Path-monotonic Upward Drawings of Plane Graphs *

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Abstract. In this paper, we introduce a new problem of finding an upward drawing of a given plane graph γ with a set $\mathcal{P}$ of paths so that each path in the set is drawn as a poly-line that is monotone in the y-coordinate. We present a sufficient condition for an instance $(\gamma, \mathcal{P})$ to admit such an upward drawing. Our results imply that every 1-plane graph admits an upward drawing.

1 Introduction

Upward planar drawings of digraphs are well studied problem in Graph Drawing [3]. In an upward planar drawing of a directed graph, no two edges cross and each edge is a curve monotonically increasing in the vertical direction. It was shown that an upward planar graph (i.e., a graph that admits an upward planar drawing) is a subgraph of a planar *st*-graph and admits a straight-line upward planar drawing [4, 12], although some digraphs may require exponential area [3]. Testing upward planarity of a digraph is NP-complete [10]; a polynomial-time algorithm is available for an embedded triconnected digraph [2].

Upward embeddings and orientations of undirected planar graphs were studied [6]. Computing bimodal and acyclic orientations of *mixed graphs* (i.e., graphs with undirected and directed edges) is known as NP-complete [13], and upward planarity testing for embedded mixed graph is NP-hard [5]. Upward planarity can be tested in cubic time for mixed outerplane graphs, and linear-time for special classes of mixed plane triangulations [8].

A *monotone drawing* is a straight-line drawing such that for every pair of vertices there exists a path that monotonically increases with respect to some direction. In an upward drawing, each directed path is monotone, and such paths are monotone with respect to the same (vertical) line, while in a monotone drawing, each monotone path is monotone with respect to a different line in general. Algorithms for constructing planar monotone drawings of trees and biconnected planar graphs are presented [1].

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In this paper, we introduce a new problem of finding an upward drawing of a given plane graph γ together with a set $\mathcal{P}$ of paths so that each path in the set is drawn as a poly-line that is monotone in the y-coordinate. Let $\gamma = (V, E, F)$ be a plane graph and D be an upward drawing of γ. We call D *monotonic to a path* P of (V, E) if D is upward in the y-coordinate and the drawing induced by path P is y-monotone. We call D *monotonic to a set of paths* $\mathcal{P}$ if D is monotonic to each path in $\mathcal{P}$. More specifically, we initiate the following problem.

Path-monotonic Upward Drawing
Input: A connected plane graph γ, a set $\mathcal{P}$ of paths of length at least 2 and two outer vertices s and t.
Output: An (s, t)-upward drawing of γ that is monotonic to $\mathcal{P}$.

We present a sufficient condition for an instance $(\gamma, \mathcal{P})$ to admit an (s, t)-upward drawing of γ that is monotonic to $\mathcal{P}$ for any two outer vertices $s, t \notin V_{\rm inl}(\mathcal{P})$ (Theorem 1). Then we apply the result to a problem of finding an upward drawing of a non-planar embedding of a graph (Theorem 2), and prove that every 1-plane graph (i.e., a graph embedded with at most one crossing per edge) admits an (s, t)-upward poly-line drawing (Corollary 1). Note that there is a 1-plane graph that admits no straight-line drawing [16], and there is a 2-plane graph with three edges that admits no upward drawing.

Figure 1(a) shows an instance $(\gamma, \mathcal{P})$ with $\mathcal{P} = \{P_1 = (v_6, u_1, v_2), P_2 = (v_1, u_1, v_5), P_3 = (v_3, u_2, v_4), P_4 = (v_3, u_3, u_4, v_9), P_5 = (v_{11}, u_5, u_4, v_8), P_6 = (v_{10}, u_5, u_3, v_7), P_7 = (v_{10}, u_6, u_4, v_7), P_8 = (v_{12}, u_7, v_{14}), P_9 = (v_{10}, u_7, v_{13})\}$. Figure 1(b) shows an (s, t)-upward drawing monotonic to $\mathcal{P}$ such that each path is drawn as a poly-line monotone in the y-coordinate for $s = v_5$ and $t = v_8$.

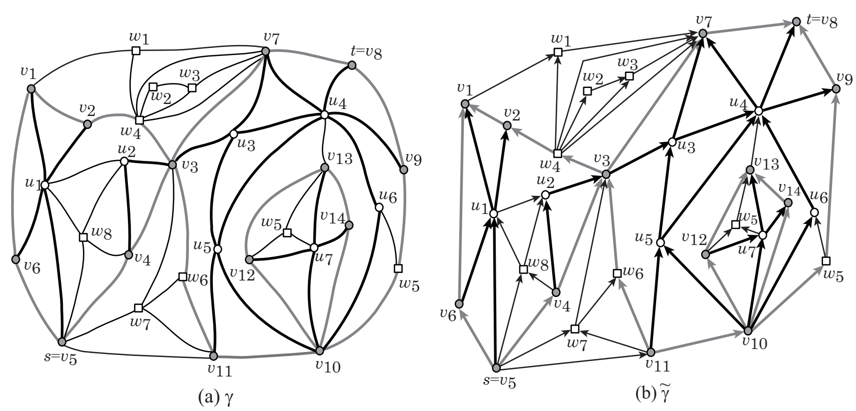

Fig. 1. (a) plane graph γ with a path set $\mathcal{P}$ and a cycle set $\mathcal{C}$, where the edges in paths in $\mathcal{P}$ (resp., cycles $\mathcal{C}$) are depicted with black thick lines (resp., gray thick lines), and the vertices in $V_{\rm inl}$ (resp., $V_{\rm end}$ and $V \setminus V_{\rm inl} \cup V_{\rm end}$) are depicted with white circles (resp., gray circles and white squares); (b) $(s = v_5, t = v_8)$-upward poly-line drawing monotonic to $\mathcal{P}$.

2 Preliminaries

Graphs In this paper, a graph stands for an undirected multiple graph without self-loops. A graph with no multiple edges is called *simple*. Given a graph $G = (V, E)$, the vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively.

A path P that visits vertices $v_1, v_2, \ldots, v_{k+1}$ in this order is denoted by $P = (v_1, v_2, \ldots, v_{k+1})$, where vertices v_1 and v_{k+1} are called the *end-vertices*. Paths and cycles are simple unless otherwise stated. A path with end-vertices $u, v \in V$ is called a *u, v-path*. A u, v-path that is a subpath of a path P is called the *sub-u, v-path* of P. Denote the set of end-vertices (resp., internal vertices) of all paths in a set $\mathcal{P}$ of paths by $V_{\mathrm{end}}(\mathcal{P})$ (resp., $V_{\mathrm{inl}}(\mathcal{P})$), which is written as $V_{\mathrm{end}}(P)$ (resp., $V_{\mathrm{inl}}(P)$) for $\mathcal{P} = \{P\}$.

Let G be a graph with a vertex set V with $n = |V|$ and an edge set E. Let $N_G(v)$ denote the set of neighbors of a vertex v in G. Let X be a subset of V. Let $G[X]$ denote the subgraph of G induced by the vertices in X. We denote by $N_G(X)$ the set of neighbors of X; i.e., $N_G(X) = \cup_{v \in X} N_G(v) \setminus X$. A connected component H of G may be denoted with the vertex subset $V(H)$ for simplicity.

For two distinct vertices $a, b \in V$, a bijection $\rho : V \to \{1, 2, \ldots, n\}$ is called an *st-numbering* if $\rho(a) = 1$, $\rho(b) = n$, and each vertex $v \in V \setminus \{a, b\}$ has a neighbor $v' \in N_G(v)$ with $\rho(v') < \rho(v)$ and a neighbor $v'' \in N_G(v)$ with $\rho(v) < \rho(v'')$. It is possible to find an st-numbering of a given graph with designated vertices a and b (if one exists) in linear time using depth-first search [7, 15]. A biconnected graph admits an st-numbering for any specified vertices a and b.

Digraphs Let $G = (V, E)$ be a digraph. The *indegree* (resp., *outdegree*) of a vertex $v \in V$ in G is defined to be the number of edges whose head is v (resp., whose tail is v). A *source* (resp., *sink*) is defined to be a vertex of indegree (resp., outdegree) 0. When G has no directed cycle, it is called *acyclic*. A digraph with n vertices is acyclic if and only if it admits a *topological ordering*, i.e., a bijection $\tau : V \to \{1, 2, \ldots, n\}$ such that $\tau(u) < \tau(v)$ for any directed edge (u, v).

We define an *orientation* of a graph $G = (V, E)$ to be a digraph $\widetilde{G} = (V, \widetilde{E})$ obtained from the graph by replacing each edge uv in G with one of the directed edge (u, v) or (v, u). A *bipolar orientation* (or *st-orientation*) of a graph is defined to be an acyclic digraph with a single source s and a single sink t [9, 14], where we call such a bipolar orientation an *(s, t)-orientation*. A graph has a bipolar orientation if and only if it admits an st-numbering. Figure 1(b) illustrates an (s, t)-orientation for $s = v_5$ and $t = v_8$.

Lemma 1. *For any vertices s and t in a biconnected graph G possibly with multiple edges, an (s, t)-orientation $\widetilde{G}$ of G can be constructed in linear time.*

We call an orientation $\widetilde{G}$ of G *compatible* to a set $\mathcal{P}$ of paths in G if each path in $\mathcal{P}$ is directed from one end-vertex to the other in $\widetilde{G}$. The orientation in Figure 1(b) is compatible to the path set $\mathcal{P}$.

Embeddings An embedding Γ of a graph (or a digraph) $G = (V, E)$ is a representation of G (possibly with multiple edges) in the plane, where each vertex in V is a point and each edge $e \in E$ is a curve (a Jordan arc) between the points representing its endvertices. We say that two edges *cross* if they have a point in common, called a *crossing*, other than their endpoints.

To avoid pathological cases, standard non-degeneracy conditions apply: (i) no edge contains any vertex other than its endpoints; (ii) no edge crosses itself; (iii) no two edges meet tangentially; and (iv) two edges cross at most one point, and two crossing edges share no end-vertex (where two edges may share the two end-vertices). In this paper, we allow three or more edges to share the same crossing. An edge that does not cross any other edge is called *crossing-free*.

Let Γ be an embedding of a graph (or digraph) $G = (V, E)$. We call Γ a *poly-line drawing* if each edge $e \in E$ is drawn as a sequence of line segments. The point where two consecutive line segments meet is called a *bend*. We call a poly-line drawing a *straight-line drawing* if it has no bend.

We call a curve in the xy-plane *y-monotone* if the y-coordinate of the points in the curve increases from one end of the curve to the other. We call Γ an *upward drawing* if (i) there is a direction d to be defined as the y-coordinate such that the curve for each edge $e \in E$ is y-monotone; and (ii) when G is a digraph, the head of e has a larger y-coordinate than that of the tail of e.

For two vertices $s, t \in V$, we call Γ an *(s, t)-upward drawing* if Γ is upward in the y-coordinate and the y-coordinate of s (resp., t) in Γ is uniquely minimum (resp., maximum) among the y-coordinates of vertices in Γ. Figure 1(b) shows an example of an (s, t)-upward poly-line drawing with $s = v_5$ and $t = v_8$.

Plane Graphs An embedding of a graph G with no crossing is called a *plane graph* and is denoted by a tuple (V, E, F) of a set V of vertices, a set E of edges and a set F of faces. We call a plane graph *pseudo-simple* if it has no pair of multiple edges e and e' such that the cycle formed by e and e' encloses no vertex.

Let $\gamma = (V, E, F)$ be a plane graph. We say that two paths P and P' in γ *intersect* if they are edge-disjoint and share a common internal vertex w, and the edges uw and wv in P and $u'w$ and wv' in P' incident to w appear alternately around w (i.e., in one of the orderings u, u', v, v' and u, v', v, u').

Let C be a cycle in γ. Define $V_{\rm enc}(C; \gamma)$, $E_{\rm enc}(C; \gamma)$ and $F_{\rm enc}(C; \gamma)$ to be the sets of vertices $v \in V \setminus V(C)$, edges $e \in E \setminus E(C)$ and inner faces $f \in F$ that are enclosed by C. The *interior subgraph* $\gamma[C]_{\rm enc}$ induced from γ by C is defined to be the plane graph $(V(C) \cup V_{\rm enc}(C; \gamma), E(C) \cup E_{\rm enc}(C; \gamma), F_{\rm enc}(C; \gamma) \cup \{f_C\})$, where f_C denotes the new outer face whose facial cycle is C. The *exterior subgraph* induced from γ by C is defined to be the plane graph $(V \setminus V_{\rm enc}(C; \gamma), E \setminus E_{\rm enc}(C; \gamma), F \cup \{f_C\} \setminus F_{\rm enc}(C; \gamma))$, where f_C denotes the new inner face whose facial cycle is C. Note that when γ is biconnected, the graph $\gamma[C]_{\rm enc}$ remains biconnected, since every two vertices $u, v \in V \setminus V_{\rm enc}(C; \gamma)$ have two internally disjoint paths without using edges in $E_{\rm enc}(C; \gamma)$.

