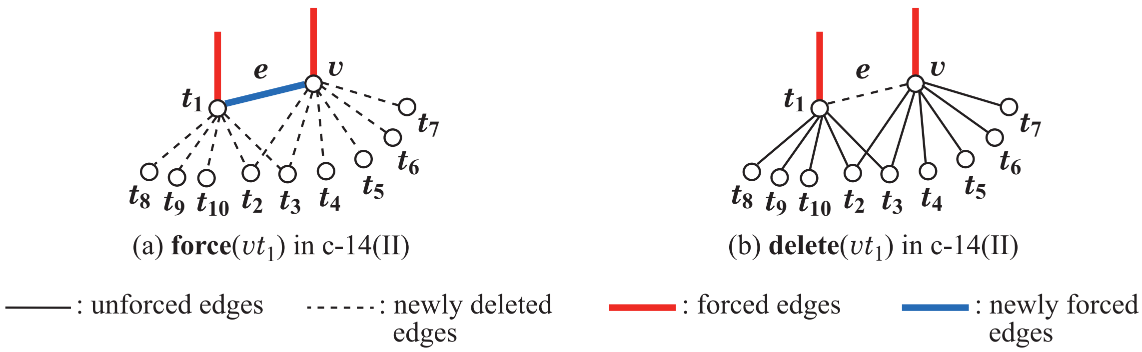


Figure 24: Illustration of branching rule c-14(II), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f7})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_8, t_1t_9 and t_1t_{10} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_2 and t_3 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{14} = \min\{\Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_4, t_5, t_6 and t_7 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8, t_9 and t_{10} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + 2m_{14} + 4m_{13} + 3m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + 2m_{14} + 4m_{13} + 3m_1,\ w'_8 - w'_6). \tag{48}$$

Case c-14(III). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 25): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

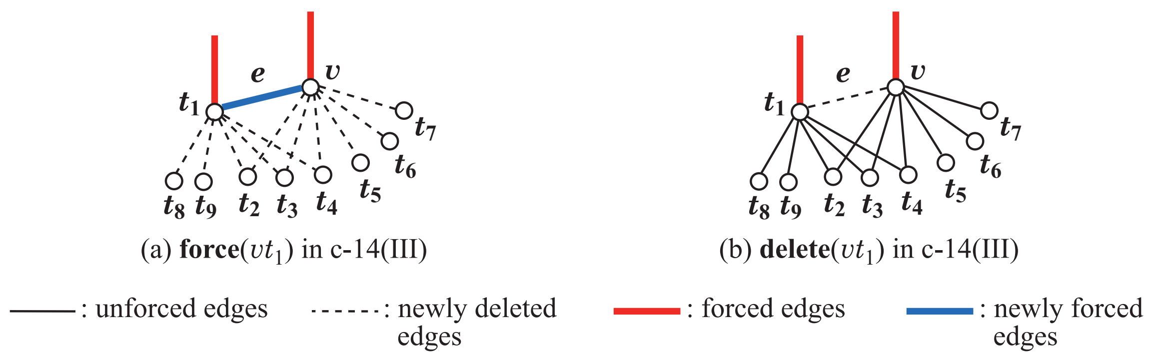


Figure 25: Illustration of branching rule c-14(III), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f7})$.

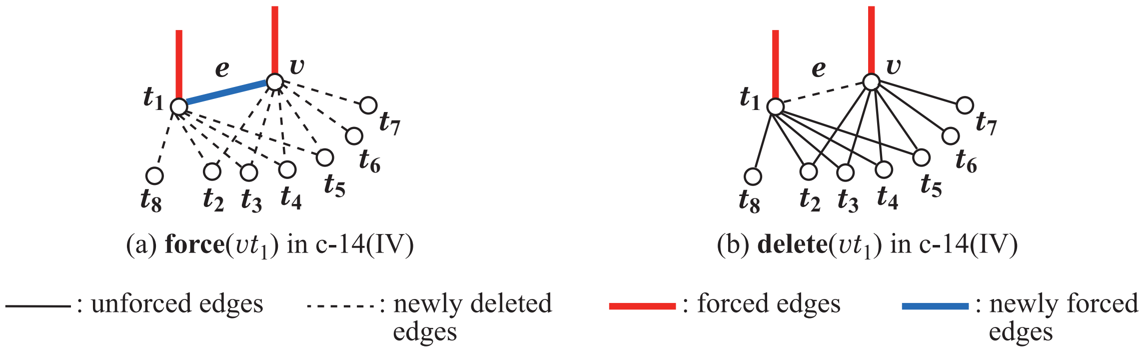
In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_8 and t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_2, t_3 and t_4 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{14} = \min\{\Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_5, t_6 and t_7 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Each of the vertices t_8 and t_9 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + 3m_{14} + 3m_{13} + 2m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + 3m_{14} + 3m_{13} + 2m_1,\ w'_8 - w'_6). \tag{49}$$

Case c-14(IV). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5\}$ (see Figure 26): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.



(a) **force**(vt_1) in c-14(IV)   (b) **delete**(vt_1) in c-14(IV)

Figure 26: Illustration of branching rule c-14(IV), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f7})$.

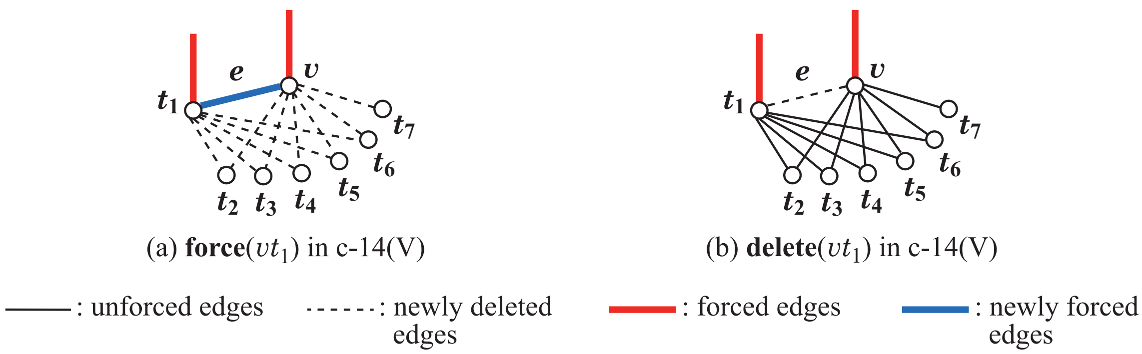
In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_5 and t_1t_8 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_2, t_3, t_4 and t_5 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{14} = \min\{\Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_6 and t_7 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Vertex t_8 can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, f8 and a u8-vertex, and its weight decreases by at least $m_1 = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + 4m_{14} + 2m_{13} + m_1$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + 4m_{14} + 2m_{13} + m_1,\ w'_8 - w'_6). \tag{50}$$

Case c-14(V). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5, t_6\}$ (see Figure 27): We branch on the edge vt_1.



