A stabilized sequential quadratic programming method for optimization problems in function spaces

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ABSTRACT

In this paper, we propose a stabilized sequential quadratic programming (SQP) method for optimization problems in function spaces. A form of the problem considered in this paper can widely formulate many types of applications, such as obstacle problems, optimal control problems, and so on. Moreover, the proposed method is based on the existing stabilized SQP method and can find a point satisfying the Karush-Kuhn-Tucker (KKT) or asymptotic KKT conditions. One of the remarkable points is that we prove its global convergence to such a point under some assumptions without any constraint qualifications. In addition, we guarantee that an arbitrary accumulation point generated by the proposed method satisfies the KKT conditions under several additional assumptions. Finally, we report some numerical experiments to examine the effectiveness of the proposed method.

KEYWORDS

Asymptotic KKT conditions; Banach space; function space; global convergence; PDE constrained optimization; stabilized SQP method;

AMS CLASSIFICATION

49K27; 49M37; 90C48

1. Introduction

In this paper, we consider the following optimization problem:

$$\begin{aligned} &\underset{x\in X}{\text{Minimize}} && f(x) \\ &\text{subject to} && g(x) = 0,\ h_j(x) \geq 0\ (j = 1, \ldots, m), \end{aligned} \tag{1}$$

where X, Y, and Z_j $(j = 1, \ldots, m)$ are real Banach spaces, W is a real Hilbert space such that X is densely and continuously embedded in W, and $f\colon W \to \mathbb{R}$, $g\colon W \to Y$, and $h_j\colon W \to Z_j$ $(j = 1, \ldots, m)$. In addition, we suppose that Z_j is densely and continuously embedded in $L^2(\Omega_j)$ for each $j \in \{1, \ldots, m\}$, where $\Omega_j \subset \mathbb{R}^{M_j}$ is a measure space. Note that the order on Z_j is induced by the natural order on $L^2(\Omega_j)$. The detailed setting of (1) is provided in Section 2.

Optimization problems in function spaces arise from a lot of fields, and there are many types of them, such as obstacle problems, optimal control problems, and so on. For these problems, many optimization methods have been proposed so far [3, 5, 6, 8–11, 17–19, 22–27, 29, 31, 34, 35, 37, 38, 41, 42, 44]. However, a large number of these

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existing methods are designed to solve problems possessing particular structures. In other words, these structures can be regarded as a restriction for such existing methods. For example, objective functionals considered in [5, 9–11, 18, 24, 25, 27, 29, 31, 37, 41] are quadratic ones, inequality constraints seen in [11, 18, 24, 25, 27, 34, 35, 37, 38, 41, 42] are the box type, and so forth.

In the field of finite dimensional optimization, there are a lot of methods for solving optimization problems [13, 21, 33]. The purpose of such existing methods is basically to obtain a Karush-Kuhn-Tucker (KKT) point which satisfies the KKT conditions. Although the KKT conditions are known as first-order necessary optimality conditions, they do not necessarily hold unless some kind of constraint qualification (CQ) is satisfied. In the early 2000s, sequential optimality conditions were introduced for finite dimensional nonlinear programming problems [28, 32]. The conditions are known as genuine optimality conditions because they always hold at local optima without CQs. For finite dimensional problems, several researchers have developed methods to find points satisfying such conditions so far [1, 2, 39, 40]. Recently, Kanzow, Steck, and Wachsmuth [23] have extended the sequential optimality conditions of finite dimensional problems into infinite ones. The extended one is called asymptotic KKT (AKKT) conditions. In [23], an augmented Lagrangian method has also been proposed, and it is designed to compute AKKT points which satisfy the AKKT conditions. Furthermore, Börgens, Kanzow, and Steck [8] have improved the previous augmented Lagrangian method so that it can be applied to more general optimization problems. To the best of the author's knowledge, the augmented Lagrangian method is the only way to find AKKT points of infinite dimensional problems. However, this method uses first-order information to update the Lagrange multipliers, that is to say, it has only the linearly convergence property. Moreover, in the case where highly accurate solutions are required, the augmented Lagrangian method may not be appropriate.

The purpose of this paper is to propose a stabilized sequential quadratic programming (SQP) method for optimization problems in function spaces and to prove its global convergence property under some mild assumptions without any CQs. Although some existing SQP-type methods [3, 17, 18, 22, 26, 35, 41, 44] have been developed for optimization problems in function spaces, the proposed method can be distinguished from them in view of the following two points:

- (i) The proposed method can solve optimization problem (1), which allows to formulate many kinds of problems in function spaces including degenerate ones. As previously mentioned, most of the existing methods are designed to solve optimization problems possessing specific structures, and hence this fact is an advantage over the existing ones.
- (ii) A sequence generated by the proposed method converges globally to a point that satisfies the KKT or AKKT conditions. If a certain CQ holds, then its arbitrary accumulation point satisfies the KKT conditions. Therefore, the proposed method also has a standard convergence property seen in a large number of the existing methods. However, convergence to an AKKT point is not seen in the existing SQP-type methods for optimization problems in function spaces, that is, the convergence result of the current paper is the first of its kind.

This paper is organized as follows. In Section 2, we first describe the detailed setting of problem (1). Secondly, we introduce optimality conditions for (1). In Section 3, we explain the stabilized SQP method and give its formal statement. Section 4 shows the global convergence of the proposed method. Section 5 provides some concrete applications of (1) and reports numerical results obtained by applying the proposed method to those applications. Finally, some concluding remarks are presented in Section 6.

In the following, we define some mathematical notation. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces. The norm on $\mathcal{X}$ is represented by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{L}(\mathcal{X},\mathcal{Y})$ be the normed space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. We use $\|\cdot\|_{\mathcal{X}\to\mathcal{Y}}$ to denote the norm on $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Moreover, we define $\mathcal{X}^* \coloneqq \mathcal{L}(\mathcal{X},\mathbb{R})$. For $\varphi \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, its adjoint operator is denoted by $\varphi^* \in \mathcal{L}(\mathcal{Y}^*,\mathcal{X}^*)$. The closed ball in $\mathcal{X}$ with radius $r > 0$ is defined by $B_{\mathcal{X}}(r) \coloneqq \{x \in \mathcal{X}; \|x\|_{\mathcal{X}} \le r\}$. Let $\langle\cdot,\cdot\rangle_{\mathcal{X}^*,\mathcal{X}}$ be the associated dual pairing. If $\mathcal{X}$ is a Hilbert space, then its inner product is denoted by $(\cdot,\cdot)_{\mathcal{X}}$, and its norm is defined by $\|\cdot\|_{\mathcal{X}} \coloneqq \sqrt{(\cdot,\cdot)_{\mathcal{X}}}$. If $\mathcal{X} \subset \mathcal{Y}$ holds and the canonical injection $I_{\mathcal{X},\mathcal{Y}}$ from $\mathcal{X}$ into $\mathcal{Y}$ is continuous, then we write $\mathcal{X} \hookrightarrow \mathcal{Y}$. Furthermore, we will omit the canonical injection $I_{\mathcal{X},\mathcal{Y}}$ if $\mathcal{X} \hookrightarrow \mathcal{Y}$ is clear. Let $\mathcal{Z} \coloneqq \mathcal{X} \times \mathcal{Y}$ be the product space. The norm on $\mathcal{Z}$ is defined by $\|z\|_{\mathcal{Z}} \coloneqq (\|x\|_{\mathcal{X}}^2 + \|y\|_{\mathcal{Y}}^2)^{\frac{1}{2}}$ for $z = (x,y) \in \mathcal{Z}$. We identify $\mathcal{Z}^*$ with $\mathcal{X}^* \times \mathcal{Y}^*$. The dual pairing between $\mathcal{Z}^*$ and $\mathcal{Z}$ is defined by $\langle\varphi,z\rangle_{\mathcal{Z}^*,\mathcal{Z}} \coloneqq \langle\phi,x\rangle_{\mathcal{X}^*,\mathcal{X}} + \langle\psi,y\rangle_{\mathcal{Y}^*,\mathcal{Y}}$ for $\varphi = (\phi,\psi) \in \mathcal{Z}^*$ and $z = (x,y) \in \mathcal{Z}$. If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then the inner product on $\mathcal{Z}$ is defined by $(z_1,z_2)_{\mathcal{Z}} \coloneqq (x_1,x_2)_{\mathcal{X}} + (y_1,y_2)_{\mathcal{Y}}$ for $z_1 = (x_1,y_1) \in \mathcal{Z}$ and $z_2 = (x_2,y_2) \in \mathcal{Z}$. Let $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$ be Fréchet differentiable at $x \in \mathcal{X}$. The Fréchet derivative of $\mathcal{F}$ is represented by $\mathcal{F}'$. If $\mathcal{X}$ is a product space such that $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ with $n \ge 2$, then $x \in X$ is expressed as $x = (x_1,\dots,x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, and we denote by $\mathcal{F}_{x_i}$ the partial Fréchet derivative of $\mathcal{F}$ with respect to $x_i \in \mathcal{X}_i$, and denote by $\mathcal{F}_{x_i x_j}$ the partial Fréchet derivative of $\mathcal{F}_{x_i}$ with respect to $x_j \in \mathcal{X}_j$. We use $\to$, $\rightharpoonup$, and $\rightharpoonup^*$ to indicate strong, weak, and weak* convergence, respectively. For $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^p$, we denote by $a \cdot b$ the inner product of a and b defined as $a \cdot b \coloneqq a^\top b$, where $\top$ means transpose. For $c \in \mathbb{R}^p$, the Euclidean norm of c is represented by $|c| \coloneqq \sqrt{c \cdot c}$. Let $F\colon S \to \mathbb{R}^n$ be a function, where $S \subset \mathbb{R}^n$. Moreover, let $F_1,\dots,F_n$ be functions from S to $\mathbb{R}$ such that $F(t) \coloneqq (F_1(t),\dots,F_n(t))$ for $t \in S$. The positive part of F is denoted by $[F]_+$, i.e., $[F]_+(t) \coloneqq ([F_1(t)]_+,\dots,[F_n(t)]_+)$ for $t \in S$, where the positive part of $r \in \mathbb{R}$ is also denoted by $[r]_+ \coloneqq \max\{r,0\}$. If S is an open set and F is differentiable at $t \in S$, we use $\nabla F(t)$ to represent the transposition of its Jacobian at t, that is, $\nabla F(t) \coloneqq [\nabla F_1(t) \cdots \nabla F_m(t)]$. Note that if $m = 1$, then $\nabla F(t)$ means the gradient of F at t. For a closed convex set C in a Hilbert space, we write P_C for the metric projector over C. Let T be a set included in a topological space. The interior and closure of T are denoted by $\mathrm{int}(T)$ and $\bar{T}$, respectively. We use 1_T to denote the characteristic function of T. We represent $\mathrm{card}(T)$ as the cardinality of T. We write $u \lesssim v$ if there exists a universal constant $c > 0$ such that $u \le cv$.

2. Preliminaries

First, we provide the detailed setting associated with problem (1). Secondly, we define several optimality conditions.

2.1. Problem setting

Throughout this paper, we use the following notation:

$$V_j \coloneqq L^2(\Omega_j)\ (j = 1,\dots,m), \quad V \coloneqq V_1 \times \cdots \times V_m,$$

where $\Omega_j \subset \mathbb{R}^{M_j}$ is a bounded open domain with a Lipschitz boundary $\partial\Omega_j$ for all $j \in \{1, \ldots, m\}$. For $\varphi \in V_j$ and $\phi \in V_j$, the inequality $\varphi \geq \phi$ means $\varphi(\tau) \geq \phi(\tau)$ almost everywhere (a.e.) $\tau \in \Omega_j$. For $\varphi_0 = (\varphi_1, \ldots, \varphi_m) \in V$ and $\phi_0 = (\phi_1, \ldots, \phi_m) \in V$, the inequality $\varphi_0 \geq \phi_0$ indicates $\varphi_j \geq \phi_j$ for all $j \in \{1, \ldots, m\}$. We suppose that the real Banach spaces X, Y, and Z_j $(j = 1, \ldots, m)$ satisfy the following assumptions:

- X is densely and continuously embedded in W;
- Y is densely and continuously embedded in some Hilbert space U;
- $Z_1, \ldots, Z_m$ are densely and continuously embedded in $V_1, \ldots, V_m$, respectively.

We define $Z \coloneqq Z_1 \times \cdots \times Z_m$. The above setting yields

$$X \hookrightarrow W \cong W^* \hookrightarrow X^*, \quad Y \hookrightarrow U \cong U^* \hookrightarrow Y^*, \quad Z \hookrightarrow V \cong V^* \hookrightarrow Z^*.$$

Let $I_j \colon Z_j \to V_j$ be the canonical injection from Z_j to V_j. We define K_{V_j} and K_{Z_j} by

$$K_{V_j} \coloneqq \{\varphi \in V_j; \varphi \geq 0\}, \quad K_{Z_j} \coloneqq \{\varphi \in Z_j; I_j(\varphi) \in K_{V_j}\},$$

respectively. We also define $K_V \coloneqq K_{V_1} \times \cdots \times K_{V_m}$ and $K_Z \coloneqq K_{Z_1} \times \cdots \times K_{Z_m}$. For $\varphi \in Z_j$ and $\phi \in Z_j$, the inequality $\varphi \geq \phi$ is often used to indicate $\varphi - \phi \in K_{Z_j}$. Similarly, for $\varphi_0 \in Z$ and $\phi_0 \in Z$, the inequality $\varphi_0 \geq \phi_0$ means $\varphi_0 - \phi_0 \in K_Z$. Let $h \colon X \to Z$ be a functional defined by

$$h(x) \coloneqq (h_1(x), \ldots, h_m(x)).$$

In addition to the above setting, no CQ is required for problem (1) as stated in Section 1.

The above setting enables us to represent many mathematical optimization models in function spaces, such as obstacle problems and elliptic control problems, as problem (1). In Section 5, we provide several concrete applications.

2.2. Optimality conditions

This section gives definitions of several first-order optimality conditions and CQs for problem (1). In these definitions, the differentiability of the functionals included in (1) is required, and hence we suppose that f, g, and h are continuously Fréchet differentiable on X. In addition, we denote the Lagrangian $L \colon X \times Y^* \times Z^* \to \mathbb{R}$ by

$$L(x, y, z) \coloneqq f(x) - \langle y, g(x)\rangle_{Y^*,Y} - \langle z, h(x)\rangle_{Z^*,Z},$$

and we denote the dual cone of K_Z by

$$K_Z^+ \coloneqq \{z \in Z^*; \langle z, \zeta\rangle_{Z^*,Z} \geq 0 \ \forall \zeta \in K_Z\}.$$

In the following, we define the KKT conditions for problem (1).

Definition 1. *If $x \in X$ is a feasible point of problem* (1)*, and there exists $(y, z) \in Y^* \times K_Z^+$ such that*

$$L_x(x, y, z) = 0, \quad \langle z, h(x)\rangle_{Z^*,Z} = 0,$$

then we say that (x, y, z) satisfies the Karush-Kuhn-Tucker (KKT) conditions.

We call x a KKT point if there exists (y, z) such that (x, y, z) satisfies the KKT conditions. As it is well known, the KKT conditions are necessary for optimality and do not make sense without some CQ. In this paper, we introduce the Robinson CQ and its extension.

Definition 2. *If $x \in X$ is a feasible point of problem (1) and satisfies*

$$0 \in \operatorname{int}\left(\begin{bmatrix} g(x) \\ h(x) \end{bmatrix} + \begin{bmatrix} g'(x) \\ h'(x) \end{bmatrix} X - \begin{bmatrix} \{0\} \\ K_Z \end{bmatrix}\right),$$

then we say that the Robinson constraint qualification (RCQ) holds at x. If $x \in X$, which is not necessarily a feasible point of problem (1), satisfies the above condition, then we say that the extended Robinson constraint qualification (ERCQ) holds at x.

It follows from [7, Theorem 3.9] that for each local optimum $x \in X$, the set $\{(y, z) \in Y^* \times K_Z^+; (x, y, z)$ satisfies the KKT conditions$\}$ is nonempty, convex, bounded, and weakly* compact in $Y^* \times Z^*$ under the RCQ. Hence, the KKT conditions make sense under the RCQ. Note that the RCQ requires that the point x is feasible. In contrast, the ERCQ is not restricted to feasible points. Such an extension is also seen in [8, 12, 14]. The ERCQ plays a crucial role in the global convergence analysis given in Section 4.

Next, we define the AKKT conditions which are first-order necessary optimality conditions. Note that they are an extension of [23, Definition 5.2] because the original definition has no equality constraint. In the following definition, we assume that the mapping $\varphi \in Z \mapsto [\varphi]_+ \in Z$ is well-defined and continuous on Z.

Definition 3. *If $x \in X$ is a feasible point of problem (1), and there exist sequences $\{x_k\} \subset X$, $\{y_k\} \subset Y^*$, and $\{z_k\} \subset K_Z^+$ such that*

$$\lim_{k\to\infty} \|x_k - x\|_X = 0, \quad \lim_{k\to\infty} \|L_x(x_k, y_k, z_k)\|_{X^*} = 0, \quad \lim_{k\to\infty} \langle z_k, [h(x_k)]_+\rangle_{Z^*,Z} = 0,$$

then we say that x satisfies the asymptotic Karush-Kuhn-Tucker (AKKT) conditions.

We call x an AKKT point if x satisfies the AKKT conditions. Moreover, we call $\{(x_k, y_k, z_k)\}$ which appears in Definition 3 an AKKT sequence corresponding to x. The next theorem states that the AKKT conditions are satisfied at each local optimum whether or not CQs hold. We omit the proof because it can be shown in a similar way to the proof of [23, Theorem 5.5].