We say that two cycles C and C' in γ *intersect* if $F_{\rm enc}(C; \gamma) \setminus F_{\rm enc}(C'; \gamma) \neq \emptyset \neq F_{\rm enc}(C'; \gamma) \setminus F_{\rm enc}(C; \gamma)$. Let $\mathcal{C}$ be a set of cycles in γ. We call $\mathcal{C}$ *inclusive* if no two cycles in $\mathcal{C}$ intersect. When $\mathcal{C}$ is inclusive, the *inclusion-forest* of $\mathcal{C}$ is defined to be a forest $\mathcal{I} = (\mathcal{C}, \mathcal{E})$ of a disjoint union of rooted trees such that (i) the cycles in $\mathcal{C}$ are regarded as the vertices of $\mathcal{I}$; and (ii) a cycle C is an ancestor of a cycle C' in $\mathcal{I}$ if and only if $F_{\mathrm{enc}}(C';\gamma) \subseteq F_{\mathrm{enc}}(C;\gamma)$. Let $\mathcal{I}(\mathcal{C})$ denote the inclusion-forest of $\mathcal{C}$.

An *st-planar graph* is defined to be a bipolar orientation of a plane graph for which both the source and the sink of the orientation are on the outer face of the graph. A directed acyclic graph G has an upward planar drawing if and only if G is a subgraph of an st-planar graph [4, 12]. Every st-planar graph can have a *dominance drawing* [3], in which for every two vertices u and v there exists a path from u to v if and only if both coordinates of u are smaller than the corresponding coordinates of v. Hence the following result is known.

Lemma 2. [3] (i) *Every simple st-planar graph admits an upward straight-line drawing;*
(ii) *Every st-planar graph with multiple edges admits an upward poly-line drawing, where each multiple edge has at most one bend; and*
(iii) *Such a drawing in* (i) *and* (ii) *can be constructed in linear time.*

We see that (ii) follows from (i) by subdividing each multiple directed edge (u, v) into a directed path (u, w, v) with a new vertex w to obtain a simple st-planar graph. Figure 1(b) illustrates an example of an st-planar graph.

3 Path-monotonic Upward Drawing

When a plane graph γ has a pair of multiple edges e and e' that encloses no vertex in the interior, we can ignore one of these edges (say e') to find an upward drawing of γ, because we can draw e' along the drawing of e in any upward drawing of the resulting plane graph. In what follows, we assume that a given plane graph is pseudo-simple.

In this paper, we present a sufficient condition for an instance $(\gamma, \mathcal{P})$ to admit an (s, t)-upward straight-line drawing of γ that is monotonic to $\mathcal{P}$ for any two outer vertices $s, t \notin V_{\mathrm{inl}}(\mathcal{P})$.

Let γ be a connected plane graph. We say that two paths P and P' in γ are *1-independent* if (i) they intersect at a common internal vertex and have no other common vertex; or (ii) they have no common vertex that is an internal vertex of one of them (where they may share at most two vertices that are end-vertices to both paths). We call a set $\mathcal{P}$ of paths *1-independent* if any two paths in $\mathcal{P}$ are 1-independent. We prove the following main result.

Theorem 1. *For a pseudo-simple connected plane graph $\gamma = (G = (V, E), F)$ and a 1-independent set $\mathcal{P}$ of paths of length at least 2 in γ, let V_{inl} denote the set of internal vertices in paths in $\mathcal{P}$, $G[V_{\mathrm{inl}}]$ denote the subgraph of G induced by V_{inl}. Assume that γ has no pair of a path $P \in \mathcal{P}$ and a cycle K with $|V(K) \setminus V_{\mathrm{inl}}| \leq 1$ such that K encloses an end-vertex of P and the internal vertices of P and the vertices in $V(K) \cap V_{\mathrm{inl}}$ belong to the same component of $G[V_{\mathrm{inl}}]$.*

Then any pair of outer vertices $s, t \notin V_{\mathrm{inl}}$ admits an (s, t)-upward drawing D monotonic to $\mathcal{P}$, where D can be chosen as a straight-line drawing if γ is simple. Such a drawing D can be constructed in linear time.

We assume that the boundary of γ forms a cycle C^o such that $V(C^o) \cap V_{\rm inl} = \emptyset$; if necessary, add two new outer edges $e_{s,t}$ and $e'_{s,t}$ joining the two outer vertices s and t to form a new outer facial cycle C^o of length 2 (see Appendix 2 for other method that is independent of choice of vertices s and t). In what follows, we assume that the boundary of a given connected planar graph γ forms a cycle.

We prove Theorem 1 by showing that every instance satisfying the condition of the theorem admits an (s,t)-orientation compatible to $\mathcal{P}$, which implies that the instance admits an (s,t)-upward straight-line (resp., poly-line) drawing monotonic to $\mathcal{P}$ by Lemma 2. To prove the existence of such an (s,t)-orientation compatible to $\mathcal{P}$, we show Theorem 1 is reduced to the following restricted case.

Lemma 3. *For a pseudo-simple connected plane graph $\gamma = (G = (V, E), F)$ and a 1-independent set $\mathcal{P}$ of paths of length at least 2 in γ, let $V_{\rm inl}$ denote the set of internal vertices in paths in $\mathcal{P}$, $\{V_i \subseteq V_{\rm inl} \mid i = 1, 2, \ldots, p\}$ denote the set of components in $G[V_{\rm inl}]$ and $\{\mathcal{P}_i \mid i = 1, 2, \ldots, p\}$ denote the partition of $\mathcal{P}$ such that $V_{\rm inl}(\mathcal{P}_i) \subseteq V_i$. Assume that γ contains an inclusive set $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of edge-disjoint cycles such that, for each $i = 1, 2, \ldots, p$, $V_i \subseteq V_{\rm enc}(C; \gamma)$ and $V_{\rm end}(\mathcal{P}_i) \subseteq V(C_i) \subseteq V \setminus V_{\rm inl}$.*

Then any pair of outer vertices $s, t \notin V_{\rm inl}$ admits an (s,t)-orientation $\widetilde{\gamma}$ of γ compatible to $\mathcal{P}$. Such an orientation $\widetilde{\gamma}$ can be constructed in linear time.

The instance in Figure 1(a) has three components $V_1 = \{u_1, u_2\}$, $V_2 = \{u_3, u_4, u_5, u_6\}$ and $V_3 = \{u_7\}$ in $G[V_{\rm inl}]$. The instance admits a cycle set $\mathcal{C} = \{C_1 = (v_1, v_2, w_4, v_3, v_4, v_5, v_6), C_2 = (v_3, v_7, v_8, v_9, w_5, v_{10}, v_{11}, w_6), C_3 = (v_{10}, v_{12}, v_{13}, v_{14})\}$, which satisfies the condition of Lemma 3. Figure 1(b) illustrates an (s,t)-orientation $\widetilde{\gamma}$ of γ in Figure 1(a) that is compatible to $\mathcal{P}$.

We prove in Section 5 that a given instance of Theorem 1 can be augmented to a plane graph so that the condition of Lemma 3 is satisfied.

4 Bipolar Orientation on Plane Graphs

This section presents several properties on bipolar orientations in plane graphs, which will be the basis to our proof of Lemma 3.

Let $g : V \to \mathbb{R}$ be a vertex-weight function in a graph $G = (V, E)$, where $\mathbb{R}$ denote the set of real numbers. We say that g is *bipolar* (or (a,b)-*bipolar*) to a subgraph $G' = (V', E')$ of G if (i) $g(u) \neq g(v)$ for the end-vertices u and v of each edge $e = uv \in E'$; (ii) V' contains a vertex pair (a, b) such that $g(a) < g(v) < g(b)$ for all vertices $v \in V' \setminus \{a, b\}$; and (iii) each vertex $v \in V' \setminus \{a, b\}$ has a neighbor $u \in N_{G'}(v)$ with $g(u) < g(v)$ and a neighbor $w \in N_{G'}(v)$ with $g(v) < g(w)$.

Observe that an (a,b)-bipolar function g to a graph G is essentially equivalent to an st-numbering of G in the sense that it admits an st-numbering $\sigma : V(G) \to \{1, 2, \ldots, |V(G)|\}$ of G such that $\sigma(a) = 1$, $\sigma(b) = |V(G)|$ and $\sigma(u) < \sigma(v)$ holds for any pair of vertices $u, v \in V$ with $g(u) < g(v)$. We observe that any bipolar function in a plane graph is bipolar to every cycle in the next lemma.

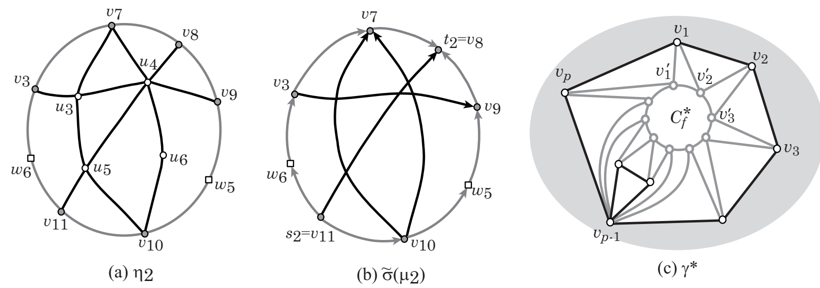

Fig. 2. (a) mesh graph $\eta_2 = (C_2, \mathrm{P}_2)$ induced from the instance γ in Figure 1(a) with cycle C_2; an instance satisfying the condition of Lemma 3: (b) $(s_2 = v_{11}, t_2 = v_8)$-orientation $\widetilde{\sigma(\mu_2)}$ of the split mesh graph $\sigma(\mu_2)$; (c) sun augmentation γ^*.

Lemma 4. *For a biconnected graph $G = (V, E)$, let $g : V \to \mathbb{R}$ be a function (s,t)-bipolar to G. If G admits a plane graph $\gamma = (V, E, F)$, then the boundary of each face $f \in F$ forms a cycle C_f and g is bipolar to C_f.*

The next lemma states that a bipolar orientation of a plane graph can be obtained from bipolar orientations of the interior and exterior subgraphs of a cycle.

Lemma 5. *For a biconnected plane graph $\gamma = (V, E, F)$ and a cycle C of the graph (V, E), let γ_C (resp., $\gamma_{\overline{C}}$) denote the interior (resp., exterior) subgraph of γ by C. For two outer vertices s and t of γ, let $\widetilde{\gamma_C}$ be an (s,t)-orientation of γ_C. Then the orientation $\widetilde{C}$ restricted from $\widetilde{\gamma_C}$ to C is an (a,b)-orientation of C for some $a, b \in V(C)$, and for any (a,b)-orientation $\widetilde{\gamma_{\overline{C}}}$ of $\gamma_{\overline{C}}$, the orientation $\widetilde{\gamma}$ of γ obtained by combining $\widetilde{\gamma_C}$ and $\widetilde{\gamma_{\overline{C}}}$ is an (s,t)-orientation of γ.*

We now examine a special type of instances of Lemma 3.

Mesh Graph A *mesh graph* is defined to be a pair $\mu = (\gamma, \mathcal{P})$ of a biconnected plane graph $\gamma = (V, E, F)$ and a 1-independent set $\mathcal{P}$ of paths in the graph (V, E) such that (i) γ consists of an outer facial cycle C and the paths in $\mathcal{P}$; and (ii) each path $P \in \mathcal{P}$ ends with vertices in C and has no internal vertex from C, where $V = V(C) \cup \bigcup_{P \in \mathcal{P}} V(P)$ and $E = E(C) \cup \bigcup_{P \in \mathcal{P}} E(P)$. We may denote a mesh graph $(\gamma, \mathcal{P})$ with an outer facial cycle C by $\mu = (C, \mathcal{P})$. Figure 2(a) illustrates an example of a mesh graph.

Let $\mu = (\gamma = (V, E, F), \mathcal{P})$ be a mesh graph with an outer facial cycle C. To find an orientation of μ compatible to $\mathcal{P}$, we treat each u, v-path $P \in \mathcal{P}$ as a single edge $e_P = uv$, which we call the *split edge* of P. The *split mesh graph* is defined to be the graph $\sigma(\mu)$ obtained from μ by replacing each path $P \in \mathcal{P}$ with the split edge e_P; i.e., $\sigma(\mu) = (V(C), E(C) \cup \{e_P \mid P \in \mathcal{P}\})$.

Let $\widetilde{\sigma(\mu)}$ be an orientation of the split mesh graph $\sigma(\mu)$. We say that $\widetilde{\sigma(\mu)}$ *induces* an orientation $\widetilde{\mu}$ of μ if each u, v-path $P \in \mathcal{P}$ is directed from u to v in $\widetilde{\mu}$ when e_P is a directed edge (u, v) in $\widetilde{\sigma(\mu)}$. Clearly $\widetilde{\mu}$ is compatible to $\mathcal{P}$. Figure 2(b) illustrates an (s,t)-orientation of the split mesh graph.

The next lemma states that an (s,t)-orientation of a mesh graph compatible to $\mathcal{P}$ can be obtained by computing an (s,t)-orientation of the split mesh graph.

Lemma 6. *For a mesh graph μ and an (s,t)-orientation $\widetilde{\sigma(\mu)}$ of the split mesh graph $\sigma(\mu)$, the orientation $\widetilde{\mu}$ of μ induced by $\widetilde{\sigma(\mu)}$ is an (s,t)-orientation of μ.*

5 Coating and Confiner

To prove that Theorem 1 implies Lemma 3, this section gives a characterization of a plane graph that can be augmented to a plane graph such that specified vertices are contained in some cycles. Let $\gamma = (G = (V,E), F)$ be a plane graph.

We call an inclusive set $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of edge-disjoint cycles in γ a *coating* of a family $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$ of subsets of V if for each $i = 1, 2, \ldots, p$, $V(C_i) \cap X = \emptyset$ and $V_{\mathrm{enc}}(C_i;\gamma) \supseteq X_i$. We say that a coating $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of $\mathcal{X}$ *covers* a given family $\{Y_1, Y_2, \ldots, Y_p\}$ of vertices if $V(C_i) \supseteq Y_i$ for each $i = 1, 2, \ldots, p$.