(a) **force**(vt_1) in c-14(V)   (b) **delete**(vt_1) in c-14(V)

Figure 27: Illustration of branching rule c-14(V), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f7})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_5 and t_1t_6 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_2, t_3, t_4 t_5 and t_6 can only be one of types f7, u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{14} = \min\{\Delta'_{7,5}, \Delta_{7,5}, \Delta'_{8,6}, \Delta_{8,6}\}$. Vertex t_7 can only be one of types f7, u7, f8 and a u8-vertex, and its weight decreases by at least $m_{13} = \min\{\Delta'_{7,6}, \Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_7 + 5m_{14} + m_{13}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_6$.

As a result, we get the following branching vector:

$$(w'_8 + w'_7 + 5m_{14} + m_{13},\ w'_8 - w'_6). \tag{51}$$

Case c-15. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u7}})$ (see Figure 28): We branch on the edge vt_1.

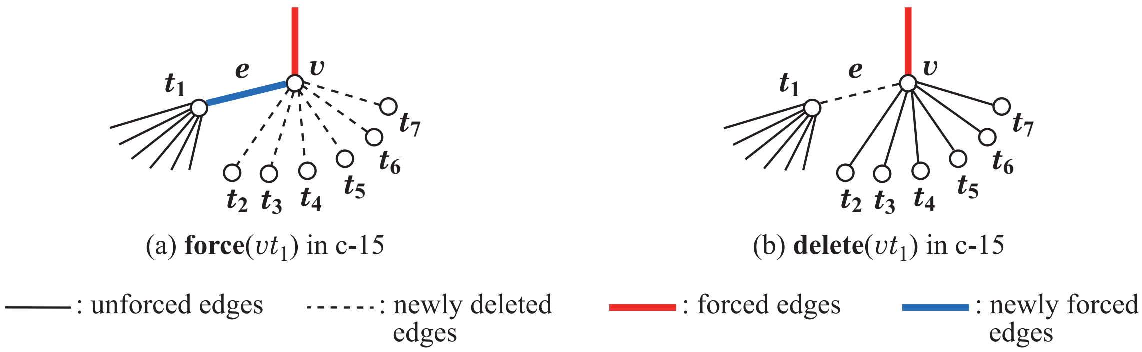


Figure 28: Illustration of branching rule c-15, where vertex $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{u7}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So, the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by Δ_7. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be one of types u7, f8 and a u8-vertex, and each of their weights decreases by at least $m_{15} = \min\{\Delta_{7,6}, \Delta'_{8,7}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w_7 - w'_7 + 6m_{15}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{7,6}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w_7 - w_6$.

As a result, we get the following branching vector:

$$(w'_8 + w_7 - w'_7 + 6m_{15},\ w'_8 - w'_7 + w_7 - w_6). \tag{52}$$

Case c-16. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{f8}}$ and $t_1 \in N_U(v; V_{\text{f8}})$ (see Figure 29): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_8, t_1t_9, t_1t_{10}, t_1t_{11}, t_1t_{12} and t_1t_{13} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight

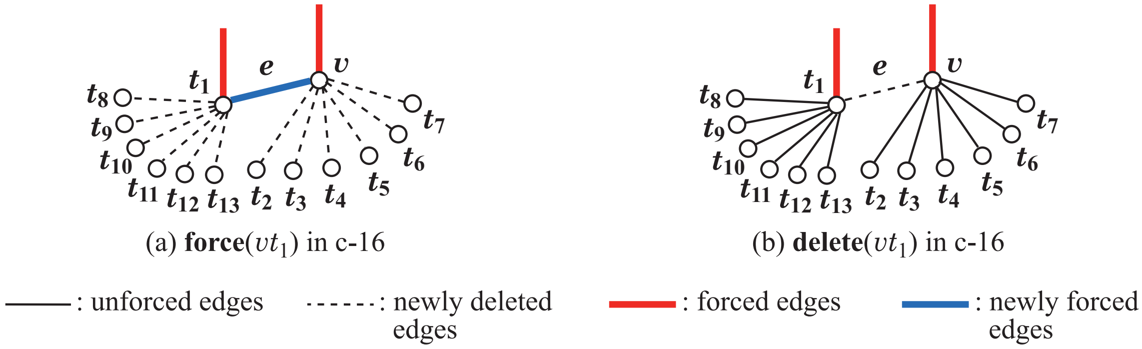


Figure 29: Illustration of branching rule c-16, where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

of vertex t_1 decreases by w'_8. Each of the vertices t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12} and t_{13} can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 12m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + 12m_{16},\ 2w'_8 - 2w'_7). \tag{53}$$

Case c-17. None of the previous cases are applicable, and there exist vertices $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$ such that $N_U(v) \cap N_U(t_1) \neq \emptyset$: We distinguish six sub-cases, according to the cardinality of the intersection $N_U(v) \cap N_U(t_1)$,

(c-14(I)) $|N_U(v) \cap N_U(t_1)| = 1$;
(c-14(II)) $|N_U(v) \cap N_U(t_1)| = 2$;
(c-14(III)) $|N_U(v) \cap N_U(t_1)| = 3$;
(c-14(IV)) $|N_U(v) \cap N_U(t_1)| = 4$;
(c-14(V)) $|N_U(v) \cap N_U(t_1)| = 5$; and
(c-14(VI)) $|N_U(v) \cap N_U(t_1)| = 6$.

Case c-17(I). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2\}$ (see Figure 30): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}, t_{12}\}$.

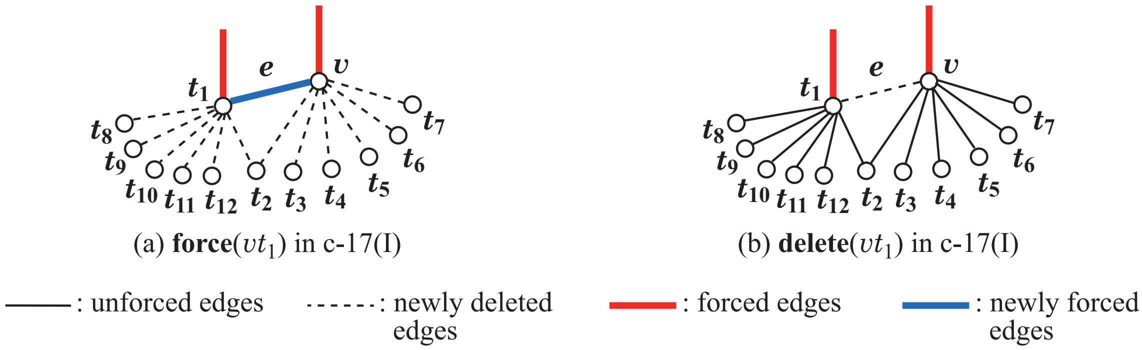


Figure 30: Illustration of branching rule c-17(I), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