Theorem 1. *Suppose that X is reflexive. Suppose also that f, g, and h are continuously Fréchet differentiable on X and that f, $\|g(\cdot)\|_U$, and $\|[-h(\cdot)]_+\|_V$ are weakly lower semicontinuous on X. If $x \in X$ is a local minimum of problem (1), then it satisfies the AKKT conditions of (1).*

The following proposition provides sufficient conditions under which an AKKT point is a KKT point.

Proposition 1. *Assume that f, g, and h are continuously Fréchet differentiable on X, and $\varphi \in Z \mapsto [\varphi]_+ \in Z$ is well-defined and continuous on Z. Assume also that Y and Z are separable. Let $x \in X$ be an AKKT point of problem (1) and let $\{(x_k, y_k, z_k)\} \subset X \times Y^* \times Z^*$ be an AKKT sequence corresponding to x. If the RCQ holds at x, then* *there exist $y \in Y^*$, $z \in Z^*$, and $\mathcal{M} \subset \mathbb{N}$ such that $y_k \rightharpoonup^* y$ and $z_k \rightharpoonup^* z$ as $k \to \infty$, $k \in \mathcal{M}$, and (x, y, z) satisfies the KKT conditions of* (1)*.*

Proof. To begin with, we show that $\{y_k\} \subset Y^*$ and $\{z_k\} \subset Z^*$ are weak* sequentially compact. Let $\mathcal{U} \coloneqq Y \times Z$ and $S(x) \coloneqq \{(u, v) \in \mathcal{U}; \exists \overline{x} \in X, \exists \overline{z} \in K_Z, u = g(x) + g'(x)\overline{x}, v = h(x) + h'(x)\overline{x} - \overline{z}\}$. Now, the RCQ holds at x, and hence there exists $r > 0$ such that $B_{\mathcal{U}}(r) \coloneqq \{w \in \mathcal{U}; \|w\|_{\mathcal{U}} \leq r\} \subset S(x)$. Let $s \in Y$ and $t \in Z$ be arbitrary elements such that $\|s\|_Y \leq 1$ and $\|t\|_Z \leq 1$. We define $u \coloneqq \frac{r}{2}s$, $v \coloneqq \frac{r}{2}t$, and $w \coloneqq (u, v) \in \mathcal{U}$. Note that $\|w\|_{\mathcal{U}} \leq \frac{\sqrt{2}}{2}r$, namely, $w \in B_{\mathcal{U}}(r) \subset S(x)$. This fact and the feasibility of x imply that there exist $\overline{x} \in X$ and $\overline{z} \in K_Z$ such that

$$u = g'(x)\overline{x}, \quad v = h(x) + h'(x)\overline{x} - \overline{z}. \tag{2}$$

Since $\{(x_k, y_k, z_k)\}$ is an AKKT sequence corresponding to x, we have $\|x_k - x\|_X \to 0$ as $k \to \infty$. Then, the continuity of f', g', h', and h means that there exist $c > 1$ and $\overline{m} \in \mathbb{N}$ such that, for all $k \geq \overline{m}$,

$$\|f'(x_k)\|_{X^*} \leq \frac{\delta r(c-1)}{4}, \quad \|g'(x) - g'(x_k)\|_{X\to Y} \leq \frac{\delta r}{8}, \tag{3}$$

$$\|h'(x) - h'(x_k)\|_{X\to Z} \leq \frac{\delta r}{16}, \quad \|h(x) - h(x_k)\|_Z \leq \frac{r}{16}, \tag{4}$$

where $\delta \coloneqq 1/\max\{\|\overline{x}\|_X, 1\}$. On the other hand, both $\{\|L_x(x_k, y_k, z_k)\|_{X^*}\}$ and $\{\langle z_k, [h(x_k)]_+\rangle_{Z^*,Z}\}$ converge to zero, and hence there exists $\overline{n} \in \mathbb{N}$ such that for all $k \geq \overline{n}$,

$$\|L_x(x_k, y_k, z_k)\|_{X^*} \leq \frac{\delta r}{8}, \quad |\langle z_k, [h(x_k)]_+\rangle_{Z^*,Z}| \leq \frac{r}{8}. \tag{5}$$

Let k be an arbitrary integer with $k > \overline{k} \coloneqq \max\{\overline{m}, \overline{n}\}$. Notice that $h(x_k) = [h(x_k)]_+ - [-h(x_k)]_+$, $z_k \in K_Z^+$, $0 \leq \langle z_k, \overline{z}\rangle_{Z^*,Z}$, and $0 \leq \langle z_k, [-h(x_k)]_+\rangle_{Z^*,Z}$. By using (2)–(5), we obtain

$$\begin{aligned}
&\langle y_k, u\rangle_{Y^*,Y} + \langle z_k, v\rangle_{Z^*,Z} \\
&\quad = \langle y_k, (g'(x) - g'(x_k))\overline{x}\rangle_{Y^*,Y} - \langle z_k, \overline{z}\rangle_{Z^*,Z} + \langle z_k, (h'(x) - h'(x_k))\overline{x}\rangle_{Z^*,Z} \\
&\qquad + \langle z_k, h(x) - h(x_k)\rangle_{Z^*,Z} + \langle z_k, [h(x_k)]_+ - [-h(x_k)]_+\rangle_{Z^*,Z} \\
&\qquad + \langle f'(x_k), \overline{x}\rangle_{X^*,X} - \langle L_x(x_k, y_k, z_k), \overline{x}\rangle_{X^*,X} \\
&\quad \leq \|g'(x) - g'(x_k)\|_{X\to Y}\|\overline{x}\|_X\|y_k\|_{Y^*} + \|h'(x) - h'(x_k)\|_{X\to Z}\|\overline{x}\|_X\|z_k\|_{Z^*} \\
&\qquad + \|h(x) - h(x_k)\|_Z\|z_k\|_{Z^*} + |\langle z_k, [h(x_k)]_+\rangle_{Z^*,Z}| \\
&\qquad + \|f'(x_k)\|_{X^*}\|\overline{x}\|_X + \|L_x(x_k, y_k, z_k)\|_{X^*}\|\overline{x}\|_X \\
&\quad \leq \frac{r}{4}\max\{\|y_k\|_{Y^*}, \|z_k\|_{Z^*}\} + \frac{r}{4}c.
\end{aligned} \tag{6}$$

Multiplying both sides of (6) by $\frac{2}{r}$ and exploiting $s = \frac{2}{r}u$ and $t = \frac{2}{r}v$ yield $\langle y_k, s\rangle_{Y^*,Y} + \langle z_k, t\rangle_{Z^*,Z} \leq \frac{1}{2}\max\{\|y_k\|_{Y^*}, \|z_k\|_{Z^*}\} + \frac{1}{2}c$. Since $s \in Y$ and $t \in Z$ are arbitrary elements satisfying $\|s\|_Y \leq 1$ and $\|t\|_Z \leq 1$,

$$\begin{aligned} \|y_k\|_{Y^*} + \|z_k\|_{Z^*} &= \sup_{\|s\|_Y \leq 1} \langle y_k, s\rangle_{Y^*,Y} + \sup_{\|t\|_Z \leq 1} \langle z_k, t\rangle_{Z^*,Z} \\ &\leq \frac{1}{2}\max\{\|y_k\|_{Y^*}, \|z_k\|_{Z^*}\} + \frac{1}{2}c. \end{aligned} \tag{7}$$

Meanwhile, it is clear that $\max\{\|y_k\|_{Y^*}, \|z_k\|_{Z^*}\} \leq \|y_k\|_{Y^*} + \|z_k\|_{Z^*}$. This fact and (7) lead to $\max\{\|y_k\|_{Y^*}, \|z_k\|_{Z^*}\} \leq c$, namely, $\|y_k\|_{Y^*} \leq c$ and $\|z_k\|_{Z^*} \leq c$ for $k > \overline{k}$. Obviously, $\|y_k\|_{Y^*} \leq \max\{\|y_1\|_{Y^*}, \dots, \|y_{\overline{k}}\|_{Y^*}\}$ and $\|z_k\|_{Z^*} \leq \max\{\|z_1\|_{Z^*}, \dots, \|z_{\overline{k}}\|_{Z^*}\}$ for $k \leq \overline{k}$, and therefore $\{y_k\}$ and $\{z_k\}$ are bounded in Y^* and Z^*, respectively. Now we recall that Y and Z are separable. By these facts and the boundedness of $\{y_k\} \subset Y^*$ and $\{z_k\} \subset Z^*$, there exist $y \in Y^*$, $z \in Z^*$, and $\mathcal{M} \subset \mathbb{N}$ such that

$$\mathrm{w}^* - \lim_{k\to\infty, k\in\mathcal{M}} y_k = y, \quad \mathrm{w}^* - \lim_{k\to\infty, k\in\mathcal{M}} z_k = z. \tag{8}$$

From now on, we prove the assertion of this proposition by exploiting the above result. It is sufficient to show that $z \in K_Z^+$, $L_x(x, y, z) = 0$, and $\langle z, h(x)\rangle_{Z^*,Z} = 0$ because x is feasible to (1). Note that $\{z_k\} \subset K_Z^+$ because $\{(x_k, y_k, z_k)\} \subset X \times Y^* \times Z^*$ is an AKKT sequence corresponding to x. It then follows from the second equality of (8) that

$$\langle z, \zeta\rangle_{Z^*,Z} = \lim_{k\to\infty, k\in\mathcal{M}} \langle z_k, \zeta\rangle_{Z^*,Z} \geq 0 \quad \forall \zeta \in K_Z.$$

Thus, we can verify $z \in K_Z^+$. Now, let $\xi \in X$ and $k \in \mathcal{M}$ be arbitrary. Then we have

$$\begin{aligned} &|\langle L_x(x, y, z), \xi\rangle_{X^*,X}| \\ &\quad \leq \|L_x(x_k, y_k, z_k)\|_{X^*}\|\xi\|_X + \|f'(x_k) - f'(x)\|_{X^*}\|\xi\|_X \\ &\qquad + \|g'(x_k) - g'(x)\|_{X\to Y}\|\xi\|_X\|y_k\|_{Y^*} + |\langle y_k - y, g'(x)\xi\rangle_{Y^*,Y}| \\ &\qquad + \|h'(x_k) - h'(x)\|_{X\to Z}\|\xi\|_X\|z_k\|_{Z^*} + |\langle z_k - z, h'(x)\xi\rangle_{Z^*,Z}|. \end{aligned} \tag{9}$$

Moreover, we obtain

$$\begin{aligned} |\langle z, h(x)\rangle_{Z^*,Z}| &= |\langle z, [h(x)]_+\rangle_{Z^*,Z}| \\ &\leq |\langle z_k, [h(x_k)]_+\rangle_{Z^*,Z}| + |\langle z - z_k, [h(x)]_+\rangle_{Z^*,Z}| \\ &\quad + \|z_k\|_{Z^*}\|[h(x)]_+ - [h(x_k)]_+\|_Z, \end{aligned} \tag{10}$$

where the first equality follows from the feasibility of x. We recall that $\|x_k - x\|_X \to 0$, $\|L_x(x_k, y_k, z_k)\|_{X^*} \to 0$, and $\langle z_k, [h(x_k)]_+\rangle_{Z^*,Z} \to 0$ as $k \to \infty$, $k \in \mathcal{M}$. Then, using (8)–(10), and the continuity of f', g', h', and $[h(\cdot)]_+$ derives $L_x(x, y, z) = 0$ and $\langle z, h(x)\rangle_{Z^*,Z} = 0$. Therefore, the assertion is proven. □

3. A stabilized SQP method

In this section, we provide a stabilized SQP method for problem (1). The proposed method consists of three main steps: computing a search direction, updating a current point, and updating Lagrange multipliers and some parameters. Before describing formal statement of the proposed method, we explain the three steps. Note that the proposed method generates two kinds of Lagrange multiplier sequences $\{(y_k, z_k)\}$ and $\{(\overline{y}_k, \overline{z}_k)\}$. Throughout this section, the functionals f, g, and h are assumed to be twice continuously Fréchet differentiable on W.

3.1. Computing a search direction

Let $(x, y, z) \in X \times U \times V$ be a given point. In the proposed method, we solve a certain subproblem to determine a search direction. To give the subproblem, we begin by considering the following:

$$\begin{aligned} \underset{(\xi,\eta,\zeta)\in\mathcal{W}}{\text{Minimize}} \quad & (f'(x), \xi)_W + \frac{1}{2}(H\xi, \xi)_W + \frac{\sigma}{2}\|\eta\|_U^2 + \frac{\sigma}{2}\|\zeta\|_V^2 \\ \text{subject to} \quad & g(x) + g'(x)\xi + \sigma(\eta - y) = 0, \\ & h(x) + h'(x)\xi + \sigma(\zeta - z) \geq 0, \end{aligned} \tag{11}$$

where $\mathcal{W} := W \times U \times V$, and $H \in \mathcal{L}(W, W)$ represents $L_{xx}(x, y, z)$ or its approximation, and $\sigma > 0$ is a penalty parameter. Problem (11) is derived from *the stabilized subproblem* used in the existing stabilized SQP methods for finite dimensional optimization problems [15, 16, 43]. By using the relation $\eta = y - \frac{1}{\sigma}(g(x) + g'(x)\xi)$, we can reformulate problem (11) as follows:

$$\begin{aligned} \underset{(\xi,\zeta)\in\mathcal{V}}{\text{Minimize}} \quad & (f'(x) - g'(x)^* s, \xi)_W + \frac{1}{2}(M\xi, \xi)_W + \frac{\sigma}{2}\|\zeta\|_V^2 \\ \text{subject to} \quad & h'(x)\xi + \sigma(\zeta - t) \geq 0, \end{aligned} \tag{12}$$

where $\mathcal{V} := W \times V$, $M := H + \frac{1}{\sigma}g'(x)^* g'(x)$, $s := y - \frac{1}{\sigma}g(x) \in U$, and $t := z - \frac{1}{\sigma}h(x) \in V$. In the proposed method, we adopt (12) as a subproblem. Let $B\colon W \times W \to \mathbb{R}$ be a bilinear form defined by $B(\xi_1, \xi_2) := (M\xi_1, \xi_2)_W$ for ξ_1, $\xi_2 \in X$. The next proposition ensures that problem (12) has the unique optimal solution under some appropriate assumptions. Its proof is given in Appendix A.

Proposition 2. *Suppose that the bilinear form B is coercive, that is, there exists $\ell_B > 0$ such that $B(\xi, \xi) \geq \ell_B\|\xi\|_W^2$ for all $\xi \in W$. Then, problem* (12) *has the unique optimum $(\xi_*, \zeta_*) \in \mathcal{V}$. Moreover, there exists $\lambda_* \in V$ such that $(\xi_*, \zeta_*, \lambda_*)$ satisfies the KKT conditions of* (12).

From now on, we give an explanation related to a search direction p. In the following argument, the bilinear form B is assumed to be coercive. Proposition 2 guarantees that problem (12) has the unique optimum (ξ_*, ζ_*). Although many of the existing SQP methods adopt ξ_* as a search direction, it is difficult to obtain such an exact optimum from practical aspects. Therefore, we consider solving problem (12) inexactly. In other words, we adopt a search direction from an appropriate neighborhood of ξ_*. To explain how to determine the search direction, we define a merit functional $F\colon W \to \mathbb{R}$ by

$$F(x; y, z, \sigma) := f(x) + \frac{1}{2\sigma}\|\sigma y - g(x)\|_U^2 + \frac{1}{2\sigma}\left\|[\sigma z - h(x)]_+\right\|_V^2.$$

Note that the functional F is the augmented Lagrangian. For the details, see [23]. It follows from [4, Corollary 12.31] that the functional F is Fréchet differentiable on W, and its Fréchet derivative at $x \in W$ is given by

$$F'(x; y, z, \sigma) = f'(x) - g'(x)^* \left(y - \frac{1}{\sigma} g(x) \right) - h'(x)^* \left[z - \frac{1}{\sigma} h(x) \right]_+ . \tag{13}$$

The functional F has the following property related to problem (12). The proof is given in Appendix A.

Proposition 3. *Suppose that the bilinear form B is coercive, that is, there exists $\ell_B > 0$ such that $B(\xi, \xi) \geq \ell_B \|\xi\|_W^2$ for every $\xi \in W$. Then, $F'(x; y, z, \sigma) = 0$ if and only if $(0, [t]_+) \in \mathcal{V}$ is the unique optimum of problem (12).*

Proposition 2 ensures the existence of a Lagrange multiplier λ_* such that $(\xi_*, \zeta_*, \lambda_*)$ satisfies the KKT conditions of problem (12). Let $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\lambda}) \in X \times V \times V$ be an element of a neighborhood of $(\xi_*, \zeta_*, \lambda_*) \in W \times V \times V$, where we note that the existence of $\widetilde{\xi} \in X$ is ensured by the fact that X is dense in W. If a pair $(\widetilde{\xi}, \widetilde{\lambda})$ satisfies

$$(F'(x; y, z, \sigma), \widetilde{\xi})_W \leq -c(M\widetilde{\xi}, \widetilde{\xi})_W - c\sigma \|\widetilde{\lambda} - [t]_+\|_V^2, \tag{14}$$

$$\|M\widetilde{\xi} + f'(x) - g'(x)^* s - h'(x)^* \widetilde{\lambda}\|_W \leq |(F'(x; y, z, \sigma), \widetilde{\xi})_W|, \tag{15}$$

then we set $p := \widetilde{\xi} \in X$ as a search direction and set $\widetilde{y} := y - \frac{1}{\sigma}(g(x) + g'(x)\widetilde{\xi}) \in U$ and $\widetilde{z} := [\widetilde{\lambda}]_+ \in V$ as trial Lagrange multipliers, where $c \in (0, 1)$ is a parameter which indicates how exactly we solve problem (12). The closer c is to 1, the more exactly $(\widetilde{\xi}, \widetilde{\zeta})$ solves problem (12). We are able to show that there exists $(\widetilde{\xi}, \widetilde{\lambda})$ satisfying (14) and (15). For its proof, see Proposition 4 given in Section 3.5. Note that the proposed method does not determine the Lagrange multiplier pair (y, z) immediately. After we compute the trial Lagrange multiplier pair $(\widetilde{y}, \widetilde{z})$ described above, we check how much the optimality conditions are improved. Based on this check, we decide whether or not to set $(\widetilde{y}, \widetilde{z})$ to be (y, z). The details are explained in Section 3.3.