For disjoint subsets $S, T \subseteq V$ in γ such that the subgraph $G[S]$ induced by S is connected, we call a cycle K of G an (S,T)-*confiner* if $|V(K) \setminus S| \leq 1$ and the interior vertex set $V_{\mathrm{enc}}(K;\gamma)$ of K contains some vertex $t \in T$.

A *plane augmentation* of a plane graph $\gamma = (V,E,F)$ is defined to be a plane embedding $\gamma^* = (V^*, E^*, F^*)$ of a supergraph (V^*, E^*) of (V,E) such that the embedding obtained from γ^* by removing the additional vertices in $V^* \setminus V$ and edges in $E^* \setminus E$ is equal to the original embedding γ.

Sun Augmentation Let $\gamma = (V,E,F)$ be a pseudo-simple connected plane graph such that the outer boundary is a cycle. We introduce *sun augmentation*, a method of augmenting γ into a pseudo-simple biconnected plane graph by adding new vertices and edges in the interior of some inner faces of γ.

For an inner face $f \in F$, let $W_f = (v_1, v_2, \ldots, v_p)$ denote the sequence of vertices that appear along the boundary in the clockwise order, where $p \geq 3$ since γ is pseudo-simple. For each inner face $f \in F$,

(i) create a new cycle $C^*_f = (v'_1, v'_2, \ldots, v'_p)$ with p new vertices v'_i, $i = 1, 2, \ldots, p$ in the interior of f so that the facial cycle of f encloses C^*_f; and
(ii) join each vertex v_i, $i = 1, 2, \ldots, p$ with v'_i and v'_{i+1} with new edges $e'_i = v_i v'_i$ and $e''_i = v_i v'_{i+1}$, where we regard v'_{p+1} as v'_1; We call the new face whose set consists of the p new edges e'_i, $i = 1, 2, \ldots, p$ a *core face* and call a vertex along a core face a *core vertex*.

Figure 2(c) illustrates how the sun augmentation γ^* is constructed.

The next lemma characterizes when a plane graph with two vertex subsets X and Y can be augmented such that a set of cycles contains vertices in Y without visiting any vertex in X.

Lemma 7. *For a pseudo-simple connected plane graph $\gamma = (G = (V,E), F)$ such that the boundary forms a cycle C^o and a subset $X \subseteq V \setminus V(C^o)$, let $\{X_i \subseteq$*

$X \mid i = 1, 2, \ldots, p\}$ denote the set of components in $G[X]$ and $Y_i \subseteq N_G(X_i)$, $i = 1, 2, \ldots, p$ be subsets of V, where possibly $Y_i \cap Y_j \neq \emptyset$ for some $i \neq j$.

Then γ contains no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$ if and only if the sun augmentation $\gamma^* = (V^*, E^*, F^*)$ of γ contains a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ that covers $\{Y_1, Y_2, \ldots, Y_p\}$.

Moreover the following can be computed in linear time: (i) *Testing whether γ contains an (X_i, Y_i)-confiner for some $i = 1, 2, \ldots, p$; and* (ii) *Finding a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ that covers $\{Y_1, Y_2, \ldots, Y_p\}$ in γ^* when γ contains no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$.*

We show how the assumption in Lemma 3 is derived from the assumption of Theorem 1 using Lemma 7. Let $\{V_i \subseteq V_{\rm inl} \mid i = 1, 2, \ldots, p\}$ denote the set of components in $G[V_{\rm inl}]$ and $\mathcal{P}_i$, $i = 1, 2, \ldots, p$ denote the partition of $\mathcal{P}$ such that $V_{\rm inl}(\mathcal{P}_i) \subseteq V_i$. We apply Lemma 7 to the plane graph γ in Theorem 1 by setting $X := V_{\rm inl}$, $X_i := V_i$ and $Y_i := V_{\rm end}(\mathcal{P}_i)$, $i = 1, 2, \ldots, p$. Note that $X \subseteq V \setminus V(C^o)$. We show from the assumption in Theorem 1 that γ has no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$.

To derive a contradiction, assume that γ has an (X_i, Y_i)-confiner K for some $i \in \{1, 2, \ldots, p\}$, where $V_{\rm enc}(K; \gamma)$ of K contains an end-vertex $y \in Y_i = V_{\rm end}(\mathcal{P}_i)$ of some path $P \in \mathcal{P}_i$. Since $|K| \geq 2$ and $|K \setminus X_i| \leq 1$, K contains a vertex $v \in K \cap X_i$. Now vertex v and the internal vertices of P belong to the same component $G[X_i] = G[V_i]$ of $G[X]$ in γ. This contradicts the assumption in Theorem 1. Hence the condition of Lemma 7 holds and the sun augmentation γ^* of γ admits a coating $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of $\{X_i = V_i \mid i = 1, 2, \ldots, p\}$ that covers $\{Y_i = V_{\rm end}(\mathcal{P}_i) \mid i = 1, 2, \ldots, p\}$. We see that such a set $\mathcal{C}$ of cycles satisfies the condition of Lemma 3.

6 Algorithmic Proof

This section presents an algorithmic proof to Lemma 3.

For a pseudo-simple biconnected plane graph $\gamma = (V, E, F)$ and a 1-independent set $\mathcal{P}$ of paths of length at least 2, we are given a partition $\{\mathcal{P}_i \mid i = 1, 2, \ldots, p\}$ of $\mathcal{P}$ and an inclusive set $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of edge-disjoint cycles that satisfy the condition of Lemma 3. For the instance $(\gamma, \mathcal{P}, \mathcal{C})$ in Figure 1(a), we obtain $\mathcal{P}_1 = \{P_1, P_2, P_3\}$, $\mathcal{P}_2 = \{P_4, P_5, P_6, P_7\}$, $\mathcal{P}_3 = \{P_8, P_9\}$ and $\mathcal{C} = \{C_1 = (v_1, v_2, w_4, v_3, v_4, v_5, v_6), C_2 = (v_3, v_7, v_8, v_9, w_5, v_{10}, v_{11}, w_6), C_3 = (v_{10}, v_{12}, v_{13}, v_{14})\}$.

Let $\mathcal{I} = (\mathcal{C}, \mathcal{E})$ denote the inclusion-forest of $\mathcal{C}$, and $\mathrm{Ch}(C)$ denote the set of child cycles C' of each cycle $C \in \mathcal{C}$ in $\mathcal{I}$, where the cycle C is called the *parent cycle* of each cycle $C' \in \mathrm{Ch}(C)$. We call a cycle $C \in \mathcal{C}$ that has no parent cycle a *root cycle* in $\mathcal{C}$, and let $\mathcal{C}_{\rm rt}$ denote the set of root cycles in $\mathcal{C}$. For a notational simplicity, we assume that the indexing of $C_1, C_2, \ldots, C_p$ satisfies $i < j$ when C_i is the parent cycle of C_j.

Based on the inclusion-forest $\mathcal{I}$, we first decompose the entire plane graph γ into plane subgraphs γ_i, $i = 0, 1, \ldots, p$ as follows. Define γ_0 to be the plane graph $\gamma - \cup_{C \in \mathcal{C}_{\rm rt}}(V_{\rm enc}(C; \gamma) \cup E_{\rm enc}(C; \gamma))$ obtained from γ by removing the vertices and edges in the interior of root cycles $C \in \mathcal{C}_{\rm rt}$. For each $i = 1, 2, \ldots, p$, define γ_i to be the plane graph $\gamma[C_i]_{\rm enc} - \cup_{C \in {\rm Ch}(C_i)}(V_{\rm enc}(C; \gamma) \cup E_{\rm enc}(C; \gamma))$ obtained from the interior subgraph $\gamma[C_i]_{\rm enc}$ by removing the vertices and edges in the interior of child cycles C of C_i.

For each cycle C_i, $i = 1, 2, \ldots, p$, we consider the mesh graph $\mu_i = (C_i, \mathcal{P}_i)$, where μ_i is a plane subgraph of γ_i. For each inner face f of μ_i, we consider the interior subgraph $\gamma_i[C_f]_{\rm enc}$ of the facial cycle C_f of f in γ_i, where we call an inner face f of μ_i *trivial* if C_f encloses nothing in γ_i; i.e., $V_{\rm enc}(C_f; \gamma_i) \cup E_{\rm enc}(C_f; \gamma_i) = \emptyset$. Let $F(\mu_i)$ denote the set of non-trivial inner faces f of μ_i.

We determine orientations of subgraphs γ_i by an induction on $i = 0, 1, \ldots, p$. For specified outer vertices $s, t \in V(C^o) \setminus V_{\rm inl}$, compute an (s, t)-orientation $\widetilde{\gamma_0}$ of γ_0 using Lemma 1. Consider the plane subgraph γ_i with $i \geq 1$, where we assume that a bipolar orientation $\widetilde{\gamma_j}$ of γ_j has been obtained for all $j < i$. Let k denote the index of the parent cycle C_k of C_i or $k = 0$ if C_i is a root cycle, where a bipolar orientation $\widetilde{\gamma_k}$ of γ_k has been obtained. In $\widetilde{\gamma_k}$, cycle C_i forms an inner facial cycle and the orientation restricted to the facial cycle C_i is a bipolar orientation, which is an (s_i, t_i)-orientation $\widetilde{C_i}$ for some vertices $s_i, t_i \in V(C_i)$ by Lemma 4. We determine an (s_i, t_i)-orientation of γ_i as follows:
Step (a) Compute an (s_i, t_i)-orientation $\widetilde{\mu_i}$ of the mesh graph $\mu_i = (C_i, \mathcal{P}_i)$;
Step (b) Extend the orientation $\widetilde{\mu_i}$ to the interior subgraph $\gamma_i[C_f]_{\rm enc}$ of each non-trivial inner face $f \in F(\mu_i)$.

At Step (a), we compute an (s_i, t_i)-orientation $\widetilde{\sigma(\mu_i)}$ of the split mesh graph $\sigma(\mu_i)$ to obtain an (s_i, t_i)-orientation $\widetilde{\mu_i}$ using Lemma 6. For Step (b), we observe that orientation $\widetilde{\mu_i}$ is (s_f, t_f)-bipolar to the facial cycle C_f of f for some vertices $s_f, t_f \in V(C_f)$ by Lemma 4. We compute an (s_f, t_f)-orientation $\widetilde{\gamma_i[C_f]_{\rm enc}}$ of the interior subgraph $\gamma_i[C_f]_{\rm enc}$ induced from γ_i by C_f using Lemma 1. An (s_i, t_i)-orientation of γ_i is obtained from the (s_i, t_i)-orientation $\widetilde{\mu_i}$ and (s_f, t_f)-orientations $\widetilde{\gamma_i[C_f]_{\rm enc}}$ for all inner faces $f \in F(\mu_i)$.

We repeat the above procedure until $i = p$. Finally construct an orientation $\widetilde{\gamma}$ of γ by combining bipolar orientations $\widetilde{\gamma_i}$ of γ_i, $i = 0, 1, \ldots, p$. By Lemma 5, $\widetilde{\gamma}$ is an (s, t)-orientation, which is compatible to $\mathcal{P}$ by construction of $\widetilde{\gamma}$. This proves the correctness of our algorithm for computing an (s, t)-orientation $\widetilde{\gamma}$ compatible to $\mathcal{P}$ (see XXXXX **Algorithm ORIENT**?? XXXXX in Appendix 7).

The inclusion-forest of an inclusive set $\mathcal{C}$ of edge-disjoint cycles can be constructed in linear time [11]. Constructing all plane subgraphs γ_i and face sets $F(\mu_i)$, $i = 1, 2, \ldots, p$ can be done in linear time, since we can find them such that each edge in γ is scanned a constant number of times. We see that a bipolar orientation of mesh graph μ_i or subgraph γ_i can be computed in time linear to the size of the graph by Lemmas 1 and 6. The total size of these graphs μ_i, $i = 1, 2, \ldots, p$ and γ_i, $i = 0, 1, \ldots, p$ is bounded by the size of input graph γ. Therefore the algorithm can be executed in linear time. This proves Lemma 3.