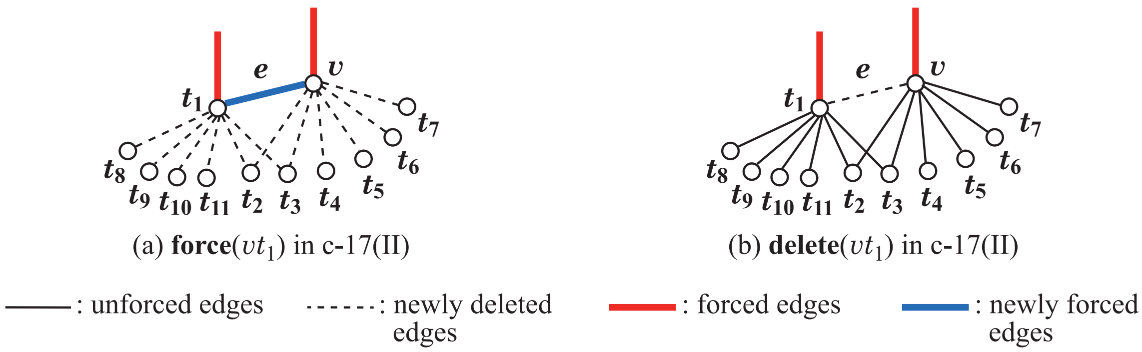
In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_8, t_1t_9, t_1t_{10}, t_1t_{11} and t_1t_{12} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_8. Vertex t_2 can only be either a type f8 or a u8-vertex, and its weight decreases by at least $m_{17} = \min\{\Delta'_{8,6},\ \Delta_{8,6}\}$. Each of the vertices t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11} and t_{12} can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7},\ \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + m_{17} + 10m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + m_{17} + 10m_{16},\ 2w'_8 - 2w'_7). \tag{54}$$

Case c-17(II). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3\}$ (see Figure 31): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}, t_{11}\}$.



(a) **force**(vt_1) in c-17(II) (b) **delete**(vt_1) in c-17(II)

Figure 31: Illustration of branching rule c-17(II), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_8, t_1t_9, t_1t_{10} and t_1t_{11} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_8. Each of the vertices t_2 and t_3 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{17} = \min\{\Delta'_{8,6},\ \Delta_{8,6}\}$. Each of the vertices t_4, t_5, t_6, t_7, t_8, t_9, t_{10} and t_{11} can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7},\ \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 2m_{17} + 8m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + 2m_{17} + 8m_{16},\ 2w'_8 - 2w'_7). \tag{55}$$

Case c-17(III). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4\}$ (see Figure 32): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9, t_{10}\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_8, t_1t_9 and t_1t_{10} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1

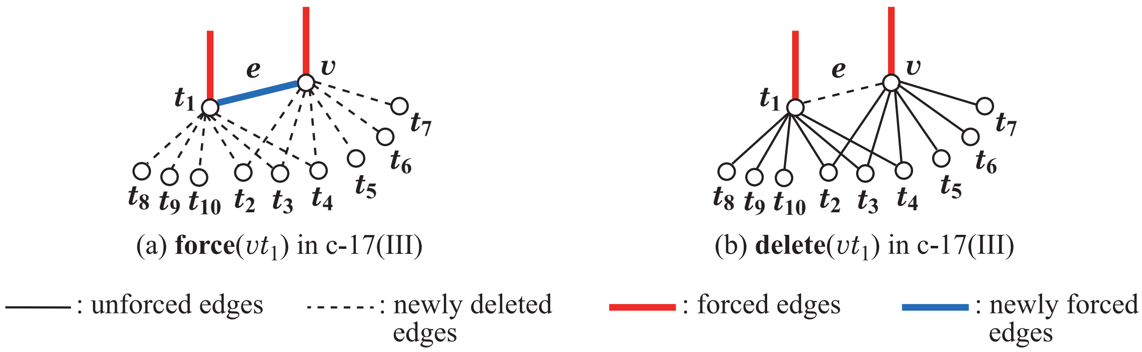


Figure 32: Illustration of branching rule c-17(III), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

decreases by w'_8. Each of the vertices t_2, t_3 and t_4 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{17} = \min\{\Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_5, t_6, t_7, t_8, t_9 and t_{10} can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 3m_{17} + 6m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + 3m_{17} + 6m_{16},\ 2w'_8 - 2w'_7). \tag{56}$$

Case c-17(IV). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5\}$ (see Figure 33): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8, t_9\}$.

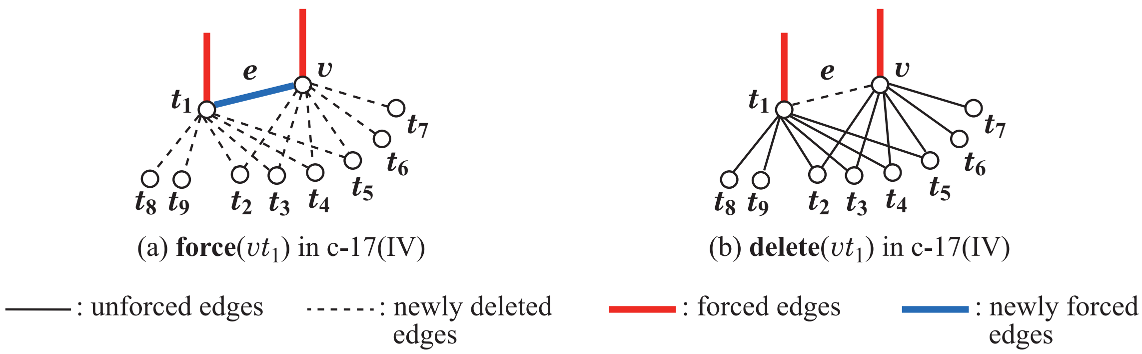


Figure 33: Illustration of branching rule c-17(IV), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_5, t_1t_8 and t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_8. Each of the vertices t_2, t_3, t_4 and t_5 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{17} = \min\{\Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_6, t_7, t_8 and t_9 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 4m_{17} + 4m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + 4m_{17} + 4m_{16},\ 2w'_8 - 2w'_7). \tag{57}$$

Case c-17(V). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5, t_6\}$ (see Figure 34): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_8\}$.