3.2. Updating a primal iterate

In what follows, a subscript k is used to denote a current iteration. This section provides a detailed explanation regarding an updating rule of a current point $x_k \in X$. To begin with, let us consider a computational process for finding the search direction $p_k \in X$ described in Section 3.1. Although the proposed method approximately solves subproblem (12) to obtain p_k, it is possible that the generated Lagrange multiplier sequence diverges as iterations progress because problem (1) does not necessarily satisfy some CQ. If we generate a search direction sequence by solving (12) with such a sequence, it might be unstable for its computational process. Hence, the proposed method generates two kinds of Lagrange multiplier sequences. The first one is a *main Lagrange multiplier sequence*, where its boundedness is not ensured as stated above. The other one is a *sub-Lagrange multiplier sequence* that is generated to be bounded and is used in order to stably compute the search direction sequence. In the following, the first and second sequences are denoted by $\{(y_k, z_k)\} \subset U \times V$ and $\{(\overline{y}_k, \overline{z}_k)\} \subset U \times V$, respectively. Furthermore, σ_k denotes the penalty parameter, H_k represents the Hessian of the Lagrangian or its approximation, and M_k, s_k, and t_k are defined as

$$
\begin{aligned}
M_k &:= H_k + \frac{1}{\sigma_k} g'(x_k)^* g'(x_k), \\
s_k &:= \overline{y}_k - \frac{1}{\sigma_k} g(x_k), \\
t_k &:= \overline{z}_k - \frac{1}{\sigma_k} h(x_k),
\end{aligned} \tag{16}
$$

respectively.

Now we recall that the search direction p_k is an approximate solution of subproblem (12) with $x := x_k$, $\sigma := \sigma_k$, $M := M_k$, $s := s_k$, and $t := t_k$. Note also that $(F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k), p_k)_W \leq 0$ by (14). We consider updating a primal iterate $x_k \in X$ so that the value of the merit functional F decreases along the search direction $p_k \in X$. For this purpose, we exploit a backtracking line-search to determine a step size $\alpha_k > 0$. This line-search computes the step size as $\alpha_k := \beta^{\ell_k}$, where $\beta \in (0, 1)$ and ℓ_k is the smallest nonnegative integer satisfying

$$
F(x_k + \beta^{\ell_k} p_k; \overline{y}_k, \overline{z}_k, \sigma_k) \leq F(x_k; \overline{y}_k, \overline{z}_k, \sigma_k) + \varepsilon \beta^{\ell_k} \Delta_k, \tag{17}
$$

$$
\Delta_k := \max\{(F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k), p_k)_W, -\rho \|p_k\|_W^2\}, \tag{18}
$$

where $\varepsilon \in (0, 1)$ and $\rho \in (0, 1)$. Notice that if $|(F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k), p_k)_W|$ is a large value, then the second term $-\rho\|p_k\|_W^2$ in (18) helps us to adopt an early iteration of the line-search. After computing the step size, we set $x_{k+1} := x_k + \alpha_k p_k$.

3.3. Updating Lagrange multipliers and some parameters

We explain details of an updating procedure regarding Lagrange multipliers and some parameters. This procedure is based on that of Gill and Robinson [15]. We denote $\widetilde{y}_{k+1}$ and $\widetilde{z}_{k+1}$ as the trial Lagrange multipliers described in Section 3.1 and call $(x_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1})$ a trial point. Moreover, we introduce the following functionals:

$$
\begin{aligned}
\Phi(x, y, z) &:= \|g(x)\|_Y + \|[-h(x)]_+\|_Z \\
&\qquad + \kappa \|L_x(x, y, z)\|_{X^*} + \kappa |\langle z, h(x)\rangle_{Z^*,Z}|, \\
\Psi(x, y, z) &:= \kappa \|g(x)\|_Y + \kappa \|[-h(x)]_+\|_Z \\
&\qquad + \|L_x(x, y, z)\|_{X^*} + |\langle z, h(x)\rangle_{Z^*,Z}|,
\end{aligned} \tag{19}
$$

where $\kappa \in (0, 1)$ is a weight parameter. It is clear that (x, y, z) satisfies the KKT conditions of (1) if and only if $\Phi(x, y, z) = \Psi(x, y, z) = 0$.

Roughly speaking, the procedure updates two kinds of the Lagrange multipliers (y_k, z_k) and $(\overline{y}_k, \overline{z}_k)$, and the parameters ϕ_k, ψ_k, and γ_k only if at least one of the following statements is satisfied:

(i) $\{(x_k, \widetilde{y}_k, \widetilde{z}_k)\}$ tends to converge to a point satisfying the KKT conditions of (1);
(ii) $\{x_k\}$ tends to converge to a stationary point of F.

Otherwise, it does not update the Lagrange multipliers (y_k, z_k) and $(\overline{y}_k, \overline{z}_k)$, and the parameters ϕ_k, ψ_k, and γ_k. Based on this concept, we present an updating procedure as Algorithm 1.

Algorithm 1 Updating procedure for Lagrange multipliers and parameters

Require: Set $C \subset U$ and $D \subset V$, where C is bounded and convex, and D is defined by $D \coloneqq \{z \in V; 0 \le z \le z_{\max}\}$ with a constant number $z_{\max} > 0$. Give $x_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}, y_k, z_k, \overline{y}_k, \overline{z}_k, \sigma_k, \phi_k, \psi_k$, and γ_k.

if $\Phi(x_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}) \le \frac{1}{2}\phi_k$, $\widetilde{y}_{k+1} \in C$, **and** $\widetilde{z}_{k+1} \in D$, **then**

  Set ▷ Step 1

$$y_{k+1} \coloneqq \widetilde{y}_{k+1},\ z_{k+1} \coloneqq \widetilde{z}_{k+1},\ \overline{y}_{k+1} \coloneqq \widetilde{y}_{k+1},\ \overline{z}_{k+1} \coloneqq \widetilde{z}_{k+1},$$
$$\phi_{k+1} \coloneqq \tfrac{1}{2}\phi_k,\ \psi_{k+1} \coloneqq \psi_k,\ \gamma_{k+1} \coloneqq \gamma_k.$$

else if $\Psi(x_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1}) \le \frac{1}{2}\psi_k$, $\widetilde{y}_{k+1} \in C$, **and** $\widetilde{z}_{k+1} \in D$, **then**

  Set ▷ Step 2

$$y_{k+1} \coloneqq \widetilde{y}_{k+1},\ z_{k+1} \coloneqq \widetilde{z}_{k+1},\ \overline{y}_{k+1} \coloneqq \widetilde{y}_{k+1},\ \overline{z}_{k+1} \coloneqq \widetilde{z}_{k+1},$$
$$\phi_{k+1} \coloneqq \phi_k,\ \psi_{k+1} \coloneqq \tfrac{1}{2}\psi_k,\ \gamma_{k+1} \coloneqq \gamma_k.$$

else if $\|F'(x_{k+1}; \overline{y}_k, \overline{z}_k, \sigma_k)\|_W \le \gamma_k$, **then**

  Set ▷ Step 3

$$y_{k+1} \coloneqq \overline{y}_k - \tfrac{1}{\sigma_k} g(x_{k+1}),\ z_{k+1} \coloneqq [\overline{z}_k - \tfrac{1}{\sigma_k} h(x_{k+1})]_+,$$
$$\overline{y}_{k+1} \coloneqq P_C(\overline{y}_k - \tfrac{1}{\sigma_k} g(x_{k+1})),\ \overline{z}_{k+1} \coloneqq P_D(\overline{z}_k - \tfrac{1}{\sigma_k} h(x_{k+1})),$$
$$\phi_{k+1} \coloneqq \phi_k,\ \psi_{k+1} \coloneqq \psi_k,\ \gamma_{k+1} \coloneqq \tfrac{1}{2}\gamma_k.$$

else

  Set ▷ Step 4

$$y_{k+1} \coloneqq y_k,\ z_{k+1} \coloneqq z_k,\ \overline{y}_{k+1} \coloneqq \overline{y}_k,\ \overline{z}_{k+1} \coloneqq \overline{z}_k,$$
$$\phi_{k+1} \coloneqq \phi_k,\ \psi_{k+1} \coloneqq \psi_k,\ \gamma_{k+1} \coloneqq \gamma_k.$$

end if

return $(y_{k+1}, z_{k+1}, \overline{y}_{k+1}, \overline{z}_{k+1}, \phi_{k+1}, \psi_{k+1}, \gamma_{k+1})$.

In Steps 1 and 2 of Algorithm 1, we check whether or not statement (i) holds. Note that statement (i) implies that $\{\Phi(x_k, \widetilde{y}_k, \widetilde{z}_k)\}$ or $\{\Psi(x_k, \widetilde{y}_k, \widetilde{z}_k)\}$ converges to zero, and $\{\widetilde{y}_k\}$ and $\{\widetilde{z}_k\}$ are bounded. These facts motivate us to adopt the if-statements of Steps 1 and 2. In this case, the trial point $(x_{k+1}, \widetilde{y}_{k+1}, \widetilde{z}_{k+1})$ has a good tendency, and hence we set $(y_{k+1}, z_{k+1}) \coloneqq (\widetilde{y}_{k+1}, \widetilde{z}_{k+1})$ and $(\overline{y}_{k+1}, \overline{z}_{k+1}) \coloneqq (\widetilde{y}_{k+1}, \widetilde{z}_{k+1})$. Moreover, we decrease ϕ_k or ψ_k to get a better point in the next iteration.

Step 3 checks whether or not statement (ii) holds. Recall that we can regard F as the augmented Lagrangian. In other words, this step tries to solve the subproblem of augmented Lagrangian methods:

$$\underset{x \in W}{\text{Minimize}}\ \ F(x; \overline{y}_k, \overline{z}_k, \sigma_k). \tag{20}$$

Hence, we update the main Lagrange multiplier sequence $\{(y_k, z_k)\}$ like them, that is, we set $(y_{k+1}, z_{k+1}) \coloneqq (\overline{y}_k - \frac{1}{\sigma_k} g(x_{k+1}), [\overline{z}_k - \frac{1}{\sigma_k} h(x_{k+1})]_+)$. On the other hand, this case has a possibility that the sub-Lagrange multipliers sequence $\{(\overline{y}_k, \overline{z}_k)\}$ diverges as iterations progress. Therefore, we adopt the updating rule with the safeguard to guarantee the boundedness, i.e., $(\overline{y}_{k+1}, \overline{z}_{k+1}) := (P_C(\overline{y}_k - \frac{1}{\sigma_k} g(x_{k+1})), P_D(\overline{z}_k - \frac{1}{\sigma_k} h(x_{k+1})))$. Moreover, we decrease γ_k in order to obtain a more accurate stationary point of F.

Step 4 means that there is no tendency of statements (i) and (ii). As stated above, Algorithm 1 does not update two types of the Lagrange multipliers (y_k, z_k) and $(\overline{y}_k, \overline{z}_k)$, and the parameters ϕ_k, ψ_k, and γ_k. In the global convergence analysis provided in Section 4, we can show that there does not occur a situation that Algorithm 1 performs Step 4 infinitely many times.

Since problem (1) does not have to satisfy any kind of CQs, there is a possibility that the proposed method cannot obtain any KKT points. Even so, the proposed method is designed so that it can obtain a stationary point of the merit functional F. Moreover, this design leads to a convergence property to AKKT points. Step 3 is devised for this purpose. As stated above, this step solves problem (20) which can be regarded as the subproblem of the augmented Lagrangian method. Therefore, it is reasonable to design an updating rule of the penalty parameter σ_k in a manner similar to the augmented Lagrangian method, namely, the following rule is adopted:

$$\sigma_{k+1} := \begin{cases} \min\{\frac{1}{2}\sigma_k, r(x_k, y_k, z_k)^{\frac{3}{2}}\} & \text{if } \|F'(x_{k+1}; \overline{y}_k, \overline{z}_k, \sigma_k)\|_W \leq \gamma_k, \\ \sigma_k & \text{otherwise.} \end{cases} \tag{21}$$

The term $r(x_k, y_k, z_k)^{\frac{3}{2}}$ in (21) helps us to achieve fast local convergence. This term is also used in that of [15].

Remark 1. *Recall that the ordinary SQP methods simultaneously update the Lagrange multipliers when determining the search direction. If the Lagrange multipliers are updated in Step* 1 *or* 2 *of Algorithm* 1*, they are set as the trial Lagrange multipliers that have already been obtained in the previous step to determine the search direction. Namely, there is no essential delay in updating the Lagrange multipliers in this case because the updating order of the primal iterate and Lagrange multipliers is the same as the ordinary SQP methods. Meanwhile, Step* 3 *is based on the updating rule of the existing AL methods as described above. Hence, there is a possibility that the Lagrange multipliers are updated after the new primal iterate has been calculated as seen in the existing AL methods, such as* [23, Algorithm 3.1]*. However, the delay in updating them plays an important role in the global convergence regarding the AKKT points.*

3.4. Formal statement of a stabilized SQP method

By summarizing the description in the above sections, we propose a stabilized SQP method for problem (1) as Algorithm 2.

Remark 2. *In Algorithm* 2*, the calculations of Lines* 2–5 *can be omitted when* $F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k) = 0$*. This is motivated by Proposition* 3*, which ensures that* $(0, [\overline{z}_k - \frac{1}{\sigma_k} h(x_k)]_+) \in \mathcal{V}$ *is the unique optimal solution of problem* (12) *if and only if* $F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k) = 0$*. In this case, we set* $p_k := 0$*,* $\widetilde{y}_{k+1} := \overline{y}_k - \frac{1}{\sigma_k} g(x_k)$*, and* $\widetilde{z}_{k+1} := [\overline{z}_k - \frac{1}{\sigma_k} h(x_k)]_+$ *and can proceed to Step* 2 *without performing Lines* 2–5 *for saving their computational cost.*

Algorithm 2

Require: Select $\beta \in (0,1)$, $\varepsilon \in (0,1)$, $\rho \in (0,1)$, and $\kappa \in (0,1)$. Take a monotonically non-decreasing sequence $\{c_k\} \subset (0,1)$. Choose $(x_0, y_0, z_0) \in X \times U \times V$ such that $y_0 \in C$ and $z_0 \in D$, where $C \subset U$ and $D \subset V$ are used in Algorithm 1. Set $\overline{y}_0 \coloneqq y_0$, $\overline{z}_0 \coloneqq z_0$, $\sigma_0 > 0$, $\phi_0 > 0$, $\psi_0 > 0$, $\gamma_0 > 0$, and $k \coloneqq 0$.

repeat

Set H_k so that $((H_k + \frac{1}{\sigma_k} g'(x_k)^* g'(x_k))\cdot, \cdot)_W$ is coercive. ▷ Step 1

Set (x, σ, M, s, t, c) as follows:

$$x \coloneqq x_k,\ \ \sigma \coloneqq \sigma_k,\ \ M \coloneqq H_k + \tfrac{1}{\sigma_k} g'(x_k)^* g'(x_k),$$
$$s \coloneqq \overline{y}_k - \tfrac{1}{\sigma_k} g(x_k),\ \ t \coloneqq \overline{z}_k - \tfrac{1}{\sigma_k} h(x_k),\ \ c \coloneqq c_k.$$

Obtain $(\widetilde{\xi}, \widetilde{\lambda}) \in X \times V$ satisfying (14) and (15) by solving (12).

Set $(p_k, \widetilde{y}_{k+1}, \widetilde{z}_{k+1})$ as follows:

$$p_k \coloneqq \widetilde{\xi},\ \ \widetilde{y}_{k+1} \coloneqq y - \tfrac{1}{\sigma}(g(x) + g'(x)\widetilde{\xi}),\ \ \widetilde{z}_{k+1} \coloneqq [\widetilde{\lambda}]_+.$$

Compute the smallest nonnegative integer ℓ_k such that (17) holds.

Set x_{k+1} as follows: ▷ Step 2

$$x_{k+1} \coloneqq x_k + \beta^{\ell_k} p_k.$$

Set $(y_{k+1}, z_{k+1}, \overline{y}_{k+1}, \overline{z}_{k+1}, \phi_{k+1}, \psi_{k+1}, \gamma_{k+1})$ by Algorithm 1. ▷ Step 3

Set σ_{k+1} by (21).

Set $k \leftarrow k + 1$.

until (x_k, y_k, z_k) satisfies some stopping criterion.

3.5. Well-definedness of Algorithm 2

In this section, we prove that Step 1 of Algorithm 2 is well defined, that is, we show the following proposition.