Figure 3 shows an execution of the algorithm applied to the instance $(\gamma, \mathcal{P}, \mathcal{C})$ in Figure 1(a). Figures 3(b), (c) and (f) show mesh graphs μ_1, μ_2 and μ_3, respectively for the instance in Figure 1(a), where $\mathcal{C}_{\rm rt} = \{C_1, C_2\}$, ${\rm Ch}(C_1) = \emptyset$, ${\rm Ch}(C_2) = \{C_3\}$, $F(\mu_1) = \{f_1\}$ $(C_{f_1} = (v_5, u_1, v_2, w_4, v_3, u_2, v_4))$, $F(\mu_2) =$

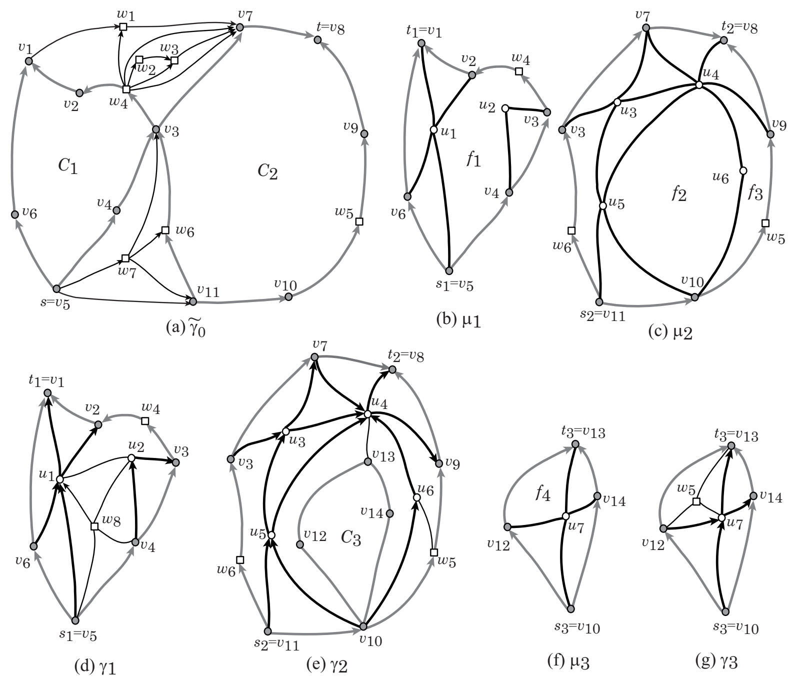

Fig. 3. (a) An $(s = v_5, t = v_7)$-orientation $\widetilde{\gamma}_0$ of γ_0; (b) Mesh graph $\mu_1 = (C_1, \mathcal{P}_1)$, where C_1 is directed as an $(s_1 = v_5, t_1 = v_1)$-orientation; (c) Mesh graph $\mu_2 = (C_2, \mathcal{P}_2)$, where C_2 is directed as an $(s_2 = v_{11}, t_2 = v_8)$-orientation; (d) Subgraph γ_1 with an (s_1, t_1)-orientation $\widetilde{\mu}_1$ of μ_1; (e) Subgraph γ_2 with an (s_2, t_2)-orientation $\widetilde{\mu}_2$ of μ_2; (f) Mesh graph $\mu_3 = (C_3, \mathcal{P}_3)$, where C_3 is directed as an $(s_3 = v_{10}, t_3 = v_{13})$-orientation; (e) Subgraph γ_3 with an (s_3, t_3)-orientation $\widetilde{\mu}_3$ of μ_3.

$\{f_2, f_3\}$ $(C_{f_2} = (v_{10}, u_5, u_4, u_6), C_{f_3} = (v_{10}, u_6, u_4, v_9, w_5))$, $F(\mu_3) = \{f_4\}$ $(C_{f_4} = (v_{12}, v_{13}, v_{14}))$. Figures 3(a), (d), (e) and (g) show subgraphs γ_0, γ_1, γ_2 and γ_3, respectively for the instance in Figure 1(a). Figure 1(b) shows an (s,t)-orientation of the instance γ in Figure 1(a).

7 Upward Drawing of a Non-plane Embedding

Let Γ be a non-plane embedding of a graph G, and E^* denote the set of crossing edges. We define a *crossing-set* to be a maximal subset $E' \subseteq E^*$ such that every two edges $e, e' \in E'$ admit a sequence of edges $e_1, e_2, \ldots, e_p$, where $e_1 = e$, $e_p = e'$ and edges e_i and e_{i+1} cross for each $i = 1, 2, \ldots, p-1$. Observe that E^* is partitioned into disjoint crossing-sets $E_1^*, E_2^*, \ldots, E_p^*$.

Let E_i^* be a crossing-set, and $\Gamma[E_i^*]$ denote the plane graph induced from Γ by the edges in E_i^*, where $\Gamma[E_i^*]$ is connected. We call E_i^* *outer* if the end-vertices of edges in E_i^* appear as outer vertices along the boundary of $\Gamma[E_i^*]$.

We apply Lemma 3 to the problem of finding an upward drawing of a non-plane embedding of a graph, and prove the following results.

Theorem 2. *Let Γ be a non-plane embedding of a graph G such that each crossing-set is outer, let $n = |V(G)|$, and let $n_{\rm c}$ denote the number of crossings in Γ. Then for any pair of outer vertices s and t in Γ, there is an (s,t)-upward drawing of Γ, and an upward poly-line drawing of Γ with $O(n + n_{\rm c})$ bends can be constructed in $O(n + n_{\rm c})$ time and space.*

Theorem 2 implies the following.

Corollary 1. *Every 1-plane graph admits an (s,t)-upward poly-line drawing for any outer vertices s and t, where each edge has at most one bend. Such a drawing can be constructed in linear time.*

References

1. P. Angelini, E. Colasante, G. Di Battista, F. Frati, M. Patrignani, Monotone drawings of graphs, J. Graph Algorithms Appl., 16 (1), pp. 5-35, 2012.
2. P. Bertolazzi, G. Di Battista, G. Liotta, C. Mannino, Upward drawings of triconnected digraphs, Algorithmica, 12(6), pp. 476-497, 1994.
3. G. Di Battista G, P. Eades P, R. Tamassia, I. G. Tollis, Graph Drawing: algorithms for the visualization of graphs, Prentice Hall, 1998.
4. G. Di Battista, R. Tamassia, Algorithms for plane representations of acyclic digraphs, Theoret. Comput. Sci., 61, pp. 175-198, 1988.
5. C. Binucci, W. Didimo, M. Patrignani, Upward and quasi-upward planarity testing of embedded mixed graphs, Theor. Comput. Sci. 526, pp. 75-89, 2014.
6. W. Didimo, M. Pizzonia, Upward embeddings and orientations of undirected planar graphs, J. Graph Algorithms Appl., 7(2), pp. 221-241, 2003.
7. S. Even, R. E. Tarjan, Computing an st-numbering, Theoret. Comput. Sci., 2 (3), pp. 339-344, 1976.
8. F. Frati, M. Kaufmann, J. Pach, C. T 坦 th, D. Wood, On the upward planarity of mixed plane graphs, J. Graph Algorithms Appl. 18(2), pp. 253-279, 2014.
9. H. de Fraysseix, P. Ossona de Mendez, P. Rosenstiehl, Bipolar orientations revisited, Disc. Applied Math., 56 (2-3), pp. 157-179, 1995.
10. A. Garg, R. Tamassia, On the computational complexity of upward and rectilinear planarity testing, SIAM J. Comput., 31(2), pp. 601-625, 1992.
11. S. Hong, H. Nagamochi, Re-embedding a 1-plane graph into a straight-line drawing in linear time, Proc. of Graph Drawing 2016, pp. 321-334, 2016.
12. D. Kelly, Fundamentals of planar ordered sets, Discrete Math., 63, pp. 197-216, 1987.
13. M. Patrignani, Finding bimodal and acyclic orientations of mixed planar graphs is NP-complete, RT-DIA-188-2011, Aug 2011.
14. P. Rosenstiehl, R.E. Tarjan, Rectilinear planar layouts and bipolar orientations of planar graphs, Disc. Comput. Geometry, 1 (4), pp. 343-353, 1986.
15. R. E. Tarjan, Two streamlined depth-first search algorithms, Fundamenta Informaticae, 9 (1), pp. 85-94, 1986.
16. C. Thomassen, Rectilinear drawings of graphs, Journal of Graph Theory, 12(3), pp. 335-341, 1988.

Appendix 1: Instances That Admit No Path-monotonic Upward Drawing

We here present some instances that cannot admit a path-monotonic upward drawing. Figure 4 illustrates three such instances. The instance $(\gamma_1, \mathcal{P}_1)$ in Figure 4(a) admits no $(s = v_1, t = v_4)$-upward drawing monotonic to a set $\mathcal{P}_1 = \{P_1 = (v_1, v_2, v_3), P_2 = (v_3, v_2, v_4)\}$ of two paths, where P_1 and P_2 share an edge v_2v_3. The instance $(\gamma_2, \mathcal{P}_2)$ in Figure 4(b) admits no $(s = v_1, t = v_5)$-upward drawing monotonic to a set $\mathcal{P}_2 = \{P_1 = (v_1, v_2, v_3), P_2 = (v_4, v_2, v_5)\}$ of two paths, where P_1 and P_2 share a common internal vertex v_2 but do not intersect. The instance $(\gamma_3, \mathcal{P}_3)$ in Figure 4(c) admits no $(s = v_1, t = v_4)$-upward drawing monotonic to a set $\mathcal{P}_3 = \{P_1 = (v_1, v_2, v_3)\}$ of a single path.

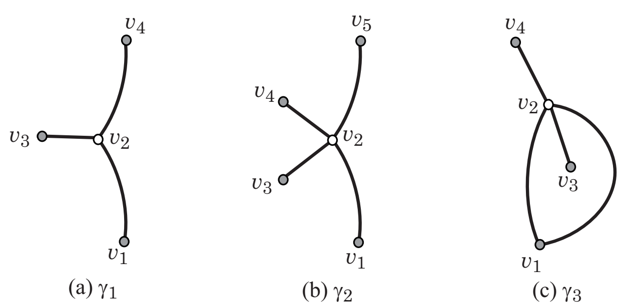

Fig. 4. Illustration for instances $(\gamma_i, \mathcal{P}_i)$, $i = 1, 2, 3$ that admit no (s, t)-upward drawing monotonic to a path set (where each vertex depicted with a gray circle indicates an end-vertex of a path in $\mathcal{P}_i$): (a) $\mathcal{P}_1 = \{P_1 = (v_1, v_2, v_3), P_2 = (v_3, v_2, v_4)\}$, $s = v_1$ and $t = v_4$; (b) $\mathcal{P}_2 = \{P_1 = (v_1, v_2, v_3), P_2 = (v_4, v_2, v_5)\}$, $s = v_1$ and $t = v_5$; (c) $\mathcal{P}_3 = \{P_1 = (v_1, v_2, v_3)\}$, $s = v_1$ and $t = v_4$.

Observe that for each instance $(\gamma_i, \mathcal{P}_i)$, $i = 1, 2$ in Figure 4(a)-(b), path set $\mathcal{P}_i$ is not 1-independent. For instance $(\gamma_3, \mathcal{P}_3)$ in Figure 4(c), path set $\mathcal{P}_3$ is 1-independent, however, cycle $K = (v_1, v_2)$ encloses an end-vertex v_3 of P_1.

Appendix 2: Preprocessing of Boundary of Instances in Theorem 1

To prove Theorem 1, we can assume that the boundary of a given connected plane graph γ forms a cycle as follows. Let $(\gamma = (G = (V, E), F), \mathcal{P} \neq \emptyset)$ be an instance that satisfies the condition in Theorem 1. If the outer boundary B of γ contains at most one vertex in $V \setminus V_{\rm inl}$, then for any path P with $V_{\rm inl}(P) \cap V(B) \neq \emptyset$, one of the end-vertices of P is enclosed by some cycle K contained in B, contradicting the assumption that there is no such pair (P, K) in Theorem 1. Hence the outer boundary B of γ contains at least two vertices in $V \setminus V_{\rm inl}$. Let $\rho(\gamma) = (v_1, v_2, \ldots, v_p)$, where $v_1 \notin V_{\rm inl}$ denote the sequence of outer vertices of

γ that appear in the clockwise order along the boundary B, where v_i and v_j for some i and j may be the same vertex $v \in V$ when v is a cut-vertex of the graph.

We first augment γ such that all vertices in V_{inl} along the boundary B will be contained in the interior of the new boundary B' as follows. For each maximal subsequence $\tau = (v_i, v_{i+1}, \ldots, v_j)$ $(i < j)$ of $\rho(\gamma)$ such that $v_i, v_j \notin V_{\text{inl}}$ and $v_{i+1}, v_{i+2}, \ldots, v_{j-1} \in V_{\text{inl}}$, create a new outer vertex v_τ together with two new outer edges $v_i v_\tau$ and $v_\tau v_j$. Let γ' denote the resulting pseudo-simple plane graph, where no vertex in V_{inl} appears as an outer vertex, and γ' is simple when γ is simple. Observe that the condition (i)-(iii) still hold in γ'.

We further augment γ' into γ'' so that the outer boundary B'' becomes a cycle as follows. If the boundary of γ' already forms a cycle, then let $\gamma'' := \gamma'$. Otherwise let $\rho(\gamma') = (u_1, u_2, \ldots, u_p)$ denote the sequence of outer vertices of γ' in the clockwise order along the boundary. For each cut-vertex v, we remove from the sequence its last appearance. Let $(u'_1, u'_2, \ldots, u'_q)$ denote the resulting sequence, where each cut-vertex removal of which from γ leaves k components appear $k-1$ times in the new reduced sequence.

For each maximal subsequence $\rho' = (u'_i, u'_{i+1}, \ldots, u'_j)$ $(i < j)$ of $\rho(\gamma')$ such that u'_i and u'_j are not cut-vertices and $u'_{i+1}, u'_{i+2}, \ldots, u'_{j-1}$ are cut-vertices, create a new outer edge $u'_i u'_j$. Let γ'' denote the resulting pseudo-simple plane graph, where the boundary forms a cycle that contains no vertex in V_{inl}. Note that γ'' is simple when γ is simple. Observe that the conditions (i)-(iii) still hold in γ'' and any (s,t)-upward straight-line (or poly-line) drawing D'' of γ'' monotonic to $\mathcal{P}$ can be modified to one for γ just by removing the newly added vertices and edges in the augmentation.

It is not difficult to see that the above augmentation can be executed in linear time.

Appendix 3: Proof of Lemma 4

Lemma 4. *For a biconnected graph $G = (V, E)$, let $g : V \to \mathbb{R}$ be a function (s,t)-bipolar to G. If G admits a plane graph $\gamma = (V, E, F)$, then the boundary of each face $f \in F$ forms a cycle C_f and g is bipolar to C_f.*

Proof. Let $f \in F$ be a face in γ. Since G is biconnected, the boundary of each face $f \in F$ forms a cycle C_f. We call a vertex v in C_f *locally maximum* (resp., *locally minimum*) if $g(v') < g(v) > g(v'')$ (resp., $g(v') > g(v) < g(v'')$) for the two neighbors $v', v'' \in N_G(v) \cap V(C_f)$. To prove the lemma, it suffices to show that C_f contains exactly one locally maximum vertex and exactly one locally minimum vertex.