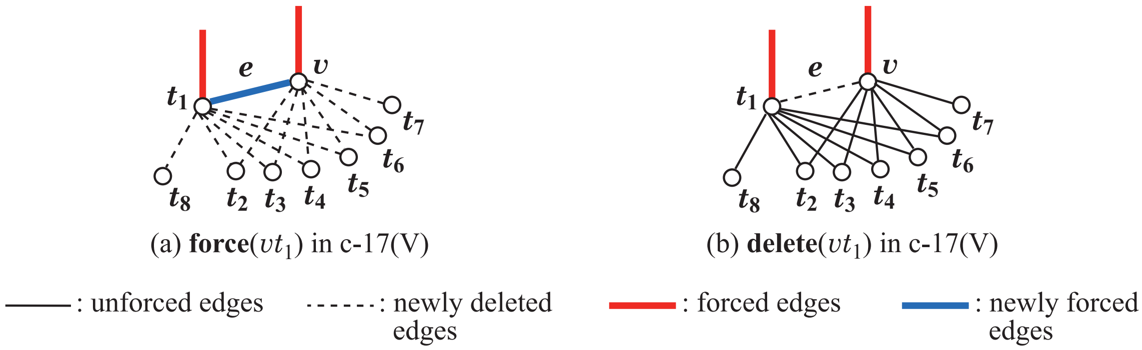


Figure 34: Illustration of branching rule c-17(V), where vertex $v \in V_{f8}$ and $t_1 \in N_U(v; V_{f8})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_5, t_1t_6 and t_1t_8 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_8. Each of the vertices t_2, t_3, t_4, t_5 and t_6 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{17} = \min\{\Delta'_{8,6}, \Delta_{8,6}\}$. Each of the vertices t_7 and t_8 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{16} = \min\{\Delta'_{8,7}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 5m_{17} + 2m_{16}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

As a result, we get the following branching vector:

$$(2w'_8 + 5m_{17} + 2m_{16},\ 2w'_8 - 2w'_7). \tag{58}$$

Case c-17(VI). Without loss of generality, assume that $N_U(v) \cap N_U(t_1) = \{t_2, t_3, t_4, t_5, t_6, t_7\}$ (see Figure 35): We branch on the edge vt_1.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6, vt_7, t_1t_2, t_1t_3, t_1t_4, t_1t_5, t_1t_6 and t_1t_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by w'_8. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be either a type f8 or a u8-vertex, and each of their weights decreases by at least $m_{17} = \min\{\Delta'_{8,6}, \Delta_{8,6}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w'_8 + 6m_{17}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w'_8 - w'_7$.

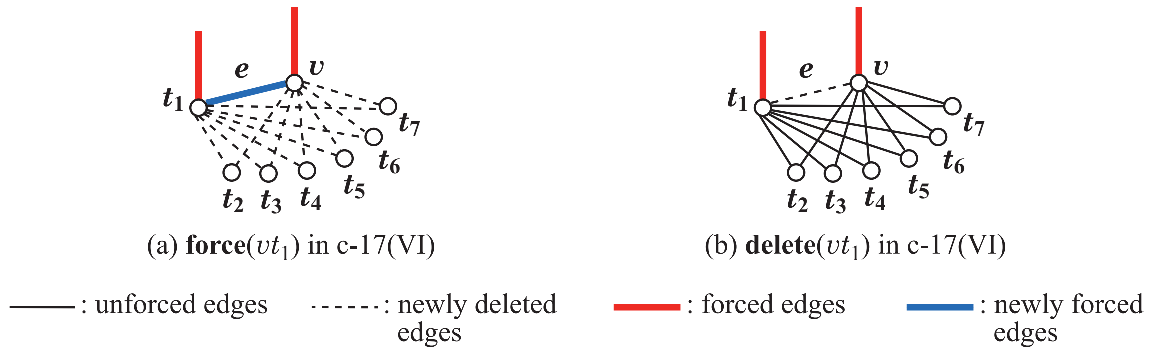


Figure 35: Illustration of branching rule c-17(VI), where vertex $v \in V_{\mathrm{f}8}$ and $t_1 \in N_U(v; V_{\mathrm{f}8})$.

As a result, we get the following branching vector:

$$(2w'_8 + 6m_{17},\ 2w'_8 - 2w'_7). \tag{59}$$

Case c-18. None of the previous cases are applicable, and there exist vertices $v \in V_{\mathrm{f}8}$ and $t_1 \in N_U(v; V_{\mathrm{u}8})$ (see Figure 36): We branch on the edge vt_1.

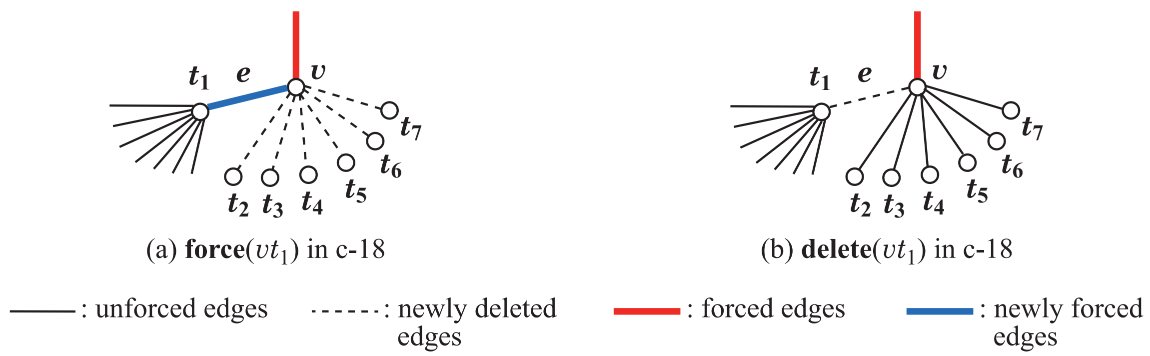


Figure 36: Illustration of branching rule c-18, where vertex $v \in V_{\mathrm{f}8}$ and $t_1 \in N_U(v; V_{\mathrm{u}8})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges vt_2, vt_3, vt_4, vt_5, vt_6 and vt_7 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by w'_8, and the weight of vertex t_1 decreases by Δ_8. Each of the vertices t_2, t_3, t_4, t_5, t_6 and t_7 can only be a u8-vertex, and each of their weights decreases by $\Delta_{8,7}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w'_8 + w_8 - w'_8 + 6(w_8 - w_7)$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta'_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w'_8 - w'_7 + w_8 - w_7$.

As a result, we get the following branching vector:

$$(7w_8 - 6w_7,\ w'_8 - w'_7 + w_8 - w_7). \tag{60}$$

4.5 Branching on Edges around u8-vertices (c-19 to c-29)

If none of the first 18 conditions can be executed, this means that the graph has no f8-vertices. But this does not mean that the maximum degree of the graph has been reduced to seven, since there might still be u8-vertices. This section derives branching vectors for branchings on an optimal edge $e = vt_1$ incident to a u7-vertex v, distinguishing the 11 cases for conditions c-19 to c-29.

Case c-19. There are no more f8-vertices, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{f3})$ (see Figure 37): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_9\}$.