Proposition 4. *If $F'(x; y, z, \sigma) \neq 0$, then there exists $(\widetilde{\xi}, \widetilde{\lambda}) \in X \times V$ such that conditions* (14) *and* (15) *hold.*

To prove the above proposition, we begin by defining some sequences. Recall that X is dense in W and Proposition 2 holds. Let $\{(\xi_j, \zeta_j, \lambda_j)\} \subset X \times V \times V$ be an arbitrary sequence such that $\|\xi_j - \xi_*\|_W \to 0$, $\|\zeta_j - \zeta_*\|_V \to 0$, and $\|\lambda_j - \lambda_*\|_V \to 0$ as $j \to \infty$, where $(\xi_*, \zeta_*, \lambda_*) \in W \times V \times V$ satisfies the following KKT conditions for problem (12):

$$\begin{gathered} M\xi_* + f'(x) - g'(x)^* s - h'(x)^* \lambda_* = 0, \quad \sigma(\zeta_* - \lambda_*) = 0, \\ h'(x)\xi_* + \sigma(\zeta_* - t) \geq 0, \quad \lambda_* \geq 0, \quad (h'(x)\xi_* + \sigma(\zeta_* - t), \lambda_*)_V = 0. \end{gathered} \tag{22}$$

In addition, we define a sequence $\{(\eta_j, \theta_j, \omega_j)\} \subset W \times V \times V$ concerning a violation error for (22) by

$$\eta_j := M\xi_j + f'(x) - g'(x)^*s - h'(x)^*\lambda_j, \tag{23}$$
$$\theta_j := \sigma(\zeta_j - \lambda_j), \tag{24}$$
$$\omega_j := h'(x)\xi_j + \sigma(\zeta_j - t) \tag{25}$$

for each $j \in \mathbb{N} \cup \{0\}$. From these definitions and (22), the sequences $\{(\xi_j, \zeta_j, \lambda_j)\}$ and $\{(\eta_j, \theta_j, \omega_j)\}$ satisfy the following conditions:

$$\begin{gathered}\lim_{j\to\infty} \|\xi_j - \xi_*\|_W = 0,\ \ \lim_{j\to\infty} \|\zeta_j - \zeta_*\|_V = 0,\ \ \lim_{j\to\infty} \|\lambda_j - \lambda_*\|_V = 0,\\ \lim_{j\to\infty} \|\eta_j\|_W = 0,\ \ \lim_{j\to\infty} \|\theta_j\|_V = 0,\ \ \lim_{j\to\infty} \|\omega_j - \omega_*\|_V = 0,\\ \lim_{j\to\infty} (\omega_j, \lambda_j)_V = 0,\ \zeta_* \geq 0,\ \lambda_* \geq 0,\ \omega_* \geq 0.\end{gathered} \tag{26}$$

Moreover, we often use

$$\begin{aligned}R_j := {}&(\eta_j, \xi_j)_W + (\omega_j, \lambda_j - [t]_+)_V\\ &+ \sigma(\lambda_j - [t]_+, t - [t]_+)_V - (\lambda_j - [t]_+, \theta_j)_V\end{aligned} \tag{27}$$

for each $j \in \mathbb{N} \cup \{0\}$. Before proving Proposition 4, we prepare two lemmas below.

Lemma 1. *Assume that $R_j \leq (1-c)((M\xi_j, \xi_j)_W + \sigma\|\lambda_j - [t]_+\|_V^2)$ holds. Then, the pair (ξ_j, λ_j) satisfies that $(F'(x; y, z, \sigma), \xi_j)_W \leq -c(M\xi_j, \xi_j)_W - c\sigma\|\lambda_j - [t]_+\|_V^2$.*

Proof. Using (13) and (23) yields

$$(F'(x; y, z, \sigma), \xi_j)_W = -(M\xi_j, \xi_j)_W + (\eta_j, \xi_j)_W + (\lambda_j - [t]_+, h'(x)\xi_j)_V. \tag{28}$$

Meanwhile, we have from (25) that

$$\begin{aligned}(\lambda_j - [t]_+, h'(x)\xi_j)_V = {}&(\lambda_j - [t]_+, \omega_j)_V + \sigma(\lambda_j - [t]_+, t - [t]_+)_V\\ &+ \sigma(\lambda_j - [t]_+, [t]_+ - \zeta_j)_V.\end{aligned} \tag{29}$$

Since $\sigma(\lambda_j - [t]_+, [t]_+ - \zeta_j)_V = -\sigma\|\lambda_j - [t]_+\|_V^2 - (\lambda_j - [t]_+, \theta_j)_V$ by (24), substituting this equality into (29) derives

$$\begin{aligned}(\lambda_j - [t]_+, h'(x)\xi_j)_V = {}&(\omega_j, \lambda_j - [t]_+)_V + \sigma(\lambda_j - [t]_+, t - [t]_+)_V\\ &- (\lambda_j - [t]_+, \theta_j)_V - \sigma\|\lambda_j - [t]_+\|_V^2.\end{aligned} \tag{30}$$

Now, we obtain $(F'(x; y, z, \sigma), \xi_j)_W = -(M\xi_j, \xi_j)_W - \sigma\|\lambda_j - [t]_+\|_V^2 + R_j$ from (27), (28), and (30). It then follows from $R_j \leq (1-c)((M\xi_j, \xi_j)_W + \sigma\|\lambda_j - [t]_+\|_V^2)$ that the desired inequality holds. □

By exploiting Lemma 1, we prove that $F'(x; y, z, \sigma) \neq 0$ is a sufficient condition under which ξ_j is a descent direction of F for sufficiently large $j \in \mathbb{N}$.

Lemma 2. *If $F'(x; y, z, \sigma) \neq 0$, then the following statements hold:*

(i) *There exists $\epsilon > 0$ such that for every $j \in \mathbb{N}$, the pair (ξ_j, λ_j) satisfies that $\epsilon \leq (M\xi_j, \xi_j)_W + \sigma\|\lambda_j - [t]_+\|_V^2$;*

(ii) *there exists* $\widetilde{m} \in \mathbb{N}$ *such that for every* $j \geq \widetilde{m}$, *the pair* (ξ_j, λ_j) *satisfies that* $(F'(x; y, z, \sigma), \xi_j)_W \leq -c(M\xi_j, \xi_j)_W - c\sigma\|\lambda_j - [t]_+\|_V^2$.

Proof. First, we prove item (i) by contradiction. Assume that there exists $\mathcal{M} \subset \mathbb{N}$ such that $\lim_{j\to\infty, j\in\mathcal{M}}((M\xi_j, \xi_j)_W + \sigma\|\lambda_j - [t]_+\|_V^2) = 0$. Hence, the coerciveness of B and (26) imply that $\xi_* = 0$ and $\lambda_* = [t]_+$. It then follows from (26), (23), and (13) that $0 = \lim_{j\to\infty, j\in\mathcal{M}} \eta_j = M\xi_* + f'(x) - g'(x)^* s - h'(x)^*\lambda_* = F'(x; y, z, \sigma)$. This contradicts $F'(x; y, z, \sigma) \neq 0$.

Next, we show item (ii). Note that (26) holds. We see that $(\omega_*, [t]_+)_V \geq 0$ by $\omega_* \geq 0$. Since $[\cdot]_+ \colon V \to K_V$ is a projection mapping, we also have $(\lambda_* - [t]_+, t - [t]_+)_V \leq 0$ from $\lambda_* \geq 0$. These facts and (27) yield that

$$R_* \coloneqq \lim_{j\to\infty} R_j = -(\omega_*, [t]_+)_V + \sigma(\lambda_* - [t]_+, t - [t]_+)_V \leq 0. \tag{31}$$

Hence, there exists $\widetilde{m} \in \mathbb{N}$ such that $|R_j - R_*| \leq (1-c)\epsilon$ for all $j \geq \widetilde{m}$, where ϵ is a positive number described in item (i). It then follows from item (i) and (31) that $R_j \leq R_* + (1-c)\epsilon \leq (1-c)((M\xi_j, \xi_j)_W + \sigma\|\lambda_j - [t]_+\|_V^2)$ for all $j \geq \widetilde{m}$. Therefore, Lemma 1 derives the desired result. □

In what follows, we prove Proposition 4 by using Lemma 2.

Proof of Proposition 4. From items (i) and (ii) of Lemma 2, there exist $\epsilon > 0$ and $\widetilde{m} \in \mathbb{N}$ such that $c\epsilon \leq c(M\xi_j, \xi_j)_W + c\sigma\|\lambda_j - [t]_+\|_V^2 \leq -(F'(x; y, z, \sigma), \xi_j)_W$ for all $j \geq \widetilde{m}$, and hence

$$c\epsilon \leq |(F'(x; y, z, \sigma), \xi_j)_W| \quad \forall j \geq \widetilde{m}. \tag{32}$$

Since $\|\eta_j\|_W \to 0$ $(j \to \infty)$ from (26), there exists $\widetilde{n} \in \mathbb{N}$ such that $\|\eta_j\|_W \leq c\epsilon$ for all $j \geq \widetilde{n}$. This fact, (23), and (32) yield that

$$\|M\xi_j + f'(x) - g'(x)^* s - h'(x)^*\lambda_j\|_W \leq |(F'(x; y, z, \sigma), \xi_j)_W| \quad \forall j \geq \widetilde{j},$$

where $\widetilde{j} \coloneqq \max\{\widetilde{m}, \widetilde{n}\}$. If we identify $(\widetilde{\xi}, \widetilde{\lambda})$ as $(\xi_{\widetilde{j}}, \lambda_{\widetilde{j}})$, then item (ii) of Lemma 2 and the above inequality ensure that the pair $(\widetilde{\xi}, \widetilde{\lambda})$ satisfies (14) and (15). □

4. Global convergence of Algorithm 2

In what follows, we prove the global convergence of Algorithm 2. To begin with, we make several assumptions and define some notation used throughout this section.

Assumption 1.

(A1) *f, g, and h are twice continuously Fréchet differentiable on W;*
(A2) *$\varphi \in Z \mapsto [\varphi]_+ \in Z$ is well-defined and continuous on Z;*
(A3) *there exists $\nu > 0$ such that for $u \in W$, $v \in W$, and $k \in \mathbb{N} \cup \{0\}$,*

$$\tfrac{1}{\nu}\|u\|_W^2 \leq ((H_k + \tfrac{1}{\sigma_k} g'(x_k)^* g'(x_k))u, u)_W, \ (H_k u, v)_W \leq \nu\|u\|_W\|v\|_W.$$

Furthermore, we suppose that Algorithm 2 generates an infinite set of iterations.

Now, we recall that M_k, s_k, and t_k are defined by (16). For simplicity, let η_k be defined by

$$\eta_k := M_k p_k + f'(x_k) - g'(x_k)^* s_k - h'(x_k)^* \lambda_k, \tag{33}$$

where λ_k is defined as $\lambda_k := \widetilde{\lambda}$, and we notice that $\widetilde{\lambda}$ is generated in Step 1 of Algorithm 2. From Step 1, it is clear that

$$(F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k), p_k)_W \leq -c_k (M_k p_k, p_k)_W - c_k \sigma_k \|\lambda_k - [t_k]_+\|_V^2, \tag{34}$$

$$\|\eta_k\|_W \leq |(F'(x_k; \overline{y}_k, \overline{z}_k, \sigma_k), p_k)_W|. \tag{35}$$

Because Step 3 of Algorithm 2 is divided into Steps 1–4 of Algorithm 1, we call them Steps 3.1–3.4, respectively. For the convergence analysis, we divide $\mathbb{N} \cup \{0\}$ into three mutually disjoint sets $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ defined by

$$\mathcal{I} := \{k \in \mathbb{N} \cup \{0\}; y_k, z_k, \overline{y}_k, \overline{z}_k, \phi_k, \psi_k, \text{ and } \gamma_k \text{ are updated by Step 3.1 or 3.2}\},$$
$$\mathcal{J} := \{k \in \mathbb{N} \cup \{0\}; y_k, z_k, \overline{y}_k, \overline{z}_k, \phi_k, \psi_k, \text{ and } \gamma_k \text{ are updated by Step 3.3}\},$$
$$\mathcal{K} := \{k \in \mathbb{N} \cup \{0\}; y_k, z_k, \overline{y}_k, \overline{z}_k, \phi_k, \psi_k, \text{ and } \gamma_k \text{ are updated by Step 3.4}\},$$

respectively.

Throughout this section, C and D denote the sets that appear in Algorithms 1 and 2, that is to say, C is the bounded convex set in U, and D is the set represented by $D = \{z \in V; 0 \leq z \leq z_{\max}\}$, where $z_{\max} > 0$ is a constant number.

The next lemma provides some properties regarding the sequences $\{\overline{y}_k\}$, $\{\overline{z}_k\}$, $\{\phi_k\}$, $\{\psi_k\}$, $\{\gamma_k\}$, and $\{\sigma_k\}$.

Lemma 3. *The following statements hold:*

(i) *If* $\mathrm{card}(\mathcal{I}) = \infty$*, then* $\phi_k \to 0$ *or* $\psi_k \to 0$ *as* $k \to \infty$*;*
(ii) *if* $\mathrm{card}(\mathcal{J}) = \infty$*, then* $\gamma_k \to 0$ *and* $\sigma_k \to 0$ *as* $k \to \infty$*;*
(iii) $\{\overline{y}_k\}$ *and* $\{\overline{z}_k\}$ *are bounded sequences included in* C *and* D*, respectively.*

Proof. To begin with, we prove item (i). If $k \in \mathcal{I}$, then $\phi_{k+1} = \frac{1}{2}\phi_k$ or $\psi_{k+1} = \frac{1}{2}\psi_k$ from Steps 3.1 and 3.2. If $k \in \mathcal{J} \cup \mathcal{K}$, then $\phi_{k+1} = \phi_k$ and $\psi_{k+1} = \psi_k$ from Steps 3.3 and 3.4. Considering these facts and $\mathrm{card}(\mathcal{I}) = \infty$ yields $\phi_k \to 0$ or $\psi_k \to 0$ as $k \to \infty$.

Next, we show item (ii). Let $k \in \mathcal{J}$. From the updating rule of Algorithm 1, we see $\gamma_{k+1} = \frac{1}{2}\gamma_k$ and $\|F'(x_{k+1}; \overline{y}_k, \overline{z}_k, \sigma_k)\|_W \leq \gamma_k$. The second inequality and (21) imply $\sigma_{k+1} \leq \frac{1}{2}\sigma_k$. Since $\{\gamma_k\}$ and $\{\sigma_k\}$ are non-increasing, it then follows from $\mathrm{card}(\mathcal{J}) = \infty$ that $\gamma_k \to 0$ and $\sigma_k \to 0$ as $k \to \infty$.

We finally provide a proof of item (iii). Since $C \subset U$ and $D \subset V$ are bounded sets, it is sufficient to show that $\overline{y}_k \in C$ and $\overline{z}_k \in D$ for all $k \in \mathbb{N} \cup \{0\}$. We prove this assertion by mathematical induction. If $k = 0$, then $\overline{y}_0 \in C$ and $\overline{z}_0 \in D$. Now, assume that $k \in \mathbb{N} \cup \{0\}$, $\overline{y}_k \in C$, and $\overline{z}_k \in D$. Note that $k + 1 \in \mathcal{I} \cup \mathcal{J} \cup \mathcal{K} = \mathbb{N}$. It then follows from Steps 3.1–3.4 that $\overline{y}_{k+1} \in C$ and $\overline{z}_{k+1} \in D$. Therefore, we arrive at the desired result. □

From now on, we focus on a situation with $\mathrm{card}(\mathcal{I}) < \infty$, $\mathrm{card}(\mathcal{J}) < \infty$, and $\mathrm{card}(\mathcal{K}) = \infty$. The next lemma shows some properties under this situation.

Lemma 4. *Suppose that* (A1) *and* (A3) *of Assumption* 1 *are satisfied, and* $\{x_k\}$ *converges to* x_* *in* X*. If* $\mathrm{card}(\mathcal{I}) < \infty$*,* $\mathrm{card}(\mathcal{J}) < \infty$*, and* $\mathrm{card}(\mathcal{K}) = \infty$*, then the* *following statements hold:*

(i) *There exist* $\widehat{k} \in \mathbb{N}$, $\widehat{y} \in U$, $\widehat{z} \in V$, $\widehat{\sigma} \in \mathbb{R}$, *and* $\widehat{\gamma} \in \mathbb{R}$ *such that* $k \in \mathcal{K}$, $\overline{y}_k = \widehat{y}$, $\overline{z}_k = \widehat{z}$, $\sigma_k = \widehat{\sigma}$, *and* $\gamma_k = \widehat{\gamma}$ *for all* $k \geq \widehat{k}$;

(ii) $\{p_k\}_{k \geq \widehat{k}}$ *is bounded in* W;

(iii) $\liminf_{k\to\infty} |\Delta_k| > 0$.

Proof. We prove item (i). Since $\mathrm{card}(\mathcal{I}) < \infty$, $\mathrm{card}(\mathcal{J}) < \infty$, and $\mathrm{card}(\mathcal{K}) = \infty$ are satisfied, there exists $\widehat{k} \in \mathbb{N}$ such that $k \in \mathcal{K}$ for all $k \geq \widehat{k}$. This fact means that Step 4 of Algorithm 1 is performed for all $k \geq \widehat{k}$, and hence we obtain $\overline{y}_k = \widehat{y}$, $\overline{z}_k = \widehat{z}$, and $\gamma_k = \widehat{\gamma}$ for all $k \geq \widehat{k}$, where $\widehat{y} := \overline{y}_{\widehat{k}}$, $\widehat{z} := \overline{z}_{\widehat{k}}$, and $\widehat{\gamma} := \gamma_{\widehat{k}}$. Furthermore, it follows from (21) that $\sigma_k = \widehat{\sigma}$ for all $k \geq \widehat{k}$, where $\widehat{\sigma} := \sigma_{\widehat{k}}$.