Consider the case where f is an inner face in γ and C_f contains two locally maximum vertices v_1^* and v_2^* (the other cases can be treated analogously). Without loss of generality assume that $g(v_2^*) \geq g(v_1^*)$. Let $u_1, u_2 \in N_G(v_1^*) \cap V(C_f)$, where $g(u_1), g(u_2) < g(v_1^*)$ and $u_1 \neq v_2^* \neq u_2$. Since g is (s,t)-bipolar to G, there is a v_i^*, t-path P_i, $i = 1, 2$ such that the function values of vertices increase along the path from v_i^* to t. This means that G contains a v_1^*, v_2^*-path P such that $g(v) \geq g(v_1^*)$ for all vertices $v \in V(P)$, since $g(v_2^*) \geq g(v_1^*)$. Also there is an s, u_i-path Q_i, $i = 1, 2$ such that the function values of vertices increase along the path from s to u_i, implying that G contains a u_1, u_2-path Q such that $g(u) < g(v_1^*)$ for all vertices $u \in V(Q)$. Since vertices v_1^* and v_2^*, and vertices u_1 and u_2 appear alternately along C_f, two paths P and Q must have a common vertex w. This, however, is impossible because $g(w) \geq g(v_1^*)$ and $g(w) < g(v_1^*)$ cannot hold at the same time. □

Appendix 4: Proof of Lemma 5

Lemma 5. *For a biconnected plane graph $\gamma = (V, E, F)$ and a cycle C of the graph (V, E), let γ_C (resp., $\gamma_{\overline{C}}$) denote the interior (resp., exterior) subgraph of γ by C. For two outer vertices s and t of γ, let $\widetilde{\gamma_C}$ be an (s, t)-orientation of γ_C. Then the orientation $\widetilde{C}$ restricted from $\widetilde{\gamma_C}$ to C is an (a, b)-orientation of C for some $a, b \in V(C)$, and for any (a, b)-orientation $\widetilde{\gamma_{\overline{C}}}$ of $\gamma_{\overline{C}}$, the orientation $\widetilde{\gamma}$ of γ obtained by combining $\widetilde{\gamma_C}$ and $\widetilde{\gamma_{\overline{C}}}$ is an (s, t)-orientation of γ.*

Proof. A topological ordering g_C of $\widetilde{\gamma_C}$ is a bipolar vertex weight to γ_C. By Lemma 4, g_C is bipolar to the inner facial cycle C in γ_C, and this means that the orientation $\widetilde{C}$ restricted from $\widetilde{\gamma_C}$ to C is an (a, b)-orientation for a source a and a sink b in $V(C)$. In the following, for a cycle H in γ and two vertices $x, y \in V(H)$, let H_{xy} (resp., H_{yx}) denote the sub-x, y-path of H that traverses H from x to y (resp., y to x) in the clockwise order.

Let $\widetilde{\gamma_{\overline{C}}}$ be an (a, b)-orientation of $\gamma_{\overline{C}}$. We consider the orientation $\widetilde{\gamma}$ of γ obtained by combining $\widetilde{\gamma_C}$ and $\widetilde{\gamma_{\overline{C}}}$. To prove that $\widetilde{\gamma}$ is an (s, t)-orientation of γ, it suffices to show that

(i) $\widetilde{\gamma}$ has no other source or sink than s and t; and
(ii) $\widetilde{\gamma}$ is acyclic.

Each vertex in $\widetilde{\gamma_C}$ is reachable from s and reachable to t; and each vertex in $\widetilde{\gamma_{\overline{C}}}$ is reachable from a and reachable to b. This implies that any vertex in $\widetilde{\gamma}$ is reachable from s and reachable to t, proving (i).

To prove (ii), we assume that $\widetilde{\gamma}$ contains a directed cycle Q to derive a contradiction. Choose Q so that the number of inner faces of γ enclosed by Q is minimized. Note that outer vertices s and t are in the exterior of Q, since Q does not contain source s or sink t. Since each of $\widetilde{\gamma_C}$ and $\widetilde{\gamma_{\overline{C}}}$ is acyclic, Q must contain some edges $e \in E_{\rm enc}(C; \gamma)$ and $e' \in E \setminus E(C) \cup E_{\rm enc}(C; \gamma)$. This means that there are vertices $u, v \in V(Q)$ such that $\{u, v\} = V(Q_{uv}) \cap V(C_{uv})$ and the edges in Q_{uv} are contained in $\widetilde{\gamma_{\overline{C}}}$. We distinguish three cases.

Case 1. Q_{uv} is a directed path from u to v (resp., v to u) and C_{uv} is a directed path from v to u (resp., u to v) in $\widetilde{\gamma_{\overline{C}}}$: In this case, Q_{uv} and C_{uv} form a directed cycle in $\widetilde{\gamma_{\overline{C}}}$, a contradiction.

Case 2. Each of Q_{uv} and C_{uv} is a directed path from u to v (or from u to v) in $\widetilde{\gamma_{\overline{C}}}$: In this case, we can modify Q by replacing Q_{uv} with C_{uv} to obtain a graph containing a directed cycle that encloses a smaller number of inner faces than Q does. This contradicts the minimality of inner faces enclosed by Q.

Case 3. $a \in V(C_{uv}) \setminus \{u, v\}$ or $b \in V(C_{uv}) \setminus \{u, v\}$: Let $b \in V(C_{uv}) \setminus \{u, v\}$ (the other case can be treated symmetrically). There is a directed b, t-path $P_{b,t}$

in $\widetilde{\gamma_C}$. Since b is the sink of the oriented cycle C and t is in the exterior of Q, paths $P_{b,t}$ and Q_{uv} intersect at some vertex $w \in V(Q_{uv})$. This implies that the sub-b, w-path of $P_{b,t}$ together with paths Q_{uv}, C_{uv} contains a directed cycle. This contradicts that $\widetilde{\gamma_C}$ is acyclic.

This proves (ii). $\square$

Appendix 5: Proof of Lemma 6

Lemma 6. *For a mesh graph μ and an (s,t)-orientation $\widetilde{\sigma(\mu)}$ of the split mesh graph $\sigma(\mu)$, the orientation $\widetilde{\mu}$ of μ induced by $\widetilde{\sigma(\mu)}$ is an (s,t)-orientation of μ.*

We prove Lemma 6 via the following two lemmas, Lemma 8 and Lemma 9.

For a mesh graph $(\gamma, \mathcal{P})$ with an outer facial cycle C and a function $g : V(C) \to \mathbb{R}$, (s,t)-bipolar to C such that $g(u) \neq g(v)$ for any two vertices $u, v \in V(C)$, an orientation $\widetilde{\gamma}$ of γ is called *g-proper* if

- the edges in C are directed from s to t; and
- the edges in each u, v-path $P \in \mathcal{P}$ are directed from u to v when $g(u) < g(v)$.

We first prove that any g-proper orientation is acyclic in Lemma 9. For this, we use the next technical lemma which facilitates a proof of Lemma 9.

Lemma 8. *For a biconnected plane graph $\gamma = (V, E, F)$ with an outer facial cycle C, let $g : V(C) \to \mathbb{R}$ be a function bipolar to C such that $g(u) \neq g(v)$ for any two vertices $u, v \in V(C)$. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a 1-independent set of a_i, b_i-paths P_i of (V, E) for some $m \geq 2$ such that, for each $i = 1, 2, \ldots, m$, $V(P_i) \cap V(C) = \{a_i, b_i\}$ and $g(a_i) < g(b_i)$. Then $g(a_2) < g(b_1)$ if*

- *P_1 and P_2 intersect; or*
- *P_1 and P_3 intersect at an inner vertex w and P_2 and P_3 intersect at a vertex in the sub-a_3, w-path of P_3.*

Proof. The sequence of vertices along C is given by $u_{\min}, u_2, u_3, \ldots, u_{p-1}, u_{\max}, v_{q-1}, v_{q-2}, \ldots, v_2$ such that $g(u_1) < g(u_2) < \cdots < g(u_p)$ and $g(v_1) < g(v_2) < \cdots < g(v_q)$ for $u_1 = v_1 = u_{\min}$ and $u_p = v_q = u_{\max}$. Without loss of generality assume that $a_1 \in \{v_1, v_2, \ldots, v_{q-1}\}$ and if $a_1 = u_{\min}$ then $b_1 \in \{u_2, u_3, \ldots, u_p\}$. We distinguish two cases.

Case 1. P_1 and P_2 intersect. We distinguish two subcases.

Case 1a. $a_1 = v_j$ with $1 \leq j \leq q-1$ and $b_1 = u_k$ with $2 \leq k \leq p$ (see Figure 5(a)): Let $X = \{v_i \mid 1 \leq i < j\} \cup \{u_i \mid 1 \leq i < k\}$. Since P_1 and P_2 intersect at an internal vertex exactly once, one of vertices a_2 and b_2 belongs to X, which means that $\min\{g(a_2), g(b_2)\} < \max\{g(a_1), g(b_1)\}$ and thereby $g(a_2) < g(b_1)$.

Case 1b. $a_1 = v_j$ and $b_1 = v_k$ with $2 \leq j < k \leq q-1$ (see Figure 5(b)): Let $Y = \{v_i \mid j < i < k\}$. Since P_1 and P_2 intersect at an internal vertex exactly once, one of vertices a_2 and b_2 belongs to Y, which implies $\min\{g(a_2), g(b_2)\} < \max\{g(a_1), g(b_1)\}$ and thereby $g(a_2) < g(b_1)$.

Case 2. P_1 and P_2 do not intersect; P_1 and P_3 intersect at an inner vertex w; and P_2 and P_3 intersect at a vertex of the sub-a_3, w-path of P_3: Since P_1 and P_3 intersect, we know that $g(a_3) < g(b_1)$ by the result in Case 1. We distinguish two subcases.
Case 2a. $a_1 = v_j$ with $1 \leq j \leq q-1$ and $b_1 = u_k$ with $2 \leq k \leq p$ (see Figure 5(a)): As in Case 1a, if a_2 or b_2 is a vertex in X, then $\min\{g(a_2), g(b_2)\} < \max\{g(a_1), g(b_1)\}$ holds and we are done. Assume that $\{a_2, b_2\} \subseteq V(C) \setminus X$. Since P_2 and P_3 intersect at a vertex of the sub-a_3, w-path of P_3, the assumption implies that $a_3 \in V(C) \setminus X$, $a_3 = v_h$ with $j < h < q-1$. Moreover, $a_2 = v_\ell$ with $j \leq \ell < h$, since otherwise $a_2 \in \{u_i \mid k < i \leq p\}$ and $b_2 \in \{v_i \mid j \leq i < h\}$ implying that $g(a_2) < g(b_2) < g(a_3) < g(b_3) < g(b_1) < g(a_2)$, a contradiction. Now $g(a_1) \leq g(a_2) < g(a_3)$ holds. Since $g(a_3) < g(b_1)$, we obtain $g(a_2) < g(a_3) < g(b_1)$, as required.
Case 2b. $a_1 = v_j$ and $b_1 = v_k$ with $2 \leq j < k \leq q-1$ (see Figure 5(b)): As in Case 1b, if a_2 or b_2 is a vertex in Y, then $\min\{g(a_2), g(b_2)\} < \max\{g(a_1), g(b_1)\}$ holds and we are done. Assume that $\{a_2, b_2\} \subseteq V(C) \setminus Y$. Since P_2 and P_3 intersect at a vertex of the sub-a_3, w-path of P_3, we see that $a_3 \in \{v_i \mid 1 \leq i < j\} \cup \{u_i \mid 1 \leq i \leq p-1\}$ and a_2 appears between a_1 and a_3 so that $g(a_2) < \max\{g(a_1), g(a_3)\}$. When $g(a_2) < g(a_1)$, we obtain $g(a_2) < g(a_1) < g(b_1)$. When $g(a_2) < g(a_3)$, we obtain $g(a_2) < g(b_1)$ by $g(a_3) < g(b_1)$. □

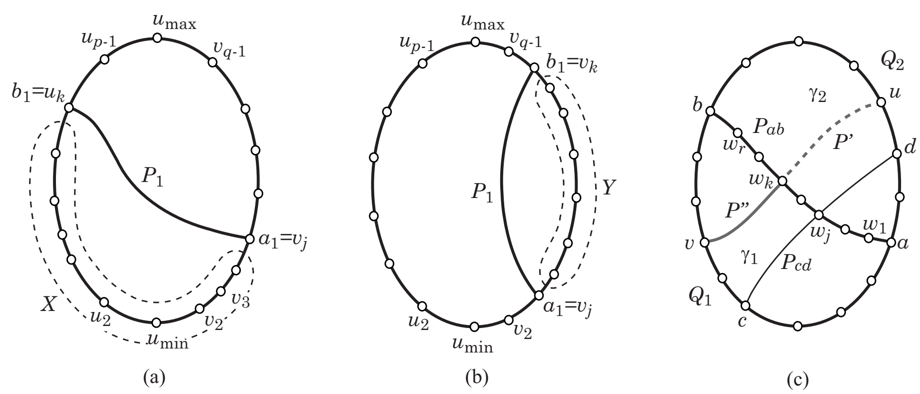

Fig. 5. Illustration for plane graphs with paths joining outer vertices: (a) An a_1, b_1-path P_1 in Cases 1a and 2a in the proof of Lemma 8; (b) An a_1, b_1-path P_1 in Cases 1b and 2b in the proof of Lemma 8; (c) An a, b-path P_{ab} intersects a u, v-path P at a vertex w_k in the proof of Lemma 9.