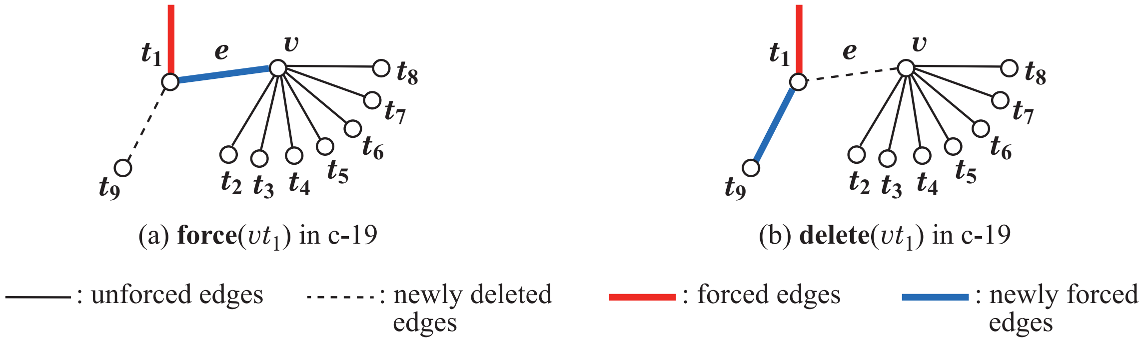


Figure 37: Illustration of branching rule c-19, where vertex $v \in V_{u8}$ and $t_1 \in N_U(v; V_{f3})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and the edge t_1t_9 will be deleted from G' by the reduction rules. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by w'_3.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation, and the edge t_1t_9 will be added to F' by the reduction rules. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by w'_3.

There are two cases for vertex t_9; 1) the vertex t_9 is of type f3,and 2) otherwise. We will analyze these two cases separately for each of branches **force**(vt_1) and **delete**(vt_1).

First, we analyze the case where vertex t_9 is an f3-vertex (see Figure 3). Recall that in this case, we denote by x the unique vertex in $N_U(t_9) \setminus \{t_1\}$. In the branch of **force**(vt_1), the edge xt_9 will be added to F' by the reduction rules. Hence the weight of vertex t_9 decreases by w'_3. If vertex x is an f3-vertex (resp., u3, f4, u4, f5, u5, f6, u6, f7, u7, or a u8-vertex), then the weight decrease α_4 of vertex x will be w'_3 (resp., Δ_3, w'_4, Δ_4, w'_5, Δ_5, w'_6, Δ_6, w'_7, Δ_7, and Δ_8). Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w'_3 + w'_3 + \alpha_4$.

In the branch of **delete**(vt_1), the edge xt_9 will be deleted from G' by the reduction rules. Hence the weight of vertex t_9 decrease by w'_3. If vertex x is an f3-vertex (resp., u3, f4, u4, f5, u5, f6, u6, f7, u7, or a u8-vertex), then the weight decrease β_4 of vertex x will be w'_3 (resp., w_3, $\Delta'_{4,3}$, $\Delta_{4,3}$, $\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, and $\Delta_{8,7}$). Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w'_3 + w'_3 + \beta_4$.

As a result, for the ordered pair (α_4, β_4) taking values in $\{(w'_3, w'_3), (\Delta_3, w_3), (w'_4, \Delta'_{4,3}), (\Delta_4, \Delta_{4,3}), (w'_5, \Delta'_{5,4}), (\Delta_5, \Delta_{5,4}), (w'_6, \Delta'_{6,5}), (\Delta_6, \Delta_{6,5}), (w'_7, \Delta'_{7,6}), (\Delta_7, \Delta_{7,6}), (\Delta_8, \Delta_{8,7})\}$, we get the following 11 branching vectors:

$$(w_8 - w'_8 + 2w'_3 + \alpha_4,\ w_8 - w_7 + 2w'_3 + \beta_4). \tag{61}$$

Next, we examine the case where vertex t_9 is not an f3-vertex. In the branch of **force**(vt_1), if vertex t_9 is a u3-vertex (resp., f4, u4, f5, u5, f6, u6, f7, u7, or a u8-vertex), then the weight decrease α_5 of vertex t_9 will be w_3 (resp., $\Delta'_{4,3}$, $\Delta_{4,3}$, $\Delta'_{5,4}$, $\Delta_{5,4}$, $\Delta'_{6,5}$, $\Delta_{6,5}$, $\Delta'_{7,6}$, $\Delta_{7,6}$, and $\Delta_{8,7}$). Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w'_3 + \alpha_5$.

In the branch of **delete**(vt_1), if vertex t_9 is a u3-vertex (resp., f4, u4, f5, u5, f6, u6, f7, u7, or a u8-vertex), then the weight decrease α_5 of vertex t_9 will be Δ_3 (resp., w'_4, Δ_4, w'_5,

Δ_5, w'_6, Δ_6, w'_7, Δ_7, and Δ_8). Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w'_3 + \beta_5$.

As a result, for the ordered pair (α_5, β_5) taking values in $\{(w_3, \Delta_3), (\Delta'_{4,3}, w'_4), (\Delta_{4,3}, \Delta_4), (\Delta'_{5,4}, w'_5), (\Delta_{5,4}, \Delta_5), (\Delta'_{6,5}, w'_6), (\Delta_{6,5}, \Delta_6), (\Delta'_{7,6}, w'_7), (\Delta_{7,6}, \Delta_7), (\Delta_{8,7}, \Delta_8)\}$, we get the following 10 branching vectors:

$$(w_8 - w'_8 + w'_3 + \alpha_5,\ w_8 - w_7 + w'_3 + \beta_5). \tag{62}$$

Case c-20. None of the previous cases are applicable, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u3})$ (see Figure 38): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_9, t_{10}\}$.

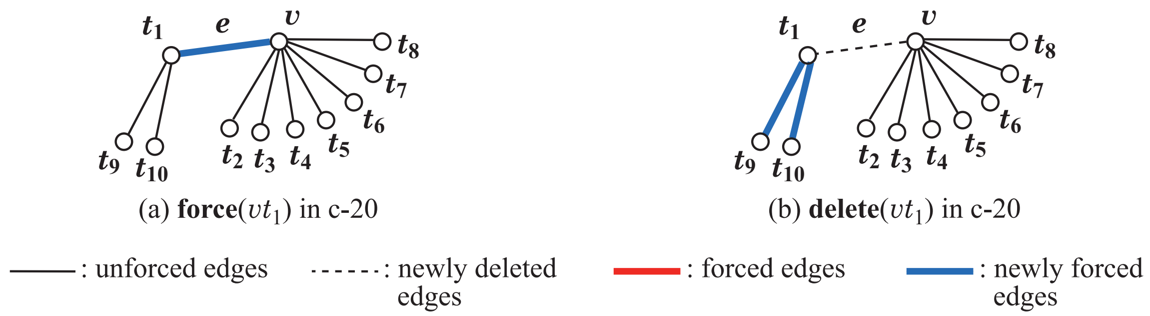


Figure 38: Illustration of branching rule c-20, where vertex $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u3})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_3. Thus the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w_3 - w'_3$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation, and edges t_1t_9 and t_1t_{10} will be added to F' by the reduction rules. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by w_3. Each of the vertices t_9 and t_{10} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, and a u8-vertex, and each of their weights decreases by at least $m_{18} = \min\{w'_3, \Delta_3, w'_4, \Delta_4, w'_5, \Delta_5, w'_6, \Delta_6, w'_7, \Delta_7, \Delta_8\}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_3 + 2m_{18}$.