Next, we show item (ii). In the following, note that $Y \hookrightarrow U \hookrightarrow Y^*$ and $Z \hookrightarrow V \hookrightarrow Z^*$ are often used. Suppose that $k \geq \widehat{k}$, namely, item (i) holds. Notice that $\{c_k\} \subset (0,1)$ is a monotonically non-decreasing sequence, that is, $c_0 \leq c_k < 1$ for all $k \in \mathbb{N}$. Then, we have from (A3) of Assumption 1 and (34) that

$$\frac{c_0}{\nu}\|p_k\|_W^2 + c_0\widehat{\sigma}\|\lambda_k - [t_k]_+\|_V^2 \leq |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}), p_k)_W|. \tag{36}$$

Exploiting (13), $s_k \in U$, and $t_k \in V$ yields

$$\begin{aligned} &|(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}), p_k)_W| \\ &\quad \leq |(f'(x_k), p_k)_W| + |\langle s_k, g'(x_k)p_k\rangle_{Y^*,Y}| + |\langle [t_k]_+, h'(x_k)p_k\rangle_{Z^*,Z}| \\ &\quad \leq |(f'(x_k), p_k)_W| + \|s_k\|_{Y^*}\|g'(x_k)p_k\|_Y + \|[t_k]_+\|_{Z^*}\|h'(x_k)p_k\|_Z \\ &\quad \lesssim \left(\|f'(x_k)\|_W + \|s_k\|_U\|g'(x_k)\|_{W\to Y} + \|t_k\|_V\|h'(x_k)\|_{W\to Z}\right)\|p_k\|_W, \end{aligned} \tag{37}$$

where (37) is derived from $\|s_k\|_{Y^*} \lesssim \|s_k\|_U$ and $\|[t_k]_+\|_{Z^*} \lesssim \|[t_k]_+\|_V \leq \|t_k\|_V$. It follows from (36) and (37) that

$$\|p_k\|_W \lesssim \|f'(x_k)\|_W + \|s_k\|_U\|g'(x_k)\|_{W\to Y} + \|t_k\|_V\|h'(x_k)\|_{W\to Z}. \tag{38}$$

Meanwhile, using $s_k = \widehat{y} - \frac{1}{\widehat{\sigma}}g(x_k)$ and $t_k = \widehat{z} - \frac{1}{\widehat{\sigma}}h(x_k)$ implies

$$\|s_k\|_U \lesssim \|s_k\|_Y \leq \|\widehat{y}\|_Y + \frac{1}{\widehat{\sigma}}\|g(x_k)\|_Y, \ \|t_k\|_V \lesssim \|t_k\|_Z \leq \|\widehat{z}\|_Z + \frac{1}{\widehat{\sigma}}\|h(x_k)\|_Z. \tag{39}$$

By $X \hookrightarrow W$, the sequential compactness of $\{x_k\} \subset X$, and (A1) of Assumption 1, there exists $\mathcal{R} > 0$ satisfying

$$\max\left\{\|f'(x_k)\|_W, \|g(x_k)\|_Y, \|g'(x_k)\|_{W\to Y}, \|h(x_k)\|_Z, \|h'(x_k)\|_{W\to Z}\right\} \leq \mathcal{R} < \infty. \tag{40}$$

We have from (38)–(40) that $\{p_k\}_{k\geq\widehat{k}}$ is bounded in W.

Finally, to show (iii), we begin by verifying $\liminf_{k\to\infty} |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}), p_k)_W| > 0$. Assume to the contrary that there exists $\mathcal{M} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty, k\in\mathcal{M}} |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}), p_k)_W| = 0. \tag{41}$$

Combining (35), (36), and (41) gives

$$\lim_{k\to\infty, k\in\mathcal{M}} \|\eta_k\|_W = 0, \quad \lim_{k\to\infty, k\in\mathcal{M}} \|p_k\|_W = 0, \quad \lim_{k\to\infty, k\in\mathcal{M}} \|\lambda_k - [t_k]_+\|_V = 0. \tag{42}$$

Let k be an arbitrary positive integer satisfying $k \geq \widehat{k}$. Since $\|F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma})\|_W = \sup\{|(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}),u)_W|; \|u\|_W \leq 1\} < \infty$, there exists $u_k \in W$ such that $\|u_k\|_W \leq 1$ and $\|F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma})\|_W < |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}),u_k)_W| + \frac{1}{k}$. Meanwhile, using (13) and (33) yields $F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}) = -(H_k + \frac{1}{\widehat{\sigma}} g'(x_k)^* g'(x_k))p_k + \eta_k + h'(x_k)^*(\lambda_k - [t_k]_+)$. By these facts and (A3) of Assumption 1, we obtain

$$\begin{aligned}
\|F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma})\|_W &< |(H_k p_k, u_k)_W| + \frac{1}{\widehat{\sigma}}|(g'(x_k)p_k, g'(x_k)u_k)_U| \\
&\qquad + |(\eta_k, u_k)_W| + |(\lambda_k - [t_k]_+, h'(x_k)u_k)_V| + \frac{1}{k} \\
&\leq \nu\|p_k\|_W + \frac{1}{\widehat{\sigma}}\|g'(x_k)p_k\|_U \|g'(x_k)u_k\|_U \\
&\qquad + \|\eta_k\|_W + \|\lambda_k - [t_k]_+\|_V \|h'(x_k)u_k\|_V + \frac{1}{k} \\
&\lesssim \left(\nu + \frac{1}{\widehat{\sigma}} \sup_{k\in\mathbb{N}} \|g'(x_k)\|^2_{W\to Y}\right)\|p_k\|_W \\
&\qquad + \|\eta_k\|_W + \|\lambda_k - [t_k]_+\|_V \sup_{k\in\mathbb{N}} \|h'(x_k)\|_{W\to Z} + \frac{1}{k},
\end{aligned} \tag{43}$$

where the last inequality is derived from $\|g'(x_k)p_k\|_U \lesssim \|g'(x_k)p_k\|_Y \leq \|g'(x_k)\|_{W\to Y}\|p_k\|_W$, $\|g'(x_k)u_k\|_U \lesssim \|g'(x_k)u_k\|_Y \leq \|g'(x_k)\|_{W\to Y}$, and $\|h'(x_k)u_k\|_V \lesssim \|h'(x_k)u_k\|_Z \leq \|h'(x_k)\|_{W\to Z}$. It follows from (40), (42), and (43) that $\|F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma})\|_W \to 0$ as $k\to\infty$, $k\in\mathcal{M}$. Hence, there exists $\overline{k}\in\mathbb{N}$ such that

$$\|F'(x_{\overline{k}+1};\widehat{y},\widehat{z},\widehat{\sigma})\|_W \leq \widehat{\gamma}, \quad \overline{k} \geq \widehat{k}. \tag{44}$$

On the other hand, item (i) shows $\overline{k}\in\mathcal{K}$, which means that the if-statement regarding Step 3.3 (Line 5 of Algorithm 1) is false, that is to say, $\|F'(x_{\overline{k}+1};\widehat{y},\widehat{z},\widehat{\sigma})\|_W > \widehat{\gamma}$. Since this result contradicts (44), we get $\liminf_{k\to\infty} |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}),p_k)_W| > 0$. Next, we verify $\liminf_{k\to\infty}\|p_k\|_W > 0$. We also assume to the contrary that there exists $\mathcal{N}\subset\mathbb{N}$ such that $\|p_k\|_W \to 0$ as $k\to\infty$, $k\in\mathcal{N}$. This assumption derives $|(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}),p_k)_W| \to 0$ as $k\to\infty$, $k\in\mathcal{N}$. Therefore, we can prove this case in a similar way to the above proof after (41). As a result, we have $\liminf_{k\to\infty} |(F'(x_k;\widehat{y},\widehat{z},\widehat{\sigma}),p_k)_W| > 0$ and $\liminf_{k\to\infty}\|p_k\|_W > 0$. These results and (18) guarantee $\liminf_{k\to\infty}|\Delta_k| > 0$. □

The following lemma guarantees that Algorithm 2 does not generate an infinite set of iterations satisfying $\mathrm{card}(\mathcal{I}) < \infty$, $\mathrm{card}(\mathcal{J}) < \infty$, and $\mathrm{card}(\mathcal{K}) = \infty$, namely, there exist infinitely many iterations included in $\mathcal{I}\cup\mathcal{J}$.

Lemma 5. *Suppose that* (A1) *and* (A3) *of Assumption* 1 *are satisfied, and* $\{x_k\}$ *converges to* x_* *in* X. *Then, there does not occur a situation such that* $\mathrm{card}(\mathcal{I}) < \infty$, $\mathrm{card}(\mathcal{J}) < \infty$, *and* $\mathrm{card}(\mathcal{K}) = \infty$.

Proof. We prove the assertion by contradiction. Assume that $\mathrm{card}(\mathcal{I}) < \infty$, $\mathrm{card}(\mathcal{J}) <$

∞, and $\mathrm{card}(\mathcal{K}) = \infty$. By item (i) of Lemma 4, there exist $\widehat{k} \in \mathbb{N}$, $\widehat{y} \in U$, $\widehat{z} \in V$, $\widehat{\sigma} \in \mathbb{R}$, and $\widehat{\gamma} \in \mathbb{R}$ such that $k \in \mathcal{K}$, $\overline{y}_k = \widehat{y}$, $\overline{z}_k = \widehat{z}$, $\sigma_k = \widehat{\sigma}$, and $\gamma_k = \widehat{\gamma}$ for all $k \geq \widehat{k}$. In what follows, we suppose that $k \geq \widehat{k}$. We can easily see that $(F'(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma}), p_k)_W \leq 0$ from (34), and hence (18) guarantees $\Delta_k \leq 0$. It then follows from (17) that

$$0 \leq -\varepsilon \beta^{\ell_k} \Delta_k \leq F(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma}) - F(x_{k+1}; \widehat{y}, \widehat{z}, \widehat{\sigma}). \tag{45}$$

The boundedness of $\{F(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma})\}_{k \geq \widehat{k}}$ is ensured by $X \hookrightarrow W$, the sequential compactness of $\{x_k\} \subset X$, and (A1) of Assumption 1. Moreover, $\{F(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma})\}_{k \geq \widehat{k}}$ is non-increasing. Combination of these facts and (45) implies $\lim_{k\to\infty} \beta^{\ell_k} \Delta_k = 0$. Therefore, there are two cases: $\liminf_{k\to\infty} \beta^{\ell_k} > 0$; $\liminf_{k\to\infty} \beta^{\ell_k} = 0$. The former case derives $\lim_{k\to\infty} \Delta_k = 0$. We further consider the latter case in the following. Since $\liminf_{k\to\infty} \beta^{\ell_k} = 0$, there exists $\mathcal{M} \subset \mathbb{N}$ such that $\lim_{k\to\infty, k\in\mathcal{M}} \ell_k = \infty$. For simplicity, we denote $\delta_k \coloneqq \beta^{\ell_k - 1} (> 0)$. Let $\widehat{F} \colon \mathbb{R} \to \mathbb{R}$ be defined by $\widehat{F}(\delta) \coloneqq F(x_k + \delta p_k; \widehat{y}, \widehat{z}, \widehat{\sigma})$. Then, note that $\widehat{F}'(\delta) = (F'(x_k + \delta p_k; \widehat{y}, \widehat{z}, \widehat{\sigma}), p_k)_W$. Without loss of generality, we can assume that $\ell_k \geq 1$ for all $k \in \mathcal{M}$ because $\lim_{k\to\infty,k\in\mathcal{M}} \ell_k = \infty$. Recall that ℓ_k is the smallest positive integer such that $\widehat{F}(\beta^{\ell_k}) \leq \widehat{F}(0) + \varepsilon \beta^{\ell_k} \Delta_k$. Since $\ell_k - 1$ does not satisfy this inequality, we obtain $\widehat{F}(0) + \varepsilon \delta_k \Delta_k < \widehat{F}(\delta_k)$. It then follows from $\widehat{F}'(0) = (F'(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma}), p_k)_W \leq \max\{(F'(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma}), p_k)_W, -\rho \|p_k\|_W^2\} = \Delta_k$ that $(\varepsilon - 1)\Delta_k < \frac{1}{\delta_k}(\widehat{F}(\delta_k) - \widehat{F}(0)) - \widehat{F}'(0)$. The mean value theorem ensures the existence of $\vartheta_k \in (0, 1)$ satisfying $\frac{1}{\delta_k}(\widehat{F}(\delta_k) - \widehat{F}(0)) = \widehat{F}'(\vartheta_k \delta_k)$, and hence we then have

$$0 \leq (\varepsilon - 1)\Delta_k < (F'(x_k + \vartheta_k \delta_k p_k; \widehat{y}, \widehat{z}, \widehat{\sigma}) - F'(x_k; \widehat{y}, \widehat{z}, \widehat{\sigma}), p_k)_W, \tag{46}$$

where the first inequality follows from $\varepsilon \in (0, 1)$ and $\Delta_k \leq 0$. The boundedness of $\{p_k\}_{k \geq \widehat{k}}$ is guaranteed by (ii) of Lemma 4. Moreover, $\lim_{k\to\infty,k\in\mathcal{M}} \delta_k = 0$ because $\delta_k = \beta^{\ell_k - 1}$, $\beta \in (0, 1)$, and $\lim_{k\to\infty,k\in\mathcal{M}} \ell_k = \infty$. These facts yield $\|(x_k + \vartheta_k \delta_k p_k) - x_*\|_W \leq \|x_k - x_*\|_W + \vartheta_k \delta_k \|p_k\|_W \lesssim \|x_k - x_*\|_X + \vartheta_k \delta_k \|p_k\|_W \to 0$ as $k \to \infty$, $k \in \mathcal{M}$. Then, the continuity of $F' \colon W \to W$ and (46) derive $\lim_{k\to\infty,k\in\mathcal{M}} \Delta_k = 0$. Therefore, we obtain $\liminf_{k\to\infty} |\Delta_k| = 0$. However, this result contradicts (iii) of Lemma 4. □

By exploiting the above lemmas, we provide some properties that play an important role in main convergence results.

Proposition 5. *Suppose that* (A1) *and* (A2) *of Assumption* 1 *hold, and* $\{x_k\}$ *converges to* x_* *in* X*. If* $\mathrm{card}(\mathcal{I}) = \infty$*, then there exist* $y_* \in U$*,* $z_* \in V$*, and* $\mathcal{M} \subset \mathbb{N}$ *such that* $y_k \rightharpoonup y_*$ *in* U *and* $z_k \rightharpoonup z_*$ *in* V *as* $k \to \infty$*,* $k \in \mathcal{M}$*, and* (x_*, y_*, z_*) *satisfies the KKT conditions of* (1)*.*

Proof. Let $\mathcal{P} \coloneqq \{k \in \mathbb{N}; k-1 \in \mathcal{I}\}$. It is clear that $\mathrm{card}(\mathcal{P}) = \infty$ by $\mathrm{card}(\mathcal{I}) = \infty$. Note that $\{x_k\} \subset X$ converges to x_* in W because $X \hookrightarrow W$. Note also that $\{y_k\}_{k\in\mathcal{P}} \subset U$ and $\{z_k\}_{k\in\mathcal{P}} \subset V$ are bounded because $\{y_k\}_{k\in\mathcal{P}} = \{\overline{y}_k\}_{k\in\mathcal{P}} \subset C$ and $\{z_k\}_{k\in\mathcal{P}} = \{\overline{z}_k\}_{k\in\mathcal{P}} \subset D$ from Steps 3.1 and 3.2 and item (iii) of Lemma 3. Hence, there exist

$\mathcal{M} \subset \mathcal{P}$, $y_* \in U$, and $z_* \in V$ such that

$$\begin{aligned} \lim_{k\to\infty, k\in\mathcal{M}} x_k &= x_* \ (W), \\ \lim_{k\to\infty, k\in\mathcal{M}} y_k &= y_* \ (U\text{–weak}), \\ \lim_{k\to\infty, k\in\mathcal{M}} z_k &= z_* \ (V\text{–weak}). \end{aligned} \tag{47}$$

From Steps 3.1 and 3.2, we can verify $\Phi(x_k, y_k, z_k) = \Phi(x_k, \widetilde{y}_k, \widetilde{z}_k) \le \frac{1}{2}\phi_{k-1} = \phi_k$ or $\Psi(x_k, y_k, z_k) = \Psi(x_k, \widetilde{y}_k, \widetilde{z}_k) \le \frac{1}{2}\psi_{k-1} = \psi_k$ for $k \in \mathcal{M}$. These results and item (i) of Lemma 3 derive $\Phi(x_k, y_k, z_k) \to 0$ or $\Psi(x_k, y_k, z_k) \to 0$ as $k \to \infty$, $k \in \mathcal{M}$, and therefore (19), (47), (A1), and (A2) imply

$$\|g(x_*)\|_Y = \lim_{k\to\infty, k\in\mathcal{M}} \|g(x_k)\|_Y = 0, \tag{48}$$

$$\|[-h(x_*)]_+\|_Z = \lim_{k\to\infty, k\in\mathcal{M}} \|[-h(x_k)]_+\|_Z = 0, \tag{49}$$

$$\lim_{k\to\infty, k\in\mathcal{M}} \|L_x(x_k, y_k, z_k)\|_{X^*} = 0, \tag{50}$$

$$\lim_{k\to\infty, k\in\mathcal{M}} |\langle z_k, h(x_k)\rangle_{Z^*,Z}| = 0. \tag{51}$$