We are ready to prove that any g-proper orientation is acyclic.

Lemma 9. *For a mesh graph $(\gamma, \mathcal{P})$ with an outer facial cycle C, let $g : V(C) \to \mathbb{R}$ be a function bipolar to C such that $g(u) \neq g(v)$ for any two vertices $u, v \in V(C)$, and $\widetilde{\gamma}$ denote the g-proper orientation of γ. Then*

– $\widetilde{\gamma}$ *is acyclic; and*
– g *can be extended to the inner vertices in* $V \setminus V(C)$ *such that* g *is bipolar to the graph* (V, E) *and* $g(u) < g(v)$ *holds for any directed edge* (u, v) *in* $\widetilde{\gamma}$.

Proof. Let $I = (\gamma, \mathcal{P}, C, g)$ denote a given instance with a mesh graph, an outer facial cycle and a function $g : V(C) \to \mathbb{R}$. We only need to prove the second statement, because any extended function bipolar to (V, E) means that $\widetilde{\gamma}$ is acyclic. We prove the second statement by an induction on the number $|\mathcal{P}|$ of paths. When $|\mathcal{P}| = 0$, the lemma is immediate. Assume that $|\mathcal{P}| \geq 1$.

Choose an a, b-path $P_{ab} \in \mathcal{P}$ for some vertices $a, b \in V(C)$ with $g(a) < g(b)$, where we assume without loss of generality that the sequence of vertices in $V(P_{ab})$ along P_{ab} is given by $a, w_1, w_2, \ldots, w_r, b$. Based on path P_{ab}, we split instance I into two smaller instances $I_i = (\gamma_i, \mathcal{P}_i, C_i, g_i)$, $i = 1, 2$.

First we define γ_i and C_i, $i = 1, 2$. Let $Q_1 = C_{ab}$ (resp., $Q_2 = C_{bc}$) denote the sub-a, b-path of C that traverses C from a to b (resp., b to a) in the clockwise order. We split γ into two plane graphs $\gamma_i = (V_i, E_i, F_i)$, $i = 1, 2$ such that γ_i is the interior subgraph of γ by C_i.

Next we define a set $\mathcal{P}_i$ of paths for each plane graph γ_i. Let P be an arbitrary u, v-path in $\mathcal{P} \setminus \{P_{ab}\}$ for some vertices $u, v \in V(C)$ with $g(u) < g(v)$. Since $\mathcal{P}$ is 1-independent, we see that path P satisfies one of following cases:

(i) $E(P) \subseteq E_1$ and $V_{\rm inl}(P) \cap V(P_{ab}) = \emptyset$, where $u, v \in V(Q_1)$;
(ii) $E(P) \subseteq E_2$ and $V_{\rm inl}(P) \cap V(P_{ab}) = \emptyset$, where $u, v \in V(Q_2)$;
(iii) $u \in V(Q_1) \setminus \{a, b\}$, $v \in V(Q_2) \setminus \{a, b\}$; and
(iv) $u \in V(Q_2) \setminus \{a, b\}$, $v \in V(Q_1) \setminus \{a, b\}$.

See Figure 5(c) for an illustration of path P_{ab}. For each u, v-path P in case (iii) or (iv), which has exactly one common internal vertex w with P_{ab}, let P' (resp., P'') denote the sub-u, w-path (resp., sub-w, v-path) of P. Define $\mathcal{P}_1$ to be the set of paths P in case (i), paths P' in case (iii) and paths P'' in case (iv). Define $\mathcal{P}_2$ to be the set of paths P in case (ii), paths P'' in case (iii) and paths P' in case (iv).

Finally we define a function $g_i : V_i \to \mathbb{R}$ for each $i = 1, 2$ so that the resulting instance $I_i = (\gamma_i, \mathcal{P}_i, C_i, g_i)$ satisfies the condition of the lemma. For this, we let $n = |V|$, $\delta = \min\{|g(u) - g(v)| \mid u, v \in V(C),\ u \neq v\}$, where $\delta > 0$ by the assumption on g, and define functions h and $g' : V_{\rm inl}(P_{ab}) \to \mathbb{R}$ as follows.

$$\begin{aligned} h(w_k) :=\ & \max\{g(u) \mid \text{a } u, v\text{-path } P \in \mathcal{P} \text{ with } g(u) < g(v) \text{ intersects } P_{a,b} \\ & \qquad \text{at some a vertex } w_j \in \{w_1, w_2, \ldots, w_k\}\}, \\ g'(w_k) :=\ & \max\{g(a), h(w_k)\} + \tfrac{k}{n}\delta. \end{aligned}$$

For each $i = 1, 2$, define a function $g_i : V(C_i) \to \mathbb{R}$ such that

$$g_i(v) := \begin{cases} g(v), & v \in V(Q_i) \\ g'(v), & vv \in V_{\rm inl}(P_{ab}). \end{cases}$$

$g_i(v) = g(v)$ for each vertex $v \in V(Q_i)$ and $g_i(w) = g'(w)$ for each vertex $w \in V_{\rm inl}(P_{ab})$.

Now we prove that for each $i = 1, 2$,

(a) g_i is bipolar to C_i; and
(b) for each u^*, v^*-path $P^* \in \mathcal{P}_i$ directed from u^* to v^* in $\overline{\gamma}$, it holds $g_i(u^*) < g_i(v^*)$, which implies that the orientation restricted from $\overline{\gamma}$ to γ_i is g_i-proper.

To prove (a), it suffices to show that

$$g(a) < g'(w_1) < g'(w_2) < \cdots < g'(w_r) < g(b), \tag{1}$$

and

$$g_i(u) \neq g'(v) \text{ for any vertices } u, v \in V(C_i),\ i = 1, 2. \tag{2}$$

By definition of h, we see that $h(w_1) \leq h(w_2) \leq \cdots \leq h(w_r)$, which implies $g(a) \leq \max\{g(a), h(w_1)\} \leq \cdots \leq \max\{g(a), h(w_r)\}$. Since $0 < \delta/n < 2\delta/n < \cdots < r\delta/n$, we have $g(a) < g'(w_1) < g'(w_2) < \cdots < g'(w_r)$. We here prove that $\max\{g(a), h(w_r)\} < g(b)$, where $g(a) < g(b)$ is immediate from the choice of P_{ab}. By Lemma 8, any u, v-path P that intersects P_{ab} at an internal vertex w_j with $1 \leq j \leq r$ satisfies $\min\{g(u), g(v)\} < \max\{g(a), g(b)\} = g(b)$. This implies that $h(w_r) < g(b)$, proving (1).

Note that $\max\{g(a), h(w_k)\} \in \{g(u) \mid u \in V(C)\}$, $k = 1, 2, \ldots, r$. By definition of $\delta > 0$, we see that $g_i(u) \neq g_i(w_k)$ for any vertices $u \in V(Q_i)$ and $w_k \in V_{\mathrm{inl}}(P_{ab})$ and that $g_i(w_j) \neq g_i(w_k)$ for any vertices $w_j, w_k \in V_{\mathrm{inl}}(P_{ab})$ with $1 \leq j < k \leq r$ by (1). This proves (2).

We prove (b) in the case where $P^* \in \mathcal{P}_1$ (the other case can be treated symmetrically). We distinguish three cases.

Case 1. $P^* = P$ for a u, v-path $P \in \mathcal{P}$ in case (i), where $g(u) < g(v)$, $u^* = u$ and $v^* = v$: In this case, $g_1(u) = g(u) < g(v) = g_1(v)$ and condition (b) holds.

Case 2. $P^* = P'$ for the sub-u, w-path P' of a u, v-path $P \in \mathcal{P}$ of case (iii), where $g(u) < g(v)$, $u^* = u$ and $v^* = w$: Since paths P and P_{ab} intersect at w, we see by definition of h that $g_1(u) = g(u) \leq h(w) \leq \max\{g(a), h(w)\} < \max\{g(a), h(w)\} + \delta/n \leq g'(w) = g_1(w)$, indicating that condition (b) holds.

Case 3. $P^* = P''$ for the sub-w, v-path P'' of a u, v-path $P \in \mathcal{P}$ of case (iv), where $g(u) < g(v)$, $u^* = w$ and $v^* = v$: See Figure 5(c) for an illustration of path $P^* = P''$. We show that $g(a) < g(v)$ and $h(w) < g(v)$. Since P and P_{ab} intersect, it holds $g(a) < g(v)$ by Lemma 8. Let $w = w_k$ and $P_{cd} \in \mathcal{P}$ be a c, d-path that attains the value of $h(w_k)$; i.e., $h(w_k) = g(c) < g(d)$ and P_{cd} contains a vertex $w_j \in \{w_1, w_2, \ldots, w_k\}$. Hence P and P_{ab} intersect at w_k and P_{cd} and P_{ab} intersect at vertex w_j of the sub-a, w_k-path of P_{ab}. By Lemma 8, it holds $h(w_k) = g(c) < g(v)$. Now $g(a) < g(v)$ and $h(w_k) = g(c) < g(v)$, where $g(v) - g(a) \geq \delta$ and $g(v) - h(w_k) = g(v) - g(c) \geq \delta$ by the definition of δ. We see that $g_1(w_k) = g'(w_k) = \max\{g(a), h(w_k)\} + \frac{k}{n}\delta < g(v) = g_1(v)$. This proves that condition (b) holds.

Observe that for each $i = 1, 2$, $g_i(u) \neq g_i(v)$ for all vertices $u, v \in V_i$. By conditions (a) and (b), each instance $I_i = (\gamma_i, \mathcal{P}_i, C_i, g_i)$, $i = 1, 2$ satisfies the condition of the lemma. Since $|\mathcal{P}_i| < |\mathcal{P}|$ for each $i = 1, 2$, we see by the induction hypothesis that function $g_i : V(C_i) \to \mathbb{R}$ can be extended to a function $g_i : V_i \to \mathbb{R}$ bipolar to the graph (V_i, E_i) such that $g_i(u) < g_i(v)$ for any directed edge (u, v) in the g_i-proper orientation $\overline{\gamma}_i$ of γ_i, where $\overline{\gamma}_i$ is the restriction of $\overline{\gamma}$ onto

(V_i, E_i). An extension of function $g : V(C) \to \mathbb{R}$ to a function $g : V \to \mathbb{R}$ is obtained by combining the extensions of g_1 and g_2 into the inner vertices of γ_1 and γ_2. We easily see that the resulting extension is a function bipolar to the entire plane graph γ such that $g(u) < g(v)$ for any directed edge (u, v) in the g-proper orientation $\overline{\gamma}$ of γ.

This completes the proof of the lemma. $\square$

Lemma 9 implies Lemma 6 as follows. For an (s,t)-orientation $\widetilde{\sigma(\mu)}$ of the split mesh graph $\sigma(\mu)$, there is an st-numbering (i.e., an (s,t)-bipolar vertex-weight function) g to the split graph $\sigma(\mu)$. Now the orientation $\widetilde{\mu}$ of μ induced by $\widetilde{\sigma(\mu)}$ is g-proper. Hence by Lemma 9, orientation $\widetilde{\mu}$ is acyclic. Obviously orientation $\widetilde{\mu}$ still has the same source s and sink t, and it is an (s,t)-orientation of μ.

This proves Lemma 6.

Appendix 6: Proof of Lemma 7

Lemma 7. *For a pseudo-simple connected plane graph $\gamma = (G = (V, E), F)$ such that the boundary forms a cycle C^o and a subset $X \subseteq V \setminus V(C^o)$, let $\{X_i \subseteq X \mid i = 1, 2, \ldots, p\}$ denote the set of components in $G[X]$ and $Y_i \subseteq N_G(X_i)$, $i = 1, 2, \ldots, p$ be subsets of V, where possibly $Y_i \cap Y_j \neq \emptyset$ for some $i \neq j$.*

Then γ contains no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$ if and only if the sun augmentation $\gamma^* = (V^*, E^*, F^*)$ of γ contains a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ that covers $\{Y_1, Y_2, \ldots, Y_p\}$.

Moreover the following can be computed in linear time:

(i) *Testing whether γ contains an (X_i, Y_i)-confiner for some $i = 1, 2, \ldots, p$; and*
(ii) *Finding a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ that covers $\{Y_1, Y_2, \ldots, Y_p\}$ in γ^* when γ contains no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$.*

We prove Lemma 7 after showing some lemma. We observe that the sun augmentation $\gamma^* = (G^* = (V^*, E^*), F^*)$ of a pseudo-simple connected plane graph γ is a pseudo-simple biconnected plane graph such that

(i) For two non-core faces f and f' sharing a core vertex u, either f and f' share a non-core vertex and an edge or a non-core face f'' contains u and the non-core vertices in f and f';
(ii) No new edge in $E^* \setminus E$ joins two original vertices in V, and $G^*[X] = G[X]$ for any subset $X \subseteq V$;
(iii) After removing the original edges in E, the resulting graph $\gamma^* - E$ remains connected;
(iv) γ^* is simple when γ is simple; and
(v) $|V^*| \leq |V| + 2|E|$, and the sun augmentation γ^* can be computed in linear time.

We here prove the next lemma on properties of coating.