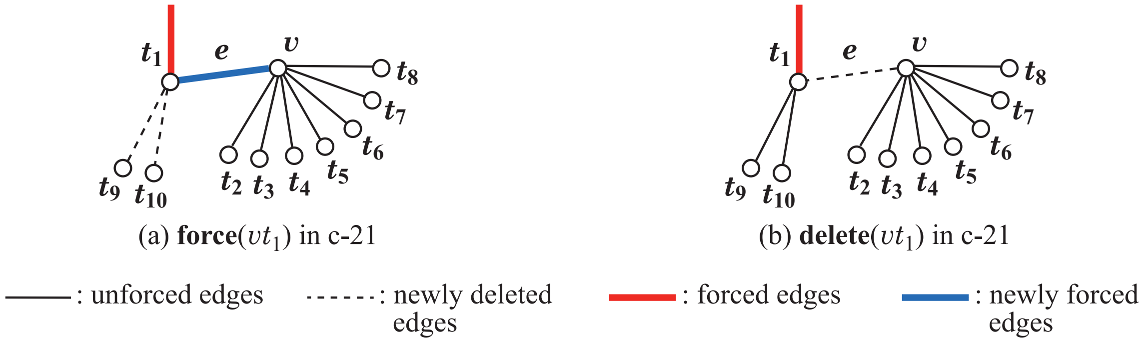
As a result, we get the following branching vector:

$$(w_8 - w'_8 + w_3 - w'_3,\ w_8 - w_7 + w_3 + 2m_{18}). \tag{63}$$

Case c-21. None of the previous cases are applicable, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{f4})$ (see Figure 39): We branch on the edge vt_1. Note that $N_U(t_1)\setminus\{v\} = \{t_9, t_{10}\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges t_1t_9 and t_1t_{10} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by w'_4. Each of the vertices t_9 and t_{10} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, and a u8-vertex, and each of their weights decreases by at least $m_{19} = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w'_4 + 2m_{19}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases



(a) **force**(vt_1) in c-21 (b) **delete**(vt_1) in c-21

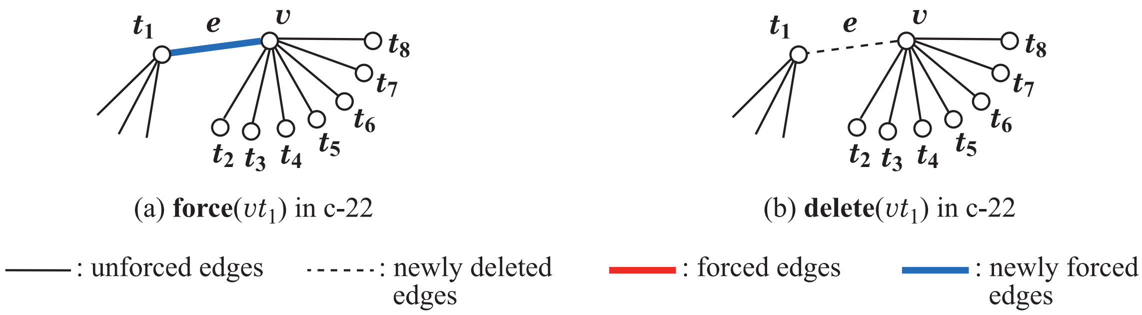
Figure 39: Illustration of branching rule c-21, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{f4}})$.

by $\Delta'_{4,3}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w'_4 - w'_3$.

As a result, we get the following branching vector:

$$(w_8 - w'_8 + w'_4 + 2m_{19},\ w_8 - w_7 + w'_4 - w'_3). \tag{64}$$

Case c-22. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u4}})$ (see Figure 40): We branch on the edge vt_1.



(a) **force**(vt_1) in c-22 (b) **delete**(vt_1) in c-22

Figure 40: Illustration of branching rule c-22, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u4}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_4. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w_4 - w'_4$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{4,3}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_4 - w_3$.

As a result, we get the following branching vector:

$$(w_8 - w'_8 + w_4 - w'_4,\ w_8 - w_7 + w_4 - w_3). \tag{65}$$

Case c-23. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{f5}})$ (see Figure 41): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_9, t_{10}, t_{11}\}$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges t_1t_9, t_1t_{10} and t_1t_{11} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by w'_5. Each of the vertices t_9, t_{10}, and t_{11} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, and

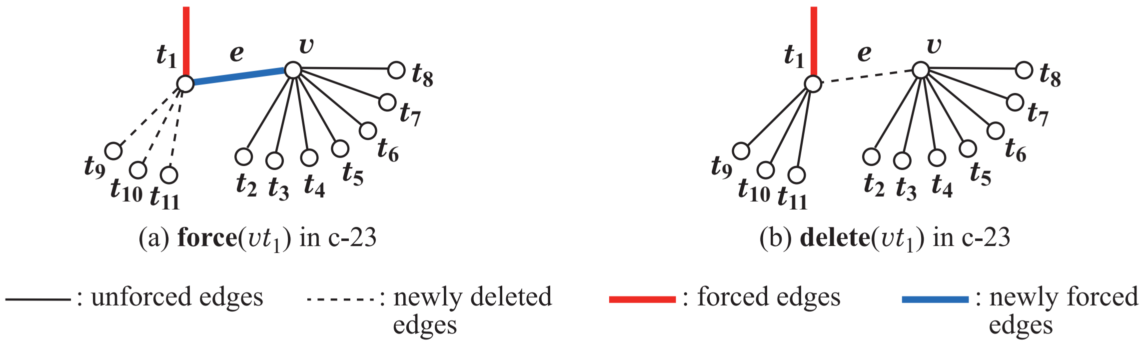


Figure 41: Illustration of branching rule c-23, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{f5}})$.

a u8-vertex, and each of their weights decreases by at least $m_{19} = \min\{w_3', w_3, \Delta_{4,3}', \Delta_{4,3}, \Delta_{5,4}', \Delta_{5,4}, \Delta_{6,5}', \Delta_{6,5}, \Delta_{7,6}', \Delta_{7,6}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w_8' + w_5' + 3m_{19}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{5,4}'$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_5' - w_4'$.