Now recall that $X \hookrightarrow W \hookrightarrow X^*$, $Y \hookrightarrow U \hookrightarrow Y^*$, and $Z \hookrightarrow V \hookrightarrow Z^*$. Let $x \in X$ and $k \in \mathcal{M}$ be arbitrary. We then obtain

$$\begin{aligned} &|\langle L_x(x_*, y_*, z_*), x\rangle_{X^*,X}| \\ &\quad \le |\langle L_x(x_k, y_k, z_k), x\rangle_{X^*,X}| + |(L_x(x_*, y_*, z_*) - L_x(x_k, y_k, z_k), x)_W| \\ &\quad \lesssim \|L_x(x_k, y_k, z_k)\|_{X^*}\|x\|_X + \|f'(x_k) - f'(x_*)\|_W\|x\|_W \\ &\qquad + \|g'(x_k) - g'(x_*)\|_{W\to Y}\|x\|_W\|y_k\|_U + |(y_k - y_*, g'(x_*)x)_U| \\ &\qquad + \|h'(x_k) - h'(x_*)\|_{W\to Z}\|x\|_W\|z_k\|_V + |(z_k - z_*, h'(x_*)x)_V| \end{aligned} \tag{52}$$

and

$$\begin{aligned} &|\langle z_*, h(x_*)\rangle_{Z^*,Z}| \\ &\quad \le |(z_k, h(x_k))_V| + |(z_* - z_k, h(x_*))_V| + |(z_k, h(x_*) - h(x_k))_V| \\ &\quad \lesssim |\langle z_k, h(x_k)\rangle_{Z^*,Z}| + |(z_* - z_k, h(x_*))_V| + \|z_k\|_V\|h(x_*) - h(x_k)\|_Z. \end{aligned} \tag{53}$$

Exploiting (47), (50)–(53), and (A1) yields

$$L_x(x_*, y_*, z_*) = 0, \quad \langle z_*, h(x_*)\rangle_{Z^*,Z} = 0. \tag{54}$$

In the following, we show $z_* \in K_Z^+$. Note that $Z \hookrightarrow V \hookrightarrow Z^*$. Let $\varphi \in K_Z$ be arbitrary, namely, $\varphi \ge 0$. Meanwhile, recall that $z_k \ge 0$ for $k \in \mathcal{M}$ because $\{z_k\} \subset D$. These facts lead to $(z_k, \varphi)_V \ge 0$ for $k \in \mathcal{M}$. Since $\{z_k\}_{k\in\mathcal{M}} \subset K_V$ converges weakly to $z_* \in V$ from (47), we have $\langle z_*, \varphi\rangle_{Z^*,Z} = (z_*, \varphi)_V = \lim_{k\to\infty, k\in\mathcal{M}}(z_k, \varphi)_V \ge 0$, i.e., $z_* \in K_Z^+$. This result, (48), (49), and (54) mean that x_* is a KKT point of (1). □

Proposition 6. *Suppose that Assumption* 1 *holds. Suppose also that* $\{x_k\}$ *converges to* x_* *in* X*, and* x_* *is feasible to* (1)*. If* $\mathrm{card}(\mathcal{I}) < \infty$*, then* x_* *is an AKKT point* *of* (1), *and there exists* $\mathcal{N} \subset \mathbb{N}$ *such that* $\{(x_k, y_k, z_k)\}_{k\in\mathcal{N}}$ *is an AKKT sequence corresponding to* x_*.

Proof. Lemma 5 ensures that $\mathrm{card}(\mathcal{J}) = \infty$ because of $\mathrm{card}(\mathcal{I}) < \infty$. Now, we define $\mathcal{Q} \coloneqq \{k \in \mathbb{N}; k-1 \in \mathcal{J}\}$. Notice that $\mathrm{card}(\mathcal{Q}) = \infty$ by $\mathrm{card}(\mathcal{J}) = \infty$ and that

$$\{y_k\} \subset Y^*, \quad \{z_k\} \subset K_Z^+. \tag{55}$$

If $k \in \mathcal{Q}$, then $\|F'(x_k; \overline{y}_{k-1}, \overline{z}_{k-1}, \sigma_{k-1})\|_W \le \gamma_{k-1}$ for $k \in \mathcal{Q}$ because $k-1 \in \mathcal{J}$. These facts, $W \hookrightarrow X^*$, and (13) derive

$$\lim_{k\to\infty, k\in\mathcal{Q}} \|L_x(x_k, y_k, z_k)\|_{X^*} = \lim_{k\to\infty, k\in\mathcal{Q}} \|F'(x_k; \overline{y}_{k-1}, \overline{z}_{k-1}, \sigma_{k-1})\|_{X^*} = 0. \tag{56}$$

In the rest of the proof, we show that there exists $\mathcal{N} \subset \mathcal{Q}$ such that

$$\lim_{k\to\infty, k\in\mathcal{N}} \langle z_k, [h(x_k)]_+\rangle_{Z^*,Z} = 0. \tag{57}$$

From now, we use the following notation:

$$\begin{aligned} h(x_*) = (h_*^{(1)}, \dots, h_*^{(m)}), \quad h(x_k) = (h_k^{(1)}, \dots, h_k^{(m)}), \\ z_k = (z_k^{(1)}, \dots, z_k^{(m)}), \quad \overline{z}_k = (\overline{z}_k^{(1)}, \dots, \overline{z}_k^{(m)}). \end{aligned}$$

For each $j \in \{1, \dots, m\}$, we denote

$$\begin{aligned} \{h_*^{(j)} > 0\} = \{\tau \in \Omega_j; h_*^{(j)}(\tau) > 0\}, \quad \{h_*^{(j)} \le 0\} = \{\tau \in \Omega_j; h_*^{(j)}(\tau) \le 0\}, \\ \{h_k^{(j)} > 0\} = \{\tau \in \Omega_j; h_k^{(j)}(\tau) > 0\}, \quad \{h_k^{(j)} \le 0\} = \{\tau \in \Omega_j; h_k^{(j)}(\tau) \le 0\}. \end{aligned}$$

Let $j \in \{1, \dots, m\}$ be an arbitrary integer. Since $\|x_k - x_*\|_W \lesssim \|x_k - x_*\|_X \to 0$ $(k \to \infty, k \in \mathcal{Q})$ because $X \hookrightarrow W$, (A1) of Assumption 1 and $Z \hookrightarrow V$ guarantee that $h_k^{(j)} \to h_*^{(j)}$ $(k \to \infty, k \in \mathcal{Q})$ in $V_j = L^2(\Omega_j)$, and hence there exist $\mathcal{Q}_j \subset \mathcal{Q}$ and $\widetilde{h}^{(j)} \in L^2(\Omega_j)$ such that

$$|h_k^{(j)}| \le \widetilde{h}^{(j)} \quad \forall k \in \mathcal{Q}_j, \tag{58}$$

$$\lim_{k\to\infty, k\in\mathcal{Q}_j} h_k^{(j)} = h_*^{(j)} \quad \text{a.e. in } \Omega_j. \tag{59}$$

Let $\widehat{h}^{(j)} \coloneqq z_{\max}\widetilde{h}^{(j)} \in L^1(\Omega_j)$ and let $k \in \mathbb{N} \cup \{0\}$ be an arbitrary integer, where we notice that $\widetilde{h}^{(j)} \in L^2(\Omega_j) \subset L^1(\Omega_j)$. By item (iii) of Lemma 3, we know $\overline{z}_k^{(j)} \le z_{\max}$. This fact and (58) imply $|z_k^{(j)}[h_k^{(j)}]_+| = [\overline{z}_{k-1}^{(j)} - \frac{1}{\sigma_{k-1}} h_k^{(j)}]_+ h_k^{(j)} \le z_{\max}\widetilde{h}^{(j)} = \widehat{h}^{(j)}$ in $\{h_k^{(j)} > 0\}$. Moreover, it is clear that $|z_k^{(j)}[h_k^{(j)}]_+| = 0 \le z_{\max}\widetilde{h}^{(j)} = \widehat{h}^{(j)}$ in $\{h_k^{(j)} \le 0\}$. These results and $\Omega_j = \{h_k^{(j)} > 0\} \cup \{h_k^{(j)} \le 0\}$ ensure that

$$\widehat{h}^{(j)} \in L^1(\Omega_j), \quad |z_k^{(j)}[h_k^{(j)}]_+| \le \widehat{h}^{(j)} \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{60}$$

Now, note that the following conditions are sufficient ones for (57):

$$\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}[h_k^{(j)}]_+ = 0 \quad \text{a.e. in } \{h_*^{(j)} > 0\}, \tag{61}$$

$$\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}[h_k^{(j)}]_+ = 0 \quad \text{a.e. in } \{h_*^{(j)} \leq 0\}. \tag{62}$$

Indeed, if (61) and (62) hold, then $\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}[h_k^{(j)}]_+ = 0$ a.e. in Ω_j because $\Omega_j = \{h_*^{(j)} > 0\} \cup \{h_*^{(j)} \leq 0\}$. Then, letting $\mathcal{N} := \cap_{j=1}^m \mathcal{Q}_j$ and combining (60) with Lebesgue's dominated convergence theorem derive

$$\lim_{k\to\infty, k\in\mathcal{N}} \langle z_k, [h(x_k)]_+\rangle_{Z^*,Z} = \sum_{j=1}^m \int_{\Omega_j} \left(\lim_{k\to\infty, k\in\mathcal{N}} z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+ \right) d\tau = 0,$$

that is, (57) holds. Therefore, we prove that (61) and (62) are satisfied. In the following, the Lebesgue measure on Ω_j is represented as μ_j.

Now, we show (61). Since (61) clearly holds when $\mu_j(\{h_*^{(j)} > 0\}) = 0$, we suppose that $\mu_j(\{h_*^{(j)} > 0\}) > 0$. Let us define $E_j \subset \{h_*^{(j)} > 0\}$ by

$$E_j := \left\{ \tau; \{h_k^{(j)}(\tau)\}_{k\in\mathcal{Q}_j} \not\to h_*^{(j)}(\tau) \text{ or } \sup_{k\in\mathbb{N}} \left\{ \overline{z}_k^{(j)}(\tau) \right\} > z_{\max} \right\}.$$

We get $\mu_j(E_j) = 0$ from item (iii) of Lemma 3 and (59). Notice that $\{h_*^{(j)} > 0\} \backslash E_j \neq \emptyset$ because $\mu_j(\{h_*^{(j)} > 0\}) > 0$. We arbitrarily take $\tau \in \{h_*^{(j)} > 0\} \backslash E_j$. Then, it can be easily seen that $\lim_{k\to\infty, k\in\mathcal{Q}_j} h_k^{(j)}(\tau) = h_*^{(j)}(\tau)$ and $\sup_{k\in\mathbb{N}}\{\overline{z}_k^{(j)}(\tau)\} \leq z_{\max}$. Moreover, item (ii) of Lemma 3 ensures $\lim_{k\to\infty, k\in\mathcal{Q}_j} \sigma_k = 0$. Hence, there exists $n_j \in \mathbb{N}$ such that $\frac{1}{2}h_*^{(j)}(\tau) \leq h_k^{(j)}(\tau)$ and $\sigma_{k-1} \leq \frac{1}{2z_{\max}} h_*^{(j)}(\tau)$ for $k \in \{k \in \mathcal{Q}_j; k \geq n_j\}$. These results guarantee that

$$\begin{aligned} z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+ &= \left[\overline{z}_{k-1}^{(j)}(\tau) - \frac{1}{\sigma_{k-1}} h_k^{(j)}(\tau) \right]_+ [h_k^{(j)}(\tau)]_+ \\ &\leq \left[z_{\max} - \frac{1}{2\sigma_{k-1}} h_*^{(j)}(\tau) \right]_+ [h_k^{(j)}(\tau)]_+ \\ &= \frac{z_{\max}}{\sigma_{k-1}} \left[\sigma_{k-1} - \frac{1}{2z_{\max}} h_*^{(j)}(\tau) \right]_+ [h_k^{(j)}(\tau)]_+ = 0 \end{aligned}$$

for all $k \in \{k \in \mathcal{Q}_j; k \geq n_j\}$. To sum up, there exists $E_j \subset \{h_*^{(j)} > 0\}$ such that $\mu_j(E_j) = 0$ and $\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+ = 0$ for any $\tau \in \{h_*^{(j)} > 0\} \backslash E_j$, namely, (61) is satisfied.

Next, we prove (62). If $\mu_j(\{h_*^{(j)} \leq 0\}) = 0$, then (62) is readily obtained, and hence we assume that $\mu_j(\{h_*^{(j)} \leq 0\}) > 0$. It follows from (59) that $\lim_{k\to\infty, k\in\mathcal{Q}_j} [h_k^{(j)}]_+ = [h_*^{(j)}]_+ = 0$ a.e. in $\{h_*^{(j)} \leq 0\}$, that is to say, there exists $F_j \subset \{h_*^{(j)} \leq 0\}$ such that

$\mu_j(F_j) = 0$ and

$$\lim_{k\to\infty, k\in\mathcal{Q}_j} [h_k^{(j)}(\tau)]_+ = 0 \quad \forall \tau \in \{h_*^{(j)} \le 0\}\backslash F_j. \tag{63}$$

Note that $\{h_*^j \le 0\}\backslash F_j \neq \emptyset$ from $\mu_j(\{h_*^{(j)} \le 0\}) > 0$. Let $\tau \in \{h_*^{(j)} \le 0\}\backslash F_j$ and $k \in \mathcal{Q}_j$ be arbitrary. We have two possible cases: $h_k^{(j)}(\tau) > 0$; $h_k^{(j)}(\tau) \le 0$. In the first case, we obtain $|z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+| = [\overline{z}_{k-1}^{(j)}(\tau) - \frac{1}{\sigma_{k-1}} h_k^{(j)}(\tau)]_+ [h_k^{(j)}(\tau)]_+ \le z_{\max}[h_k^{(j)}(\tau)]_+$. In the second case, the same inequality is verified as follows: $|z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+| = 0 \le z_{\max}[h_k^{(j)}(\tau)]_+$. These two cases imply $|z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+| \le z_{\max}[h_k^{(j)}(\tau)]_+$. Taking the limit in both of this inequality and using (63) yield $\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+ = 0$. As a result, we can verify the existence of $F_j \subset \{h_*^{(j)} \le 0\}$ which satisfies $\mu_j(F_j) = 0$ and $\lim_{k\to\infty, k\in\mathcal{Q}_j} z_k^{(j)}(\tau)[h_k^{(j)}(\tau)]_+ = 0$ for all $\tau \in \{h_*^{(j)} \le 0\}\backslash F_j$, that is, (62) holds.

Therefore, we have from (55)–(57) that $\{(x_k, y_k, z_k)\}_{k\in\mathcal{N}}$ is an AKKT sequence corresponding to x_*. □

Proposition 7. *Suppose that Assumption* 1 *is satisfied. If* $\{x_k\}$ *converges to* x_* *in* X*, then* x_* *is a stationary point of* $\min\{J(x) := \frac{1}{2}\|g(x)\|_U^2 + \frac{1}{2}\|[-h(x)]_+\|_V^2; x \in X\}$*, that is, it satisfies* $J'(x_*) = g'(x_*)^* g(x_*) - h'(x_*)^*[-h(x_*)]_+ = 0$*. Moreover, if* x_* *satisfies the ERCQ, then it is feasible to* (1)*.*

Proof. If $\mathrm{card}(\mathcal{I}) = \infty$ occurs, then Proposition 5 implies that x_* is a KKT point of (1). The fact means that x_* is a global optimum of $\min\{J(x); x \in X\}$, namely, it satisfies the stationary condition. We consider the case where $\mathrm{card}(\mathcal{I}) < \infty$. It then follows from Lemma 5 that $\mathrm{card}(\mathcal{J}) = \infty$. Let us define $\mathcal{Q} := \{k \in \mathbb{N}; k-1 \in \mathcal{J}\}$. Notice that $\mathrm{card}(\mathcal{Q}) = \infty$ by $\mathrm{card}(\mathcal{J}) = \infty$ and that $\|F'(x_k; \overline{y}_{k-1}, \overline{z}_{k-1}, \sigma_{k-1})\|_W \le \gamma_{k-1}$ for $k \in \mathcal{Q}$. Now, items (ii) and (iii) of Lemma 3 guarantee that $\gamma_k \to 0$ and $\sigma_k \to 0$ as $k \to \infty$, $k \in \mathcal{Q}$, and $\{\overline{y}_k\} \subset U$ and $\{\overline{z}_k\} \subset V$ are bounded, respectively. We have from (13) that $\sigma_{k-1} F'(x_k; \overline{y}_{k-1}, \overline{z}_{k-1}, \sigma_{k-1}) = \sigma_{k-1} f'(x_k) - g'(x_k)^*(\sigma_{k-1}\overline{y}_{k-1} - g(x_k)) - h'(x_k)^*[\sigma_{k-1}\overline{z}_{k-1} - h(x_k)]_+$. Since $\{x_k\}$ converges to x_* in W because $X \hookrightarrow W$, the above facts, (A1), and (A2) yield

$$\|J'(x_*)\|_W = \lim_{k\to\infty, k\in\mathcal{Q}} \sigma_{k-1} \|F'(x_k; \overline{y}_{k-1}, \overline{z}_{k-1}, \sigma_{k-1})\|_W = 0. \tag{64}$$

This shows that the former assertion holds.