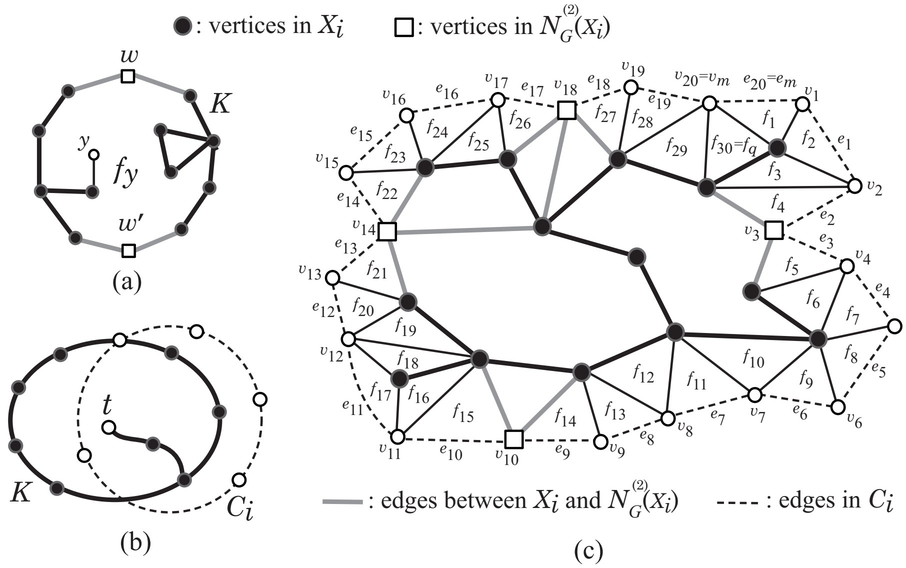

Fig. 6. (a) A fictitious configuration where the boundary of face f_y contains at least two vertices $w, w' \in N_G^{(2)}(X_i) \subseteq V(B_i)$; (b) A fictitious configuration where an (X_i, T)-confiner K intersects a cycle C_i such that $t \in V(C_i)$, $V(C_i) \cap X = \emptyset$ and $V_{\rm enc}(C_i; \gamma^*) \supseteq X_i$; (c) Plane subgraph $\eta_i^{(2)} = (W_i^{(2)} = X_i \cup N_G^{(2)}(X_i), E_i^{(2)}, F_i^{(2)})$ and cycle $C_i = (v_1, v_2, \ldots, v_m)$.

Lemma 10. *For a pseudo-simple plane graph $\gamma = (G = (V, E), F)$ such that the boundary forms a cycle C^o and a subset $X \subseteq V \setminus V(C^o)$, let $\{X_i \subseteq X \mid i = 1, 2, \ldots, p\}$ denote the set of components in $G[X]$, $E[X_i]$ denote the set of edges in the component $G[X_i]$, E_i^+ denote the set of edges in E between X_i and $N_G(X_i)$, $\eta_i = (X_i \cup N_G(X_i), E[X_i] \cup E_i^+, F_i)$, $i = 1, 2, \ldots, p$ denote the plane subgraph of γ induced by the vertices in $X_i \cup N_G(X_i)$ and the edges in $E[X_i] \cup E_i^+$, and denote by B_i the outer boundary of η_i.*

(i) *Let y be a vertex in $N_G(X_i) \setminus V(B_i)$ for some i. Then there is a $(X_i, \{y\})$-confiner;*
(ii) *For a subset $T \subseteq N_G(X_i)$ for some $i \in \{1, 2, \ldots, p\}$, assume that γ has an (X_i, T)-confiner. Then no plane augmentation γ^* of γ admits a coating $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of $\{X_1, X_2, \ldots, X_p\}$ such that $T \subseteq V(C_i)$;*
(iii) *The sun augmentation $\gamma^* = (G^* = (V^*, E^*), F^*)$ of γ contains a coating $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of $\{X_1, X_2, \ldots, X_p\}$ such that $N_G(X_i) \cap V(B_i) \subseteq V(C_i)$, $i = 1, 2, \ldots, p$; and*
(iv) *A coating $\mathcal{C}$ of the sun augmentation γ^* in* (iii) *can be computed in linear time.*

Proof. Let $N_G^{(2)}(X)$ denote the set of neighbors $u \in N_G(X)$ incident to more than one vertex in X; i.e., $N_G^{(2)}(X) = \{u \in N_G(X) \mid |N_G(u) \cap X| \geq 2\}$. For each

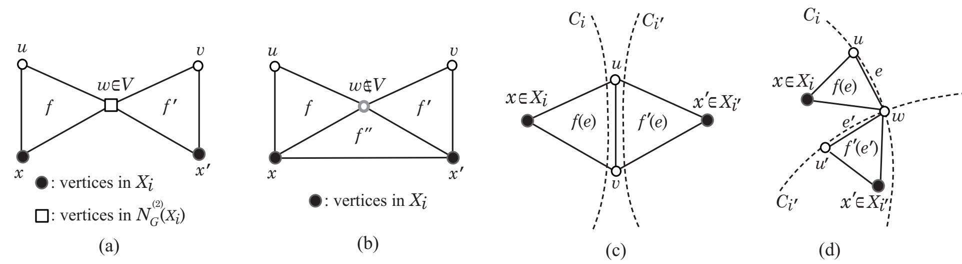

Fig. 7. (a) Two faces $f, f' \in F_i^*$ with $E(f) \cap E(f') = \emptyset$ and $V(f) \cap V(f') = \{w\} \subseteq V$; (b) Two faces $f, f' \in F_i^*$ with $E(f) \cap E(f') = \emptyset$ and $V(f) \cap V(f') = \{w\} \subseteq V^* \setminus V$; (c) A fictitious configuration where C_i and $C_{i'}$ share an edge $e = uv$; (d) A fictitious configuration where C_i and $C_{i'}$ intersect at a vertex w.

$i = 1, 2, \ldots, p$, let $E[X_i]$ denote the set of edges in the component $G[X_i]$ and $E_i^{(2)}$ denote the set of edges in E_i^+ between X_i and $N_G^{(2)}(X_i)$.

(i) Note that $E(G[X_i])$ is the set of edges between two vertices in X_i. Consider the graph $\eta_i^\dagger = \eta_i - (E_i^+ \setminus E(B_i))$ obtained from η_i by removing all inner edges in E_i^+, where $\eta_i^\dagger$ and η_i have the same boundary B_i. Observe that any vertex in $V(B_i) \cap N_G(X_i)$ belongs to $N_G^{(2)}(X_i)$. There is an inner face f_y of the plane graph $\eta_i^\dagger$ such that the interior of f_y contains vertex $y \in N_G(X_i) \setminus V(B_i)$. The boundary of f_y contains at most one vertex in $N_G^{(2)}(X_i)$ ($\subseteq V(B_i)$), since otherwise $G[X_i]$ cannot be connected, as illustrated in Figure 6(a). The boundary of f_y may not be a cycle, but it contains a cycle K that encloses y. We see that y has a neighbor $y' \in X_i$ which is connected to a vertex in K in $G[X_i]$. Since $|K \setminus X_i| \leq 1$, we see that K is an $(X_i, \{y\})$-confiner.

(ii) Let K be an (X_i, T)-confiner that encloses a vertex $t \in T$. To derive a contradiction, assume that there is a cycle C_i in some plane augmentation γ^* of γ such that $t \in V(C_i)$, $V(C_i) \cap X = \emptyset$ and $V_{\text{enc}}(C_i; \gamma^*) \supseteq X_i$. This implies that two cycles K and C_i cannot share two or more vertices in the plane, as illustrated in Figure 6(b). Note that C_i contains vertex $t \in V_{\text{enc}}(K; \gamma^*)$. We see that cycle C_i cannot have X_i as part of its interior without sharing two or more vertices with K, a contradiction. Therefore no plane augmentation γ^* of γ admits a coating $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of $\{X_1, X_2, \ldots, X_p\}$ such that $T \subseteq V(C_i)$.

(iii) We introduce some notations. For each $i = 1, 2, \ldots, p$, let $\eta_i^{(2)} = (X_i \cup N_G^{(2)}(X_i), E[X_i] \cup E_i^{(2)}, F_i^{(2)})$ denote the plane subgraph of γ induced by the vertices in $X_i \cup N_G^{(2)}(X_i)$ and the edges in $X_i \cup N_G^{(2)}(X_i)$, and denote by $B_i^{(2)}$ the outer boundary of $\eta_i^{(2)}$. Figure 6(c) illustrates plane subgraph $\eta_i^{(2)}$.

Let $\gamma^* = (G^* = (V^*, E^*), F^*)$ denote the sun augmentation of γ. For an inner face f in γ^*, let $V(f)$ and $E(f)$ denote the sets of vertices and edges of the facial cycle C_f of f, respectively.

For each $i = 1, 2, \ldots, p$, let F_i^* denote the set of faces f in γ^* such that $V(f)$ contains a vertex in X_i and $E(f)$ contains an edge outside the boundary $B_i^{(2)}$. Each face $f \in F_i^*$ is a non-core face with $|V(f) \cap X_i| = 1, 2$. Note that no edge in $E^* \setminus E$ joins two vertices in V in γ^*. Hence no face $f \in F_i^*$ contains any vertex in X_j with $i \neq j$; i.e., $V(f) \cap X_j = \emptyset$, since otherwise X_i and X_j would belong to the same component of $G[X]$. Let $\psi = (f_1, f_2, \ldots, f_q)$, $q = |F_i^*|$ denote the circular sequence of the faces in F_i^* in the order that they appear along $B_i^{(2)}$, where f_j and f_{j+1}, $j = 1, 2, \ldots, q$ share a vertex in $X_i \cup N_G^{(2)}(X_i)$. Let $v_1, v_2, \ldots, v_m$ denote the sequence of vertices in $\cup_{j=1,2,\ldots,q} V(f_j) \setminus X_i$ in γ^* in the order that they appear in the sequence $f_1, f_2, \ldots, f_q$, where $\{v_1, v_2, \ldots, v_m\} \cap X = \emptyset$ since $V(f) \cap X = \emptyset$ for all faces $f \in F_i^*$. Since each non-core face in F_i^* is a triangle, γ^* contains an edge e_j joining two vertices v_j and v_{j+1} (or an edge e_m joining v_m and v_1). Let $C_i = (v_1, v_2, \ldots, v_m)$ denote the subgraph that consists of vertices $v_1, v_2, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_m$. See Figure 6(c) for an illustration of cycle C_i.

From $\{v_1, v_2, \ldots, v_m\} \cap X$, $V(C_i) \cap X = \emptyset$ holds. Note that $G[X_i]$ and C_i are both connected graphs, where $V(C_i)$ contains a vertex not enclosed by the boundary $B_i^{(2)}$. Hence $V(C_i) \cap X = \emptyset$ implies that $V_{\rm enc}(C_i; \gamma) \supseteq X_i$.

Note that $V(B_i^{(2)}) \subseteq V(B_i)$ holds and the boundary $B_i^{(2)}$ contains all vertices in $X_i \cap V(B_i)$ and the neighbors in $N_G^{(2)}(X_i) \cap V(B_i)$. Each neighbor $v \in N_G(X_i) \cap V(B_i) \setminus N_G^{(2)}(X_i)$ is adjacent to a vertex $x \in X_i \cap V(B_i^{(2)})$. This implies that $N_G(X_i) \cap V(B_i) \subseteq V(C_i)$.

We show that C_i is a simple cycle. Consider two faces $f, f' \in F_i^*$ that share a vertex $w \in V(C_i)$. When $E(f) \cap E(f') \neq \emptyset$, f and f' are indexed consecutively as f_j and f_{j+1} in the sequence ψ. Assume that $E(f) \cap E(f') = \emptyset$ and $V(f) \cap V(f') = \{w\}$. Note that each of f and f' contains a vertex in X_i, say $x \in V(f) \cap X_i$ and $x' \in V(f') \cap X_i$. First consider the case where $w \in V$. Since each of f and f' contains a vertex in X_i, we see that $w \in N_G^{(2)}(X_i)$. In this case, $E(B_i^{(2)})$ contains exactly two edges incident to w, which must be $wx \in E(f)$ and $wx' \in E(f')$, as shown in Figure 7(a). This implies that no other face $f'' \in F_i^* \setminus \{f, f'\}$ can contain such a vertex w by the definition of F_i^*.

Next consider the case where w is a core vertex in $V^* \setminus V$. By construction of the sun augmentation γ^*, each of $V(f)$ and $V(f')$ contains exactly one vertex in V, which are x and x', respectively, and the set $\{x, w, x'\}$ forms a non-core face f'', where $f'' \in F_i^*$ holds, as shown in Figure 7(b). Hence faces f, f'' and f' are indexed consecutively as f_j, f_{j+1} and f_{j+2} in the sequence ψ. From these two cases, we see that C_i is a simple cycle. Also each inner face f with $V(f) \cap V(C_i) \neq \emptyset$ in the interior $\gamma^*[C_i]_{\rm enc}$ belongs to F_i^*, and satisfies $V(f) \cap X = V(f) \cap X_i \neq \emptyset$. In particular, each edge $e \in E(C_i)$ is contained in a non-core face $f(e) \in F_i^*$.

Finally we prove that the cycles C_i and $C_{i'}$ with $1 \leq i < i' \leq p$ are edge-disjoint and do not intersect. Assume that C_i and $C_{i'}$ share an edge $e = uv$. Note that e is contained in a non-core face $f(e) \in F_i^*$ and in a non-core face $f'(e) \in F_{i'}^*$, where $f(e)$ and $f'(e)$ contain a vertex $x \in X_i$ and a vertex $x' \in X_{i'}$, respectively, and $V(f(e)) = \{u, v, x\}$ and $V(f'(e)) = \{u, v, x'\}$, as shown in

Figure 7(c). However, when two non-core faces share an edge $e = uv$, one of the faces cannot have a vertex in $V \setminus \{u, v\}$ in the sun augmentation γ^*. This implies that C_i and $C_{i'}$ are edge-disjoint.