As a result, we get the following branching vector:

$$(w_8 - w_8' + w_5' + 3m_{19},\ w_8 - w_7 + w_5' - w_4'). \tag{66}$$

Case c-24. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u5}})$ (see Figure 42): We branch on the edge vt_1.

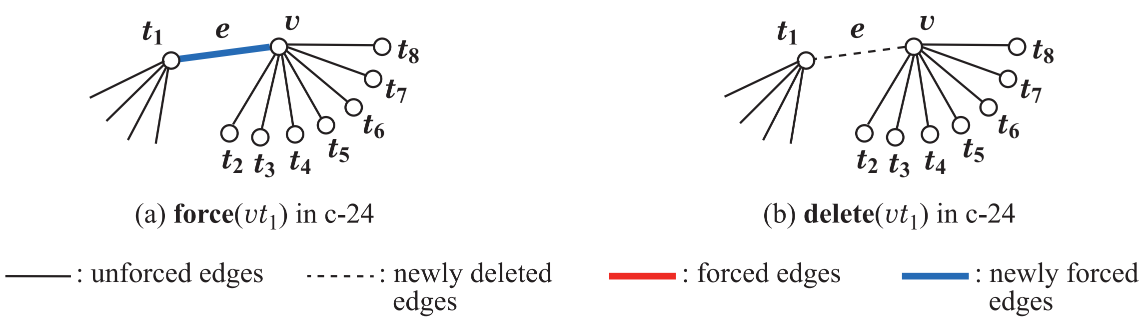


Figure 42: Illustration of branching rule c-24, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u5}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_5. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w_8' + w_5 - w_5'$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{5,4}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_5 - w_4$.

As a result, we get the following branching vector:

$$(w_8 - w_8' + w_5 - w_5',\ w_8 - w_7 + w_5 - w_4). \tag{67}$$

Case c-25. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{f6}})$ (see Figure 43): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_9, t_{10}, t_{11}, t_{12}\}$.

(a) **force**(vt_1) in c-25 (b) **delete**(vt_1) in c-25

: unforced edges : newly deleted edges : forced edges : newly forced edges

Figure 43: Illustration of branching rule c-25, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{f6}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges t_1t_9, t_1t_{10}, t_1t_{11} and t_1t_{12} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by w_6'. Each of the vertices t_9, t_{10}, t_{11} and t_{12} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, and a u8-vertex, and each of their weights decreases by at least $m_{19} = \min\{w_3'$, w_3, $\Delta_{4,3}'$, $\Delta_{4,3}$, $\Delta_{5,4}'$, $\Delta_{5,4}$, $\Delta_{6,5}'$, $\Delta_{6,5}$, $\Delta_{7,6}'$, $\Delta_{7,6}$, $\Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w_8' + w_6' + 4m_{19}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{6,5}'$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_6' - w_5'$.

As a result, we get the following branching vector:

$$(w_8 - w_8' + w_6' + 4m_{19},\ w_8 - w_7 + w_6' - w_5'). \tag{68}$$

Case c-26. None of the previous cases are applicable, and there exist vertices $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u6}})$ (see Figure 44): We branch on the edge vt_1.

(a) **force**(vt_1) in c-26 (b) **delete**(vt_1) in c-26

: unforced edges : newly deleted edges : forced edges : newly forced edges

Figure 44: Illustration of branching rule c-26, where vertex $v \in V_{\text{u8}}$ and $t_1 \in N_U(v; V_{\text{u6}})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_6. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w_8' + w_6 - w_6'$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{6,5}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_6 - w_5$.

As a result, we get the following branching vector:

$$(w_8 - w_8' + w_6 - w_6',\ w_8 - w_7 + w_6 - w_5). \tag{69}$$

Case c-27. None of the previous cases are applicable, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{f7})$ (see Figure 45): We branch on the edge vt_1. Note that $N_U(t_1) \setminus \{v\} = \{t_9, t_{10}, t_{11}, t_{12}, t_{13}\}$.

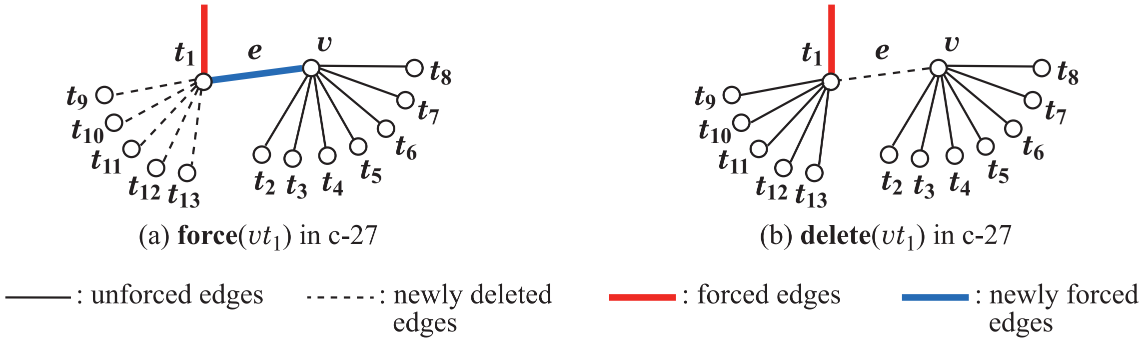


Figure 45: Illustration of branching rule c-27, where vertex $v \in V_{u8}$ and $t_1 \in N_U(v; V_{f7})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation, and edges t_1t_9, t_1t_{10}, t_1t_{11}, t_1t_{12} and t_1t_{13} will be deleted from G' by the reduction rules. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by w'_7. Each of the vertices t_9, t_{10}, t_{11}, t_{12} and t_{13} can be any of the possible vertex types f3, u3, f4, u4, f5, u5, f6, u6, f7, u7, and a u8-vertex, and each of their weights decreases by at least $m_{19} = \min\{w'_3, w_3, \Delta'_{4,3}, \Delta_{4,3}, \Delta'_{5,4}, \Delta_{5,4}, \Delta'_{6,5}, \Delta_{6,5}, \Delta'_{7,6}, \Delta_{7,6}, \Delta_{8,7}\}$. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w'_7 + 5m_{19}$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta'_{7,6}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w'_7 - w'_6$.

As a result, we get the following branching vector:

$$(w_8 - w'_8 + w'_7 + 5m_{19},\ w_8 - w_7 + w'_7 - w'_6). \tag{70}$$

Case c-28. None of the previous cases are applicable, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u7})$ (see Figure 46): We branch on the edge vt_1.