To prove the latter part, we suppose that the ERCQ holds at x_*. Since x_* satisfies the ERCQ, there exist $\widehat{\xi} \in X$ and $\widehat{\zeta} \in K_Z$ such that $0 = g(x_*) + g'(x_*)\widehat{\xi}$ and $0 = h(x_*) + h'(x_*)\widehat{\xi} - \widehat{\zeta}$, i.e.,

$$-g'(x_*)\widehat{\xi} = g(x_*), \tag{65}$$

$$\widehat{\zeta} = h'(x_*)\widehat{\xi} + h(x_*). \tag{66}$$

In what follows, we represent $h(x_*)$ and $\widehat{\zeta}$ as

$$h(x_*) = (h_*^{(1)}, \dots, h_*^{(m)}), \quad \widehat{\zeta} = (\widehat{\zeta}^{(1)}, \dots, \widehat{\zeta}^{(m)}),$$

respectively. From $\widehat{\zeta} \in K_Z$, it is clear that $0 \geq -\widehat{\zeta}^{(j)}$ for all $j \in \{1, \ldots, m\}$. Hence, we get $0 \geq -\sum_{j=1}^{m} \int_{\Omega_j} [-h_*^{(j)}(\tau)]_+ \widehat{\zeta}^{(j)}(\tau) d\tau = -([-h(x_*)]_+, \widehat{\zeta})_V$. This inequality and equality (66) yield

$$\begin{aligned} 0 &\geq -([-h(x_*)]_+, h'(x_*)\widehat{\xi})_V + \sum_{j=1}^{m} \int_{\Omega_j} [-h_*^{(j)}(\tau)]_+ (-h_*^{(j)}(\tau)) d\tau \\ &= -\langle h'(x_*)^* [-h(x_*)]_+, \widehat{\xi} \rangle_{X^*, X} + \|[-h(x_*)]_+\|_V^2, \end{aligned} \tag{67}$$

where the equality of (67) is derived from $[-h_*^{(j)}(\tau)]_+(-h_*^{(j)}(\tau)) = |[-h_*^{(j)}(\tau)]_+|^2$ for $j \in \{1, \ldots, m\}$. Meanwhile, we recall that $J'(x_*) = g'(x_*)^* g(x_*) - h'(x_*)^* [-h(x_*)]_+$. It then follows from (64) and (65) that

$$-\langle h'(x_*)^* [-h(x_*)]_+, \widehat{\xi} \rangle_{X^*, X} = (g(x_*), -g'(x_*)\widehat{\xi})_U = \|g(x_*)\|_U^2. \tag{68}$$

Combining (67) and (68) means $0 \geq \|g(x_*)\|_U^2 + \|[-h(x_*)]_+\|_V^2$. Therefore, we conclude that x_* is a feasible point. □

Propositions 5 and 6 derive the following theorems associated with the global convergence of Algorithm 2.

Theorem 2. *Suppose that Assumption* 1 *holds. Let* $\{(x_k, y_k, z_k)\} \subset X \times U \times V$ *be an infinite sequence generated by Algorithm* 2. *Then, any feasible accumulation point* $x_* \in X$ *of* $\{x_k\}$ *satisfies at least one of the following two statements:*

- (i) *There exist* $y_* \in U$, $z_* \in V$, *and* $\mathcal{M} \subset \mathbb{N}$ *such that* $y_k \rightharpoonup y_*$ *in* U *and* $z_k \rightharpoonup z_*$ *in* V *as* $k \to \infty$, $k \in \mathcal{M}$, *and* (x_*, y_*, z_*) *satisfies the KKT conditions of* (1);
- (ii) x_* *is an AKKT point of* (1) *and there exists* $\mathcal{N} \subset \mathbb{N}$ *such that* $\{(x_k, y_k, z_k)\}_{k \in \mathcal{N}}$ *is an AKKT sequence corresponding to* x_*.

Proof. Without loss of generality, we can assume that $\{x_k\}$ converges to x_* in X. There is a possibility that $\mathrm{card}(\mathcal{I}) = \infty$ or $\mathrm{card}(\mathcal{I}) < \infty$. If the first case occurs, then Proposition 5 ensures that statement (i) is satisfied. On the other hand, it follows from Proposition 6 that the second case means statement (ii). □

Theorem 3. *Suppose that Assumption* 1 *holds. Suppose also that* Y *and* Z *are separable. Let* $\{(x_k, y_k, z_k)\} \subset X \times U \times V$ *be an infinite sequence generated by Algorithm* 2. *If any accumulation point* $x_* \in X$ *of* $\{x_k\}$ *satisfies the ERCQ, then there exist* $y_* \in Y^*$, $z_* \in Z^*$, *and* $\mathcal{N} \subset \mathbb{N}$ *such that* $y_k \rightharpoonup^* y_*$ *in* Y^* *and* $z_k \rightharpoonup^* z_*$ *in* Z^* *as* $k \to \infty$, $k \in \mathcal{N}$, *and* (x_*, y_*, z_*) *satisfies the KKT conditions of* (1).

Proof. Assume without loss of generality that $\{x_k\}$ converges to x_* in X. Since x_* satisfies the ERCQ, Proposition 7 ensures that x_* is feasible to (1), and hence the RCQ holds at x_*. Note that f, g, and h are continuously Fréchet differentiable on X from (A1) and $X \hookrightarrow W \hookrightarrow X^*$. Note also that $U \hookrightarrow Y^*$ and $V \hookrightarrow Z^*$. It then follows from Theorem 2 that x_* is an AKKT point and there exists $\mathcal{M} \subset \mathbb{N}$ such that $\{(x_k, y_k, z_k)\}_{k \in \mathcal{M}} \subset X \times Y^* \times Z^*$ is an AKKT sequence corresponding to x_* because every KKT point is also an AKKT point. Since Y and Z are separable and the RCQ holds at x_*, Proposition 1 implies that $\{y_k\}_{k \in \mathcal{M}}$ and $\{z_k\}_{k \in \mathcal{M}}$ have respectively weakly convergent subsequences $\{y_k\}_{k\in\mathcal{N}}$ and $\{z_k\}_{k\in\mathcal{N}}$ such that $y_k \rightharpoonup^* y_*$ in Y^* and $z_k \rightharpoonup^* z_*$ in Z^* as $k \to \infty$, $k \in \mathcal{N}$, and (x_*, y_*, z_*) satisfies the KKT conditions of (1). □

Remark 3. *Theorem* 2 *guarantees that the sequence* $\{x_k\}$ *generated by Algorithm* 2 *globally converges to a point that satisfies the KKT or AKKT conditions. Such a property is not found in the existing SQP-type methods for optimization problems in function spaces.*

Remark 4. *As mentioned in Theorem* 2*, there is a possibility that Algorithm* 2 *finds an infeasible point of problem* (1)*. This property can be also seen in the existing augmented Lagrangian method proposed in* [23]*. However, Proposition* 7 *also ensures that the infeasible point certainly satisfies the stationary condition of* $\min\{J(x); x \in X\}$*. This fact indicates that Algorithm* 2 *tends to find a feasible point of* (1)*. It would be rare for Algorithm* 2 *to find an infeasible point because there are indeed no such cases in numerical experiments provided in Section* 5*.*

Remark 5. *In Theorems* 2 *and* 3*, we suppose that the generated sequence* $\{x_k\}$ *converges to some accumulation point* $x_* \in X$*. Actually, the same assumption can be also seen in the global convergence analysis of the existing literature* [23]*. However, it is difficult to verify whether the assumption is satisfied or not before running Algorithm* 2*. In future research, it is worthwhile providing some sufficient conditions for the assumption or proving the global convergence under a weaker assumption that* $\{x_k\}$ *weakly converges to* x_**.*

5. Applications and numerical experiments

In this section, we provide some applications related to problem (1) and apply Algorithm 2 to them. The details of those applications are found in the existing papers and textbooks, such as [8, 9, 20, 23, 30, 31, 36, 38].

Example 1 We consider an obstacle problem:

$$\begin{aligned} &\underset{u}{\text{Minimize}} \quad \int_\Omega |\nabla u|^2 d\tau \\ &\text{subject to} \quad u \geq \psi \text{ in } \Omega, \end{aligned} \tag{69}$$

where the set $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with a Lipschitz boundary and the function $\psi \in H_0^1(\Omega)$ is given. In this case, we regard f and h as

$$f(x) := \int_\Omega |\nabla u|^2 d\tau, \quad h(x) := u - \psi,$$

respectively. Furthermore, we consider X, Z, and W as

$$X := H_0^1(\Omega), \quad Z := H_0^1(\Omega), \quad W := H_0^1(\Omega),$$

respectively.

Example 2 As a standard application in the optimal control, we provide an elliptic control problem:

$$\begin{aligned} \underset{y,u}{\text{Minimize}} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & A(y) + \varphi(\cdot, y(\cdot)) = au \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \\ & y \geq y_c \text{ in } \Omega, \quad u_a \leq u \leq u_b \text{ in } \Omega. \end{aligned} \tag{70}$$

Here, the set $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with a Lipschitz boundary $\partial\Omega$, the parameter α is positive, the functions $y_d \in L^2(\Omega)$, $a \in L^\infty(\Omega)$, $u_a \in L^\infty(\Omega)$, $u_b \in L^\infty(\Omega)$, and $y_c \in C(\bar{\Omega})$ are given, the operator A is defined as

$$A(y) \coloneqq -\operatorname{div}(M\nabla y) = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial \tau_j}\left(M_{ij}\frac{\partial}{\partial \tau_i} y\right) \tag{71}$$

for $y \in H^1_0(\Omega)$, the matrix-valued function $M \colon \Omega \to \mathbb{R}^{d\times d}$ satisfies that $M(\tau) = [M_{ij}(\tau)] \in \mathbb{R}^{d\times d}$ is symmetric for all $\tau \in \Omega$, $M_{ij} \in C^{0,1}(\bar{\Omega})$ for each $i, j \in \{1, \ldots, d\}$, and there exists $\delta > 0$ such that $\xi^\top M(\tau)\xi \geq \delta|\xi|^2$ for $\tau \in \Omega$ and $\xi \in \mathbb{R}^d$. Moreover, the function $\varphi \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable with respect to $\tau \in \Omega$ for each fixed $\theta \in \mathbb{R}$, is continuous and monotonically increasing with respect to $\theta \in \mathbb{R}$ for a.e. $\tau \in \Omega$, and satisfies the following two conditions:

(i) There exists $K > 0$ such that $|\varphi(\tau, 0)| \leq K$ for a.e. $\tau \in \Omega$;
(ii) for each $M > 0$, there exists $L_M > 0$ such that $|\varphi(\tau, \theta) - \varphi(\tau, \vartheta)| \leq L_M|\theta - \vartheta|$ for a.e. $\tau \in \Omega$ and all $\theta, \vartheta \in [-M, M]$.

It follows from [36, Theorem 4.7] that for each control u, the state equation has the unique solution $y = G(u) \in H^1_0(\Omega) \cap C(\bar{\Omega})$. This example can be expressed as problem (1) by the following setting: The functionals f, g, and h are respectively regarded as

$$\begin{aligned} f(x) &\coloneqq \tfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)}, \\ g(x) &\coloneqq A(y) + \varphi(\cdot, y(\cdot)) - au, \\ h(x) &\coloneqq (y - y_c, u - u_a, u_b - u), \end{aligned}$$

and the function spaces X, Y, Z, W, and U are respectively set to

$$X \coloneqq \mathcal{Y} \times L^2(\Omega), \quad Y \coloneqq L^2(\Omega), \quad Z \coloneqq C(\bar{\Omega}) \times L^2(\Omega) \times L^2(\Omega),$$
$$W \coloneqq L^2(\Omega) \times L^2(\Omega), \quad U \coloneqq L^2(\Omega).$$

Here $\mathcal{Y} \coloneqq \{y \in H^1_0(\Omega) \cap C(\bar{\Omega}); B(y) \in L^2(\Omega)\}$ is a Banach space equipped with a norm $\|y\|_{\mathcal{Y}} \coloneqq \|y\|_{H^1_0(\Omega)} + \|y\|_{C(\bar{\Omega})} + \|B(y)\|_{L^2(\Omega)}$ for $y \in \mathcal{Y}$, where the operator B is defined as $B(y) \coloneqq A(y) + \varphi(\cdot, y(\cdot))$, and the completeness of $\mathcal{Y}$ is proven in Appendix B. Note that the RCQ does not hold for all feasible points of problem (70) due to the box constraint $u_a \leq u \leq u_b$.

Example 3 We also give an optimal control problem with a control complementarity constraint:

$$\begin{aligned}
\underset{y,u,v}{\text{Minimize}} \quad & \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha_1}{2}\|u\|_{L^2(\Omega)}^2 + \frac{\alpha_2}{2}\|v\|_{L^2(\Omega)}^2 \\
& \qquad\qquad + \frac{\beta}{2}\|u\|_{H^1(\Omega)}^2 + \frac{\beta}{2}\|v\|_{H^1(\Omega)}^2 \qquad (72) \\
\text{subject to} \quad & A(y) + ay = bu + cv \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \\
& (u, v)_{L^2(\Omega)} = 0, \quad u \geq 0 \text{ in } \Omega, \quad v \geq 0 \text{ in } \Omega,
\end{aligned}$$

where the set $\Omega \subset \mathbb{R}^n$ is a bounded and open domain with a Lipschitz boundary $\partial\Omega$, the parameters α_1, α_2, and β are positive, the functions $y_d \in L^2(\Omega)$, $a \in L^\infty(\Omega)$, $b \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$ are given, and the operator A is the same one defined by (71). As described in Example 2, it is clear that a solution of the state equation satisfies $y = G(u, v) \in H_0^1(\Omega) \cap C(\bar{\Omega})$. To represent problem (72) as the proposed model, we should define the functionals f, g, and h by

$$\begin{aligned}
f(x) &:= \tfrac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \tfrac{\alpha_1}{2}\|u\|_{L^2(\Omega)}^2 + \tfrac{\alpha_2}{2}\|v\|_{L^2(\Omega)}^2 \\
&\qquad + \tfrac{\beta}{2}\|u\|_{H^1(\Omega)}^2 + \tfrac{\beta}{2}\|v\|_{H^1(\Omega)}^2, \\
g(x) &:= (A(y) + ay - bu - cv, (u, v)_{L^2(\Omega)}), \\
h(x) &:= (u, v),
\end{aligned}$$

respectively, and adopt the following function spaces:

$$\begin{aligned}
X &:= \mathcal{Z} \times H^1(\Omega) \times H^1(\Omega), \quad Y := L^2(\Omega) \times \mathbb{R}, \quad Z := L^2(\Omega) \times L^2(\Omega), \\
W &:= L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \quad U := L^2(\Omega) \times \mathbb{R},
\end{aligned}$$

where $\mathcal{Z} := \{y \in H_0^1(\Omega) \cap C(\bar{\Omega}); A(y) \in L^2(\Omega)\}$ is a Banach space equipped with a norm $\|y\|_{\mathcal{Z}} := \|y\|_{H_0^1(\Omega)} + \|y\|_{C(\bar{\Omega})} + \|A(y)\|_{L^2(\Omega)}$ for $y \in \mathcal{Z}$, and the completeness of $\mathcal{Z}$ can be shown in a manner similar to Appendix B. Notice that all feasible points of problem (72) do not satisfy the RCQ because of the complementarity constraint $(u, v)_{L^2(\Omega)} = 0$.

In the following, we report some numerical experiments to confirm the practical validity of Algorithm 2. Throughout the experiments, test problems given later were approximated as finite dimensional ones by discretizing them, and those approximate problems were solved. The program was written in MATLAB R2020b.

We explain the setting of Algorithm 2. First of all, we give the stopping criteria used in Step 1. From Theorem 2, there are two possible cases for the sequence $\{(x_k, y_k, z_k)\}$ generated by Algorithm 2: (i) $\{x_k\}$ converges to a KKT point; (ii) $\{x_k\}$ converges to an AKKT point. Case (i) means that there exists $\mathcal{M} \subset \mathbb{N}$ such that $\{r(x_k, y_k, z_k)\}_{k\in\mathcal{M}}$ converges to zero, where r is defined by

$$r(x_k, y_k, z_k) := \max\{|g(x_k)|, |[-h(x_k)]_+|, |\nabla_x L(x_k, y_k, z_k)|, |z_k \cdot h(x_k)|\}$$

with $\nabla_x L(x_k, y_k, z_k) = \nabla f(x_k) - \nabla g(x_k) y_k - \nabla h(x_k) z_k$. Moreover, case (ii) implies that Algorithm 2 performs Step 3.3 (Step 3 of Algorithm 1) infinitely many times, that is, there exists $\mathcal{N} \subset \mathbb{N}$ such that $\{\gamma_k\}_{k\in\mathcal{N}}$ converges to zero. By considering these facts, we adopted the following stopping conditions:

$$r(x_k, y_k, z_k) \le 10^{-6}, \quad \gamma_k \le 10^{-6}, \quad \text{or} \quad k = 100,$$

where a run was considered to have failed if $k = 100$. The parameters were set as $\beta := 0.5$, $\varepsilon := 10^{-4}$, $\rho := 10^{-4}$, $\kappa := 10^{-5}$, $\phi_0 := 10^3$, $\psi_0 := 10^3$, $\gamma_0 := 10^{-3}$, and $\sigma_0 := 10^{-3}$. The sets C and D were chosen as $C := \{y; -y_{\max}e \le y \le y_{\max}e\}$ and $D := \{z; 0 \le z \le z_{\max}e\}$, where $y_{\max} := 10^6$, $z_{\max} := 10^6$, and e denotes the all-ones vector whose dimension is defined by the context. The initial point was selected as $(x_0, y_0, z_0) := (0, 0, 0)$.