Next assume that C_i and $C_{i'}$ intersect at a vertex $w \in V(C_i) \cap V(C_{i'})$. Let $e = uw \in E(C_i)$ and $e' = u'w \in E(C_{i'})$ be edges incident to w, where we choose e' in the interior $\gamma^*[C_i]_{\rm enc}$. Then e and e' are contained in faces $f(e) \in F_i^*$ and $f'(e) \in F_{i'}^*$, respectively, as shown in Figure 7(d). Recall that a vertex $x \in X_i$ and a vertex $x' \in X_{i'}$ are contained in $f(e)$ and $f'(e')$, respectively. However, we have observed that any inner face in $\gamma^*[C_i]_{\rm enc}$ containing a vertex in $V(C_i)$ contains a vertex $x \in X_i$, contradicting that such a face $f'(e')$ contains a vertex $\tilde{x} \in X_i$ other than $x' \in X_i$, since $\tilde{x}$ and x' must have been in the same component of $G[X]$. Hence C_i and $C_{i'}$ do not intersect.

Therefore $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ is a coating of $\{X_1, X_2, \ldots, X_p\}$ such that $N_G(X_i) \cap V(B_i) \subseteq V(C_i)$, $i = 1, 2, \ldots, p$.

(iv) We show that, for each $i = 1, 2, \ldots, p$, the cycle $C_i \in \mathcal{C}$ can be computed in $O(|V(C_i)|)$ time after some linear-time preprocessing.

As observed, the sun augmentation γ^* of γ can be constructed in linear time. For each vertex $v \in V^*$, let $E^*(v)$ denote the set of edges in E^* incident to v in γ^*, where we assume that the edges in $E^*(v)$ are stored in a linked-list in the clockwise order around v.

We compute the set $E[X]$ of edges in E that join two vertices in X, the components $X_1, X_2, \ldots, X_p$ in the induced graph $G[X] = (V, E[X])$ and the edge set $E[X_i] \cup E_i^+$, $i = 1, 2, \ldots, p$ in linear time. For each edge $e \in E$, we also compute $\mathrm{id}(e)$ as the index i of the component X_i in $G[X]$ such that $e \in E[X_i] \cup E_i^+$.

For each vertex $v \in N_G(X)$, let $E_i^+(v)$, $i = 1, 2, \ldots, p$ denote the set of edges in E_i^+ incident to v. We show how to compute each non-empty set $E_i^+(v)$ so that the edges in $E_i^+(v)$ are stored in a linked-list in the clockwise order around v. Prepare a 1-dimensional array A with entries $A[i] = (a, b)$, $i = 1, 2, \ldots, p$ such that a stores a vertex (or null) and b stores an edge (or null), which is initialized as $A[i]$:=(null,null). We choose each vertex $v \in N_G(X)$ in some order, and traverse the edges in the linked-list for $E^*(v)$. When we encounter an edge $e \in E^*(v) \cap E$ with $\mathrm{id}(e) = j$ in the list, update the current entry $A[j] = (a, b)$ as follows. If $a \neq v$ then set $A[j] := (v, e)$; and if $a = v$ then $b \in E_j^+(v)$ holds and we set $A[j] := (v, e)$ and let the edge b be linked to the current edge e in the linked-list for $E_j^+(v)$. After this, the linked-list for each non-empty set $E_i^+(v)$, $v \in N_G(X)$, $i = 1, 2, \ldots, p$ is computed in linear time since the number of edges scanned in this procedure is a constant times for each edge. Also the set $N_G^{(2)}(X_i)$ is obtained as the set of vertices $v \in N_G(X)$ with $|E_i^+(v)| \geq 2$.

Finally we find some edge e_i^* incident to a vertex $x \in X_i \cap V(B_i^{(2)})$ not from the interior of the graph $\eta_i^{(2)} = (V_i^{(2)}, E_i^{(2)}, F_i^{(2)})$ for each $i = 1, 2, \ldots, p$. We call such an edge the *first edge* of i. Let $E_{(2)}^+$ denote the set of edges between X_i and $N_G^{(2)}(X_i)$ for all $i = 1, 2, \ldots, p$, and remove the edges in $E_{(2)}^+$ from γ^* to obtain a graph $\gamma^* - E_{(2)}^+$, which remains connected by the construction of γ^*. Then compute a spanning tree T^* of $\gamma^* - E^+_{(2)}$ as follows. First construct a spanning tree T_i of component $G[X_i]$ for each $i = 1, 2, \ldots, p$, and choose a spanning tree T^* in $\gamma^* - E^+_{(2)}$ such that T^* contains all spanning trees T_i, $i = 1, 2, \ldots, p$. Regard T^* as a digraph rooted at some vertex $s \in V(C^o)$ wherein each edge uv in T^* is directed from the parent u to the child v. Note that T^* contains exactly one incoming edge $e = (u, v)$ for each tree T_i such that v belongs to X_i and u is the parent of v in T^*. Note that $v \in V(B_i^{(2)})$ and $u \notin N_G^{(2)}(X_i)$ since the edges in $E^+_{(2)}$ are removed. We set this edge e to be the first edge e_i^* of i.

In the following, i is an index $i \in \{1, 2, \ldots, p\}$. We are ready to generate the sequence ψ of faces in F_i^*. In the following, we find the edges in $E(f)$ of these faces incident to X_i, from these edges we can find the sequence $\psi = (f_1, f_2, \ldots, f_q)$. See Figure 6(c) for an illustration of the sequence $f_1, f_2, \ldots, f_q$. For the first edge $e_i^* = x_1 v$ with $x_1 \in X_i$, we initialize $e := x_1 v$ and $x := x_1$. Then we repeat the following:

> Trace(e, x): traverse edges in the linked-list of $E^*(x)$ starting from the edge e until we encounter an edge $e' = xu \in E^*(x)$ such that $u \in X_i$ or $u \in N_G^{(2)}(X_i)$ for the first time.

In the former, we execute Trace$(e' = xu, u)$; In the latter, we traverse the linked-list for $E_i^+(u)$ to find the next edge $e' = ux'$ with $x' \in X_i$ in $O(1)$ time and then execute Trace$(e' = ux', x')$.

We see that the above procedure can correctly find the edges in $E(f)$ of the faces in the sequence $\psi = (f_1, f_2, \ldots, f_q)$ in $O(q)$ time. Based on sequence ψ, we can construct the cycle C_i in $O(|V(C_i)|) = O(q)$ time. The total time for computing all cycles C_i, $i = 1, 2, \ldots, p$ is linear to the size of γ. □

Now we prove Lemma 7 by using Lemma 10.

Given a pseudo-simple connected plane graph $\gamma = (G = (V, E), F)$ and a subset $X \subseteq V \setminus V(C^o)$ in Lemma 7, we construct the sun augmentation γ^* of γ and a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ such that $N_G(X_i) \cap V(B_i) \subseteq V(C_i)$ in Lemma 10(ii). We distinguish two cases:
(a) $V(C_i) \supseteq Y_i$ for each $i = 1, 2, \ldots, p$; and
(b) there is a vertex $y \in Y_i \setminus V(C_i)$ for some $i \in \{1, 2, \ldots, p\}$.

In (a), the obtained coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ covers $\{Y_1, Y_2, \ldots, Y_p\}$, and there is no (X_i, Y_i)-confiner for any i by Lemma 10(ii). In (b), $y \in Y_i \setminus V(C_i) \subseteq N_G(X_i) \setminus V(C_i) \subseteq N_G(X_i) \setminus (N_G(X_i) \cap V(B_i)) = N_G(X_i) \setminus V(B_i)$. Hence by Lemma 10(i), γ has an $(X_i, \{y\})$-confiner, which is also an (X_i, Y_i)-confiner by the definition of confiners.

The arguments in (a) and (b) imply that γ contains no (X_i, Y_i)-confiner for any $i = 1, 2, \ldots, p$ if and only if the sun augmentation γ^* of γ contains a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ that covers $\{Y_1, Y_2, \ldots, Y_p\}$. Computing the sun augmentation γ^* and a coating $\mathcal{C}$ of $\{X_1, X_2, \ldots, X_p\}$ in Lemma 10(ii) can be done in linear time by Lemma 10(iv).

This proves Lemma 7.

Appendix 7: Algorithm ORIENT for Proving Lemma 3

An entire algorithm for proving Lemma 3 is described as follows.

Algorithm ORIENT
Input: A pseudo-simple biconnected plane graph $\gamma = (V, E, F)$, an 1-independent set $\mathcal{P}$ of paths of length at least 2, a partition $\mathcal{P}_i$, $i = 1, 2, \ldots, p$ of $\mathcal{P}$ and an inclusive set $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ of edge-disjoint cycles satisfying the condition of Lemma 3 and two outer vertices s and t of γ.
Output An (s,t)-orientation $\widetilde{\gamma}$ of γ compatible to $\mathcal{P}$.

Compute the inclusion-forest $\mathcal{I} = (\mathcal{C}, \mathcal{E})$ of $\mathcal{C}$ and the set $\mathcal{C}_{\rm rt}$ of root cycles in $\mathcal{C}$, letting the indexing of $C_1, C_2, \ldots, C_p$ satisfy $i < j$ when C_i is the parent cycle of C_j, the plane subgraphs γ_i and the sets $F(\mu_i)$ of non-trivial inner faces, $i = 1, 2, \ldots, p$;
Compute an (s,t)-orientation $\widetilde{\gamma}_0$ of γ_0 using Lemma 1;
for each $i = 1, 2, \ldots, p$ **do**
    /* Now orientation $\widetilde{\gamma}_k$ of the parent cycle C_k of non-root cycle C_i or $\widetilde{\gamma}_k = \widetilde{\gamma}_0$ of root cycle C_i is (s_i, t_i)-bipolar to C_i for some $s_i, t_i \in V(C_i)$ by Lemma 4; Execute Step (a) */
    Compute an (s_i, t_i)-orientation $\widetilde{\mu}_i$ of mesh graph $\eta_i = (C_i, \mathcal{P}_i)$ using Lemmas 1 and 6;
    for each inner face $f \in F(\mu_i)$ **do**
        /* Now orientation $\widetilde{\mu}_i$ is (s_f, t_f)-bipolar to the facial cycle C_f of f for some vertices $s_f, t_f \in V(C_f)$ by Lemma 4; Execute Step (b) */
        Compute an (s_f, t_f)-orientation $\widetilde{\gamma_i[C_f]_{\rm enc}}$ of the interior subgraph $\gamma_i[C_f]_{\rm enc}$ induced from γ_i by C_f using Lemma 1.
    endfor
endfor;
Output the orientation $\widetilde{\gamma}$ of γ by combining bipolar orientations $\widetilde{\gamma}_i$ of γ_i, $i = 1, 2, \ldots, p$.

Appendix 8: Proof of Theorem 2

Theorem 2 *Let Γ be a non-plane embedding of a graph G such that each crossing-set is outer, and let $n = |V(G)|$ and $n_{\rm c}$ denote the number of crossings in Γ.*

Then for any pair of outer vertices s and t in Γ, there is an (s,t)-upward drawing of Γ, and an upward poly-line drawing of Γ with $O(n + n_{\rm c})$ bends can be computed in $O(n + n_{\rm c})$ time and space.

Proof. We show that Theorem 1 can be applied to the planarization of an instance of Theorem 2. Let Γ be a non-plane embedding of a graph G. Assume that each crossing-set E_i^* is outer.

We construct the plane graph by planarizing Γ, i.e., replacing each edge crossing as a graph vertex. If the resulting planarization is not connected, then we add a least number of new edges to make it connected while keeping planarity.

Let $\gamma = (G = (V, E), F)$ denote the resulting connected plane graph, V_{inl} denote the set of crossings in $\mathit{\Gamma}$, and V_{end} denote the set of end-vertices of crossing edges in E^*. Clearly $V_{\text{inl}} \cap V_{\text{end}} = \emptyset$. In γ, each crossing edge $e = uv \in E^*$ in $\mathit{\Gamma}$ is replaced with a path $P_e = (u_1, u_2, \ldots, u_k)$ in γ such that $u_1 = u$, $u_k = v$, and $u_2, u_3, \ldots, u_{k-1} \in V_{\text{inl}}$.

Define a path set $\mathcal{P} = \{P_e \mid e \in E^*\}$, where we see that $\mathcal{P}$ is 1-independent since $\mathit{\Gamma}$ satisfies the standard non-degeneracy conditions. Let $E_1^*, E_2^*, \ldots, E_p^*$ be the partition of E^* into crossing-sets. For each crossing-set E_i^*, let $\mathcal{P}_i$ denote the set of paths P_e with $e \in E_i^*$, where $V_{\text{inl}}(\mathcal{P}_i)$ is a component in $G[V_{\text{inl}}]$. Since each crossing-set E_i^* is outer, we see that γ has no $(V_{\text{inl}}, V_{\text{end}})$-separator K.

By Theorem 1, there exists an (s, t)-upward poly-line drawing of $\mathit{\Gamma}$ for any outer vertices s and t, and such a drawing can be constructed in linear time, where the total number of bends is at most $|E(G)| + |V_{\text{inl}}| = O(n + n_{\text{c}})$.

This proves Theorem 2. $\square$