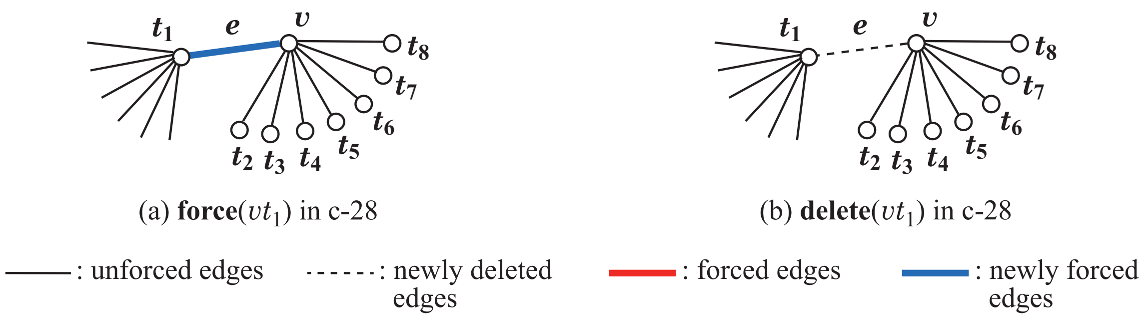


Figure 46: Illustration of branching rule c-28, where vertex $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u7})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_7. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $w_8 - w'_8 + w_7 - w'_7$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{7,6}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $w_8 - w_7 + w_7 - w_6$.

As a result, we get the following branching vector:

$$(w_8 - w_8' + w_7 - w_7',\ w_8 - w_6). \tag{71}$$

Case c-29. None of the previous cases are applicable, and there exist vertices $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u8})$ (see Figure 47): We branch on the edge vt_1.

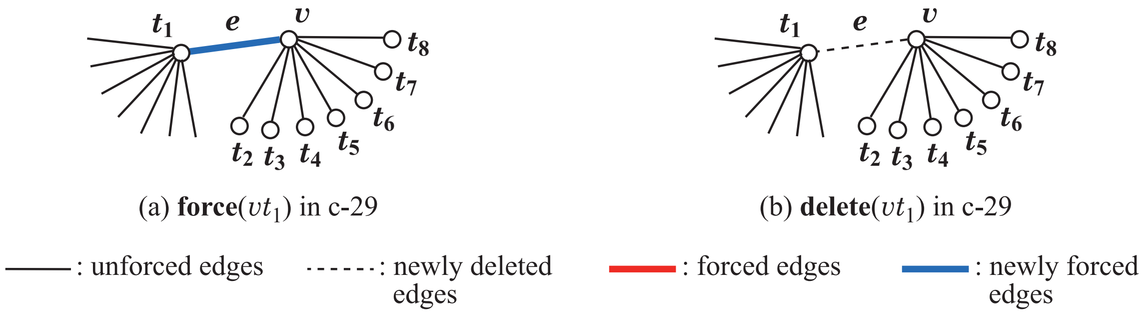


Figure 47: Illustration of branching rule c-29, where vertex $v \in V_{u8}$ and $t_1 \in N_U(v; V_{u8})$.

In the branch of **force**(vt_1), the edge vt_1 will be added to F' by the branching operation. So the weight of vertex v decreases by Δ_8, and the weight of vertex t_1 decreases by Δ_8. Thus, the total weight decrease for this case in the branch of **force**(vt_1) is at least $2w_8 - 2w_8'$.

In the branch of **delete**(vt_1), the edge vt_1 will be deleted from G' by the branching operation. So the weight of vertex v decreases by $\Delta_{8,7}$, and the weight of vertex t_1 decreases by $\Delta_{8,7}$. Thus, the total weight decrease for this case in the branch of **delete**(vt_1) is at least $2w_8 - 2w_7$.

As a result, we get the following branching vector:

$$(2w_8 - 2w_8',\ 2w_8 - 2w_7). \tag{72}$$

4.6 Switching to the TSP in Degree-7 Graphs

If none of the 29 cases of Figures 1 and 2 apply, this means that all vertices in the graph have degree seven or less. In that case, we can use a fast algorithm for the TSP in degree-7 graphs, called tsp7(G, F) to solve the remaining instances. Xiao and Nagamochi [14, Lemma 3] have shown how to leverage results obtained by a measure-and-conquer analysis, and that an algorithm can be used as a sub-procedure. We can get a non-trivial time bound on this sub-procedure if we know the respective weight setting mechanism. We calculate the maximum ratio of the vertex weights for the TSP in degree-7 graphs and the TSP in degree-8 graphs, and this will become a constraint in the quasiconvex program whose solution gives us the respective vertex weights.

Here we use the $O^*(3.5939^n)$-time algorithm for the TSP in degree-7 graphs by Md Yunos et al. [10], where the weight $\hat{w}_3'$ for an f3-vertex is 0.129815, the weight $\hat{w}_3$ for a u3-vertex is 0.231850, the weight $\hat{w}_4'$ for an f4-vertex is 0.285517, the weight $\hat{w}_4$ for a u4-vertex is 0.503746, the weight $\hat{w}_5'$ for an f5-vertex is 0.378022, the weight $\hat{w}_5$ for a u5-vertex is 0.707555, the weight $\hat{w}_6'$ for an f6-vertex is 0.449136, the weight $\hat{w}_6$ for a u6-vertex is 0.867483, the weight $\hat{w}_7'$ for an f7-vertex is 0.508069, and the weight $\hat{w}_7$ for a u7-vertex is 1. Let $\hat{w}$ denote the weight of vertices in degree-7 graphs, and let $\kappa = \max\{\frac{\hat{\omega}(v)}{\omega(v)} \mid v \in V_{\mathrm{f}i} \cup V_{\mathrm{u}i}, i = 3, 4, \ldots, 7\}$. For this step, the running time bound is

$$T(\mu(I)) \leq O\left(3.5939^{\kappa}\right). \tag{73}$$

4.7 Overall Analysis

As a result, by solving all branching vectors from Eqs. (27) to (72) and the switching constraint of Eq. (73) in a quasiconvex program according to the method introduced by Eppstein [1], the branching factor of each of the branching vectors from Eqs. (27) to (72) and the switching constraint of Eq. (73) does not exceed 4.148449, and the tight constraints are in conditions c-10, c-21, c-23, c-25, and the switching constraint of Eq. (73). This completes a proof of Theorem 1.

5 Conclusion

In this paper, we presented an exact algorithm for the TSP in degree-8 graphs. We use a similar technique as in the algorithm of the TSP in degree-5 graphs by Md Yunos et al. [8]. Even though the result does not give an advantageous algorithm for the TSP in degree-8 graphs over Gurevich and Shelah's algorithm for the TSP in general, it gives a limit as to the applicability of our choice of branching rules and analysis method for designing a polynomial-space exact algorithm for the TSP in degree bounded graphs. Perhaps, a different set of the branching rules and improving the analysis technique not only for the algorithm of the TSP in degree-8 graphs, but improving the running time bound of the algorithm for the TSP in degree-5, 6 and 7 graphs should be sought for to achieve better results.

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