In the experiments, three test problems related to Examples 1–3 were solved. To begin with, we give the following obstacle Bratu problem which is obtained by changing the objective function of (69):

$$\begin{aligned} \underset{u}{\text{Minimize}} \quad & \int_\Omega \left(|\nabla u|^2 - \alpha e^{-u}\right) d\tau \\ \text{subject to} \quad & u \ge \psi \text{ in } \Omega, \end{aligned} \tag{73}$$

where $\Omega := (0,1)^2$, $\alpha := 0.3$, and $\psi(\tau_1, \tau_2) := \max\{0.1 - 30(\tau_1 - 0.5)^4 - 30(\tau_2 - 0.5)^4, 0\}$. Note that this problem is nonlinear and nonconvex.

Regarding Example 2, we consider the following optimal control problem with a semilinear PDE constraint:

$$\begin{aligned} \underset{y,u}{\text{Minimize}} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to} \quad & -\Delta y + y^3 = u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \\ & y \ge y_c \text{ in } \Omega, \end{aligned} \tag{74}$$

where $\Omega := (0,1)^2$, $\alpha := 0.002$, $y_d(\tau_1, \tau_2) := -1$, and $y_c(\tau_1, \tau_2) := -0.6 + 0.5 \min\{\tau_1 + \tau_2, 1 + \tau_1 - \tau_2, 1 - \tau_1 + \tau_2, 2 - \tau_1 - \tau_2\}$. This problem is also nonlinear and nonconvex.

Finally, we present the following problem associated with Example 3:

$$\begin{aligned} \underset{y,u,v}{\text{Minimize}} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{H^1(\Omega)} + \frac{\alpha}{2}\|v\|^2_{H^1(\Omega)} \\ \text{subject to} \quad & -\Delta y + y = 1_{\Omega_1} u + 1_{\Omega_2} v \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \\ & (u, v)_{L^2(\Omega)} = 0, \quad u \ge 0 \text{ in } \Omega, \quad v \ge 0 \text{ in } \Omega, \end{aligned} \tag{75}$$

where $\Omega := (0,1)^2$, $\Omega_1 := \{(\tau_1, \tau_2) \in \Omega; \tau_2 < 0.25\}$, $\Omega_2 := \{(\tau_1, \tau_2) \in \Omega; \tau_2 > 0.75\}$, $\alpha := 0.001$, and $y_d(\tau_1, \tau_2) := \cos(\pi\tau_1)\cos(2\pi\tau_2)$. As stated in Example 3, it is known that the RCQ does not hold at any feasible point of (75) because the complementarity constraint exists.

Tables 1–3 indicate computational results that Algorithm 2 solved the three test problems with the mesh size being changed. Note that x_*, y_*, and z_* described in each table denote the final iteration points of $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$, respectively. Moreover, numerical results for problems (73)–(75) are shown in Figures 1–3, respectively. For each mesh size, Algorithm 2 succeeded in solving all the problems, and its iteration numbers seem to be nearly constant regardless of the mesh size. In addition, the values of $\max\{|y_*|, |z_*|\}$ given in Table 3 indicate that the Lagrange multipliers $\{y_k\}$ and $\{z_k\}$

did not diverge even though problem (75) is degenerate. Therefore, the effectiveness of Algorithm 2 was also shown for the degenerate problem.

Table 1. Performance of Algorithm 2 on problem (73)

mesh size	iteration	$r(x_*, y_*, z_*)$	$\max\{\lvert y_*\rvert, \lvert z_*\rvert\}$
2^{-4}	3	3.7832e−07	1.2891e−01
2^{-5}	3	1.0054e−07	6.1570e−02
2^{-6}	3	4.3314e−08	3.2603e−02
2^{-7}	3	6.8796e−07	1.6394e−02

Table 2. Performance of Algorithm 2 on problem (74)

mesh size	iteration	$r(x_*, y_*, z_*)$	$\max\{\lvert y_*\rvert, \lvert z_*\rvert\}$
2^{-4}	3	4.1339e−08	1.8920e−01
2^{-5}	3	5.7983e−07	1.8812e−01
2^{-6}	3	1.4023e−07	1.8745e−01
2^{-7}	3	5.9678e−07	1.8708e−01

Table 3. Performance of Algorithm 2 on problem (75)

mesh size	iteration	$r(x_*, y_*, z_*)$	$\max\{\lvert y_*\rvert, \lvert z_*\rvert\}$
2^{-4}	13	6.4322e−07	8.2135e−01
2^{-5}	12	5.2823e−07	1.1438e−00
2^{-6}	11	7.0608e−07	7.5793e−01
2^{-7}	13	8.8749e−07	5.0120e−01

6. Concluding remarks

In this paper, we have proposed Algorithm 2 for solving problem (1). Problem (1) has a general form and does not need to satisfy any CQs. The setting allows us to formulate many kinds of optimization problems in function spaces including degenerate ones. Algorithm 2 produces a sequence converging to a point that satisfies the KKT or AKKT conditions. We have proven that Algorithm 2 globally converges without assuming any CQs. In the numerical experiments, we have confirmed that Algorithm 2 performs well for all the test problems, which include degenerate ones.

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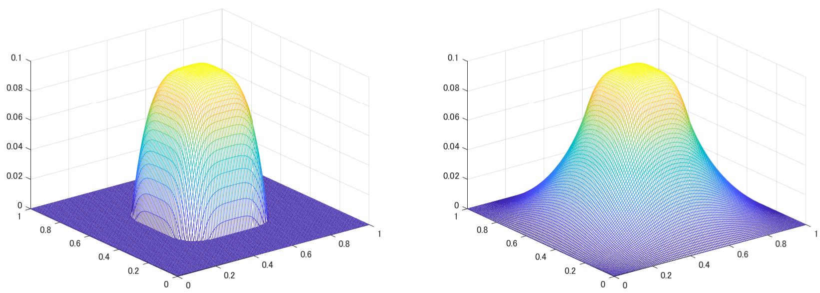
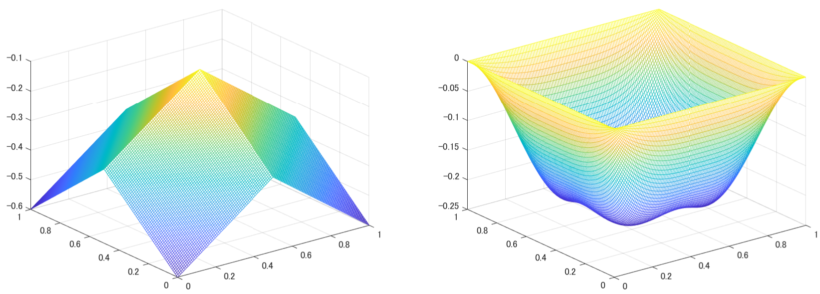
Declarations

Conflict of interest The author declares no competing interests.

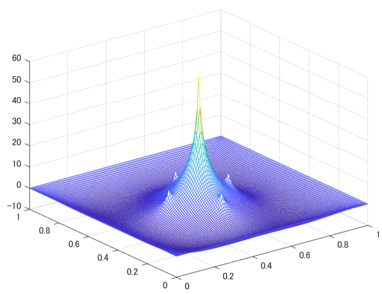


(a) Constraint function ψ (b) Solution u

Figure 1. Numerical results for problem (73) with the mesh size 2^{-7}

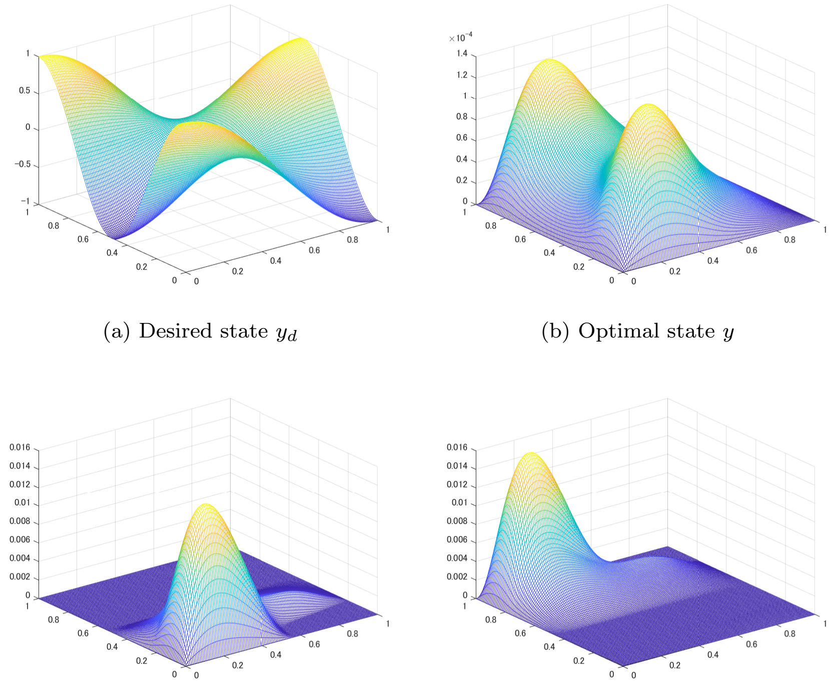


(a) Constraint function y_c (b) Optimal state y



(c) Optimal control u

Figure 2. Numerical results for problem (74) with the mesh size 2^{-7}



(a) Desired state y_d (b) Optimal state y

(c) Optimal control u (d) Optimal control v

Figure 3. Numerical results for problem (75) with the mesh size 2^{-7}

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Appendix A.

Proof of Proposition 2. Let us define $\mathcal{F}(v) := (f'(x)-g'(x)^*s,\xi)_W+\frac{1}{2}(M\xi,\xi)_W+\frac{\sigma}{2}\|\zeta\|_V^2$ and $\mathcal{S} := \{(\xi,\zeta) \in \mathcal{V}; h'(x)\xi + \sigma(\zeta - t) \geq 0\}$. For each $v := (\xi,\zeta) \in \mathcal{S}$, we can evaluate $\mathcal{F}(v)$ as follows: $\mathcal{F}(v) \geq \frac{\ell_B}{2}(\|\xi\|_W - \frac{1}{\ell_B}\|f'(x) - g'(x)^*s\|_W)^2 - \frac{1}{2\ell_B}\|f'(x) - g'(x)^*s\|_W^2 + \frac{\sigma}{2}\|\zeta\|_V^2$. Thus, the coerciveness of $\mathcal{F}$ is verified. In addition, we obtain $-\infty < \inf\{\mathcal{F}(v); v \in \mathcal{S}\}$, that is, there exists $\{v_j\} \subset \mathcal{S}$ such that $\mathcal{F}(v_j) \to \inf\{\mathcal{F}(v); v \in \mathcal{S}\}$ as $j \to \infty$. It then follows from the coerciveness of $\mathcal{F}$ that $\{v_j\} \subset \mathcal{V}$ is bounded. Meanwhile, W and V are Hilbert spaces, and hence so is $\mathcal{V}$. By these facts, there exist $v_* := (\xi_*,\zeta_*) \in \mathcal{V}$ and $\mathcal{M} \subset \mathbb{N}$ such that $v_j \rightharpoonup v_*$ as $j \to \infty$, $j \in \mathcal{M}$. Since $\mathcal{S}$ is convex and strongly closed, it is weakly closed, i.e., $v_* \in \mathcal{S}$. Now, we can easily see that $\mathcal{F}$ is weakly lower semicontinuous because it is proper convex. Hence, $F(v_*) = \inf\{\mathcal{F}(v); v \in \mathcal{S}\}$, which implies that $v_* = (\xi_*, \zeta_*)$ is an optimum of problem (12). The uniqueness of v_* follows from the strict convexity of $\mathcal{F}$.

Note that $(\xi, \zeta) \mapsto h'(x)\xi + \sigma\zeta$ is a surjective mapping from $\mathcal{V}$ to V. This fact means that the RCQ holds at each feasible point of (12). Therefore, there exists $\lambda_* \in V$ such that $(\xi_*, \zeta_*, \lambda_*)$ satisfies the KKT conditions of (12). □

Proof of Proposition 3. Since the bilinear form B is coercive, Proposition 2 ensures that problem (12) has the unique optimum $(\xi_*, \zeta_*) \in \mathcal{V}$ to be also a KKT point of (12). Therefore, it can be easily verified that

$$M\xi_* + f'(x) - g'(x)^* s - h'(x)^* \zeta_* = 0, \tag{A.1}$$

$$(\zeta_*, h'(x)\xi_* + \sigma(\zeta_* - t))_V = 0, \tag{A.2}$$

$$h'(x)\xi_* + \sigma(\zeta_* - t) \geq 0, \quad \zeta_* \geq 0. \tag{A.3}$$

We have from (13), (A.1), and (A.2) that

$$\begin{aligned}(F'(x; y, z, \sigma), \xi_*)_W &= -(M\xi_*, \xi_*)_W + (\zeta_*, h'(x)\xi_*)_V - ([t]_+, h'(x)\xi_*)_V \\ &= -(M\xi_*, \xi_*)_W - \sigma(\zeta_*, \zeta_* - t)_V - ([t]_+, h'(x)\xi_*)_V.\end{aligned} \tag{A.4}$$

The first inequality of (A.3) and $[t]_+ \geq 0$ derive $0 \leq ([t]_+, h'(x)\xi_* + \sigma(\zeta_* - t))_V$, i.e.,

$$-([t]_+, h'(x)\xi_*)_V \leq \sigma([t]_+, \zeta_* - t)_V = \sigma([t]_+ - \zeta_*, \zeta_* - t)_V + \sigma(\zeta_*, \zeta_* - t)_V. \tag{A.5}$$

The third term in the right-hand side of (A.4) can be evaluated by (A.5), and therefore we obtain $(F'(x; y, z, \sigma), \xi_*)_W \leq -(M\xi_*, \xi_*)_W - \sigma\|\zeta_* - [t]_+\|_V^2 + ([t]_+ - \zeta_*, [t]_+ - t)_V$. Since $\zeta_* \geq 0$ from the second inequality of (A.3), the well-known property of the projection $[\cdot]_+ \colon V \to K_V$ guarantees that $([t]_+ - \zeta_*, [t]_+ - t)_V \leq 0$, namely,

$$(F'(x; y, z, \sigma), \xi_*)_W \leq -(M\xi_*, \xi_*)_W - \sigma\|\zeta_* - [t]_+\|_V^2. \tag{A.6}$$

Now, we suppose that $F'(x; y, z, \sigma) = 0$. It follows from (A.6) and the coerciveness of B that $\xi_* = 0$ and $\zeta_* = [t]_+$, and hence $(0, [t]_+)$ is the unique optimum of (12). Conversely, we assume that $(0, [t]_+)$ is the unique optimum of (12), that is, $\xi_* = 0$ and $\zeta_* = [t]_+$. Combining (13) and (A.1) yields $F'(x; y, z, \sigma) = 0$. □

Appendix B.

Proof of the completeness of $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. Let $\{y_j\}$ be a Cauchy sequence in $\mathcal{Y}$. The definition of the norm $\|\cdot\|_{\mathcal{Y}}$ implies that $\{y_j\}$ and $\{B(y_j)\}$ are also Cauchy sequences in $H_0^1(\Omega) \cap C(\bar{\Omega})$ and $L^2(\Omega)$, respectively. Hence, there exist $y \in H_0^1(\Omega) \cap C(\bar{\Omega})$ and $z \in L^2(\Omega)$ such that

$$\lim_{j\to\infty} \left(\|y_j - y\|_{H_0^1(\Omega)} + \|y_j - y\|_{C(\bar{\Omega})}\right) = 0, \quad \lim_{j\to\infty} \|B(y_j) - z\|_{L^2(\Omega)} = 0. \tag{B.1}$$

Since $\{y_j\}$ is bounded in $C(\bar{\Omega})$, there exists $M_0 > 0$ such that $\|y_j\|_{C(\bar{\Omega})} \leq M_0$ for all $j \in \mathbb{N} \cup \{0\}$. Let us define $M \coloneqq \max\{M_0, \|y\|_{C(\bar{\Omega})}\} < \infty$, where notice that $y \in C(\bar{\Omega})$.

We readily have

$$\|y\|_{L^\infty(\Omega)} \le M, \quad \|y_j\|_{L^\infty(\Omega)} \le M \quad \forall j \in \mathbb{N} \cup \{0\}. \tag{B.2}$$

Recall that the function φ satisfies conditions (i) and (ii) mentioned in Example 2. It then follows from (B.2) and [36, Lemma 4.11] that

$$\|\varphi(\cdot, y_j(\cdot)) - \varphi(\cdot, y(\cdot))\|_{L^2(\Omega)} \le L_M \|y_j - y\|_{L^2(\Omega)} \quad \forall j \in \mathbb{N} \cup \{0\}. \tag{B.3}$$

Now, exploiting $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and (B.3) yields

$$\begin{aligned}
&\|B(y) - z\|_{H^{-1}(\Omega)} \\
&\quad \le \|A(y_j) - A(y)\|_{H^{-1}(\Omega)} + \|\varphi(\cdot, y_j(\cdot)) - \varphi(\cdot, y(\cdot))\|_{H^{-1}(\Omega)} + \|B(y_j) - z\|_{H^{-1}(\Omega)} \\
&\quad \lesssim \|A(y_j) - A(y)\|_{H^{-1}(\Omega)} + \|\varphi(\cdot, y_j(\cdot)) - \varphi(\cdot, y(\cdot))\|_{L^2(\Omega)} + \|B(y_j) - z\|_{L^2(\Omega)} \\
&\quad \le \|A(y_j) - A(y)\|_{H^{-1}(\Omega)} + L_M \|y_j - y\|_{H_0^1(\Omega)} + \|B(y_j) - z\|_{L^2(\Omega)}.
\end{aligned} \tag{B.4}$$

The continuity of A, (B.1), and (B.4) ensure that $B(y) = z \in L^2(\Omega)$. Then, using (B.1) again implies that the Cauchy sequence $\{y_j\}$ converges to $y \in \mathcal{Y}$. □